UNIVERSITY OF CALIFORNIA

Los Angeles

# Mathematical Models and Techniques for Medical Imaging

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Biomedical Engineering

by

# Tin Man Lee

2008

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University of California, Los Angeles 2008 To my parents Kwok-Kwong Lee and Sai-Chi Lam and my mentors Howard and Norma Lee

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#### ACKNOWLEDGMENTS

I would like to express my greatest gratitude to Professor Tony Chan. His intelligence, knowledge, insights and energy have improved me tremendously. I am indebted to him for all his great help in many aspects of my life. During my four years of Ph.D. studies, he was Dean of Physical Science at UCLA and then became Assistant Director of the National Science Foundation. With such a busy schedule in an executive position, he still always give me feedbacks and advice promptly whenever I need help. He constantly motivates me to keep trying in the face of obstacles in research. I would like to thank Professor Usha Sinha for her help and guidance in the last 4 years. Her deep knowledge in medical imaging and MR physics have greatly helped my research. I want to give special thanks to Professor Xue-Cheng Tai for his guidance and insights, especially in the past year during the wonderful collaboration with Professor Chan and me in the tracking project, which leads to chapters 3 and 4 in this dissertation. His great insights in the contribution of the original idea, and careful attention to details, and daily skype meetings with me from Norway have pushed the project into new heights, and I truly enjoyed it. Thanks Professor Andy Yip for the great collaboration on the deblurring project, which resulted in Chapter 5 of the thesis. Thanks to Oddvar Christensen and Johan Lie for the enjoyable collaboration especially on the DTI tensor denoising project.

Special thanks to Dr. Hooshang Kangarloo. I have high admiration for his leadership, kindness, resilience and vision. His constant reminder for me to be excellent, and be a person and a great scientist deeply touches me. Thanks Professor Ricky Taira for his guidance in research and always taking care of me in life. Special thanks to Professor Achi Brandt. His kindness, wisdom, great intuition and his very challenging questions for me in our weekly meetings have helped me to think critically and tackle any problems effectively. Thanks to Professor Alan Yuille for the stimulating discussions and great insights. I would like to thank my Ph.D. committee members Professors Luminita Vese, Henry Huang and Paul Thompson. Thanks to my bosses Sarang Lakare and Matthias Wolf at Siemens Medical Solutions for the enjoyable and stimulating summer internship.

I would like to thank the wonderful professors and staff at Medical Imaging Informatics, who provided me with a wonderful working environment and constant help in research: Professors Alex Bui, Deni Aberle, Craig Morioka, Suzie El-Saden, James Sayre, Isabel Rippy, Kerry Engber, and Patrick Langdon.

I also wish to thank my colleagues. Great thanks to Luis Selva for the motivation especially in the last few months to finish the thesis. Thanks also to Will Hsu, my dear officemate, for his great help and his tolerating my huge stacks of books and papers, and my sleeping bag, which even took up some of his space. Thanks to my colleagues Ronald Lui, Mingqiang Zhu, Connie Ni for the motivation and stimulating discussions, fighting together to finish the defense. Thanks to Igor Yanovsky for the help and support to finish the thesis. Thanks to Hui Sun and Rashid Fahmi for their encouragements after we met at Siemens. Thanks to the colleagues in medical informatics Emily Watt, Neda Jahanshad, Lenny D'Avolio, Nam Yu, Jung Choi, Siamak Ardekani, Rod Sun, Frank Mung, Lew Andrada, Hussain Tameem, Corey Arnold, Shishir Dube, Vijay Bashyam, Winston Wu, and Eugenio Iglesias.

I would like to thank my parents whose love and encouragement made the impossible "inevitable". This work would have been impossible without my mentors Howard and Norma Lee, who founded the Sammy Lee Scholarship in which I was one of the recipients, and who have been constantly giving me care and advice over the years. Special thanks also go to Professors James Tong for his great support. Thanks Barbara Lee, Gerald Lee for constantly taking care of me. Thanks also go to Patsy Ho, the scholarship from Fan Foundation, Gwyneth Van, Marita Tong, Thomas Wu and Grace Wu.

This dissertation was supported in part by the Medical Imaging Informatics Training Grant of the National Library of Medicine (NLM) under Contract LM007356, the National Science Foundation (NSF) under Contract DMS-0610079 and ONR Contract N0014-06-1-0345, the National Institutes of Health through the NIH Roadmap for Medical Research, grant U54 RR021813 entitled Center for Computational Biology.

## Abstract of the Dissertation

# Mathematical Models and Techniques for Medical Imaging

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Doctor of Philosophy in Biomedical Engineering University of California, Los Angeles, 2008 Professor Tony F. Chan, Chair Professor Usha Sinha, Co-chair

Partial differential equations (PDE) and variational methods have been researched vigorously and applied to medical image processing in the last two decades. The acquired medical imaging data such as Magnetic Resonance Imaging have reached high isotropic resolution in 3D, and can be in scalar, vector or matrix form at each grid point. Automatic extraction of useful information from these huge high dimensional multi-valued datasets requires advanced computational algorithms, especially in the presence of image noise and blur. In this thesis, we proposed several variational and level set based methods for medical data denoising, deblurring and segmentation. For matrix valued image denoising in Diffusion Tensor Imaging(DTI), we extended the idea of channel coupling in the vector valued total variation paper by Blomgren and Chan and incorporated a term in the constrained optimization to guarantee positive definiteness. For DTI fiber tracking and segmentation, we proposed a new approach based on the level set framework which can automatically capture multiple curves with branching. For image deblurring, we developed a model to segment and recover blurred image where different regions of an image is blurred by different point spread functions. We showed that we can recover the image if the initial guess is tuned appropriately even though the problem is very ill-posed. Finally, we discussed an improved version of the combined wavelet and total variation method and discuss its use in medical image denoising.

# CHAPTER 1

# Introduction

The rapid development of advanced medical imaging technologies such as Magnetic Resonance Imaging(MRI), Positron Emitted Tomography (PET) and Computer Tomography (CT) have provided us an unprecedented way to diagnose illness in-vivo and non-invasively. Some of the most advanced techniques based on these imaging modality remain in the research stage but never reach routine clinical usage. One of the bottleneck is due to low Signal-to-Noise Ratio(SNR) which require long and repeated acquisition of the same subject to reduce noise and blur. For example, a high SNR DTI dataset requires an hour of data acquisition [CLL07]. A high SNR of HARDI data can take 13 hours [Tuc04]. To recover noisy and blurry image without lengthy repeated scans, post-processing of data plays a critical role in two aspects. First, automatic denoising and deblurring algorithms can restore the data computationally to achieve a quality that might take much longer otherwise. Secondly, computational object segmentation techniques can perform intelligent extraction of data directly and automatically from noisy observations. We proposed variational methods and non-linear partial differential equations for automatic denoising, deblurring and segmentation in an attempt to shorten the bridge of medical imaging research towards routinely clinical practice.

We first review the background knowledge of human brain central nervous system, data acquisition, and some fundamental mathematical image processing techniques related to the techniques we developed in this thesis.

## 1.1 Human Brain, Axons and Fiber Tracks/Bundles

Human brain contains billions of neurons, which are the basic signaling units. These neurons communicate with each other through chemical and electrical signals known as synapses. Understanding how these neurons are connected with each other can help us understand cognitive process and behavior. Certain long processes of the nerve cells are known as nerve fibers. Nerve fibers are found universally in the peripheral nerves, and in the white substance of the brain and medulla spinalis[Gra00].

Figure 1.1 shows the structure of a neuron and its histology. The axon, which is the main transmission line of the nervous system, is a very thin, cable-like projection which can extend to a meter in length or even longer[Wik05b]. Each neuron usually has one axon, but the axon can undergo extensive branching, enabling communication with many target cells. The axons in central nervous system are sheathed in myelin formed by glial cells. Myelin is opaque and white in color. The whole axon is enclosed in a delicate membrane, called the white nucleated sheath of Schwann. The central portion of the axon is called the axis-cylinder, which goes from its origin in the nerve center to its peripheral termination without interruption. The myelin sheath, which is composed of fluidic fatty matter, insulates and protects the central part of the axon, and limits the passage of water molecules across its boundary, which contribute to the the restricted diffusion property being utilized in DTI. Many neuro-degenerative diseases such as Multiple Sclerosis and Alzheimer's disease, are believed to be caused by the demyelination of axons. The thickness of axons in the central nervous system is about 1 micrometer. Therefore, thousands of axons can pass through a 1.3mm



Figure 1.1: Structure of a neuron [Wik05a] (a), Diagram of longitudinal sections of medullated nerve fibers. Osmic acid. [Gra00] (b).

by 1.3mm by 1.3mm voxel in a typical high resolution DTI, which causes severe partial volume effect. Here, we only focus on describing fiber tracks in the white matter of the brain.

To explain where the fiber tracks are connected to, we first have to describe the functional areas of the brain. Each hemisphere of the human brain can be divided into six lobes based on fissures and sulci: the frontal, the parietal, the temporal, the occipital, the limbic, and the insula. Each lobe has its corresponding function and is explained in [Gra00]. On the left hand side of Figure 1.2, we show the lobes in the hemisphere. These lobes are connected to different part of the brain by fiber tracks, as shown in the right hand side of Figure 1.2. The white matter consists of nerve fibers which are different in size and arranged in bundles separated by neuroglia. Neuroglia is the supporting ground tissue found in the brain. Fiber tracks may be divided, according to their course and connections, into three distinct systems[Gra00]:

(1) Projection fibers connecting the hemisphere with the lower parts of the brain and the medulla spinalis. For example, the motor tract occupies the genu and anterior two-thirds of the occipital part of the internal capsule. The geniculate fibers, which decussate and end in the motor nuclei of the cranial nerves



Figure 1.2: Principal fissures and lobes of the cerebrum viewed laterally. [Gra00] (a), Diagram showing principal systems of association fibers in the cerebrum. [Gra00] (b).

of the opposite side. The cerebrospinal fibers prolongs through the pyramid of the medulla oblongata into the medulla spinalis. The optics nerve track passes to the occipital lobe while the acoustic fiber track passes to the temporal lobe. The motor fiber tracks are at first widely diffused and they gradually approach each other descending through the corona radiata.

(2) Transverse or commissural fibers connecting the two hemispheres. Examples include the transverse fibers of the corpus callosum, the anterior and the posterior commissure.

(3) Association fibers connecting different structures in the same hemisphere, such as the collateral branches of the projection fibers. For example, the short association fiber connects adjacent gyri.

Figure 1.3 shows the optics fiber track where crossing can be seen near the optics chaisma, which is a challenge for DTI to capture. The motor fiber track also shows a lot of branching when it branches out into the motor cortex. Therefore, branching and crossing are important aspects to be taken care of in fiber tractography.



Figure 1.3: The optics track. [Gra00] (a), The motor track. [Gra00] (b).

## 1.2 Diffusion Tensor Imaging(DTI)

A conventional MRI brain scan of a schizophrenic patient looks normal even to an experienced neurologist, even though the patient suffers from profound mental problems. The differences are usually subtle in structural MRI. There is obviously a need to study the neuroanatomy of the patients, which was not possible until the development of DTI [BML94]. DTI is an emerging technology which has the potential to allow scientists to, through the restricted random diffusion of water molecules, indirectly look at the neuronal connection between different brain areas in-vivo, i.e. the wiring of the axonal fibers that constitute the brain's white matter. The highly anisotropic nature of white matter water movement allows researchers to infer location, orientation and possibly connections of neuronal fibers. DTI provides more information than structural MR, and was well documented that Diffusion Weighted Images (DWI) allows early detection of stroke which cannot be seen in structural MRI. With this technology, not only Schizophrenia, but any neuro-related diseases, e.g. autism, addiction, epilepsy, traumatic brain injury, brain tumors, stroke, multiple sclerosis, and various neurodegenerative diseases such as Alzheimer's disease, can be analyzed in a way that was not previously possible. These connections, as approximated by DTI, can shed lights on understand these brain diseases.

Here we describe the calculation of diffusion tensor from the diffusion weighted images and the baseline image without diffusion sensitization gradient. In each voxel, we form a *diffusion tensor* D based on a series of k direction-specific MR measurements  $S_k$ . The matrix  $D \in \mathbb{R}^{3\times 3}$  is a symmetric and positive definite matrix. The diffusion tensor model assumes that each diffusion weighted image  $S_k$ at a particular diffusion sensitization direction relates to the baseline acquisition  $S_0$  by the Stejskal-Tanner equation [ST65]:

$$S_k = S_0 e^{-bg_k^T Dg_k}.$$
 (1.1)

Here  $g_k \in \mathbb{R}^3$  denotes the direction associated with  $S_k$ , and b is a positive scalar defined in the scanner protocol. The value of b represents the amount of diffusion weighting (level of sensitivity to diffusion). If the b-value equals 0, there is no diffusion weighting and we obtain the baseline scan  $S_0$ . If the diffusion value is very high, we can get greater resolution, but also more noise. The standard b-value for adults is 1000 and we use this value throughout our experiments on human brains. Since  $D \in \mathbb{R}^{3\times 3}$  is symmetric, it has six degrees of freedom. Thus at least six measurements  $S_k$  are required to estimate the tensor, as well as the baseline measurement  $S_0$ . In the case of more than six measurements  $S_k$ , we can use a least squares minimization[WMK99]. From the eigenvalue decomposition of the diffusion tensor, we can reveal properties like the dominant diffusion direction and the anisotropy of diffusing water molecules [WMK99].

The human subject data used in our experiments were acquired using a 3.0 T scanner (Magnetom Trio<sup>®</sup>), Siemens Medical Solutions, Erlangen, Germany) with a 8-element head coil array and a gradient subsystem with the maximum gradient strength of 40 mT/m and maximum slew rate of 200 mT/m/ms. The DTI data were based on spin-echo single shot EPI acquired utilizing generalized auto calibrating partially parallel acquisitions (GRAPPA) technique with acceleration factor of 2 and 64 reference lines. The DTI acquisition consists of one baseline EPI and six diffusion weighted images (b-factor of 1000 s/mm2) along following gradient directions:  $G1 = 1/\sqrt{2}(1,0,1)T$ ,  $G2 = 1/\sqrt{2}(-1,0,1)T$ ,  $G3 = 1/\sqrt{2}(0,1,1)T$ ,  $G4 = 1/\sqrt{2}0(0,1,-1)T$ ,  $G5 = 1/\sqrt{2}(1,1,0)T$ ,  $G6 = 1/\sqrt{2}(1,1,0)T$ . Each acquisition had the following parameters: TE/TR/averages was 91 ms/10000 ms/2, FOV was 256 mm x 256 mm, slice thickness/gap was 2 mm/0 mm, acquisition matrix was 192 x 192 pixels and partial Fourier encoding was 75%.

#### 1.2.1 Noise in DTI

Typically, DTI suffers from poor Signal-to-Noise Ratio(SNR). Noise in DTI has been studied by works such as [BAM98, BP00, SLN00, CH05]. The acquisition of DTI is based on Echo-Planar Imaging(EPI)[Man77, DGM77, SHP08] which can acquire the whole image in a single shot with a speed 10,000 times faster than conventional MRI. EPI has the advantage that, without affecting the total imaging time, the whole longitudinal magnetization can be used for image formation, thus reducing any motion-induced artifacts. However, due to the need to use a high gradient, there is a huge penalty for bandwidth. The low bandwidth in EPI represents the smaller difference in adjacent pixels frequencies which results in an image of low contrast. SNR of EPI can be 2/3 lower than conventional MRI. In DTI, a longer TE is needed to allow the sensitization gradient pulses, thus further reduces the SNR of EPI. The diffusion encoding also tends to attenuate the signals. The noise in DTI is further aggravated by partial volume(or volumeaveraging) effects where each voxel contains multiple fiber directions averaged. Errors due to noise in the estimation of the diffusion tensor can create adverse effects in two areas: voxel-by-voxel anisotrpy indices comparison between subjects and fiber tractography. In order to enhance the quality of DTI for routine clinical practice, SNR of DTI needs to be improved. SNR is one of the limitations for the automatic detection of subtle abnormalities. There are two hardware methods to overcome low SNR. The simplest one is to repeatedly acquire the same subject and take the average. Another method is to simply use more diffusion sensitization directions or encoding levels, and use least square to find the diffusion tensor. The main drawback is the long acquisition time, which can take longer than an hour and hinders the popularity for routine clinical practice. In Chapter 2, we describe our post-processing method which can improve the quality of the image without sacrificing acquisition time.

## **1.3** Mathematical Image Processing

#### 1.3.1 Total Variation(TV) and Color TV

Removing noise in an image has always been a challenging task because of the difficulty of identifying noise from useful information, especially in medical imaging where some information is critical for diagnosis. A lot of linear filtering algorithms have been used, such as simple Gaussian smoothing/Fourier domain low pass filtering, to more advanced Tikonov regularization and wavelet denoising. Although these methods are very fast, they tend to smooth out edges of an image, or create artifacts at the edges such as the Gibbs phenomena. Edges in an image are very critical in object boundary detection or medical diagnosis. A number of non-linear filters were thus developed to remedy the edge smearing effects.

Total Variation (TV) method was invented by Rudin, Osher and Fatemi[ROF92]. Define the Total Variation (TV) semi-norm for scalar valued data u as

$$TV(u) = \int_{\Omega} |\nabla u| dx \tag{1.2}$$

Throughout this paper,  $\nabla$  denotes the spatial gradient, while  $\nabla$  denotes the divergence operator. In the ROF model, the TV semi-norm with an added  $L_2$  fidelity norm is minimized

$$\min_{u} \left\{ G(u, f) = TV(u) + \frac{\lambda}{2} ||u - f||_2^2 \right\}.$$
 (1.3)

Note that we can write the functional G more abstractly as

$$G = R + \frac{\lambda}{2}F,\tag{1.4}$$

where R is a regularization functional and F is a fidelity functional. The regularization term is a geometric functional measuring smoothness of the estimated solution. The fidelity term is a least squares measure of fitness of the estimated solution compared to the data.

This leads to the Euler-Lagrange equation

$$\partial_u G = -\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + \lambda(u - f). \tag{1.5}$$

We can find a minimum by searching for a steady state of

$$\frac{\partial u}{\partial t} = -\partial_u G,\tag{1.6}$$

which is the way the ROF model was first solved [ROF92], or by directly attacking the zero of the Euler-Lagrange equation

$$-\partial_u G = 0, \tag{1.7}$$

for example by a fixed-point iteration [BC96]. This is in general less time consuming than solving the equation using the method of steepest descent, but more tedious to carry out numerically.

Blomgren and Chan [BC96] generalized this approach to vector (color) image regularization by proposing a new TV norm:

$$TV\mathbf{u} = \sqrt{\sum_{i} TV(u_i)^2}.$$
(1.8)

where the Euler-Lagrange equation can be solved by time marching,

$$\frac{\partial u_i}{\partial t} = \alpha_i \nabla \cdot \frac{\nabla u_i}{|\nabla u_i|} - \lambda (u_i - f_i)$$
(1.9)

with

$$\alpha_i = \frac{TV(u_i)}{TV(\mathbf{u})},\tag{1.10}$$

The weight  $\alpha_i$  in the first term acts as a coupling between the geometric part of the three image channels. The new TV norm is rotationally invariant, reduces to the usual TV term in scalar case, and does not penalize discontinuities. For a detailed treatment of TV regularization methods we refer the reader to the recent book by Chan and Shen [CS05].

### 1.3.2 Level Set Methods

An image as a function can be interpreted as a collection of isophotes, or level sets. Therefore, we can represent an image using level set method proposed by Osher and Sethian [OS88]. The level set method is a very powerful numerical technique for tracking interfaces and shapes. Given an interface  $\Gamma$  in  $\mathbb{R}^n$  bounding a open region  $\Omega$ , the level set method represent this interface as the zero level set of one higher dimension Lipschitz continuous function  $\phi$ :

$$\Gamma = \{ (x, y) | \phi(x, y) = 0 \}, \tag{1.11}$$

The interface  $\Gamma$  is a curve in 2D or a surface in 3D. By this representation, an image is divided by the zero level set into two regions: the region where  $\phi > 0$  represents the background (outside region) and the region where  $\phi < 0$  represents the object inside the interface  $\Gamma$  (inside region). Let the interface x(t) be a closed curve parameterized by t, the level set representation is:

$$\phi(x(t), t) = 0, \tag{1.12}$$

Differentiate the above equation with respect to t, we have:

$$\phi_t + \nabla \phi \cdot x'(t) = 0, \tag{1.13}$$

Let the speed function of the interface be F(t), then,

$$\phi_t + \nabla \phi \cdot F = 0, \tag{1.14}$$

If F is the speed at the outward normal direction, we obtain the fundamental representation of the level set method:

$$\frac{\partial \phi}{\partial t} + F |\nabla \phi| = 0, \qquad (1.15)$$

The level set method allows numerical computations involving curves and surfaces to be computed on a fixed Cartesian grid without having to parameterize these objects. It is very easy to follow shapes that change topology, such as branching, and analyze and compute its subsequent motion under a velocity which can depend on position, time or the gemoetry of the interface. The level set method has been widely applied in many areas including computational fluid dynamics, computer graphics, computer vision and image processing. Most of the level set function uses the zero level set as the interface for object detection. Therefore, these techniques focus on fast computation by narrow band method, where the level set function within a band near the interface is computed. The potential of utilizing all the level sets of a level set function has not been researched vigorously due to the conventional understanding that anywhere further away from the zero level set is un-interesting. In Chapter 3 and 4, we extend the zero level set method to use all level set of one function for tracking.

#### 1.3.3 Chan-Vese Model

Chan and Vese [CV99] proposed a region-based image segmentation model based on curve evolution, the Mumford and Shah functional, and level set method that has revolutionized the automatic segmentation of images such as MRI. This segmentation method is motivated by the fact that the boundaries of some objects are smeared and cannot be defined accurately by the gradient. The idea of Chan and Vese is to consider the information inside the regions, and not just at their boundaries. The model can detect objects whose boundaries are not necessarily defined by gradient because the stopping term does not depend on the gradient of the image, as in the classical active contour models, but is instead related to a particular segmentation of the image. It allows initial guess to be at any location and it can detect interior contours. It is also robust to noise. The model can be described by

$$\inf_{c_1, c_2, \Gamma} F(c_1, c_2, \Gamma) = \mu |\Gamma| + \int_{inside(\Gamma)} |u_0 - c_1|^2 dx + \int_{outside(\Gamma)} |u_0 - c_2|^2 dx \quad (1.16)$$

where  $\mu$  is a positive parameter,  $u_0$  is the original image to be segmented,  $\Gamma$  is the evolving curve, and  $c_1, c_2$  are the two unknown constants. The model finds the best approximation of the image  $u_0$  which is a set of regions with two different intensities, where the region with intensity  $c_1$  denotes the region to be detected (inside), and the region with intensity  $c_2$  denotes the background (outside).  $\Gamma$ is the boundary between the two regions. This model is implemented using the level set formulation by using Heaviside function H,

$$F(c_1, c_2, \phi) = \mu \int_{\Omega} |\nabla(H(\phi))| + \int_{\Omega} |u_0 - c_1|^2 H(\phi) dx + \int_{\Omega} |u_0 - c_2|^2 (1 - H(\phi)) dx$$
(1.17)

where  $\phi$  is the level set function. By approximating H into  $H_{\epsilon}$  as described in [CV99], derivatives can be computed to find the minimum. Figure 1.4 shows that the energy function only gives a minimum at the exact segmentation. Multiphase model has been proposed by [VC02] where  $\log n$  number of level set functions are used to segment n regions. We utilized the Chan-Vese model for our spatial-varying deblurring project in Chapter 5.



Figure 1.4: Illustration of the Chan-Vese Model.

## 1.4 Contributions and Organization of This Dissertation

The main goal of this thesis is to show that variational PDE methods, when correctly designed and adapted, can be very powerful in solving challenging medical image processing problems.

Semi-positive definiteness(SPD) of diffusion tensor is a very critical physical requirement in DTI. Other denoising methods try to enforce SPD explicitly by turning negative eigenvalues into zero, we have introduced a method that can do it automatically and structurally in Chapter 2. We also extended the Color TV model by Bloomgren and Chan [BC96] to matrix-valued DTI data, so that discontinuities would not be penalized.

For neuronal fiber tracking, typical level set methods use only one level set function to implicitly define a single fiber track, we have developed a method that uses ALL level sets of one level set function to track all the data in Chapter 3. Our method can also track fibers with branching and crossing naturally, without any explicit parameterization of the tracks. Our approach is global which can avoid local tracking errors by streamline methods[XZC99]. Our algorithm is completely automatic. We do not require user inputs of seed points or end points. We extend the algorithm to 3D in Chapter 4, where the idea of using 2 level set functions [BCM01a] is further developed to perform tracking in 3D.

We introduced a simultaneous segmentation and deblurring method which can recover an image blurred by spatially varying point spread functions in Chapter 5, based on total variation and level set method.

In chapter 6, we showed that an adaptive Bayesian threshold can further improve the current wavelet TV combined results in denoising medical images.

# CHAPTER 2

# **Total Variation Denoising of Tensor**

## 2.1 Enforcing Semi-Positive Definite

One of the major differences between DTI denoising and traditional image denoising is the importance of enforcing Semi-Positive Definiteness(SPD) in DTI. Each diffusion tensor can be decomposed into 3 pairs of eigenvalues with its corresponding eigenvectors, which models the diffusion process of water molecules. The magnitude of these eigenvalues are proportional to the standard deviations of the motion of water molecules along each direction of the corresponding eigenvectors. Since the standard deviation of water motion should always be equal to or bigger than zero, so does each of the eigenvalue, and thus the diffusion tensor should be SPD. However, due to noise in collecting diffusion weighted images, many of the computed eigenvalues are negative which is physically impossible. Other methods such as [TD02] turn the eigenvalues from negative to zero explicitly before performing any further image processing, which is a bit ad-hoc. We proposed a method that can do it automatically and structurally. We automatically enforce SPD by treating the tensor D implicitly as the product  $D = LL^T$ . We work with the elements of L as variables, instead of working directly on D. Since every symmetric positive definite (SPD) matrix has a factorization on this form, we are guarantee to obtain D which is SPD. Our method is related to but different from [ZVC03]. It is similar in the sense that we both use Cholesky

Factorization. However, we regularize the elements of the diffusion tensor D, while they regularize the elements of the lower triangular matrix L. Intuitively, regularizing the elements of D is more direct than regularizing the elements of L.

Here we introduce the functional that we minimize in order to regularize tensor valued data. Let L be a lower triangular matrix. We define D by

$$D = LL^T. (2.1)$$

This has immediate implications on D; symmetry, positive definiteness and orthogonality of eigenvectors. These properties are required by the diffusion tensor model. Thus D is a natural choice. We define  $\ell_{ij}$  as the element in the *i*'th row and *j*'th column of L. The elements  $d_{ij}$  are defined in the same manner.

Let us look at the algebraic equation expressing D as functions of  $\ell_{ij}$ . We derive the expressions for  $D \in \mathbb{R}^{3 \times 3} \subset SPD$ . We explicitly write out the matrix multiplication:

$$D = \begin{pmatrix} \ell_{11}^2 & \ell_{11}\ell_{21} & \ell_{11}\ell_{31} \\ \ell_{11}\ell_{21} & \ell_{21}^2 + \ell_{22}^2 & \ell_{21}\ell_{31} + \ell_{22}\ell_{32} \\ \ell_{11}\ell_{31} & \ell_{21}\ell_{31} + \ell_{22}\ell_{32} & \ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 \end{pmatrix}.$$
 (2.2)

Also, [ZVC03] used L as the data fidelity term when compared with the K-space data. Their method is elegant in that they utilized the K-space MRI data which are less noisy. However, our focus here is to denoise routinely-acquired DTI data where normally the K-space data are not available from the patient dataset.

## 2.2 Total Variation Denoising on Matrix

Regularization of tensor valued data has been studied during the last couple of years, see [CTD04, TD02, TD05] for vectored valued anisotropic diffusion methods.

By enforcing SPD as described above, we put the constraint into a Total Variation model where we extended and generalized Blomgren and Chan's Color TV model [BC96] to matrix-valued images. We choose TV because of it smooth noise while preserving any discontinuities.

We propose a new TV norm for matrix  $D \in \mathbb{R}^3 \times \mathbb{R}^3$  as

$$TV(D) = \left( TV[d_{11}(\ell_{ij})]^2 + 2TV[d_{21}(\ell_{ij})]^2 + TV[d_{22}(\ell_{ij})]^2 + 2TV[d_{31}(\ell_{ij})]^2 + TV[d_{32}(\ell_{ij})]^2 + TV[d_{33}(\ell_{ij})]^2 \right)^{\frac{1}{2}}.$$
(2.3)

This is a natural generalization of the total variation norm of a vector [BC96].

Therefore, the constrained minimization problem is in terms of  $\ell_{ij}$ . For each unique  $\ell_{kl}$  we minimize

$$\min_{\ell_{kl}} \sqrt{\sum_{ij} TV d_{ij}(\ell_{kl})^2} + \frac{\lambda}{2} \sum_{ij} \|d_{ij} - \hat{d}_{ij}\|_2^2,$$
(2.4)

where  $\{kl\} \in \{11, 21, 31, 22, 23, 33\}$  and  $\hat{d}_{ij}$  denotes the elements of the tensor estimated from the noisy data. Similar to the scalar model, the functional (2.4) has the abstract form (1.4).

We also note here that the functional used by Wang et. al. [ZVC03] is different from our model in that they used a norm of p > 12/7. They mentioned that TV norm is computational intensive and not feasible. We successfully introduced the matrix TV regularization which is feasible and not computationally intensive.

### 2.2.1 Euler-Lagrange Equations

Now we derive the Euler-Lagrange equations corresponding to the minimization functional (2.4). We first differentiate the fidelity functional

$$\frac{\partial F}{\partial \ell_{ij}} = \frac{\partial}{\partial \ell_{ij}} \sum_{ij} \|d_{ij} - \hat{d}_{ij}\|_2^2$$

$$= 2 \sum_{ij} \left( d_{ij} - \hat{d}_{ij} \right) \frac{\partial d_{ij}}{\partial \ell_{ij}}.$$
(2.5)

We differentiate D with respect to  $\ell_{ij}.$ 

$$\frac{\partial D}{\partial \ell_{11}} = \begin{pmatrix} 2\ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{21} & 0 & 0 \\ \ell_{31} & 0 & 0 \end{pmatrix}, \qquad (2.6)$$

$$\frac{\partial D}{\partial \ell_{12}} = \begin{pmatrix} 0 & \ell_{11} & 0\\ \ell_{11} & 2\ell_{21} & \ell_{31}\\ 0 & \ell_{31} & 0 \end{pmatrix}$$
(2.7)

$$\frac{\partial D}{\partial \ell_{31}} = \begin{pmatrix} 0 & 0 & \ell_{11} \\ 0 & 0 & \ell_{21} \\ \ell_{11} & \ell_{21} & 2\ell_{31} \end{pmatrix}, \qquad (2.8)$$

$$\frac{\partial D}{\partial \ell_{22}} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 2\ell_{22} & \ell_{32}\\ 0 & \ell_{32} & 0 \end{pmatrix}, \qquad (2.9)$$
$$\frac{\partial D}{\partial \ell_{32}} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & \ell_{22}\\ 0 & \ell_{22} & 2\ell_{32} \end{pmatrix}, \qquad (2.10)$$

$$\frac{\partial D}{\partial \ell_{33}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\ell_{33} \end{pmatrix}.$$
 (2.11)

Writing out (2.5), we find the derivative of the fidelity functional

$$\frac{\partial F}{\partial \ell_{kl}} = 2 \Big[ (d_{11} - \hat{d}_{11}) \frac{\partial d_{11}}{\partial \ell_{kl}} + 2(d_{21} - \hat{d}_{21}) \frac{\partial d_{21}}{\partial \ell_{kl}} \\
+ (d_{22} - \hat{d}_{22}) \frac{\partial d_{22}}{\partial \ell_{kl}} + 2(d_{31} - \hat{d}_{31}) \frac{\partial d_{31}}{\partial \ell_{kl}} \\
+ 2(d_{32} - \hat{d}_{32}) \frac{\partial d_{32}}{\partial \ell_{kl}} + (d_{33} - \hat{d}_{33}) \frac{\partial d_{33}}{\partial \ell_{kl}} \Big]$$
(2.12)

Using the chain rule, we find the derivatives of the regularization functional

$$\frac{\partial R}{\partial \ell_{kl}} = -\sum_{ij} \alpha_{ij} \nabla \cdot \left(\frac{\nabla d_{ij}}{|\nabla d_{ij}|}\right) \frac{\partial d_{ij}}{\partial \ell_{kl}},\tag{2.13}$$

with

$$\alpha_{ij} = \frac{TV(d_{ij})}{TV(D)}.$$
(2.14)

We use finite difference schemes to implement the above equations. For the details of numerical implementation, please refer to [CLL07].

# 2.3 Experimental Results

In this section we perform numerical experiments on synthetically constructed tensor fields and real tensor fields from a human brain. For the synthetical fields, we have constructed clean tensor fields which are degraded with noise having an a priori known distribution. Thus we are able to measure in the SNR sense how well the method performs on synthetical data. For the real human brain datasets the true solution is of course not known a priori. In this case we measure the performance of the method in terms of a reference solution where a large series of acquisitions are averaged. This is explained in detail in Section 2.3.2. For the numerical implementation of the regularization and some of the visualizations we have used Matlab.

#### 2.3.1 Synthetical Tensor Field

In the first numerical experiment, displayed in Figure 2.1, we test the performance of the proposed method on a simple tensor field. This field is mapping a square domain  $\Omega \subset \mathbb{R}^2$ , with four distinct regions, to  $\mathbb{R}^{2 \times 2}$ . We construct the clean tensor valued data by prescribing the eigenvalues and corresponding eigenvectors. The values of each element of L is in the range [0, 1]. Then we add normally distributed noise  $\eta(\sigma)$  to each element of the matrix, that is  $\hat{D} = D + \eta(\sigma)$ . Finally, the degraded diffusion tensor is constructed by  $\hat{D} = \hat{L}\hat{L}^{T}$ . The noise level in the simulation in Figs. 2.1 and 2.2 are given by  $\sigma = 0.35$  and  $\sigma = 0.25$  respectively. The time step parameter is  $\Delta t = 0.001$ . Notice that the discontinuities in the data are preserved in the solution, i.e. the edge preserving property of scalar and vector valued TV flow is kept in the proposed matrix valued flow. In the first example, the diffusion is anisotropic in the whole domain. To show how the proposed method differentiate between isotropic and anisotropic diffusion, we show a similar example, but with one of the four regions interchanged with a region of isotropic diffusion. The isotropic region is constructed by considering the orthogonal matrix Q of the QR factorization of a random  $2 \times 2$  matrix. The



Figure 2.1: A synthetically produced purely anisotropic tensor field with four distinct regions is degraded with normally distributed noise. The noisy field is then processed with our proposed method. (a) The clean vector field  $D_0$ . (b) The noisy field  $\hat{D} = D_0 + \eta$ . (c) The recovered field D.



Figure 2.2: Visualization of the true vector field (a), the noisy field (b), and the recovered field (c). In this example, the tensor field is isotropic in the lower left corner, anisotropic in the other parts.

columns of the matrix Q are considered to be the eigenvectors of the diffusion tensor. We specify two random numbers in the range [0, 1] as the eigenvalues of the tensor. Thus the diffusion is random in the isotropic region.

From these numerical examples we see that the proposed method gives encouraging results on the synthetical data.

#### 2.3.2 Human Brain DTI

We also perform numerical experiments on DTMRI acquisitions of a healthy human brain. For validating the proposed regularization method on real data, we construct a reference solution  $D^*$  by averaging 18 replications. Each replication consists of six direction weighted and one nonweighted acquisition. This reference solution is compared to solutions where averages of two, four and six replications are postprocessed with the proposed regularization method. As a measure of the distance between the reference solution and the processed solution, we use the Frobenius norm

$$m(D, D^{*}) = \left( [d_{11} - d_{11}^{*}]^{2} + 2[d_{12} - D_{12}^{*}]^{2} + 2[d_{13} - d_{13}^{*}]^{2} + [d_{22} - d_{22}^{*}]^{2} + 2[d_{23} - d_{23}^{*}]^{2} + [d_{33} - d_{33}^{*}]^{2} \right)^{1/2}$$

$$(2.15)$$

For each simulation, we report the regularization parameter  $\lambda$ , and the metric  $m(\cdot, \cdot)$  in Table2.1. As can be seen from Table2.1, the error reduced by an average of over 30% for each case. We also performed numerous region-of-interest (ROI) studies at different homogenious regions of the human brain to analyze the voxel by voxel difference quantitatively between noisy, denoised and ground truth (18-averaged) results[CLL07]. We also calculated the denoised major eigen vectors at these ROIs. The results were analyzed by domain experts and the denoised results were proven to be superior than the noisy data and has a resemblance to the ground truth. We display the result of applying the proposed method on real DTMRI data. In the figures we display a 2D slice of the scalar fractional anisotropy (FA). We used the DTMRI software tool DTIStudio [JZK06] to construct the visualizations.

The noise level is different for each simulation due to the number of acquisitions. Consequently, the regularization parameter is different for each simulation. In practice, the regularization parameter is estimated once for each imaging pro-

Replications	$\lambda$	$\operatorname{reg} m(D, D^*)$	non reg $m(D, D^*)$
1	9	136	208
2	13	113	154
3	19	84	105

Table 2.1: The distance  $m(D, D^*)$  from the numerical examples shown in Figs. 2.3.



Figure 2.3: The noisy 2 averaged acquisition (a) is compared with the denoised acquisition (b) and a reference solution at 9 averaged.

tocol. When this is done, the same regularization parameter can be used for subsequent applications of the same imaging protocol.

# 2.4 Conclusion

In this chapter we have generalized the color TV regularization method of Blomgren and Chan [BC96] to yield a novel regularization method for tensor valued data. We have shown that the proposed method is applicable as a regularization method with the important property of preserving both edges in the data and positive definiteness of the diffusion tensor. Numerical experiments on synthetically produced data and data from DTMRI of a human brain indicate good performance of the proposed method. The results showed that TV-based matrix denoising has the potential to restore a 4-minute acquisition to rival the quality of those taken much longer time, thus facilitating routine clinical practice.

# CHAPTER 3

# Capturing Multiple Curves in a Noisy Vector Field

Other level set based fiber tracking algorithms consider fiber tracking as a front propagation problem, where tracks are found by local propagation. In this chapter, we propose a new fiber tracking algorithm. Our approach can capture all individual fiber tracks by using only one single level set function in 2D. The main contribution is the capability of capturing branching curves without explicit parameterization for each curve. The algorithm is based on variational PDEs in a level set framework. The vector field is given naturally from the major eigenvector from Diffusion Tensor Imaging. We move all level curves of a function until all the normals of these level curves align with the normals of the vector field while doing a total-variation based denoising on the data. The level curves will align with the underlying curves, thus creating a high gradient at the level set function. The regions where a fiber track is not present will have a flat surface due to no alignment of level curves. Because we utilize all level curves of a function, different underlying tracks will be aligned to different level curves, allowing the segmentation of different tracks easily. Our method provides a global view of the curves, which can act as a complementary tool for the existing local streamline approach.

## 3.1 Introduction

Fiber tracking has been a very hot topic since the invention of DTI. There are two main streams for fiber tracking: Streamline based or probabilistic based. Streamline starts from a user-defined seed point or end point and propagate voxel by voxel based on the major eigenvector of the tensor[BPP00, XZC99]. Fast marching and level set based methods[PWB02, MKQ04] improve on streamline by allowing splitting and merging of individual fiber. Probabilistic approach refers to tracking from all the seeds, allowing a random walk onto the whole dataset to show the probability of paths[IFS01, BBK02, HTV02]. The advantage of streamline is its simplicity and speed, but a local error can lead to completely wrong track. Also an expert is needed to hand pick the seed point and end point locations. The advantage of probabilistic approach is the global property. But it is time consuming and cannot give a deterministic tracks.

Using all level curves of a function for image denoising have been described by [OF02]. Manipulating the vector field generated by the normals of level curves of an image, powerful image processing techniques can be carried out. [LOT04] proposed a two-stage method to solve a Fourth Order PDE which can reduce stair-case effects of TV where the first stage is smoothing the normal of level curves of the noisy image. The second stage of their algorithm mentioned the fitting of a surface onto the denoised vector field, which is related to our fiber tracking model. Later, [RT007] proposed smoothing tangent fields of the image by geometrically explaining the angle minimization between tangent field and the tangents of the level curves of a function. Using more than just the zero level of a level set function for image segmentation has been proposed by Chung and Vese[CV05]. Instead of using more than one level set function for the multiphase segmentation, they used multiple levels of one or multiple level set functions. [TC03] also proposed using multiple levels of a level set in their work on piecewise constant level set. Our method uses all levels of a level set function, so no user input is needed to specify number of levels. There are also previous attempts of finding a best fitting line based on a global energy, such as RANSAC[FB81] and multiscale line integrals[SBB00], but these are for straight lines but not curves, and that they explicitly find the lines while we represent the curves implicitly. Our main idea is to smooth the major eigenvector (tangent) in a noisy vector field and find a surface where the tangents of the level lines best matches the vectors. In this way, branching is handled by a higher dimensional function. The method removes noise at the same time. The local errors due to point by point fiber tracking can be avoided. Here in our case, the DTI data naturally provides the tangent vectors and we manipulate the normal vectors of the level set function to fit the data.

### 3.2 2D Curves Capturing

Our main goal is to relate the vector field to a gradient of the level set function. We want the normal of the vector field to align with the gradient of the level set function. This is equivalent to aligning the tangent of the level lines to the vector field. Ideally,  $\mathbf{n} \cdot \nabla_{\phi} = |\mathbf{n}| |\nabla \phi| \cos \theta$ . Here,  $\theta$  is the acute angle between the two vectors. If  $\theta = 0$ , we have  $|\mathbf{n}| |\nabla \phi| = |\nabla \phi|$  since  $|\mathbf{n}|$  is equal to 1. So, we investigate the possibility of using the minimization problem

$$\min_{\phi} \int_{\Omega} \left( |\nabla \phi| - \nabla \phi \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right) d\Omega \tag{3.1}$$

for tracking of the underlying curves. The vector field  $\mathbf{n}$  is the input vector field which we want to track. The idea behind the method is to find an image  $\phi$ , whose gradient approximately matches the vector field  $\mathbf{n}$ . In this case the tracks will be the level curves of  $\phi$ . We are trying to minimize the angle difference between the level line tangent and the vector tangent, thus we are trying to maximize the dot product.

Assume there is a 2D vector field, where each grid point contains a unit vector. If there is a "hidden" curve passing through certain grid points, then the unit vectors of these grid points indicate the respective tangential direction of this hidden curve. For those remaining grid points where the "hidden" curve does not pass through, their unit vectors are completely random. Assume further that the grid points where the curve passes through is also contaminated with noise, and more than one curve might be present. These curves may even branch. We want a method that can "denoise" the random vectors and "segment" these noisy hidden curves. Generally, in  $\mathbb{R}^n$ , subdomains are *n*-dimensional, while the interface has dimension *n*-1. We say the interface has codimension one[OF02]. Thus, curves in 2D are co-dimension 1 object that we can view as "interface" between two objects. The level set method was originally developed for curves in  $\mathbb{R}^2$  and surfaces in  $\mathbb{R}^3$ . Problems involving motion of curves in  $\mathbb{R}^3$  are usually done by front tracking[OP03]. However, merging and splitting may occur. [BCM01b] proposed using two level set functions to model 3D curve motion.

The corresponding Euler-Lagrange equations are:

$$\nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|} - \frac{\mathbf{n}}{|\mathbf{n}|}\right) = 0 \ in \ \Omega \tag{3.2}$$

with Neumann boundary condition

$$\left(\frac{\nabla\phi}{|\nabla\phi|} - \frac{\mathbf{n}}{|\mathbf{n}|}\right) \cdot \nu = 0 \text{ in } \partial\Omega \tag{3.3}$$

We try to avoid  $\nabla \phi$  to become too small so we want  $\phi$  to fulfill two conditions:

$$\begin{cases} \int \phi = 0, \\ \int |\phi|^2 = 1 \text{ or constant}, \end{cases}$$

In the original implementation to achieve uniqueness, the method is:

$$\phi_{new} = c \times \frac{\phi - \bar{\phi}}{(\int |\phi - \bar{\phi}|^2)^{1/2}}$$
(3.4)

where  $\bar{\phi}$  is the average of  $\phi$  and c is a constant. We pick c to be 20.

# 3.3 Numerical Scheme

Numerical implementation is as follows,

$$\frac{d^{n+1} - d^n}{\Delta t} = D_x^- \left(\frac{D_x^+ d^n}{T_1^n} - n_1\right) + D_y^- \left(\frac{D_y^+ d^n}{T_2^n} - n_2\right)$$
(3.5)

where

$$T_1 = \sqrt{(D_x^+ d^n)^2 + (A_x(C_y^h d^n))^2 + \epsilon}$$
(3.6)

$$T_2 = \sqrt{(D_y^+ d^n)^2 + (A_y(C_x^h d^n))^2 + \epsilon}$$
(3.7)

where  $A_x w = (w(x, y) + w(x + h, y))/2$  and  $A_y w = (w(x, y) + w(x, y + h))/2$ 

One of the interesting point to note in this equation is the first term which has the property of total variation regularization. Minimizing this term enables the  $\phi$ function to be piecewise smooth, thus it automatically have denoising property. The second term is to align the normal of the level lines to the normal of the vector field. Each point in all level lines are moving until the level lines are aligned with the vectors of the dataset. Therefore, at the true underlying bundles, there are more level lines being attracted to, and thus the gradient is higher in those regions. For noisy regions, the bad alignment would not give a minimum for the energy function so no level lines are situated at the noisy region. So this is equivalent to a denoising process.

# 3.4 Tracking and Segmentation

The resulting level set function provide us a powerful tool to track and segment the fibers. Since the level curves are aligned with the fibers, regions of the fibers appear to be high gradient, while the other non-fiber regions are flat after the total-variation-like denoising. A simple thresholding of the image gradient of  $\phi$ can capture ALL the fibers. In addition, the level set function captures different fibers at different levels. To track an individual fibers, all we need is to choose a bandwidth between two flat regions of the level set function. In summary, we can use thresholding to capture ALL fibers, and we can use bands between two flat levels to capture distinct fiber bundles. This is a really interesting property that we can obtain from the level set function based on the gradients and the intensity levels to obtain distinct connected bundles.

#### 3.5 Experimental Results

We first performed tests on synthetic vector fields.

#### 3.5.1 One Track

All the numerical computations are carried out in MATLAB by a Intel dual core 2.0 processor with 2G RAM. The time step is 0.0001 and the number of iterations



Figure 3.1: Clean vector field (a), noisy field (b). There is one underlying curve in the vector field.

is 60000. The result becomes stable after that. We start with an underlying straight line crossing diagonally in a 2D vector field. Each grid point along the line has a unit vector indicating the tangent of the line. We add random noise to the component of all vectors to create a noisy vector field. The original clean vector field and the noisy vector field is shown in Figure 3.1. We then perform our minimization process onto the noisy dataset.

Our algorithm does not require the user to indicate any seed point. As seen in Figure 3.2, the underlying line information is captured by a 2D level set function. The characteristics of the level set function is that all the level curves are aligned with the tangent of the underlying curves, thus creating a high gradient in those regions. The regions without curve contains only random noise but no structures, therefore, no level curve can be aligned with these random structures to minimize the function. Thus the noisy region is smoothed and gives a flat region.

From the level set function, it can be seen immediately that a bundle is located between level 0.47 to 0.51, which is between the levels of two flat region. The level lines between these levels are considered as the correct line bundle we are looking for, as in Figure 3.3.



Figure 3.2: Surface plot of the calculated level set function (a), image of the calculated level set function (b). The bundle is captured by the level curves of the level set function.



Figure 3.3: The level curves of the level set function are shown.



Figure 3.4: Clean vector field (a), noisy field (b). There is a thin track crossing a thick track in the vector field.

#### 3.5.2 Crossing Thick and Thin Tracks

Two-third of the voxels in white matter of human brain contain crossing fibers. We showed that our algorithm can handle fiber crossing in the synthetic data in the presence of noise. Sometimes the crossing bundles have different thickness. Traditional tracking method fails to capture the thinner bundle. Our algorithm successfully handle this case. In Figure 3.4, we show the original and noisy vector field where an underlying thinner track is crossing with a thicker track.

As can be seen from Figure 3.5, the level set function captures the crossing by 4 different levels of the level set function.

We can easily capture the crossing tracks from the level set function, as in Figure 3.5

#### 3.5.3 Branching Tracks

Next, we show the main motivation of our method, which is to allow a level set function to handle topological changes during branching. In Figure 3.6, we show



Figure 3.5: Surface plot of the calculated level set function (a), image of the calculated level set function (b). The crossing tracks are captured by different levels of one level set function. By selecting the appropriate levels, the tracks can be found easily through the level set function.

the original and noisy vector field where an underlying line bundle has branching. Traditional point by point tracking may have problem at those intersections, and parameterization is not straight forward to handle splitting/merging. Our algorithm handle the topological changes naturally by a higher dimensional level set function. Therefore, even during branching, the higher dimensional function can capture it by allowing multiple levels to handle the topological change in a lower dimension.

As can be seen from Figure 3.7, the level set function captures the branching by two different levels of the level set function.

We can easily track the branching level set function from the distinct levels where the bundle appears, as in Figure 3.7

#### 3.5.4 Two Tracks with Branching

Another powerful property of using our apporach is that more than one bundle can be captured with just one level set function. Moreover, the user do not need



Figure 3.6: Clean vector field (a), noisy field (b). There is one branching track in the vector field.



Figure 3.7: Surface plot of the calculated level set function (a), image of the calculated level set function (b). The branching track is captured by different levels of one level set function. By selecting the appropriate levels, the tracks can be found easily through the level set function.



Figure 3.8: Clean vector field (a), noisy field (b). There is two underlying curves in the vector field, one of them is a branching curve.

to specify in advance how many bundles they are expecting from the dataset. The user also does not need to specify the upper bound of the bundles, as the level set essentially have infinite number of levels to handle multiple bundles. Figure 3.8 shows the clean and noisy vector field. There are 2 bundles in the vector field, while one of them has branching.

Figure 3.9 shows that the level set function effectively capture the 2 bundles through different levels.

Figure 3.10 shows another interesting property of level set tracking, where we can distinguish different bundles easily by the level difference in the level set function. Connectivity is immediately known because of the level lines. Therefore, this algorithm has the potential for classification of bundles in terms of connectivity, thickness, length, total curvature, etc, with the inherent properties of level set method.



Figure 3.9: Surface plot of the calculated level set function (a), image of the calculated level set function (b). The curves are captured by the level curves of the level set function.



Figure 3.10: Tracking of track 1 (a), track 2 (b). The potential of selecting different curve to track is shown because the level set function captures the curves in distinctive levels.



Figure 3.11: Clean vector field (a), noisy field (b). There is one underlying bending curve in the vector field.

#### 3.5.5 Two High Curvature Tracks

When the bundle undergoes high curvature corner, the directional information is averaged out by adjacent voxels and give an isotropic direction. Since our method is global, the level curve can fit in the missing region globally. We show that the algorithm can capture not only straight line by bending curve bundles, as can be seen in Figure 3.11. Clean and noisy vector field are shown. This is another property of using level set. The level curve is not limited to a polynomial of fixed order such as a B-Spline. The curvature of the final curve depends on the global information of the fibers, thus our algorithm can capture curves of any curvature based on the energy function.

Figure 3.12 shows the result of the calculated level set function. We prove that circular bending bundle can be captured by our method.

Figure 3.13 shows the tracking result of the bundle.

We also show that multiple circular bending bundles can be captured. Figure 3.14 shows the clean and noisy vector field of two underlying bending curves.

Figure 3.15 shows the resulting level set function where we see two distinct



Figure 3.12: Surface plot of the calculated level set function (a), image of the calculated level set function (b). The bending curve is captured by the level curves of the level set function.



Figure 3.13: The level curves of the level set function are shown.



Figure 3.14: Clean vector field (a), noisy field (b). There are two underlying bending tracks in the vector field.

part with obvious high gradients indicating the presence of bundles.

Figure 3.16 again shows that we can capture whatever bundle we like by taking certain band of level lines from the level set function.

If we look at [WGJ97] and [Des08], there are short and long association fibers in the brain as shown in Figure 3.17, where the curvatures and locations are very similar to the synthetic data we just tested.

#### 3.5.6 Multiple Tracks

Figure 3.18 shows 5 bundles in a clean and noisy vector field.

Figure 3.19 shows 5 distinct gradient jumps in one level set function. We therefore can conclude that these five distinct bundles can be tracked individually through their specific level lines.

#### 3.5.7 Real Human Brain DTI Data Projected in 2D

We looked at two region of interest(ROI) from real DTI data projected into a 2D plane. The data we acquired was obtained by an average of 18 dataset. The first



Figure 3.15: Surface plot of the calculated level set function (a), image of the calculated level set function (b). The tracks are captured by the level curves of the level set function.



Figure 3.16: Tracking of track 1 (a), track 2 (b). The potential of selecting different tracks to track is shown because the level set function captures the curves in distinctive levels.



Figure 3.17: We show that two of the association fibers in human brain is similar to the synthetic case we just tested.



Figure 3.18: Clean vector field (a), noisy field (b). There are two underlying bending bundles in the vector field.



Figure 3.19: Surface plot of the calculated level set function (a), image of the calculated level set function (b). All the five bundles are captured by the level curves of the level set function and can be tracked easily by distinctive levels.



Figure 3.20: We show that two ROIs in human brain DTI data. Left: ROI 2, Right: ROI 1

region is near the corpus callosum area which has strong directional information. The second region shows fiber branchings. Figure 3.20 shows the location of the ROIs. The slice number is 53.

Figure 3.21 shows ROI 1 which is the corpus callosum. The region is characterized by horizontal tracks communicating between the left and right hemisphere, thus giving strong directional information at that region.

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Figure 3.21: We show the vector field of ROI 1



Figure 3.22:  $\phi$  (a), contours of  $\phi$  (b). There is one underlying fiber bundle in the vector field.



Figure 3.23: We show the vector field of ROI 2

Figure 3.22 shows the resulting  $\phi$  and the contours of  $\phi$  after 50000 iterations of time step 1e-5. As we can see, the algorithm successfully captures the fiber tracks.

Figure 3.23 shows ROI 2.

Figure 3.22 shows the resulting  $\phi$  and the contours of  $\phi$  after 100000 iterations of time step 1e-5. The algorithm successfully converges and captures the branching fiber tracks.

It is difficult to validate the accuracy of the results on real data. From visual



Figure 3.24:  $\phi$  (a), contours of  $\phi$  (b). There are multiple underlying fiber bundles in the vector field, one of them contains branches.

inspection, the result agrees with experts. The algorithm shows that it is possible to represent multiple fiber tracks with branchings in just one function.

## 3.6 Discussion

In our synthetic examples we assumed the vector field consists of only unit vectors. In reality, there are additional of information in DTI, which is the anisotropy of the diffusion tensor. The fractional anisotropy is measured by the ratio of the major eigenvalue versus the minor eigenvalues. A big anisotropy value represents a high magnitude of water diffusion in that direction. With this additional weight associated with each unit vector, we can easily extend our model to deal with weighted vector field. For the case of unit vector field, geometrically we are maximizing  $|\nabla \phi| |\nabla n| \cos \theta$  which is equal to  $|\nabla \phi| \cos \theta$  where  $\theta$  is the angle between the gradient of  $\phi$  and the normal of the vector, while minimizing the TV of  $\phi$ . When n is no longer a unit vector, the equation is  $|\nabla \phi| |\nabla n| \cos \theta$ . Therefore, the level curves of  $\phi$  can align even better with the underlying fiber because a bigger |n| can maximize the equation more favorably. Experimental results also proved the explanation.

# 3.7 Conclusion on 2D Curves Capturing

We successfully utilized all level curves of a function to capture the fibers, in the presence of noise and fiber branching and crossing. Previous attempts have tried to capture the motion of a single curve in an image. We capture all curves in a noisy vector field. The approach is new, and has the nice features including the edge preserving property of TV denoising and topology handling property of level set methods.

#### 3.8 Extension to 3D

A simple extension of our 2D algorithm to 3D is by adding a z component into the model. The model stays the same except  $\phi$  is a 3D function and n is a 3Dvector. The only difference is the numerical implementation,

Numerical implementation is as follows,

$$\frac{d^{n+1} - d^n}{\Delta t} = D_x^- \left(\frac{D_x^+ d^n}{T_1^n} - n_1\right) + D_y^- \left(\frac{D_y^+ d^n}{T_2^n} - n_2\right) + D_z^- \left(\frac{D_z^+ d^n}{T_3^n} - n_3\right) \quad (3.8)$$

where

$$T_1 = \sqrt{(D_x^+ d^n)^2 + (A_x(C_y^h d^n))^2 + \epsilon}$$
(3.9)

$$T_2 = \sqrt{(D_y^+ d^n)^2 + (A_y(C_x^h d^n))^2 + \epsilon}$$
(3.10)

$$T_3 = \sqrt{(D_z^+ d^n)^2 + (A_z(C_x^h d^n))^2 + \epsilon}$$
(3.11)

where  $A_x w = (w(x, y, z) + w(x + h, y, z))/2$ ,  $A_y w = (w(x, y, z) + w(x, y + h, z))/2$ and  $A_z w = (w(x, y, z) + w(x, y, z + h))/2$  However, level set methods are used to model codimension-one objects such as points in  $R^1$ , curves in  $R^2$ , and surfaces in  $R^3$ . Therefore, a surface is represented at each level set of  $\phi$  in 3D and we cannot track the curves in 3D. In order to model curves in  $R^3$  which is codimension-two, we need to represent it as the intersection of two level set functions as described in the next chapter.

# CHAPTER 4

# **3D** Level Curves Tracking

### 4.1 Introduction

We extend the 2D concept into tracking curves in 3D. The difference is we align tangent vector given by the intersection between two level set functions to the tangent vector of the given vector field. Using level set on 3D curve has been studied by Girogi[Gio94] and Ambrosio and Soner[AS96]. They used the squared distance as the distance function from one level set to capture the curve. The problems include accurate determination of the location of zero level sets, and the thickening problem during curve merging. Later, [BCM01a] used two level set functions to capture the motion of 3D curve, where the intersection of the zero level sets of two functions is used to represent the curve. No one has attempted to use two level set function to capture multiple curves in a noisy 3D vector field. We believe that the good properties, including edge preserving denoising and handling of topology changes can be extended from 2D to 3D.

## 4.2 Our 3D Model

Instead of analyzing the motion of a single curve, we again posted this as a minimization problem where we use ALL level lines of two level set functions to capture ALL curves. As in [BCM01a], we see that the tangent of a vector can be defined as the cross product of the normal of two level set functions:  $T = |\nabla \phi \times \nabla \psi|$ . Therefore, a natural extension from the 2D case plus a little change on the notion where we are trying to align the tangent T with exactly the vector field, instead of the normal of the vector field in the 2D case:

$$\min_{\phi,\psi} \int_{\Omega} \left( |\nabla\phi \times \nabla\psi| - \nabla\phi \times \nabla\psi \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right) d\Omega$$
(4.1)

To find the Euler-Lagrange equations, we first let

$$F(\phi,\psi) = \int_{\Omega} \left( |\nabla\phi \times \nabla\psi| - \nabla\phi \times \nabla\psi \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right) d\Omega$$
(4.2)

First, we want to find

$$\frac{\partial F}{\partial \phi} \cdot \mu = \lim_{\epsilon \to 0} \frac{F(\phi + \epsilon \mu, \psi) - F(\phi, \psi)}{\epsilon}$$
(4.3)

For a given vector in  $\mathbb{R}^3$ , we define the project operator to be:

$$P_w = I - \frac{w \otimes w}{|w|^2}.$$

It is known that

$$|P_w v| = \frac{|w \times v|}{|w|}, \quad \forall v, w \in \mathbb{R}^3.$$

$$(4.4)$$

In addition, we know that

$$(v_1 \times v_2) \cdot v_3 = v_1 \cdot (v_2 \times v_3), \quad \forall v_1, v_2, v_3 \in \mathbb{R}^3$$

Therefore, we have

$$\frac{\partial F}{\partial \phi} \cdot \mu = \int_{\Omega} \frac{\left( |P_{\nabla \psi}(\nabla \phi + \epsilon \nabla \mu)| - |P_{\nabla \psi} \nabla \phi| \right)}{\epsilon} |\nabla \psi| - \nabla \mu \times \nabla \psi \cdot \frac{\mathbf{n}}{|\mathbf{n}|}$$
(4.5)

As we know:

$$\frac{|x+\epsilon y| - |x|}{\epsilon} = \frac{(x+\epsilon y)^2 - x^2}{\epsilon(|x+\epsilon y|^2 + |x|^2)} \to \frac{2xy}{2x^2} = \frac{x}{|x|} \cdot y$$
(4.6)

For our case,

$$x = P_{\nabla \phi} \nabla \psi \tag{4.7}$$

$$y = P_{\nabla\psi} \nabla\mu \tag{4.8}$$

Therefore, by plugging in x and y, we have

$$\frac{\partial F}{\partial \phi} \cdot \mu = \int_{\Omega} \frac{P_{\nabla \psi} \nabla \phi \cdot P_{\nabla \psi} \nabla \mu}{|P_{\nabla \psi} \nabla \phi|} |\nabla \psi| - \nabla \mu \cdot (\nabla \psi \times \frac{\mathbf{n}}{|\mathbf{n}|})$$
(4.9)

So, writing the equation in this form, we have:

$$\frac{\partial F}{\partial \phi} \cdot \mu = \int_{\Omega} \left( \frac{P_{\nabla \psi} \nabla \phi \cdot P_{\nabla \psi} \nabla \mu}{|P_{\nabla \psi} \nabla \phi|} |\nabla \psi| - \nabla \mu \cdot (\nabla \psi \times \frac{\mathbf{n}}{|\mathbf{n}|}) \right)$$
(4.10)

Note the property of:

$$P_w u \cdot P_w v = P_w u \cdot v = u \cdot P_w v \tag{4.11}$$

So now we have:

$$= \int_{\Omega} \frac{P_{\nabla\psi} \nabla \phi \cdot \nabla \mu}{|P_{\nabla\psi} \nabla \phi|} |\nabla \psi| - \nabla \mu \cdot (\nabla \psi \times \frac{\mathbf{n}}{|\mathbf{n}|})$$
(4.12)

And now we can use the integration by parts:

$$\int_{\Omega} \mathbf{v} \cdot \nabla \mu = \int_{\partial \Omega} \nabla \mathbf{v} \cdot \nu \mu - \int_{\Omega} \nabla \cdot \mathbf{v} \mu$$
(4.13)

In the above,  $\nu$  is the unit out normal vector on  $\partial\Omega$ . So now the equation becomes:

$$= \int_{\partial\Omega} \left[ \left( \frac{P_{\nabla\psi} \nabla\phi}{|P_{\nabla\psi} \nabla\phi|} |\nabla\psi| - \nabla\psi \times \frac{\mathbf{n}}{|\mathbf{n}|} \right) \right] \cdot \nu\mu - \int_{\Omega} \nabla \cdot \left[ \left( \frac{P_{\nabla\psi} \nabla\phi}{|P_{\nabla\psi} \nabla\phi|} |\nabla\psi| - \nabla\psi \times \frac{\mathbf{n}}{|\mathbf{n}|} \right) \right] \cdot \mu$$
(4.14)

So,

$$\phi_t = \nabla \cdot \left[ \left( \frac{P_{\nabla \psi} \nabla \phi}{|P_{\nabla \psi} \nabla \phi|} |\nabla \psi| - \nabla \psi \times \frac{\mathbf{n}}{|\mathbf{n}|} \right) \right]$$
(4.15)

Boundary condition:

$$\left[\left(\frac{P_{\nabla\psi}\nabla\phi}{|P_{\nabla\psi}\nabla\phi|}|\nabla\psi|-\nabla\psi\times\frac{\mathbf{n}}{|\mathbf{n}|}\right)\right]\nu=0$$
(4.16)

Note that

$$\frac{P_{\nabla\psi}\nabla\phi}{|P_{\nabla\psi}\nabla\phi|}|\nabla\psi| = \frac{P_{\nabla\psi}\nabla\phi}{|\nabla\psi\times\nabla\phi|}|\nabla\psi|^2$$
(4.17)

For a given w and v, we evaluate the projection operator here by plugging in  $P_w$ :

$$(P_w v)|w|^2 = \begin{pmatrix} w_2^2 + w_3^2 & -w_1 w_2 & -w_1 w_3 \\ -w_1 w_2 & w_1^2 + w_3^2 & -w_2 w_3 \\ -w_1 w_3 & -w_2 w_3 & w_1^2 + w_2^2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$
(4.18)

For our problem, we need to set  $w = \nabla \psi, v = \nabla \phi$ .

So for the x-component, we have: So for the x-component, we have:

$$\frac{\left[(w_2^2 + w_3^2) \cdot v_1 + (-w_1w_2) \cdot v_2 + (-w_1w_3) \cdot v_3}{|w \times v|} - \frac{w_2 \cdot \mathbf{n}_3 - w_3 \cdot \mathbf{n}_2}{|\mathbf{n}|}$$
(4.19)

Similarly, to minimize with respect to  $\psi$ , with special attention to the positive and negative sign of cross product, we have:

$$\frac{\partial F}{\partial \psi} \cdot \mu = \lim_{\epsilon \to 0} \frac{F(\phi, \psi + \epsilon \mu) - F(\phi, \psi)}{\epsilon}$$
(4.20)

$$= \int_{\Omega} \frac{\left( |\nabla \phi \times \nabla (\psi + \epsilon \mu)| - |\nabla \phi \times \nabla \psi| - \nabla \phi \times \nabla \psi| - \nabla \phi \times \nabla \mu \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right)$$
(4.21)

$$\rightarrow \int_{\Omega} \frac{\left(\nabla\phi \times \nabla\psi\right) \cdot \left(\nabla\phi \times \nabla\mu\right)}{\left|\nabla\phi \times \nabla\psi\right|} + \nabla\mu \times \nabla\phi \cdot \frac{\mathbf{n}}{|\mathbf{n}|}$$
(4.22)

Note that there is a change of negative to positive sign on second term because the order of cross product has changed:

By using the same approach, we have

$$\frac{\partial F}{\partial \psi} \cdot \mu = \int_{\partial \Omega} \left[ \left( \frac{P_{\nabla \phi} \nabla \psi}{|P_{\nabla \phi} \nabla \psi|} | \nabla \phi | + \nabla \phi \times \frac{\mathbf{n}}{|\mathbf{n}|} \right) \right] \cdot \nu \mu - \int_{\Omega} \nabla \cdot \left[ \left( \frac{P_{\nabla \phi} \nabla \psi}{|P_{\nabla \phi} \nabla \psi|} | \nabla \phi | + \nabla \phi \times \frac{\mathbf{n}}{|\mathbf{n}|} \right) \right] \cdot \mu$$
(4.23)

So,

$$\psi_t = \nabla \cdot \left[ \left( \frac{P_{\nabla \phi} \nabla \psi}{|P_{\nabla \phi} \nabla \psi|} |\nabla \phi| + \nabla \phi \times \frac{\mathbf{n}}{|\mathbf{n}|} \right) \right]$$
(4.24)

Boundary condition:

$$\left[\left(\frac{P_{\nabla\phi}\nabla\psi}{|P_{\nabla\phi}\nabla\psi|}|\nabla\phi|+\nabla\phi\times\frac{\mathbf{n}}{|\mathbf{n}|}\right)\right]\cdot\nu=0\tag{4.25}$$

We can use  $(P_w v)|w|^2$  again, but this time we need to set  $w = \nabla \phi, v = \nabla \psi$ .

# 4.3 Numerical Implementation

Discretization for  $\phi_t$ 

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = D_x^{-1}(T_1^n) + D_y^{-1}(T_2^n) + D_z^{-1}(T_3^n)$$
(4.26)

The discritization of  $\nabla\psi\times\nabla\phi$  :

$$= (\psi_y \phi_z - \psi_z \phi_y, \ \psi_z \phi_x - \psi_x \phi_y, \ \psi_x \phi_y - \psi_y \phi_x)$$

$$(4.27)$$

So for example,

$$T_{1}^{n} = \left(\frac{P_{1} + P_{2} + P_{3}}{\sqrt{(F_{x})^{2} + (F_{y})^{2} + (F_{z})^{2} + \epsilon}} - \frac{D_{y}^{+}\psi\mathbf{n}_{3} - D_{z}^{+}\psi\mathbf{n}_{2}}{\sqrt{\mathbf{n}_{1}^{2} + \mathbf{n}_{2}^{2} + \mathbf{n}_{3}^{2} + \epsilon}} \right)$$
(4.28)

where

$$P_1 = \left(\nabla \psi_y^2 + \nabla \psi_z^2\right) \cdot \nabla \phi_x \tag{4.29}$$

$$P_2 = -\nabla \psi_x \cdot \nabla \psi_y \cdot \nabla \phi_y \tag{4.30}$$

$$P_3 = -\nabla \psi_x \cdot \nabla \psi_z \cdot \nabla \phi_z \tag{4.31}$$

$$F_{x} = D_{y}^{+}\psi D_{z}^{+}\phi - D_{z}^{+}\psi D_{y}^{+}\phi$$
(4.32)

$$F_{y} = D_{z}^{+}\psi D_{x}^{+}\phi - D_{x}^{+}\psi D_{y}^{+}\phi$$
(4.33)

$$F_z = D_x^+ \psi D_y^+ \phi - D_y^+ \psi D_x^+ \phi \tag{4.34}$$

For our numerical discretization, we use a finite difference scheme. We use forward difference for the spatial derivative and backward difference for the divergence. The boundary condition is tricky here because at each direction in  $R^3$ , we have cross products involving the other two directions. We use Neumann boundary condition and the result looks fine. Another challenge is the initial guess for  $\phi$  and  $\psi$ . We use an initial  $\phi$  such that its gradient in all three directions are constants, and we use a randomly generated  $\psi$  to run the iterations.


Figure 4.1: The top right shows the original vector field in one of the 10 slices of the 3D vector field. The bottom right shows the recovered tangent (blue) versus the original (red) vector. We show that the recovered tangent align with the original vector after many iterations.

#### 4.4 3D Numerical Examples

We first started with the simplest example where the 3D vector field is unidirectional and without noise. We want to check if we start with a pre-calculated  $\phi$  and  $\psi$  and the results will stay the same after many iterations. Figure 4.1 shows the original and recovered vectors after many iterations, which prove that the algorithm stays at the correct vector if the calculated tangents align with the vector field. Figure 4.2 shows one slice of the surface plot of the two level set functions. As shown, the two functions stay the same after many iterations.

Next we created a 3D vector field of size  $10 \times 10 \times 10$  with 2 fiber bundles. The first bundle is located at slice 3 and 4, with a width of 3 voxel, height of 2 voxels and length of 14 voxels passing through the slice horizontally and diagonally.



Figure 4.2: Surface plots of the final  $\phi$  and  $\psi$  after 40000 iterations at time step 1e-4.

The second bundle is located at slice 7 and 8 with the same dimensions. We filled the data with Gaussian noise with zero mean and 0.5 variance. We also normalized the vector field so each voxel contains a unit vector. Figure 4.3 shows two representative slice of the vector field. The 1,2,5,6,9 and 10th slices look like the vector field on the left. The 3 and 4, and 7 and 8 slices look like the vector field on the right. Our task is to see if our algorithm can track these two bundles from its noisy vector field. Figure 4.3 shows the recovered tangent vector from the cross product of  $\nabla \phi$  and  $\nabla \psi$ . On the left, we see that the tangent vectors do not align with the noisy vectors, and the magnitude of tangent vectors every voxel in that slice are significantly small, which indicates that the level set functions do not intersect in those regions. On the right hand side, we can see that the tangent vectors do not align with the align perfectly with the fiber bundle, with a big magnitude, and the tangent vectors do not align with the noise vectors. This is a natural extension of the 2D case where level lines are attracted to regions of



Figure 4.3: Noisy vector field with a track (slice 3, 4, 7 and 8) and without a fiber bundle (slice 1, 2, 5, 6, 9 and 10).

real fiber bundles. In Figure 4.4 we show the recovered tangent vectors computed from  $\phi$  and  $\psi$ . And we proved that our 3D algorithm works for multiple fiber bundle segmentation. Based on the magnitude of the tangents, we can use simple thresholding to find the location of the tracks from the tangent magnitudes (from 1 to 5) which are significantly bigger than the noisy regions(smaller than 0.5), as shown in Figure 4.5.

#### 4.5 Further Work

We presented the basic formulation of the 3D case but the details are not fully developed yet. Tracking 3D curves using 2 level set functions are much trickier than in 2D, due to the cross products. In the future, we plan to analyze the convergence and uniqueness of the 3D formulation.



Figure 4.4: Recovered tangents (blue) versus the original noisy vector field (Red). In this example, the tangents aligned with the vectors in the track, and do not align with the noise.



Figure 4.5: Surface plots of the detected bundles. We show that the two tracks are detected by our algorithm.

## CHAPTER 5

### Spatial-Varying Blind Image Deblurring

#### 5.1 Introduction

In this chapter, we proposed a method to simultaneously segment regions blurred by different blur kernels and restore the image. The method is an extension to the Chan-Wong Total Variation Blind Deconvolution [CW96]. We assume an image is blurred by different blur kernels at different part of the image, not necessary along the object boundaries. Our model is also based on an energy function similar to the Chan-Vese model [CV99] which is used for region-based segmentation of the blurred regions, and total variation is used to regularize the restored image and the point spread functions. There are three unknowns we need to minimze in our model: the boundaries of blurred regions, the respective blur kernels, and the sharp image. We first show that each of these unknowns can be found exactly given another two unknowns. We then analyze the possibility of recovering a sharp image and the segmentation of blurred regions simultaneously given the blur kernels. Finally, we investigate the simultaneous recovery of all three unknowns. We also perform numerical experiments on synthetic data, natural images as well as medical images.

Medical images are sometimes degraded by blurs during the acquisition process. One image may have more than one blur at different regions, and the type of point spread function (PSF) may also be unknown. For example, when a fixed camera shooting a road of multiple moving cars with slow shutter speed. (motion blur), the problem becomes a spatially varying blur. Sometimes we try to take a photograph of a person but mistakenly focused on the background and we want to deblur the photograph, this is also a spatially varying deblurring because the background is in focus while the person is out of focus. Besides natural images, medical images such as SPECT and MRI (blurred by T2\* decay) and astronomical images are often blurred by spatially varying blurs.

Most of the previous work on spatially varying blur assumes the blurring function is known. Nagy and O'Leary [NO97, JO98, JO98] developed a fast deconvolution algorithm via matrix-vector multiplication involving banded Toeplitz matrices, where the assumption is that the blurring support is small. Many other spatially varying deblurring methods have been proposed[CS05, CCS, MK08, BSK06, BSK07a]

Only very few works have addressed the spatially varying deblurring of unkown blurs, even though most of the images in daily encounter belong to this type of blur. The most representative work was by You and Kaveh[YK96] who proposed to use a parametric space-variant PSF with piecewise smooth space variance constraint in a variational framework. They utilized a parametric spacevariant PSF with piecewise smooth space variance constraint. The model suffers from stopping at local minima. The Laplacian method to detect blur by [BSK06] may not be robust to noise.

#### 5.2 Spatial Varying Deblurring Method

Chan and Wong[CW96] proposed a blind deconvolution algorithm which can simultaneously recover the image and the point spread function, while the blurring kernel is spatially invariant. The model is as follows,

$$\min_{u} f(u,h) = \min_{u,h} \frac{1}{2} ||h * u - z||_{L^{2}\Omega}^{2} + \alpha_{1} \int_{\Omega} |\nabla u| + \alpha_{2} \int_{\Omega} |\nabla h|$$
(5.1)

where  $\alpha_1$  and  $\alpha_2$  are positive parameters that determines the weight of regularity. The novelty is the addition of a TV regularization term to the point spread function and the use of alternating minimization to recover the image and the point spread function.

In the case of spatial varying blur, we assume an image is divided by two regions and each region is blurred by a distinct blur kernel. Therefore, a level set based formulation similar to Chan-Vese is used to locate the outside region blurred by  $k_1$  and the inside region blurred by  $k_2$ , for the case of two blurring kernels. The novelty is the TV regularization term for each point spread function and also a length term for the boundary. The objective function is:

$$E(\phi, u, k_1, k_2) = \frac{\alpha_{fit,1}}{2} \int_{\phi>0} (k_1 * u - f)^2 + \frac{\alpha_{fit,2}}{2} \int_{\phi<0} (k_2 * u - f)^2$$
(5.2)  
+ 
$$\int_{\Omega} |\nabla u| + \alpha_{k,1} \int_{\Omega_k} |\nabla k_1| + \alpha_{k,2} \int_{\Omega_k} |\nabla k_2| + \alpha_{\phi} \int_{\Omega} |\nabla H(\phi)|_3)$$

We have three unknowns to recover: the image u, the regions $\phi$ , and the blur kernels k. We use an alternating minimization scheme. Restoration of a blurred and noisy image is very difficult especially with an unknown blur regions because one has to mulitply the reciprocals of the vanishing multipliers from the low-pass filtered image, which is very unstable to noises or other high frequency small perturbations in the image data[CS05]. Tikhonov regularization leads to oversmoothed edges so we utilize TV regularization. We used the lagged diffusivity fixed point method and conjugate gradient as suggested in [VO96]. If the regions of blur is not known, the problem is even more difficult. We used alternating minimization to find the three unknowns. To find the image, we keep  $\phi$  and k fixed. We get the Euler-Lagrange equation as follows,

$$(\lambda K^* K + Lu)u = K^* f \tag{5.4}$$

Here, K is the spatially varying blur matrix introduced by [NO97].

To find the segmentation  $\phi$ , we obtain the Euler-Lagrange equation as follows,

$$\phi_t = \delta_\epsilon(\phi) \left[ -(k \ast u - f)^2 + (k \ast u - f)^2 + \alpha \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|} \right]$$
(5.5)

To find k, here we assume the blurs are parameterized out-of-focus blurs, so we need to find  $r_1$  and  $r_2$ . The Euler Lagrange equations are:

$$E(r_1) = \int_{\phi>0} k(r_1) * u - f)^2 + \frac{\mu_1}{k_1}$$
(5.6)

$$E(r_2) = \int_{\phi>0} k(r_2) * u - f)^2 + \frac{\mu_2}{k2}$$
(5.7)

## 5.3 Numerical Results to Recover the Sharp Image with Known PSF and Known Regions of Blurs

Figure 5.1 shows the results of image u when  $\phi$  and k are fixed.



Figure 5.1: Restoring the image with given  $\phi$  and k.

## 5.4 Numerical Results to Find Segmentation of Blurred Regions with Known Sharp Image and Known PSF

Figure 5.2 shows the results of segmentation  $\phi$  when u and k are fixed. Results also show that the radius of  $k_1$  and  $k_2$  are correctly determined if given u and  $\phi$ .

# 5.5 Numerical Results to Find Segmentation of Blurred Region with Unknown Regions of Blurs and Known PSF

Figure 5.3 shows the test when only k are given.

To find the segmentation of blurred region given only the blur kernels is very challenging and has not been done before. [BSK07b] proposed using a log laplacian method to find the blurred region, but the method is ad-hoc and not robust to noise. We performed an alternating minimization method where we first alternatively estimate the deblurred image and the segments. We perform our

Contours Juxtaposed



Contours Juxtaposed



Figure 5.2: Restoring the  $\phi$  with given sharp image and k.



Figure 5.3: Restoring the  $\phi$  and the image given k.

test on a synthetic image which has piecewise linear regions separated by edges. We blurred the outside region of the image by an out-of-focus blur of radius 1 and inside region by radius 4. Our initial guess is a circle 10 units outside the true segmentation. The alternating minimization algorithm is able to stop the curve evolution after the curve is aligned with the true segmentation. Thus the correct image is recovered. Figure 5.4 shows the test when only k are given. The image is correctly recovered.

#### 5.6 Discussion

We attempted to tackle the fundamentally important blind deblurring of spatialvarying out-of-focus blurs. We presented a variational PDE model for simultaneous segmentation of blurred region and recovery of image degraded by spatially varying blurs. The model is based on Chan-Wong which was originally used to find the point spread function and recover the image in the case of blind decon-



Figure 5.4: Resulting  $\phi$  and the image.

volution with spatially invariant blur. We extend it to solve spatially varying blur by adopting a Chan-Vese model to segment the regions of blurs. The functional is iteratively solved by alternating minimization. In our example we used parameterized blur kernels, in particular two out-of-focus blurs where the radii of the kernels are known. In the case of piecewise smooth or piecewise constant images, our model is able to segment the blurred regions and recover the sharp image from the blurred image with known point spread functions but unknown boundaries, as validated by successful example. We presented the basic formulation to extend the Chan-Wong model, but there are more work to be done in the future to add in more priors to find an unique solution.

## CHAPTER 6

### Wavelet and Total Variation Denoising in DTI

In this chapter, we introduce a combined BayesShrink and total variation method, which uses translational invariant BayesShrink wavelet thresholding with total variation regularization as an extension to [CZ07, DF], which successfully removes image noise and pseudo-Gibbs phenomena while preserving most texture and edges. We compare our results with other denoising methods proposed for DTI images based on visual and quantitative metrics.

#### 6.1 Introduction

The availability of fast wavelet transform allows wavelet denoising to perform in a high speed, which popularize the usage of wavelet denoising in medical imaging in the past decade. However, due to it's linear nature, it tends to smooth edges and creates artifacts known as the Gibbs Phenomena around edges. [Don95] introduced a soft thresholding scheme which is non-linear, but Gibbs phenomena still exists.

Edges are very important in medical diagnosis. Therefore, we hope to utilize the power of total-variation based method to regularize the wavelet coefficients. We applied an improved wavelet and total variation denoising scheme on Diffusion Tensor Imaging. Here we seek to have a direct image denoising method on the diffusion weighted images. Wavelet is not good for approximating piecewise smooth function because it penalize jumps and cause oscillations around edges known as Gibbs Phenomena. PDE-based methods have a very strong foundation in theoretical aspects such as convergence and stability properties, numerical schemes for implementation, and have been shown to work well when dealing with local features in images. Most importantly, TV allows discontinuities. However, TV sometimes suffers from loss of texture leading to the well-known 'staircase' effect.

Wavelet methods are local in nature and play an important role in image processing areas such as denoising[Don95]. Wavelet-based methods suffer from pseudo-Gibbs artifacts while PDE based methods such as Total variation (TV) preserve edges but suffer from loss of texture leading to the well-known 'staircase' effect. In this chapter, we first review wavelet denoising. We then propose an approach for noise reduction of DTI images based on combining adaptive Bayes-Shrink thresholding and Total Variation. We demonstrate our technique on DTI brain images obtained with sixteen averages and compare the denoised images both visually as well as with a quantitative metric with either denoising technique alone. Preliminary evaluation is performed by estimating the standard deviations of the signals in uniform tissue regions in the diffusion weighted images as well in the calculated Fractional Anisotropy (FA) images; the latter are very sensitive to noise in the raw diffusion weighted images. Denoising is performed separately on each diffusion-weighted image and the following techniques are compared: TV, wavelet methods with different thresholding schemes applied independently as well as methods based on the combination of wavelet and TV approach. The images denoised with the proposed combined method (wavelet denoising with adaptive Bayes-Shrink thresholding and TV-based) performed better than all other techniques and could potentially reduce the total DTI acquisition time by a factor of 2 or more.

#### 6.2 Related Work

Chan and Zhou[CZ07] used TV regularization to recover over-compressed images by aggressive wavelet thresholding. Durand and Froment[DF] claimed that the thresholded wavelet coefficients have some important structures that should not be thrown away. So they proposed to use TV to regulaize the thresholded coefficients. Weickert[MW07], Coifman and Sowa[CS01] and Malgouyres[Mal] proposed related ideas in a more general framework. The combination approaches [CZ07, DF] of wavelets and total variation, proposed so far, did not use BayesShrink adaptive thresholding. We will show that BayesShrink is a better choice due to the generalized Gaussian distribution in the wavelet domain.

#### 6.3 Wavelet + TV Model

We denoise the acquired images: baseline and images with diffusion sensitization along 6 directions. The noise is modeled as additive Gaussian, i.e. x = s + n, where x is the noisy image, s is the image we want to recover and n is the noise. The SNR of magnitude MR images is known to be approximately Gaussian for high SNR [GN] and since the images that were being denoised here were already acquired with six averages, an approximate Gaussian distribution was assumed.

Noise can be analyzed easily when the signals are decomposed into different scales. Assume an image x is an N by N signal degraded by a zero-mean Gaussian noise n. Given the noisy observations, our task is to find the best estimate of the original signal s as closely in terms of Mean-Square-Error as possible. We used a Daubechies-4 wavelet basis because it is orthogonal and provides good resolution in both spatial and frequency domain with its compactly supported basis. One problem of 2-D discrete wavelet transform (DWT) is that the transform is not

shift-invariant due to the downsampling operation. In this paper, we used, for all the different methods, an undecimated DWT[GLO95] which is shift-invariant.

Further, in order to apply the adaptive Bayes-Shrink thresholding scheme, it is important to confirm that the distribution of wavelet coefficients at each higher subband follows a Generalized Gaussian Distribution (GGD)[DV99].We analyzed the noise at each subband and verified that it does follow a Generalized Gaussian Distribution.

After applying the DWT to a noisy image, the equation becomes

$$X = W(s) + W(n) \tag{6.1}$$

where W(.) is the DWT, X, S, N are the wavelet decomposition of the original image x, clean image s, and noise n. X comprises of a few large coefficients (mostly useful signals such as edges of an image) and many small coefficients (mostly noise) due to the special properties of wavelet transform on piece-wise smooth functions. Wavelet denoising is therefore just simply keeping the large coefficients and turning the small coefficients (noise) into zero. The soft thresholding[Don95] operation can be represented as:

$$D(X,\lambda) = \begin{cases} sign(X)(||X|| - \lambda), & \text{for } ||X|| > \lambda; \\ 0, & \text{otherwise.} \end{cases}$$

where X is the wavelet coefficient and D is the threshold value. The threshold is  $\lambda = \sigma \sqrt{2 \log M}$  where M is the length of the signal and  $\sigma$  is the variance of the wavelet coefficients in the most detailed subband.

A further improvement to soft thresholding was proposed in [CYV] which used an adaptive threshold for a given subband based on the data and/or some a priori information. This subband adaptive threshold known as Bayes-Shrink was proposed for 2-D signals with detail subband coefficients having a generalized Gaussian (GG) distribution. For a large class of images, the wavelet coefficients of the detailed subbands obey a GG distribution for all levels. The PDF of a generalized Gaussian random variable is defined as

$$f_{\alpha,\beta} = \frac{\beta}{2\alpha\Gamma 1/\beta} e^{-\left(\frac{|x|}{\alpha}\right)^{\beta}} \tag{6.2}$$

where  $\Gamma(x) = \int_0^\infty e^{-z} z^{x-1} dx$ , x > 0 is the gamma function.

The two parameters of the generalized Gaussian density, and , control the overall form of the GGD and may vary from subband to subband. Bayes-Shrink proposed the optimal soft threshold for a given detail subband is

$$\lambda_B = \frac{\sigma_N^2}{\sigma_S} \tag{6.3}$$

where  $\sigma_N^2$  is the variance in finest scale and  $\sigma_S$  is the standard deviation of the wavelet coefficients in the subband of interest for the signal, provided that  $\beta$ lies between 0.5 and 4.  $\lambda_B$  is then used to threshold the wavelet coefficients. We verified that all the detail subbands of the DTI images satisfy the GGD as shown in Figure 6.1.

While texture information is well preserved, wavelet soft-thresholding using Bayes-Shrink still introduces artifacts because, as all wavelet coefficients are lowered, local averages are not preserved, thus leading to peaks being eroded. Therefore, edges (features) might not be well preserved in wavelet soft-thresholding. Two groups have proposed the combination of wavelet and TV minimization to remove the pseudo-Gibbs phenomenon generated by wavelet thresholding or reconstruct the edges being thrown away by wavelet thresholding respectively. Chan and Zhou [CZ07] proposed to solve

$$\min_{c_{j,n}} \alpha \int_{\Omega} |\nabla x_{C_{j,n}}| dx + \frac{1}{2} ||x - s||_{L^{2}(\Omega)}^{2}$$
(6.4)

where  $u_{C_{j,n}}$  is the function reconstructed using the wavelet coefficients;  $C_{j,n}$ are the retained coefficients after wavelet thresholding. Durand and Froment [DF] proposed a method where retained wavelet coefficients are not modified while the cancelled coefficients are used to perform TV minimization to reduce pseudo-Gibbs phenomena. The result works well in denoising and reduces pseudo-Gibbs phenomenon. Coifman and Sowa and Malgouyres proposed a similar idea in a more general framework. The combination approaches of wavelets and total variation, purposed so far, did not use BayesShrink adaptive thresholding, instead they only used Donoho's soft-thresholding. In this paper, we extend the combined schemes so that Bayes-Shrink adaptive thresholding is first performed before TV regularization is performed either on the retained or cancelled coefficients. We compare the results on 1-D single channel synthetic data and then compared and analyzed the effects of these denoising schemes on DTI images. Figure 6.8 and Figure 6.9 shows one channel of DTI and the computed Fractional Anistropy map respectively. Figure 6.10 shows the Region of Interest(ROI) studies. Table 6.1 shows the table that computes the standard deviation of ROIs.

#### 6.4 Numerical Results

Figure 6.2 shows the noisy and wavelet BayesSrhink denoised 1-D signals. The original signal is a constant intensity of 50 with a jump at location 60 to an intensity value of 100. The noise is partly suppressed by BayesShrink. Figure 6.3 shows the noisy and total variation denoised 1-D signals. The number of iterations

is 100. The noise is suppressed better than the BayesShrink but there are staircase effects. Figure 6.4 shows the noisy and the denoised image by wavelet BayesSrhink and total variation combined scheme. Compared with Figure 6.2 and 6.3, the noise is better suppressed with reduced staircase effects. Figure 6.5, 6.6 and 6.7 shows the 2D synthetic data with the wavelet, total variation and combined denoising schemes respectively. Gaussian noise is added to the synthetic data. As can be seen from Figure 6.7, the staircase effect in Figure 6.6 is reduced. Figure 6.8 shows the denoising schemes to one slice of one angle of DTI of one patient. Figure 6.9 shows the FA maps calculated from the denoised DTI data. Examination of the profiles near edges in the wavelet Bayes-Shrink scheme clearly shows that edges are a little blurred and some pseudo-artifacts are seen. Careful examination of the total variation method in Figure 6.6 shows the presence of the 'staircase' artifact in the boundaries of different constant regions which give the image an artificial appearance. Our combined approach using Bayes-Shrink with either TV on the retained or discarded coefficients is visually superior to either technique alone, with the proposed Bayes-Shrink with total variation on retained wavelet coefficients having the best visual results. This combined approach clearly do not show either the staircase or the edge artifacts while image noise is reduced. A comparison to an acquired image from 18 averages clearly shows that the image quality of the denoised image from 6 averages using the proposed method approaches that of the higher average (18 averages) acquisition. Further, visually we estimated the absence of the staircase and edge artifacts present in the original separate methods.

Table 6.1 shows one result of the 19 patients we tested our algorithms. It shows the standard deviations of different Region of Interest (ROI) which are 3 by 3 pixel windows. The locations of ROI are shown in Figure 6.10. Our combined scheme shows the lowest standard deviation as compared to wavelet and total variation methods alone. This visual analysis was also confirmed by ROI analysis (3x3 pixels) on five uniform tissue regions. Table 1 shows the quantitative results of the wavelet soft-thresholding, total variation and our proposed method. The standard deviation measure was used since in uniform tissue regions this value is minimized when noise is reduced. The noise in the background is not a reliable measure since different methods affect background noise in different ways. The standard deviations, which were calculated by averaging 5 regions of ROIs of all denoising schemes, have a lower value than the noisy image. Our proposed method has the best visual results and lower standard deviations than the 20-averaged image, although the standard deviations are higher than the total variation scheme. More detailed analysis using quantitative metrics are required to assess the artifacts in the denoised images. We computed the fractional anisotropic (FA) values after we denoised the whole brain data set of 455 images (65 slices, each slice 7 images). FA maps generated from the original and denoised images using the proposed method are shown in Figure 6.9. Again, the decrease in noise is clearly seen, especially in areas of low FA values which are particularly sensitive to noise in the images. We also performed an ROI analysis on FA images calculated from the raw images since FA is very sensitive to noise and can be used to assess the efficacy of closely performing denoising techniques. Preliminary quantitative results show a reduction of standard deviations in regions of ROIs of FA maps using our proposed method when compared to the 20-averaged FA. Currently, we have implemented all the denoising schemes separately on each channel (each diffusion weighted or baseline image). We plan to work with the complex data (instead of the routine magnitude reconstructed images) since complex images have true Gaussian noise for all signal values.



Figure 6.1: Generalized Gaussian distribution of HH level 3 subband of the DTI image.



Figure 6.2: Top: noisy signal. Bottom: Wavelet BayesShrink denoised signal.

Patient1	Original 20 average	Wavelet BayesShrink	Total Variation	Combined
ROI 1	502.40	502.40	357.71	397.25
ROI 2	349.52	349.52	335.58	329.26
ROI 3	486.50	486.50	428.11	436.71
ROI 4	836.98	836.98	720.47	775.25
ROI 5	308.86	308.08	245.53	251.23

Table 6.1: Standard deviation of ROIs in Figure 6.10.



Figure 6.3: Top: noisy signal. Bottom: Total vaviation denoised image.



Figure 6.4: Top: noisy signal. Bottom: signal denoised by combined wavelet and total variations scheme.



Figure 6.5: 2D step function. Top: original noisy image. Bottom: wavelet BayesShrink denoisied.



Figure 6.6: 2D step function. Top: original noisy image. Bottom: Total variation denoised.



Figure 6.7: 2D step function. Top: original noisy image. Bottom: Combined denoised.



Figure 6.8: DTI image of one slice at one angle. From left to right: original 20-averaged image, wavelet BayesShrink denoised image, total variation denoised image and combined-denoised image.



Figure 6.9: FA maps. From left to right: original 20-averaged image, wavelet BayesShrink denoised image, total variation denoised image and combined-denoised image.



Figure 6.10: Locations of ROIs of patient 1.

#### 6.5 Conclusion

We reviewed and compared several optimal image denoising schemes developed in recent years and applied them to noisy DTI data. We also extended and proposed an algorithm that showed superior results when denoising DTI images. The successful development of robust denoising algorithms will allow DTI images with high quality to be acquired in clinically relevant times. This will enable DTI images to be collected on a routine basis on patients with neurological conditions. Population based comparisons of diffusion tensor indices calculated from the denoised images will then allow automated detection of subtle abnormalities.

We used a Daubechies-4 wavelet basis because it is orthogonal and provides good resolution in both spatial and frequency domain with its compactly supported basis. One problem of 2-D discrete wavelet transform (DWT) is that the transform is not shift-invariant due to the downsampling operation.

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