

Augmented Lagrangian Method, Dual Methods and Split Bregman Iteration for ROF Model

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Abstract. In the recent decades the ROF model (total variation (TV) minimization) has made great successes in image restoration due to its good edge-preserving property. However, the non-differentiability of the minimization problem brings computational difficulties. Different techniques have been proposed to overcome this difficulty. Therein methods regarded to be particularly efficient include dual methods of CGM (Chan, Golub, and Mulet) [7] Chambolle [6] and split Bregman iteration [14], as well as splitting-and-penalty based method [28] [29]. In this paper, we show that most of these methods can be classified under the same framework. The dual methods and split Bregman iteration are just different iterative procedures to solve the same system resulted from a Lagrangian and penalty approach. We only show this relationship for the ROF model. However, it provides a uniform framework to understand these methods for other models. In addition, we provide some examples to illustrate the accuracy and efficiency of the proposed algorithm.

1 Introduction

Image restoration such as denoising and deblurring is one of the most fundamental task in image processing and is in general based on regularization. To preserve image edges and features during image regularization procedures is difficult but very desired. Recently the ROF model [23] has been demonstrated to be very successful in edge-preserving image restoration; see [9] [11] and references therein. Consequently the model attracted much attention and has been extended to high order models [8] [31] [18] [19] [16] [25] and vectorial models [24] [2] [10] for color image restoration [17] [27].

However, the computation of the ROF model suffers from serious nonlinearity and non-differentiability. In [23], the authors proposed an artificial time marching strategy to the associated Euler-Lagrange equation. This method is slow due to strict stability constraints in the time step size. Besides, the artificial time marching method computes solutions of not the exact ROF model, but

its approximation, say, regularized ROF model. Different techniques have been proposed to overcome this difficulty.

There are several methods regarded as particularly efficient. One approach is the dual methods [7] [5] [6], which is based on various dual formulations of the model. The other is split Bregman iteration [14], which uses functional splitting and Bregman iteration for constrained optimization [20] [30]. Similar to split Bregman iteration, another approach based on splitting and then alternating minimization of the penalized cost function was proposed in [28] [29]. In this paper, we present augmented Lagrangian method to solve the model and show that the dual method and split Bregman iteration can actually be either deduced from, or equivalent to our method.

2 ROF model and related numerical solvers

Assume $\Omega \subset \mathbb{R}^2$ is a bounded open subset (usually a rectangle in image processing) and $f : \Omega \rightarrow \mathbb{R}$ is an observed image. f often contains various degradation and can be noisy and blurred, which is usually modelled as

$$f = Ku + n, \quad (1)$$

where u is the true image, and K, n are the linear operator and noise respectively. The K operator may stand for the identity operator, or various blur operations such as Gaussian blur and motion blur. The noise n may denote Gaussian noise or salt-pepper noise or even others. Image restoration aims to recover u from f with some information of K and n .

In this paper we assume that n is some Gaussian white noise and K is a general blur operator. Since the variance of n and the blur kernel of K can usually be estimated, we further assume we know K and the variance of n exactly. With these knowledge, it's still difficult to recover u from f . Even in the pure denoising case ($K = I$), it's not an easy task to get u since we only know the variance of the random noise n . For pure deblur case in which $K \neq I$ and $n = 0$, we cannot directly solve $f = Ku$ to get u due to the compactness of K . The problem $f = Ku$ is ill-posed, and the solution would be highly oscillatory. Regularization on the solution should be considered. The restoration problem is thus presented using some regularity $R(u)$ as

$$\begin{aligned} \min_u R(u) \\ \text{s.t. } \|f - Ku\|^2 = \sigma^2, \end{aligned} \quad (2)$$

where σ is the variance of n . The constrained minimization problem is often solved approximately using Tikhonov regularization as follows

$$\min_u F(u) = R(u) + \frac{\lambda}{2} \|Ku - f\|^2, \quad (3)$$

for some parameter λ .

There are many choices for the regularity term $R(u)$. One of the most basic and successful choice of the regularity is due to Rudin, Osher, and Fatemi [23] in which $R(u)$ was chosen to be the total variation of u . The so-called ROF model reads

$$u = \arg \min_u F_{\text{rof}}(u) = \int_{\Omega} |\nabla u| + \frac{\lambda}{2} \|Ku - f\|^2. \quad (4)$$

In [23] the authors considered the image denoising problem ($K = I$) and presented a gradient descent method to solve (4). (Here the method is described for general K .) The artificial time marching was introduced to the associated Euler-Lagrange equation as follows

$$\begin{aligned} u_t &= \nabla \cdot \left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \beta}} \right) + K^*(f - Ku), \\ u(0) &= f \end{aligned}, \quad (5)$$

where β is a small positive number to avoid zero division and K^* is the L_2 adjoint of K .

There are mainly two drawbacks for the gradient descent method (5). At first, the method computes the solution of (4) not exactly, but approximately. On the second, the method is slow due to strict constraints on the time step size. The choice of β affects both aspects. Larger the β , more efficient the scheme is, whereas worse the approximation will be. There is a tradeoff between the accuracy and efficiency in choosing β .

Many algorithms have been proposed to improve this method. Those regarded as particularly efficient include dual methods and split Bregman iteration, as well as splitting-and-penalty based method, as mentioned before.

Before we go on, we present here an obviously equivalent formulation of the restoration problem (4), which will play an important roll in our derivation. The difficulty to solve the ROF restoration model (4) is due to the non-differentiability of the total variation norm. We introduce an auxiliary variable q for ∇u to separate the calculation of the non-differentiable term and the fidelity term. The model (4) is thus equivalent to

$$\begin{aligned} \min_{u,q} G_{\text{rof}}(u, q) &= \int_{\Omega} |q| + \frac{\lambda}{2} \|Ku - f\|^2 \\ \text{s.t.} \quad q &= \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = \nabla u \end{aligned}, \quad (6)$$

a constrained optimization problem.

2.1 CGM dual method

In [7] Chan *et al* presented a primal-dual method for the TV minimization. They introduced a new variable

$$\omega = \frac{\nabla u}{|\nabla u|} \quad (7)$$

to the Euler-Lagrange equation of the model (4), yielding

$$\begin{aligned} -\nabla \cdot \omega + \lambda K^*(Ku - f) &= 0 \\ \nabla u - \omega |\nabla u| &= 0 \end{aligned}, \quad (8)$$

to remove some of the singularity caused by the non-differentiability of the object functional.

Different from the original Euler-Lagrange equation for u , this system contains both u and ω variables. In [7], u and ω are called the primal and dual variables, respectively. Again the authors approximate this primal-dual system using a regularized TV norm for real calculation. Newton's linearization technique for both the primal and dual variables is used to solve the discrete version.

2.2 Chambolle's dual method

Another work based on dual formulation with a slightly different derivation is due to Chambolle. In [6] Chambolle used Legendre-Fenchel transform and a key result from optimization theory to get an original and efficient algorithm for total variation minimization. The primal variable of the image data is expressed explicitly with the dual variable and only the dual variable is iteratively computed. The primal variable u is obtained from the final result of the dual variable. However, the algorithm dose not consider general K operators.

Specifically, Chambolle adopted the following definition of total variation for general (not necessary to be smooth) function u :

$$\text{TV}(u) = \sup\left\{\int_{\Omega} u(x)\nabla \cdot \xi(x) : \xi \in C_c^1(\Omega; \mathbb{R}^2), |\xi(x)| \leq 1, \forall x \in \Omega\right\}. \quad (9)$$

Denoting

$$S = \text{Closure}\{\nabla \cdot \xi(x) : \xi \in C_c^1(\Omega; \mathbb{R}^2), |\xi(x)| \leq 1, \forall x \in \Omega\}, \quad (10)$$

Chambolle showed that the ROF restoration model (4) with $K = I$ (Note the slight difference between Eqn. (4) and the model in [6] about the parameter λ) yields

$$u = f - \frac{1}{\lambda} \pi_S(\lambda f) = f - \pi_{\frac{S}{\lambda}}(f), \quad (11)$$

where $\pi_S(\cdot)$ is the L_2 norm projection operator to S , which reads

$$\pi_S(\cdot) = \arg \min_{\text{div}\xi(x)} \{\|\text{div}\xi(x) - \cdot\|^2 : |\xi(x)| \leq 1, \forall x \in \Omega\}. \quad (12)$$

Since S is not a linear space, this projection is nonlinear. From the KKT conditions and with a careful observation, it was shown in [6] that $\xi(x)$ for $\pi_S(\lambda f)$ satisfies

$$-\nabla(\text{div}\xi(x) - \lambda f) + |\nabla(\text{div}\xi(x) - \lambda f)|\xi(x) = 0, \quad (13)$$

which can be solved by a semi-implicit gradient descent algorithm. Note here we present the continuous case instead of the discrete version used in [6].

2.3 Split Bregman iteration

Recently (split) Bregman iteration attracts much attention in signal recovery and image processing community. The basic idea is to transform a constrained optimization problem to a series of unconstrained problems. In each unconstrained problem, the object function is defined by Bregman distance [3] of a convex functional.

The Bregman distance of a convex functional $J(u)$ is defined as the following (nonnegative) quantity

$$D_J^p(u, v) \equiv J(u) - J(v) - \langle p, u - v \rangle, \quad (14)$$

where $p \in \partial J(v)$.

When $J(u)$ is a continuously differentiable functional, its sub-differential $\partial J(v)$ has a single element for each v , and consequently the Bregman distance is unique. In this case the distance is just the difference at the point u between $J(\cdot)$ and its first order approximation at the point v . For non-differentiable functionals, the sub-differential may contain none or multiple values. Therefore, the Bregman distance between u and v can be ill-defined or multivalued. However, this poses no difficulty for the iterative algorithms as the algorithms automatically choose a unique sub-gradient in each iteration as long as the fidelity term for the constraints is differentiable (this condition holds usually). We also want to emphasize here that Bregman distance of a functional is not a distance in the usual sense since, in general, $D_J^p(u, v) \neq D_J^p(v, u)$ and the triangle inequality does not hold. See [20] [30] for more details.

To find the solution of the ROF model (4), or equivalently the constrained problem (6), split Bregman iteration (In [14] algorithms for $K = I$, say, TV denoising are presented) solves a sequence of unconstrained problems taking the form as

$$(u^{k+1}, q^{k+1}) = \arg \min_{u, q} D_{G_{\text{rof}}}^{(p_u^k, p_q^k)}((u, q), (u^k, q^k)) + \frac{r}{2} \int_{\Omega} |q - \nabla u|^2, \quad (15)$$

where p_u^k, p_q^k , sometimes written together to be (p_u^k, p_q^k) , are the sub-gradients of G_{rof} at (u^k, q^k) with respect to u and q , respectively. Taking the update of the sub-gradients into consideration, the iteration procedure can be formulated as Algorithm 1. For the computation of (u^{k+1}, q^{k+1}) , we refer to Algorithm 3 for more details.

Algorithm 1 Split Bregman iteration for the ROF model

1. *Initialization:* $q^0 = 0, u^0 = 0, p_q^0 = 0, p_u^0 = 0$;
2. *For* $k=0, 1, 2, \dots$: Compute (u^{k+1}, q^{k+1}) using Eqn. (15), and update

$$\begin{aligned} p_u^{k+1} &= p_u^k - r \operatorname{div}(q^{k+1} - \nabla u^{k+1}) \\ p_q^{k+1} &= p_q^k - r(q^{k+1} - \nabla u^{k+1}) \end{aligned} \quad (16)$$

3 Augmented Lagrangian method, and relations to dual methods and split Bregman iteration

In this section we present augmented Lagrangian method [15] [21] [22] for the ROF model, or equivalently the constrained problem (6). Augmented Lagrangian method has many advantages over other methods such as penalty method [1], and has been successfully applied to nonlinear PDE and mechanics [13]. We also show that the dual methods and split Bregman iteration can be either deduced from, or equivalent to augmented Lagrangian method.

3.1 Augmented Lagrangian method

In augmented Lagrangian method, one solves the constrained optimization problem (6) by

$$\min_{u,q} \max_{\mu} L_{\text{rof}}(u, q, \mu) = \int_{\Omega} |q| + \frac{\lambda}{2} \|Ku - f\|^2 + \int_{\Omega} \mu \cdot (q - \nabla u) + \frac{r}{2} \int_{\Omega} |q - \nabla u|^2, \quad (17)$$

where $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ is the Lagrange multiplier and r is a positive constant. That is, the method is to seek a saddle point of the augmented Lagrangian functional $L_{\text{rof}}(u, q, \mu)$. The system of optimality conditions is thus

$$\frac{\partial L_{\text{rof}}}{\partial u} = \lambda K^*(Ku - f) + \nabla \cdot \mu + r \nabla \cdot (q - \nabla u) = 0, \quad (18)$$

$$\frac{\partial L_{\text{rof}}}{\partial q} = \frac{q}{|q|} + \mu + r(q - \nabla u) = 0, \quad (19)$$

$$\frac{\partial L_{\text{rof}}}{\partial \mu} = q - \nabla u = 0. \quad (20)$$

We now have two ways to solve the problem (17). One is using optimization techniques to directly minimize/maximize corresponding functionals; while the other is solving the associated system of optimality conditions.

The augmented Lagrangian method uses an iterative procedure to solve (17); see Algorithm 2. The iterative scheme runs until some stopping condition is satisfied.

To solve the problem (21), we separate it to the following two sub-problems ([28] [29]):

$$\arg \min_u \frac{\lambda}{2} \|Ku - f\|^2 - \int_{\Omega} \mu^k \cdot \nabla u + \frac{r}{2} \int_{\Omega} |q - \nabla u|^2, \quad (23)$$

for given q , and

$$\arg \min_q \int_{\Omega} |q| + \int_{\Omega} \mu^k \cdot q + \frac{r}{2} \int_{\Omega} |q - \nabla u|^2, \quad (24)$$

Algorithm 2 Augmented Lagrangian method for the ROF model

1. *Initialization:* $u^0 = 0, q^0 = 0, \mu^0 = 0$;
2. For $k=0, 1, 2, \dots$: compute (u^{k+1}, q^{k+1}) as a minimizer of the augmented Lagrangian method for the Lagrange multiplier μ^k , i.e.,

$$(u^{k+1}, q^{k+1}) = \arg \min_{u, q} L_{\text{rof}}(u, q, \mu^k), \quad (21)$$

where $L_{\text{rof}}(u, q, \mu^k)$ is defined in Eqn. (17); and update

$$\mu^{k+1} = \mu^k + r(q^{k+1} - \nabla u^{k+1}). \quad (22)$$

for given u . Sub-problems (23) and (24) can be efficiently solved. For (23), the optimality condition gives a linear equation

$$\lambda K^*(Ku - f) + \text{div} \mu^k + r \text{div} q - r \Delta u = 0$$

for u , which allows us to use Fast Fourier transforms. Denoting $\mathcal{F}(u)$ as the Fourier transform of u , we get u from

$$u = \mathcal{F}^{-1} \left(\frac{\lambda \mathcal{F}(K^*) \mathcal{F}(f) - \mathcal{F}(\text{div}) \cdot \mathcal{F}(\mu^k) - r \mathcal{F}(\text{div}) \cdot \mathcal{F}(q)}{\lambda \mathcal{F}(K^*) \mathcal{F}(K) - r \mathcal{F}(\Delta)} \right), \quad (25)$$

where applying Fourier transform to a vector such as div and μ^k means applying Fourier transform to its components, respectively; and Fourier transforms of operators such as $K, \partial_x, \partial_y, \Delta$ are regarded as the transforms of their corresponding convolution kernels (for differential operators, the kernels will be approximated by kernels of difference operators). For (24), we actually have the following closed form solution

$$q = \begin{cases} \frac{1}{r} \left(1 - \frac{1}{|w(x, y)|}\right) w(x, y), & |w(x, y)| > 1, \\ 0, & |w(x, y)| \leq 1, \end{cases} \quad (26)$$

where $w = r \nabla u - \mu^k$, since we can reformulate the problem to be

$$\arg \min_q \int_{\Omega} |rq| + \frac{1}{2} \int_{\Omega} |rq - (r \nabla u - \mu^k)|^2.$$

Based on these observation, we can use Algorithm 3 to solve (21). Here N

Algorithm 3 Augmented Lagrangian method for the ROF model – solve the sub-problem of Eqn. (21)

1. *Initialization:* $u^{k+1,0} = u^k, q^{k+1,0} = q^k$;
 2. For $n = 0, 1, 2, \dots, N$: Compute $u^{k+1, n+1}$ from Eqn. (25) for $q = q^{k+1, n}$; and then compute $q^{k+1, n+1}$ from Eqn. (26) for $u = u^{k+1, n+1}$;
 3. $u^{k+1} = u^{k+1, N}, q^{k+1} = q^{k+1, N}$.
-

can be chosen using some convergence test techniques. In common augmented Lagrangian method, one usually sets $N = 1$.

As for the second approach to solve the problem (17), people can use some other iterative procedures to solve the corresponding optimality system. Actually the optimality system naturally infers CGM and the dual method of Chambolle as shown in the following.

3.2 Relations between augmented Lagrangian method and dual methods as well as split Bregman iteration

In this sub-section we show that CGM and Chambolle's dual methods for the ROF model can be deduced naturally from the augmented Lagrangian method. This is a much simpler derivation of the dual methods. Also split Bregman iteration is demonstrated to be equivalent to Algorithm 2.

Connection to CGM dual method We first show that CGM dual method can be deduced from the augmented Lagrangian method. The optimality conditions for the augmented Lagrangian approach are given in (18)–(20). From Eqn. (20), we get $q = \nabla u$. Combining this with (19), we see that

$$\mu = -\frac{\nabla u}{|\nabla u|}. \quad (27)$$

Therefore, the dual variable in CGM dual method is nothing but the Lagrange multiplier μ with a different sign. Hence, the system of optimality conditions (18)–(20) is equivalent to

$$\begin{aligned} \nabla \cdot \mu + \lambda K^*(Ku - f) &= 0 \\ \nabla u + \mu |\nabla u| &= 0, \end{aligned} \quad (28)$$

which is just the primal-dual system of CGM dual method if one replaces $-\mu$ with ω .

Connection to Chambolle's dual method We now further derive Chambolle's dual method. From the first equation of (28), we get u as:

$$u = (\lambda K^* K)^{-1}(\lambda K^* f - \operatorname{div} \mu), \quad (29)$$

yielding the equation for the dual variable

$$\nabla((K^* K)^{-1}(\lambda K^* f - \operatorname{div} \mu)) + |\nabla((K^* K)^{-1}(\lambda K^* f - \operatorname{div} \mu))| \mu = 0. \quad (30)$$

For image denoising problems where $K = I$, (30) and (29) are just the equations used by Chambolle in [6] to solve the dual variable and recover the primal variable u , respectively. The equation (30) for the dual variable in [6] was obtained through a not well-known KKT conditions for inequalities constrained optimization problems, whereas here we deduce this equation very naturally from the augmented Lagrangian method. This is a generic formulation and is not discussed in [6]. We also point out here that some connections between CGM and Chambolle's dual methods have been noticed in [32].

Connection to split Bregman iteration The split Bregman iteration is actually equivalent to the augmented Lagrangian method. Considering the zero initialization for the sub-gradients and the Lagrange multiplier and letting

$$(p_u^k, p_q^k) = -(\operatorname{div} \mu^k, \mu^k) \quad (31)$$

for each k , we have

$$\begin{aligned} & (u^{k+1}, q^{k+1}) \\ &= \arg \min_{u, q} D_{G_{\text{rof}}}^{(p_u^k, p_q^k)}((u, q), (u^k, q^k)) + \frac{r}{2} \int_{\Omega} |q - \nabla u|^2 \\ &= \arg \min_{u, q} \int_{\Omega} |q| + \frac{\lambda}{2} \|Ku - f\|^2 + \int_{\Omega} u \operatorname{div} \mu^k + \int_{\Omega} \mu^k \cdot q + \frac{r}{2} \int_{\Omega} |q - \nabla u|^2 \\ &= \arg \min_{u, q} \int_{\Omega} |q| + \frac{\lambda}{2} \|Ku - f\|^2 - \int_{\Omega} \mu^k \cdot \nabla u + \int_{\Omega} \mu^k \cdot q + \frac{r}{2} \int_{\Omega} |q - \nabla u|^2 \\ &= \arg \min_{u, q} L_{\text{rof}}(u, q, \mu^k), \end{aligned}$$

indicating the equivalence between split Bregman iteration and the iterative procedure for augmented Lagrangian method. In the context of compressive sensing, this equivalence has been pointed out in [30].

3.3 Remark

We want to emphasize that our observations can be extended to many other models including anisotropic TV, high order nonlinear PDE filters (e.g. fourth order models), vectorial TV, and even general models. Similarly, we can use FFT-based fast solvers and closed form solutions to solve the sub-problems for the corresponding algorithms. In addition, one can also derive naturally the dual methods [12] [26] [4] from the system of optimality conditions of augmented Lagrangian functionals for these models. Furthermore, the equivalence between split Bregman iteration and augmented Lagrangian method is also valid for these models. More details will be given in a forthcoming paper.

4 Examples

Two numerical examples are provided in Fig. 1 and Fig. 2 to illustrate the accuracy and efficiency of our method. We compare our method with some built-in Matlab functions, i.e. *deconvwnr.m*, *deconvreg.m* and *deconvlucy.m* in Fig. 1. As one can see, our method generates much better restoration than these built-in Matlab functions in comparable (or even less) CPU time costs. We also compare our method (with increasing parameter r) in Fig. 2 with the recently developed FTVd package based on pure splitting-and-penalty, which is one of the most efficient approaches as compared to other existing methods as discussed in [29]. From Fig. 1 and 2 people can also compare FTVd with our method with fixed parameter r .

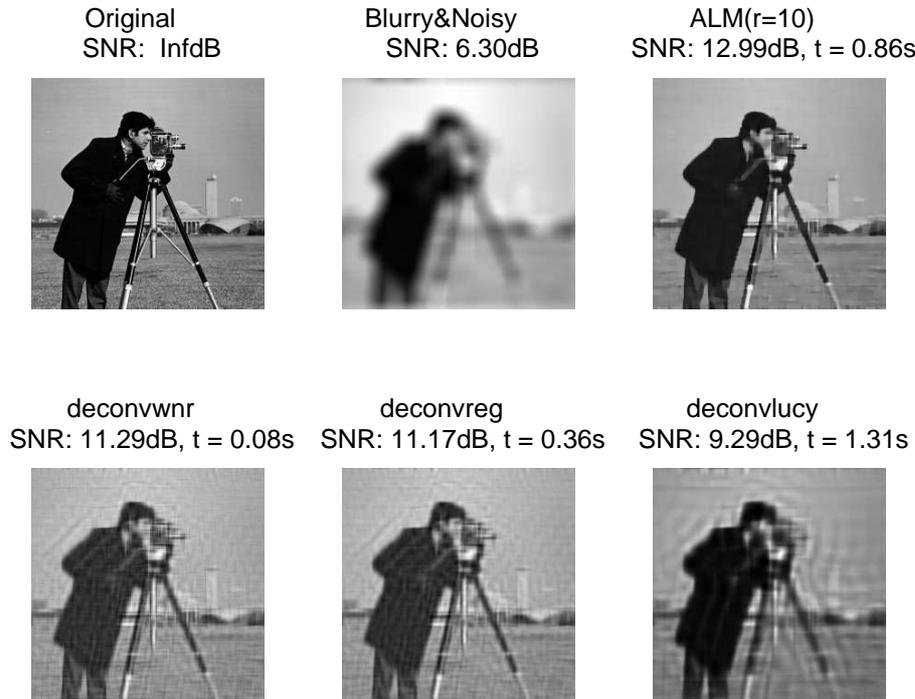


Fig. 1. Augmented Lagrangian method for ROF restoration, and comparisons to built-in Matlab functions. In the sub-figures, SNR and t denote signal-noise-ratio and the CPU time usage, respectively.

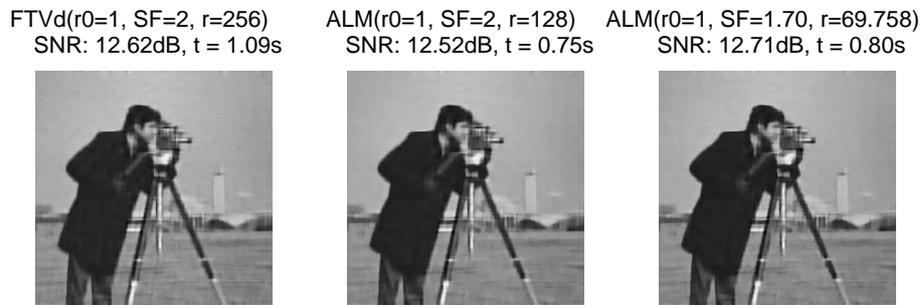


Fig. 2. Comparisons between FTVd package (splitting-and-penalty) and augmented Lagrangian method with increasing penalty parameters for ROF restoration. In the sub-figures, r_0 , SF, r stand for the initial value, the scaling factor and the final value of the penalty parameter of methods, respectively; And SNR and t denote signal-noise-ratio and the CPU time usage, respectively.

5 Conclusions

In this paper we use an approach based on augmented Lagrangian method for ROF model. The algorithm benefits from FFT-based fast solvers and closed form solution. We also show that our method gives a uniform framework to understand the approaches currently regarded to be particularly efficient for ROF model, such as dual methods and split Bregman iteration. The CGM and Chambolle's dual methods can be naturally derived from our method and the dual variables in these methods are nothing but the Lagrange multiplier. Split Bregman iteration is actually equivalent to augmented Lagrangian method. Numerical examples demonstrate the accuracy and efficiency of our approach. The method can be extended to many other restoration models.

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