
Discovering Point Sources in Unknown Environments

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Abstract: We consider the inverse problem of discovering the location of a source from very sparse point measurements in a bounded domain that contains impenetrable (and possibly unknown) obstacles. We present an adaptive algorithm for determining the measurement locations, and ultimately, the source locations. Specifically, we investigate source discovery for the Laplace operator, though the approach can be applied to more general linear partial differential operators. We propose a strategy for the case when the obstacles are unknown and the environment has to be mapped out using a range sensor concurrently with source discovery.

1 Introduction

This work is motivated by robotic applications in which a robot, sent into an unknown environment, is supposed to discover the location of a signal source and place it under its line-of-sight in an efficient manner. The unknown environment contains non-penetrable solid obstacles and should be avoided along the robot's path. The robot can gather measurements from two different sensors: a range sensor that gives distance from the robot to the surrounding obstacles, and a sensor that measures the signal strength that is being emitted from the yet-to-be-located source. We will refer to the information from the range sensor as the visibility of the robot and to the information from the signal strength sensor as the signal. While measurements can be taken anywhere, we are interested in having the robot take very few measurements with its sensors. Our goal is to design an algorithm that determines how the robot should navigate through the environment and where along its path it should take measurements.

This problem is classified as an inverse problem, however, it differs greatly from many typical inverse problems, which assume simple domains and dense arrays of sensors at fixed locations. In [6], Ling et al explore such a situation in which they recover the exact locations of multiple sources in a Poisson

equation, given an initial guess for the locations and Dirichlet data collected on the boundary of the domain. To accomplish this, they use the special form of the free space Green's function for the Laplacian. For inverse problems related to the heat equations with sources, see [1, 2].

When the obstacles are unknown, the environment needs to be mapped out as the robot moves so that attempts to take measurements inside the obstacles are avoided and the robot's path does not intersect obstacles. The previous work of [3] on mapping of obstacles in unknown domains using visibility is useful in this regard. In it the authors propose an algorithm to construct a high-order accurate representation of the portions of the solid surfaces that are visible from a vantage point and to generate the corresponding occlusion volume. Also they propose an algorithm to construct a piecewise linear path so that any point on the solid surfaces is seen by at least one vertex of the path and an accurate representation of the solids is constructed from the point clouds that are collected at the vertices of the path.

The [3] algorithm was motivated by the work of LaValle, Tovar et al. [7, 5, 8]. In [8], a single robot (observer) must be able to navigate through an unknown simply or multiply connected piecewise-analytic planar environment. The robot is equipped with a sensor that detects discontinuities in depth information (called gaps) and their topological changes in time. As a result of exploration, the region is characterized by the number of gaps and their relative positions. No distance or angular information is accumulated. In contrast, the [3] algorithm maps the obstacles in Cartesian coordinates as the observer proceeds through the environment, and utilizes the recovered information for further path planning. At the termination of the path all the obstacles' boundaries are reconstructed. Thus a complete representation of the environment is obtained.

From our experience, the discovery of signal sources may be very insensitive to the presence of (or parts of) obstacles in sub-regions of the domain, possibly due to the decay of the signal strength. This suggests that the visibility path algorithm in [3] should be modified adaptively according to the previously obtained measurements of the signal and estimations of the signal source location.

To illustrate the main ideas in this paper, consider the following problem

$$\begin{aligned} \Delta u(x) &= \delta(x - y) & \text{in } D_\Omega \equiv D \setminus \Omega \\ u &= 0 & \text{on } \Gamma, \end{aligned} \tag{1}$$

where u denotes the signal strength, D denotes a bounded domain, $\Omega \subsetneq D$ denotes the solid obstacles in the domain, $\Gamma = \partial D_\Omega$, and y denotes the (unknown) source location. Furthermore, let $\psi(\cdot; z) : D \mapsto \mathbb{R}$ describe the visibility of the domain from an observing location $z \in D_\Omega$. We require that $\psi(\cdot; z)$ be a signed distance function such that the set $W_z := \{x \in D : \psi(x; z) < 0\}$ corresponds to the region, including the interior of the solid obstacles, that is occluded from the observing location z . This means that the line segment connecting z and any point in W_z must intersect with the obstacles Ω . Such

visibility functions can be computed efficiently using the algorithms described in [10, 9, 4, 3].

A first, rather simple approach, would be to use gradient descent to determine the sample locations via the ordinary differential equation,

$$\frac{dX}{ds} = -\nabla u, \text{ with } X(0) = z_0. \quad (2)$$

However, there are two drawbacks to this method. First, it only works in cases where the Green's function has a specific structure, such as in the case for the Laplace operator. Second, even for the Laplace operator, one can come up with a pathological configuration for the obstacles where the gradient vanishes at points other than the source.

We continue with the method proposed in this paper. At an observing location z_1 , we can measure the signal strength $I_1 = u(z_1)$. We look at the solution to the adjoint problem,

$$\begin{aligned} \Delta v_1 &= \delta(x - z_1) & \text{in } D_\Omega \\ v_1 &= 0 & \text{on } \Gamma. \end{aligned} \quad (3)$$

Now, for $y \neq z_1$ we have

$$\begin{aligned} v_1(y) &= \int_{D_\Omega} \delta(x - y) v_1 dx = \int_{D_\Omega} v_1 \Delta u dx \\ u(z_1) &= \int_{D_\Omega} \delta(x - z_1) u dx = \int_{D_\Omega} u \Delta v_1 dx, \end{aligned} \quad (4)$$

and thus, by Green's identity,

$$v_1(y) = u(z_1) = I_1. \quad (5)$$

Therefore the source must lie on the I_1 level set of v_1 ,

$$y \in \{x \in D_\Omega : v_1(x) = I_1\}. \quad (6)$$

Next, based on the visibility information, we select the next observing location z_2 from the region that is not occluded to z_1 , that is $z_2 \in D \setminus W_{z_1}$. Denote the signal strength at z_2 by $I_2 = u(z_2)$. The function v_2 can be computed and we can narrow down the possible locations of y ,

$$y \in \{v_1 = I_1\} \cap \{v_2 = I_2\}. \quad (7)$$

We can repeat this procedure for more measurements.

In the case when the obstacles Ω are unknown, the visibility functions $\psi(x; z_k)$ provide a convenient over approximation of Ω , since $\Omega \subseteq W_{z_k}$. This can be used in conjunction with the maximum principle for Poisson's equation to estimate the location of y . We will discuss this in greater detail in later sections.

2 Mathematical Formulation

In the most general setting that we will consider, the inverse source problem can be formulated as follows. Let $u(x)$ satisfy,

$$\begin{aligned} Lu &= \alpha\delta(x - y) & \text{in } D_\Omega \equiv D \setminus \Omega \\ Bu &= 0 & \text{on } \Gamma, \end{aligned} \quad (8)$$

where D is a bounded domain, Ω are the (possibly unknown) obstacles, $\Gamma = \partial D_\Omega$, L is a linear partial differential operator, B is an operator specifying the boundary conditions, and $\alpha > 0$. We will assume that at a given point $z \in D_\Omega$ we can sample u and the domain. That is, at a given z we can measure $u(z)$ and its derivatives and the visibility function $\psi(x; z)$. The inverse source location problem is to recover the source location y and the source strength α from a sequence of sample locations z_k .

The main approach that we will use in this problem is to look at the adjoint operator, L^* , with the appropriate boundary conditions B^* :

$$\begin{aligned} L^*v &= F_z & \text{in } D_\Omega \\ B^*v &= 0 & \text{on } \Gamma, \end{aligned} \quad (9)$$

for some distribution F_z with support $\{z\}$. Now, using the properties of the adjoint and assuming that $z \neq y$,

$$(Lu, v) - (u, L^*v) = 0, \quad (10)$$

and hence,

$$\alpha v(y) = F_z[u]. \quad (11)$$

For example, if we use $F = \delta$, we get

$$\alpha v(y) = u(z). \quad (12)$$

Similary, for $F = -\partial_{x_1}\delta$, we get

$$\alpha v(y) = \partial_{x_1}u(z), \quad (13)$$

and so on.

For unknown domains, we will use the methods developed in [3] to find the visibility function and use it determine the sequence of sample locations z_k .

3 Poisson's Equation

In this section we consider the case when $L = \Delta$ in 2 dimensions with Dirichlet boundary conditions:

$$\begin{aligned} \Delta u &= \alpha \delta(x - y) & \text{in } D_\Omega \\ u &= 0 & \text{on } \Gamma. \end{aligned} \quad (14)$$

We note that this operator is self-adjoint and that the following standard maximum principle holds:

Theorem 1. *Let D_Ω be bounded and w satisfy*

$$\begin{aligned} \Delta w &= 0 & \text{in } D_\Omega \\ w &\leq 0 & \text{on } \Gamma, \end{aligned} \quad (15)$$

then

$$w \leq 0 \quad \text{in } D_\Omega. \quad (16)$$

Now, supposed that we have an over-estimate for the obstacles Ω^+ , so that $\Omega \subseteq \Omega^+$ and let v and v^+ satisfy (14) with obstacle sets Ω and Ω^+ respectively. Then, since the fundamental solution for the Laplacian for any domain is non-positive, $v \leq 0$ on $\Gamma^+ = \partial D_{\Omega^+}$. Let $w = v - v^+$, so that w satisfies the conditions of Theorem 1 for D_{Ω^+} . Thus, $w = v - v^+ \leq 0$ in D_{Ω^+} . Furthermore, if we extend v^+ to D_Ω by 0, we have that $v^+ \leq v$ in D_Ω . This fact will be used in the case of unknown obstacles.

3.1 Known Environment

At a sample location z_k , equation (11) gives us

$$\begin{aligned} \alpha v_k(y) &= u(z_k) \\ \alpha w_{k1}(y) &= \partial_{x_1} u(z_k) \\ \alpha w_{k2}(y) &= \partial_{x_2} u(z_k), \end{aligned} \quad (17)$$

where v_k , w_{k1} and w_{k2} satisfy (9) with F equal to $\delta(x - z_k)$, $-\partial_{x_1} \delta(x - z_k)$ and $-\partial_{x_2} \delta(x - z_k)$, respectively. Since v_k is non-zero except at the boundary, we can also form,

$$\begin{aligned} \frac{w_{k1}(y)}{v_k(y)} &= \frac{\partial_{x_1} u(z_k)}{u(z_k)} \\ \frac{w_{k2}(y)}{v_k(y)} &= \frac{\partial_{x_2} u(z_k)}{u(z_k)}. \end{aligned} \quad (18)$$

Thus, y is in the intersection of the $u(z_k)$ level set of v , the $\partial_{x_1} u(z_k)$ level set of w_1 and so on. Note that in the last two equations, α does not appear. As we take more and more measurements z_k , the intersection of all of these sets will be smaller and smaller. Furthermore, for a pair of measurements j and k , we can form,

$$\frac{v_k(y)}{v_l(y)} = \frac{u(z_k)}{u(z_l)}, \quad (19)$$

and so on. Note that these are also independent of α .

3.2 Unknown Environment

At a sample location z_k , we have the visibility function $\psi(x; z_k)$. From this function we can obtain an over-estimate of the obstacles, Ω_k^+ , such that $\Omega \subseteq \Omega_k^+$. After K measurements, we let

$$\Omega^+ = \bigcap_{k=1}^K \Omega_k^+ ,$$

and

$$\begin{aligned} \Delta v_k^+ &= \delta(x - z_k) & \text{in } D_{\Omega^+} \equiv D \setminus \Omega^+ \\ v_k^+ &= 0 & \text{on } \Gamma^+ , \end{aligned} \quad (20)$$

where $\Gamma^+ = \partial D_{\Omega^+}$.

The results from section 3.1 apply and for the v_k defined in that section, we have

$$\alpha v_k(y) = u(z_k) ,$$

but since we don't know Ω , we cannot find v_k . However, the maximum principle gives us that

$$\alpha v_k^+(y) \geq \alpha v_k(y) = u(z_k) .$$

Thus,

$$y \in \bigcap_{k=1}^K \{x | \alpha v_k^+(x) \geq u(z_k)\} .$$

In the case when α is unknown, we would like to find an α independent set which includes y . From section 3.1, for a pair of samples k and l , we have

$$\frac{v_k(y)}{v_l(y)} = \frac{u(z_k)}{u(z_l)} ,$$

and thus,

$$\frac{v_k^+(y)}{v_l(y)} \geq \frac{u(z_k)}{u(z_l)} .$$

Now, let Ω^- be an under-estimate of the obstacles, so that $\Omega^- \subseteq \Omega$. A simple choice for Ω^- is the empty set (no obstacles). Also, let v^- satisfy (20) for Ω^- . By the maximum principle, $v_l \geq v^-$. Thus,

$$\frac{v_k^+(y)}{v^-(y)} \geq \frac{u(z_k)}{u(z_l)} ,$$

which is independent of α as desired.

3.3 Numerical Experiments and the Proposed Algorithm

We model the δ -source as Gaussian in simulations. To determine the location of the source, we build a probability density as follows. For each measurement, we let the probability density, $p_k(x)$, be constant in the possible region (it may be a curve) and have Gaussian drop-off away from this region. After k different measurements, we let the probability density be

$$p(x) = \frac{\left(\prod_{j=1}^k p_j(x)\right)^{\frac{1}{k}}}{\int_{\Omega} \left(\prod_{j=1}^k p_j(x)\right)^{\frac{1}{k}} dx} . \quad (21)$$

Known Strength, Known Environment

For this experiment, we assume that the source has strength $\alpha = 1$ and use Algorithm 1. We sample $u(x)$ at 3 locations, which results in 9 level sets if we use the level sets given by (17). The domain and results are given in Figure 1.

Algorithm 1 Source detection in known environment.

- 1: $u(z)$: solution of equation (14) that can be measured for any z .
 - 2: $k = 1$
 - 3: z_k : vantage point outside the occluding objects
 - 4: compute v_k : solution of Equation (14) and any of the w_{k1}, \dots , that are available
 - 5: compute p as in Equation (21)
 - 6: **while** p is not localized **do**
 - 7: $k = k + 1$
 - 8: choose z_k to be outside of the set $\{x : v_{k-1} > u(z_{k-1})\}$
 - 9: compute v_k : solution of Equation (14) and any of the w_{k1}, \dots , that are available
 - 10: re-compute p as in Equation (21)
 - 11: **end while**
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Alternatively, we can use only v_k along with all pairs (19). The results for 3 measurements (total of 6 level sets after the 3rd measurements) are shown in Figure 2.

Unknown Strength, Known Environment

For this experiment, we assume that the strength α is unknown and we use Algorithm 1. We sample $u(x)$ at 3 locations. Since the strength is unknown, we use equations (19). After locating the source, its strength can be approximated using,

$$\alpha = \frac{u(z_k)}{v_k(y)} . \quad (22)$$

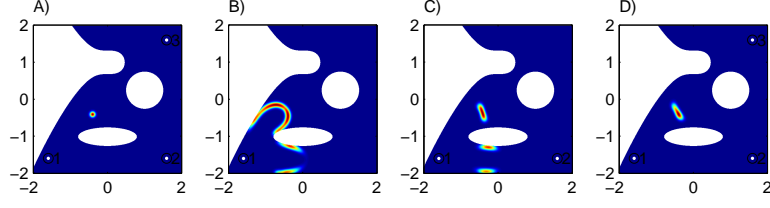


Fig. 1. Location of a source with known strength in a known environment for Poisson's equation. Location based on 3 measurements with v_k , w_{k1} and w_{k2} . Figures: A) The known environment, source and sample locations; B) $p(x)$ after 1 measurement; C) $p(x)$ after 2 measurements; D) $p(x)$ after 3 measurements.

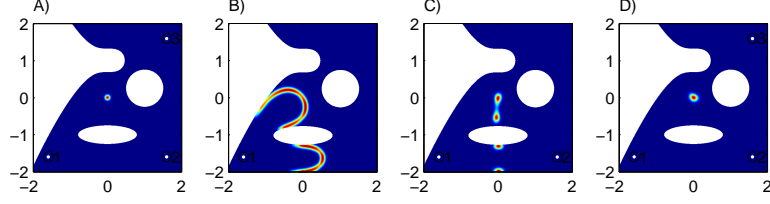


Fig. 2. Location of a source with known strength in a known environment for Poisson's equation. Location based on 3 measurements with v_k and pairwise combinations. Figures: A) The known environment, source and sample locations; B) $p(x)$ after 1 measurement; C) $p(x)$ after 2 measurements; D) $p(x)$ after 3 measurements.

The results are shown in Figure 3. The actual source location is $(0.200, 0.400)$ and its strength is $\alpha = 1.817547$. The probability p after 3 measurements has a maximum at $(0.208, 0.400)$. The strength estimates (22) are 1.861106, 1.837540, 1.767651, and the average is 1.822099.

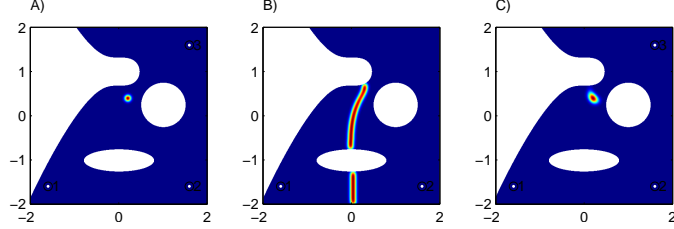


Fig. 3. Location of a source with unknown strength in a known environment for Poisson's equation. Location based on 3 measurements with v_k pairwise combinations. Figures: A) The known environment, source and sample locations; B) $p(x)$ after 2 measurements; C) $p(x)$ after 3 measurements. Actual parameters: source location $(0.200, 0.400)$ with strength 1.817547. Location results: $(0.208, 0.400)$. Strength estimates, 1) 1.861106, 2) 1.837540, 3) 1.767651. Averaged $\alpha = 1.822099$.

Known Strength, Unknown Environment

In order to detect a source of given strength in an unknown environment the observer utilizes visibility information to proceed through the environment and to narrow down the region of possible source locations. In particular, let $\psi(\cdot, z_k)$ be the visibility level set function corresponding to the vantage point z_k . Then, $\{\psi(\cdot; z_k) \geq 0\}$ is the visible portion of D and $\{\psi(\cdot; z_k) < 0\}$ is invisible. Let Ψ_k denote joint visibility along the path. In the level set framework, $\Psi_k = \max_{j=1, \dots, k} \{\psi(\cdot, z_j)\}$. The remaining occluded set $\{x \in D : \Psi_k(x) < 0\}$ may be used as an over-approximation of the obstacles Ω_k^+ . Note, as the observer explores more of D , Ω_k^+ becomes a better approximation of the obstacles Ω . Thus, the $u(z_k)$ level set of v_k^+ would pass closer to the source location y .

Furthermore, let $\{q_j\}_{j=1}^M$ be the filtered out visible points on the boundaries of the obstacles, collected along the observer's path $\{z_1, \dots, z_k\}$. To construct an under-approximation of the obstacles Ω_k^- , we take the union of all ϵ -balls $B_\epsilon(q_j)$, touching the visible points, such that $B_\epsilon(q_j) \subseteq \{\Psi_k \leq 0\}$. Dirichlet boundary conditions are enforced at the union of the boundaries of $B_\epsilon(q_j)$. Then, using the maximum principle, Theorem 1, we may “sandwich” the location of the source $y \in \{x \in D : v_k^-(x) \leq u(z_k) \leq v_k^+(x)\}$.

As the observer proceeds through the environment, the next step along the path is chosen in the currently visible region, so that the resulting path is continuous and consists of a finite number of steps as in [3]. We adopt the algorithm in [3] to navigate through the unknown environment, in which the observer approaches one of the visible horizons, or edges on the piecewise-smooth visibility map, defined in [3]. The next step z_{k+1} is obtained by overshooting the horizon location by the amount inversely proportional to the curvature of the obstacle's boundary near the horizon.

To optimize the search, we choose a direction so that $z_{k+1} \in \{W_k \geq 0\}$, if the continuity of the path can be preserved. Otherwise, we simply proceed towards the nearest horizon, as was proposed in [3]. The algorithm terminates when the entire set $\{W_k \geq 0\}$ is visible from the vertices along the path. Note, that in most cases the proposed location algorithm would terminate prior to full mapping of the environment. However, if the environment has been fully explored before the source was located, the algorithm for the known environment may be applied.

Finally, we would like to remark that according to [3], the environment is considered to be completely explored when all the horizons detected along the path have been cleared. The observer may return to an earlier vantage point along the path to see other horizons. Therefore, the resulting path may branch out. The complete search strategy is described in Algorithm 2 below.

Note that v_k^+ and v_k^- are the level set functions. Then, for a given k , the set $\{x \in D : v_k^-(x) \leq u(z_k) \leq v_k^+(x)\}$, containing the source, is defined by another level set function W_k , positive in the interior of the set and negative outside. Numerically, W_k is defined in step 10 of the above algorithm. As the

Algorithm 2 Source detection in unknown environment. Source strength is known.

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1:  $u(z)$ : solution of equation (14) that can be measured for any  $z$ .
2:  $k = 1$ 
3:  $z_k$ : vantage point outside the occluding objects
4:  $\psi(\cdot, z_k)$ : visibility with respect to  $z_k$ 
5:  $\Psi_k$ : joint visibility along the path
6: construct  $\Omega_k^+$ : over-estimate of  $\Omega$  with respect to  $z_k$ 
7: construct  $\Omega_k^-$ : under-estimate of  $\Omega$  with respect to  $z_k$ 
8: compute  $v_k^+$ : solution of Equation (14) with obstacles  $\Omega^+$ 
9: compute  $v_k^-$ : solution of Equation (14) with obstacles  $\Omega^-$ 
10: set  $W_k(x) := -(v_k^+(x) - u(z_k))(v_k^-(x) - u(z_k)), x \in D$ 
11: while  $\{W_k \geq 0\} \not\subseteq \{\Psi_k \geq 0\}$  do
12:    $k = k+1$ 
13:   set  $\Psi_k = \max\{\psi(\cdot, z_k), \Psi_{k-1}\}$ 
14:   construct  $\Omega_k^+, \Omega_k^-$ 
15:   compute  $v_k^+, v_k^-$ 
16:   set  $W_k(x) := \min\{-(v_k^+(x) - u(z_k))(v_k^-(x) - u(z_k)), W_{k-1}(x)\}, x \in D$ 
17:   if  $\{W_k \geq 0\} \cap \{\psi(\cdot, z_k) > 0\} \neq \emptyset$  then
18:     choose  $z_k \in \{W_k \geq 0\} \cap \{\psi(\cdot, z_k) > 0\}$ 
19:   else
20:     choose  $z_k \in \{\psi(\cdot, z_k) > 0\}$  according to the original exploration algorithm
       in [3]
21:   end if
22: end while

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observer proceeds through the environment, we take the intersection of all such sets corresponding to each observing location. In the level set framework, this translates to $\min_{j=1,\dots,k} \{W_j\}$, computed in step 16. Similarly, joint visibility along the path Ψ_k is computed as $\max_{j=1,\dots,k} \{\psi(\cdot, z_j)\}$ in step 13.

Figures 4, 5, and 6 demonstrate the performance of Algorithm 2. In all these figures, the over-approximation of the obstacles Ω^+ , based on joint visibility, is depicted by the orange contour, and the under-approximation Ω^- , based on ϵ -balls around the visible boundary points, is depicted by the magenta contour. The $u(z_k)$ level set of v_k^+ is shown in green and the $u(z_k)$ level set of v_k^- is shown in blue. The blue region is the set $\{W_k \geq 0\}$. The location of the source is marked by the red star and the path is shown in black, with circles indicating the discrete steps.

Figure 4 shows a simple environment with three disk-shaped obstacles. The source is located at $(0.75, 0.75)$. The observer may not see the source from its initial position at $(-0.82, -0.91)$. The blue region $\{W_1 \geq 0\}$ almost overlaps with the invisible set $\{\psi_1 < 0\}$. The next vantage point is chosen to be inside the blue region. One can see that after two steps the region $\{W_2 \geq 0\}$, containing the source, has shrunk significantly. Finally, after three steps, $u(z_k)$ level sets of v_k^+ and v_k^- coincide, and the source is located somewhere on the curve $\{W_3 = 0\}$. Since this set is entirely visible from the observer's

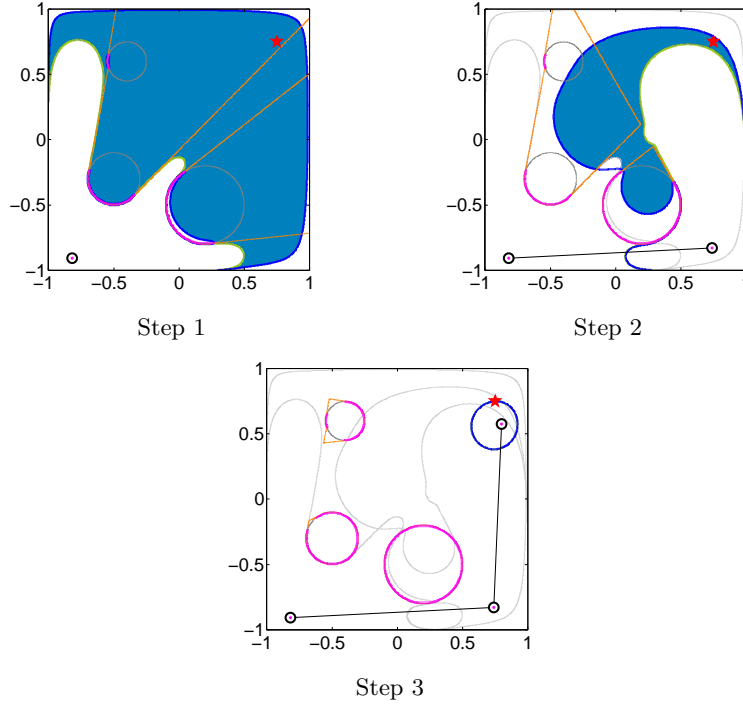


Fig. 4. Unknown environment, known source strength. The source is located at $(0.75, 0.75)$. Orange contour is boundary of Ω_k^+ and the magenta contour is the boundary of Ω_k^- . The blue region is W , the green contour is the $u(z_k)$ level set of v_k^+ and the blue contour is the $u(z_k)$ level set of v_k^- .

position, the search is complete. Note that the environment has not been entirely explored up to this point.

Figure 5 depicts a much more complex example. Here, the region is constructed from a slice of Grand Canyon elevation data¹, which has a much more complex geometrical structure comparing to the example with three circles. We further increased the complexity of the Grand Canyon terrain by adding two disk-shaped holes to the interior of the region. The source is concealed in a small bay with coordinates $(0.25, -0.55)$. At step 1 the blue region $\{W_1 \geq 0\}$, containing the source overlaps with the invisible set $\{\psi_1 < 0\}$. Therefore the observer simply approaches the nearest edge to arrive at z_2 . From now on there is a preferable direction to approach. The next observing position z_3 is chosen according to step 18 of the algorithm. Now there are two possible directions to investigate. The observer chooses the nearest one to arrive at z_4 . Since there are no new horizons at z_4 , the observer backtracks to z_3 and

¹ The terrain data were obtained from:

<ftp://ftp.research.microsoft.com/users/hhoppe/data/gcanyon/>

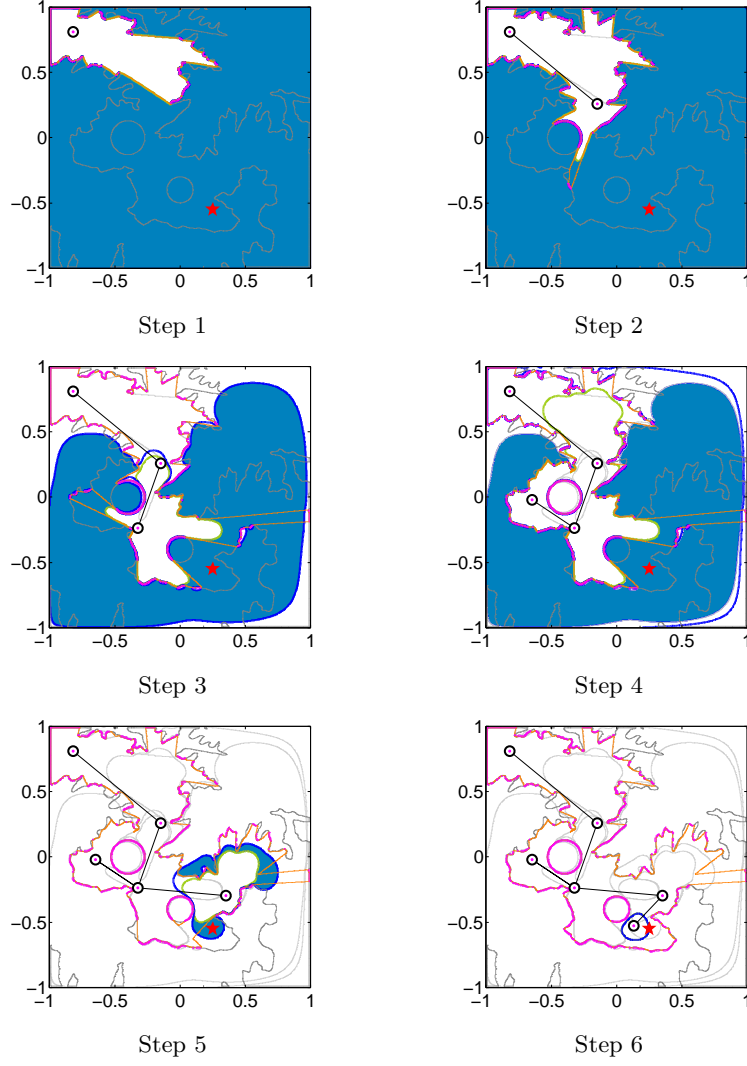


Fig. 5. Unknown environment, known source strength. The source is located at $(0.25, -0.55)$. Orange contour is boundary of Ω_k^+ and the magenta contour is the boundary of Ω_k^- . The blue region is W , the green contour is the $u(z_k)$ level set of v_k^+ and the blue contour is the $u(z_k)$ level set of v_k^- .

explores the second choice horizon. As the observer approaches the source, the blue region shrinks. At z_5 the observer chooses the nearest of three possible horizons. Finally, the entire set of possible source locations is visible from z_6 and, therefore, the algorithm terminates. We remark that the source has been found long before the entire environment has been explored.

Finally, Figure 6 depicts the most complicated example. The source is concealed in a small cave at $(0.112, 0.876)$. Steps 1 through 12 are chosen according to the original [3] exploration algorithm, since the sets $\{W_2 \geq 0\}$ and $\{\Psi_k \geq 0\}$ coincide for $k = 1, \dots, 12$. Finally, the observer backtracks to z_2 to clear previously unexplored horizons. At z_{14} the set containing the source becomes visible. In this example, the observer must explore almost the entire region to finally locate the source.

4 Conclusion

In this paper, we have developed an algorithm that can locate a source of unknown strength for a generic partial differential operator in a bounded domain with obstacles. The algorithm relies on the solution of the adjoint problem and the reciprocity that exists between the operator and its adjoint. We have shown examples for the case of Poisson's equation.

In the case of unknown obstacles, we have proposed a method for locating the source which is based on previous unknown environment exploration methods and relies on the maximum principle to determine a set of possible source locations. This algorithm also works in the case of unknown source strength. Several examples for Poisson's equation were shown.

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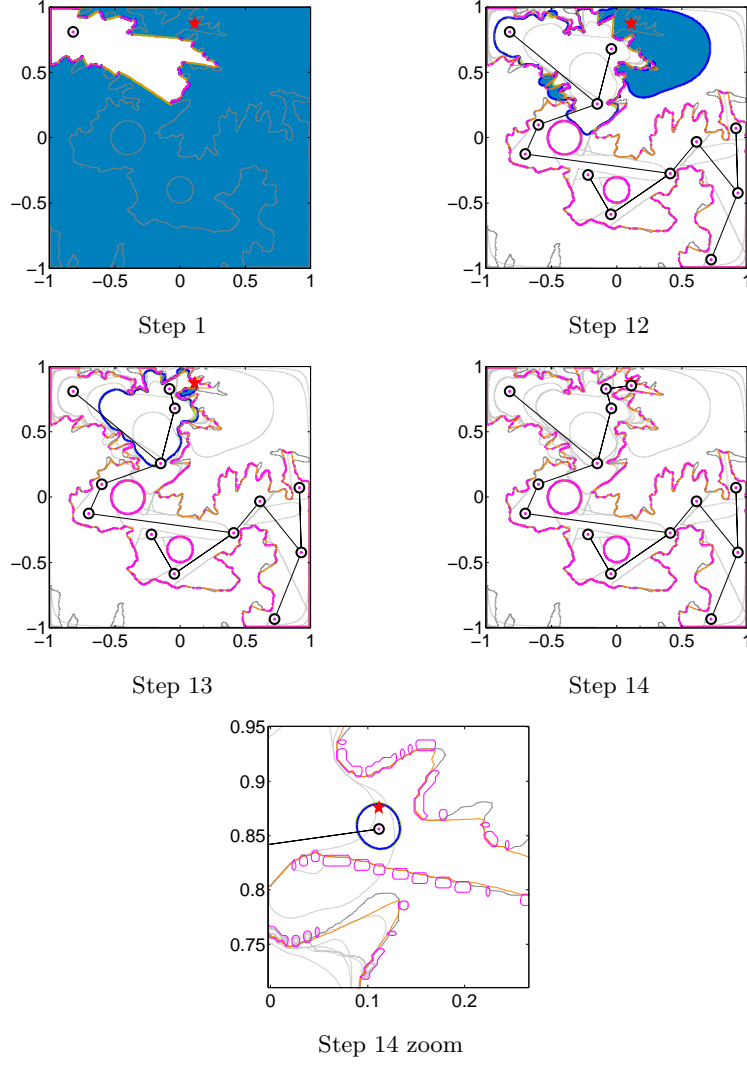


Fig. 6. Unknown environment, known source strength. The source is located at $(0.112, 0.876)$. Orange contour is boundary of Ω_k^+ and the magenta contour is the boundary of Ω_k^- . The blue region is W , the green contour is the $u(z_k)$ level set of v_k^+ and the blue contour is the $u(z_k)$ level set of v_k^- . Steps 2 through 11 are skipped since no information regarding the source location is available.

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