REPRESENTATION AND PROPERTY ADJUSTMENTS OF SURFACE DIFFEOMORPHISMS USING BELTRAMI HOLOMORPHIC FLOW

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Abstract. In this work, we propose a novel idea of representing surface diffeomorphisms using Beltrami coefficients, which are complex-valued functions defined on the surface that describe local non-conformal distortions of the surface map. According to Quasiconformal Teichmuller theory, there is an one-to-one correspondence between the set of surface diffeomorphisms fixing three points and the set of Beltrami coefficients with L^{∞} -norm strictly less than 1. Every surface diffeomorphism is associated with a unique Beltrami coefficient. Conversely, given every such coefficient, we can reconstruct the associated diffeomorphism exactly using a flow, called the Beltrami holomorphic flow, which solves the Beltrami equation. The use of Beltrami coefficients to represent surface diffeomorphisms is a powerful method because it captures the most essential features of surface maps, such as conformality distortions, rotational changes and dilations. By adjusting the Beltrami coefficient, we can adjust the surface diffeomorphism accordingly to obtain desired properties of the map. Moreover, the Beltrami holomorphic flow guarantees to give a smooth sequence of surface diffeomorphisms. Therefore, a sequence of surface diffeomorphisms can be represented by a sequence of Beltrami coefficient and can be reconstructed by the Beltrami holomorphic flow. Using this approach, we propose several applications to properties adjustment of surface maps. It includes the accurate alignment of landmark curves, the reconstruction of surface diffeomorphisms, the construction of Riemann maps and the restoration of diffeomorphic and conformality properties. We apply our algorithms on different Riemann surfaces. Experimental results show that the Beltrami coefficient can effectively assist us to represent and adjust surface diffeomorphisms.

Key words. Beltrami coefficient/equation, holomorphic flow, genus zero surface diffeomorphism, conformal map, landmark matching/alignment, Riemann map

1. Introduction. In shape analysis, it has become increasingly popular to find diffeomorphisms of surfaces satisfying certain properties for further study [4, 5, 14, 17]. Besides, finding special diffeomorphisms such as the conformal parameterizations of surfaces is also of great theoretical and practical interest. Two types of surfaces commonly occur in the study of shapes, especially in medical imaging and face recognition. The first type is simply connected closed surface like hippocampal or cortical surface in medical research. The second type is simply connected open surface like human face in face recognition or a patch of an internal organ in medical research.

In various situations of surface registration, different constraints like landmarkmatching and conformality-preserving are enforced to get desirable diffeomorphisms [16, 11, 14]. A good diffeomorphism usually preserves local geometry well. However, getting these diffeomorphisms is often a time-consuming process. An increasing number of constraints tend to cause an increasing number of distortions or misaligned landmarks in the final map. To deal with these problems, it is desirable to adjust the diffeomorphism in a certain region while keeping the outside region fixed. In the rest of this paper, closed surface means simply-connected closed surfaces with genus zero, open surface means simply-connected surface patch with smooth boundary, and surface means an instance of one of these two types of surfaces.

In this paper, we propose to represent diffeomorphisms of both closed and open surfaces using the Beltrami coefficient, which is an easily computable complex-valued function defined on its domain. It measures the local non-conformal distortion of the diffeomorphism at every point of the domain. Given the set of all diffeomorphisms of a closed or open surface S fixing three pairs of corresponding points, there is a oneto-one correspondence between them and the set of Beltrami coefficients on S with L^{∞} -norm strictly less than 1. Each such diffeomorphism is the unique solution of the Beltrami equation, which specifies the value of the Beltrami coefficient at every point of the domain. On both types of surfaces, we show that it is possible to solve the Beltrami equation exactly using the Beltrami holomorphic flow introduced in Section 3. In other words, we can obtain a diffeomorphism of a surface with any Beltrami coefficient or distortion we specify.

Motivated by the above facts, we propose the use of Beltrami coefficient to finely adjust the conformality and other properties of surface maps. This is achieved by perturbing the original Beltrami coefficient and flowing the diffeomorphism to satisfy the new Beltrami coefficient. Moreover, we can adjust the mapping properties locally by changing the Beltrami coefficient in the region we are interested while fixing the outside region. By damping the Beltrami coefficient outside the region and flowing the diffeomorphism, we can preserve the shape of the image of the region under the diffeomorphism while mapping the exterior of the region conformally to the target surface. In this way, a precise diffeomorphism fixing a particular region can be obtained with the degree of conformality fine-tuned by the Beltrami coefficient. This approach provides a powerful tool to regulate surface diffeomorphisms.

Using this approach, we propose several applications for adjusting surface maps. The first application is the restoration of surface diffeomorphisms from distorted nondiffeomorphic maps. The second application is the restoration of conformality of surface diffeomorphisms. The third application is the construction of conformal parameter from any simply connected surface patch. This application is also a novel method of constructing the generalized Riemann maps from surfaces as in the Riemann mapping theorem.

In Section 2, we present some previous work in this area. In Section 3, we introduce the mathematical theories behind this work. Section 5 illustrates our proposed methods in solving the above problems. Finally, we conclude our work and some future directions in Section 6.

2. Previous Work. Computing diffeomorphisms between surfaces has been extensively studied by different research groups for shape analysis. Grenander et al. [3] proposed a framework based on constructing diffeomorphisms between shapes to perform computational inference on population and disease testing. Vaillant et al. [13] proposed that diffeomorphisms matching landmarks points between surfaces be computed before statistics is performed using a tangent space representation of diffeomorphisms.

On the other hand, conformal maps have been widely used since they preserve local geometry of the surface well. This method is motivated by the generalized Riemann mapping theorem, stating that every genus zero closed or open surface is conformally equivalent to the Riemann sphere or a unit disk respectively. For cases in 2D, Symm [12] proposed computing conformal mappings by solving integral equations. Levin et al. [9] proposed the use of the Bergman Kernel Method to compute conformal mapping of simply-connected domains.

Conformal maps between Riemann surfaces are also widely studied. For genus zero surfaces, Yau et al. [6] proposed a variational approach to compute the conformal parameterization by minimizing harmonic energy. The method is applied in medical research to map brain surfaces conformally onto a canonical domain before study is made [5]. As for higher genus surfaces, they proposed the use of holomorphic 1-form to conformally parameterize the surfaces onto 2D rectangles [7]. In order to deal with the problem of aligning landmarks, they further proposed the idea of slit conformal

parameterization [15]. They used the Ricci flow to conformally flow an old metric to a new metric with a constant Gaussian curvature. The resulting map is a conformal map that aligns landmarks to horizontal slits in the planar domain. Besides, Wang et al [14] proposd a variational approach to locally distort the conformality of the parameterization so as to better align landmarks. The resulting map is not conformal but the landmark mismatch error is greatly reduced. However, as the number of landmark lines increases, it becomes increasingly difficult to obtain well-behaved maps that are free from distortions. Very often, a diffeomorphic map which is one-to-one and onto cannot be obtained.

In this paper, we propose to solve this problem by solving the Beltrami equation. It has been of great theoretical interest for a long time because of its deep connection with geometry, especially quasiconformal geometry and Teichmüller theory [2]. Solving the Beltrami equation is also a more general problem than finding the Riemann map, as the distortion of different area of the map can be specified. Daripa [1] proposed two fast algorithms to solve the Beltrami equation in the complex plane and the interior of a unit disk based on the fast evaluation of two integrals. He [8] proposed an efficient discrete algorithm to solve Beltrami equations using circle packing, which can also be used to find Riemann maps in 2D.

3. Mathematical Theories. In this section, we discuss the mathematical theories related to orientation-preserving diffeomorphisms of the Riemann sphere and the unit disk, which we denote by $\overline{\mathbb{C}}$ and \mathbb{D} respectively. The Riemann sphere can be realized as the complex plane compactified by a point at infinity. These theories give us basic tools to work with genus zero closed surfaces and surface patches as they are conformally equivalent to $\overline{\mathbb{C}}$ and \mathbb{D} respectively.

3.1. The Beltrami Equation and Beltrami Coefficient. Let S_1 and S_2 be two simply connected closed surfaces and f be an orientation-preserving diffeomorphism between them. By the uniformization theorem, S_1 and S_2 are conformally equivalent to the Riemann sphere. Let $\varphi_1: S_1 \to \overline{\mathbb{C}}$ and $\varphi_2: S_2 \to \overline{\mathbb{C}}$ be their conformal parameterizations. Then $\tilde{f} = \varphi_2 \circ f \circ \varphi_1^{-1}$ is an orientation-preserving diffeomorphism of $\overline{\mathbb{C}}$. Without loss of generality, we may assume that \tilde{f} fixes 0, 1 and ∞ . This can be done by choosing three points a, b and c on S_1 and adjusting φ_1 and φ_2 so that they map a, b, c to $0, 1, \infty$ and f(a), f(b), f(c) to $0, 1, \infty$ respectively. To study diffeomorphisms between simply connected closed surfaces, it suffices to understand their corresponding diffeomorphisms of $\overline{\mathbb{C}}$ fixing 0, 1 and ∞ .

Let f be a complex-valued function on a domain in \mathbb{C} with continuous partial derivatives. f is said to be quasiconformal if it is orientation-preserving and satisfies the Beltrami equation

$$\frac{\partial f}{\partial \overline{z}} = \mu(z) \frac{\partial f}{\partial z},\tag{3.1}$$

where μ is some complex-valued Lebesgue measurable function satisfying $\sup |\mu| < 1$. If we write f = u + iv, where u and v are real-valued functions defined on $\overline{\mathbb{C}}$, then f_z and $f_{\overline{z}}$ are defined as:

$$f_{z} = \frac{1}{2} [u_{x} + v_{y} + (v_{x} - u_{y})i]$$

$$f_{\overline{z}} = \frac{1}{2} [u_{x} - v_{y} + (u_{y} + v_{x})i]$$
(3.2)



FIG. 3.1. The unit circle with an upright cross and its image under the linear quasiconformal map $z \mapsto 0.8(z + 0.25e^{\pi i/4}\overline{z})$.

The Beltrami equation admits a geometrical interpretation locally. Around every $z_0 \in \mathbb{C}$, we can write

$$f(z) \approx f(z_0) + T(z - z_0) = f(z_0) + A [z - z_0 + \mu(z_0)(\overline{z - z_0})],$$
(3.3)

where T is the differential of f at z_0 and A is a complex constant. Here, $T: \mathbb{C} \to \mathbb{C}$ is the real-linear map defined as

$$T(w) = A(w + \mu \overline{w}), \tag{3.4}$$

where $\mu = \mu(z_0)$. It is easy to check that T is an orientation-preserving bijection if an only if $|\mu| < 1$. Therefore, a quasiconformal map induces an orientation-preserving homeomorphism from its domain onto its image. Note that T is the composition of the stretch map $S(w) = w + \mu \overline{z}$ and a conformal multiplication by A. Therefore all of the distortion caused by T is expressed by the complex number μ , which is called the complex dilation or the Beltrami coefficient. An illustration of the effect of μ is shown in Figure 3.1. From the stretch map S, we can find the angle of maximal magnification of T to be $(\arg \mu)/2$ with magnifying factor $1 + |\mu|$, and its angle of maximal shrinking to be $(\arg \mu - \pi)/2$ with shrinking factor $1 - |\mu|$. The distortion or dilation of T is defined as:

$$K(T) = \frac{1 + |\mu|}{1 - |\mu|}.$$
(3.5)

4. Representation of surface diffeomorphisms with Beltrami coefficients. In order to adjust the properties of surface diffeomorphism, we firstly have to find an effective way to represent them. The representation should be simple and capture the properties of the surface diffeomorphism well. It turns out that Beltrami coefficient, which is a complex-valued function defined on the surface, is a powerful and useful tool for us to represent surface diffeomorphism. According to Quasiconformal Teichmuller theory, given the set of all diffeomorphisms fixing three pairs of corresponding points, there is a one-to-one correspondence between these diffeomorphisms and Beltrami coefficients with L^{∞} -norm strictly less than 1. Each such diffeomorphism is the unique solution of the Beltrami equation. Therefore, given a surface diffeomorphism $f : S_1 \to S_2$, we can get a representation of f by computing its associated Beltrami coefficient $\mu_f = \frac{f_{\overline{x}}}{f_z}$. Conversely, given a Beltrami coefficient μ_f that is associated to a certain surface diffeomorphism f, we can reconstruct f by solving the Beltrami equation. In order to solve the Beltrami equation, we can use the *Beltrami holomorphic flow* as described in the following subsections. **4.1. Beltrami holomorphic flow on Genus zero Closed Surfaces.** In the last section, it is highlighted that studying orientation-preserving diffeomorphisms between simply connected closed surfaces is equivalent to studying orientation-preserving diffeomorphisms of $\overline{\mathbb{C}}$ fixing 0, 1 and ∞ . In this section, we consider the theory of such diffeomorphisms.

Suppose $\mu(z)$ is a measurable complex-valued function defined on \mathbb{C} such that sup $|\mu| = k < 1$. It was shown by Bojarski [2] that there is a quasiconformal map fsatisfying the Beltrami equation. In fact, the following theorem asserts the existence of normalized solutions to the Beltrami equation on $\overline{\mathbb{C}}$ and the dependence of the solution f on μ :

THEOREM 4.1 (The Mapping Theorem). The equation (3.1) gives a one-to-one correspondence between the set of quasiconformal homeomorphisms of $\overline{\mathbb{C}}$ that fix the points 0, 1, and ∞ and the set of measurable complex-valued functions μ on $\overline{\mathbb{C}}$ for which $\sup |\mu| = k < 1$. Furthermore, the normalized solution f^{μ} to (3.1) depends holomorphically on μ . Let $\{\mu(t)\}$ be a family of Beltrami coefficients depending on a real or complex parameter t. Suppose also that $\mu(t)$ is differentiable at t = 0, that is, $\mu(t)$ can be written in the form

$$\mu(t)(z) = \mu(z) + t\nu(z) + t\epsilon(t)(z)$$
(4.1)

for $z \in \mathbb{C}$, with suitable μ in the unit ball of $L^{\infty}(\mathbb{C})$, $\nu, \epsilon(t) \in L^{\infty}(\mathbb{C})$ such that $\| (\epsilon(t) \|_{\infty} \to 0 \text{ as } t \to 0$. Then

$$f^{\mu(t)}(w) = f^{\mu}(w) + t\dot{f}^{\mu}[\nu](w) + o(|t|)$$
(4.2)

locally uniformly on \mathbb{C} as $t \to 0$, for $w \in \mathbb{C}$, and where

$$\dot{f}[\nu](w) = -\frac{f^{\mu}(w)(f^{\mu}(w)-1)}{\pi} \int_{\mathbb{C}} \frac{\nu(z)((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))} \, dx \, dy.$$
(4.3)

This theorem motivates us to represent an orientation-preserving genus zero surface diffeomorphism using its Beltrami coefficient and three-point correspondence. Indeed, using equation (4.3), we can accurately flow a diffeomorphism between two such surfaces to another diffeomorphism with a slightly perturbed μ . This gives us a way to solve the Beltrami equation on $\overline{\mathbb{C}}$ with an arbitrary Beltrami coefficient satisfying the condition of the theorem. Starting from the identity map on $\overline{\mathbb{C}}$, we can adjust the map iteratively using equation (4.3) with a slightly different μ until we finally get the diffeomorphism with our desired μ . Moreover, in every iteration of the flow, it is easy to require that $\sup |\mu| < 1$. By our previous discussion, this automatically ensures that every intermediate and the final map in the process is an orientation-preserving genus zero surface diffeomorphism. Besides reconstructing genus zero surface diffeomorphisms, equation (4.3) also allows us to finely adjust the conformality and other properties of genus zero surface maps.

Specifically, given two genus zero closed surface S_1 and S_2 respectively. We first map them conformally onto the Riemann sphere. This can be done using various approaches such as harmonic energy minimization. To reconstruct the diffeomorphism associated with the prescribed Beltrami coefficient, we start with the identity map from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$. Using equation (4.3), we can adjust the Beltrami coefficient of the map slight each iteration to get a diffeomorphism of $\overline{\mathbb{C}}$ fixing 0, 1 and ∞ . We continue to iterate until the desired Beltrami coefficient is satisfied. In this way, the required diffeomorphism from S_1 to S_2 can also be obtained via the composition maps.

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4.2. Beltrami holomorphic flow on Surface Patches. Let \mathcal{K}_1 and \mathcal{K}_2 be two surface patches and f be an orientation-preserving diffeomorphism between them. By the uniformization theorem, \mathcal{K}_1 and \mathcal{K}_2 are conformally equivalent to the unit disk. Let $\varphi_1 \colon \mathcal{K}_1 \to \mathbb{D}$ and $\varphi_2 \colon \mathcal{K}_2 \to \mathbb{D}$ be their conformal parameterizations. Then $\tilde{f} = \varphi_2 \circ f \circ \varphi_1^{-1}$ is an orientation-preserving diffeomorphism of \mathbb{D} . Without loss of generality, we may assume that \tilde{f} fixes 0 and 1. In what follows, we consider the theory of orientation-preserving diffeomorphisms of $\mathbb D$ fixing 0 and 1, which helps us to finely adjust maps between surface patches. In order to do so, we need to modify equation (4.3) so that it works on the unit disk instead of the Riemann sphere.

PROPOSITION 4.2. Let $f: \mathbb{D} \to \mathbb{D}$ be a diffeomorphism of the unit disk fixing 0 and 1, and satisfies the Beltrami equation $f_{\overline{z}} = \mu f_z$ with μ defined on \mathbb{D} . Let f be the extension of f to $\overline{\mathbb{C}}$ defined as

$$\tilde{f}(z) = \begin{cases} f(z), & \text{if } |z| \le 1, \\ \frac{1}{f(1/\overline{z})}, & \text{if } |z| > 1. \end{cases}$$
(4.4)

Then \tilde{f} satisfies the Beltrami equation

$$\tilde{f}_{\overline{z}} = \tilde{\mu}\tilde{f}_z \tag{4.5}$$

on $\overline{\mathbb{C}}$, where the Beltrami coefficient $\tilde{\mu}$ is defined as

$$\tilde{\mu}(z) = \begin{cases} \mu(z), & \text{if } |z| \le 1, \\ \frac{z^2}{\overline{z^2}} \overline{\mu(1/\overline{z})}, & \text{if } |z| > 1. \end{cases}$$
(4.6)

Proof. First of all, we prove \tilde{f} satisfies the Beltrami equation:

$$\tilde{f}_{\overline{z}} = \tilde{\mu}\tilde{f}_z$$

Clearly, \tilde{f} satisfies equation (4.5) on \mathbb{D} . Outside \mathbb{D} , we consider f and \tilde{f} as functions in z and \overline{z} . Note that:

$$\frac{\partial}{\partial z}\overline{f(z,\overline{z})}=\overline{\frac{\partial}{\partial z}f(z,\overline{z})}$$

we have:

$$\begin{aligned} \frac{\partial \tilde{f}(z,\overline{z})}{\partial z} &= \frac{\partial}{\partial z} \frac{1}{\overline{f(1/\overline{z},1/z)}} = -\overline{f(1/\overline{z},1/z)}^{-2} \frac{\partial}{\partial z} \overline{f(1/\overline{z},1/z)} \\ &= -\overline{f(1/\overline{z},1/z)}^{-2} \overline{\frac{\partial}{\partial \overline{z}}} f(1/\overline{z},1/z) = -\overline{f(1/\overline{z},1/z)}^{-2} \overline{(-1/\overline{z}^2)} f_z(1/\overline{z},1/z) \\ &= z^{-2} \overline{f(1/\overline{z},1/z)}^{-2} \overline{f_z(1/\overline{z},1/z)}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial \tilde{f}(z,\overline{z})}{\partial \overline{z}} &= \frac{\partial}{\partial \overline{z}} \frac{1}{\overline{f(1/\overline{z},1/z)}} = -\overline{f(1/\overline{z},1/z)}^{-2} \frac{\partial}{\partial \overline{z}} \overline{f(1/\overline{z},1/z)} \\ &= -\overline{f(1/\overline{z},1/z)}^{-2} \overline{\frac{\partial}{\partial z}} f(1/\overline{z},1/z) = -\overline{f(1/\overline{z},1/z)}^{-2} \overline{(-1/z^2)} \overline{f_{\overline{z}}(1/\overline{z},1/z)} \\ &= \overline{z}^{-2} \overline{f(1/\overline{z},1/z)}^{-2} \overline{f_{\overline{z}}(1/\overline{z},1/z)} = \overline{z}^{-2} \overline{f(1/\overline{z},1/z)}^{-2} \overline{\mu(1/\overline{z})} \overline{f_{z}(1/\overline{z},1/z)} \end{aligned}$$

Now,

$$\overline{f_z(1/\overline{z}, 1/z)} = z^2 \overline{f(1/\overline{z}, 1/z)}^2 \frac{\partial \tilde{f}(z, \overline{z})}{\partial z}$$

Thus, we have,

$$\begin{split} \frac{\partial \tilde{f}(z,\overline{z})}{\partial \overline{z}} &= \overline{z}^{-2} \overline{f(1/\overline{z},1/z)}^{-2} \overline{\mu(1/\overline{z})f_z(1/\overline{z},1/z)} \\ &= \overline{z}^{-2} \overline{f(1/\overline{z},1/z)}^{-2} \overline{\mu(1/\overline{z})} z^2 \overline{f(1/\overline{z},1/z)}^2 \frac{\partial \tilde{f}(z,\overline{z})}{\partial z} \\ &= \frac{z^2}{\overline{z^2}} \overline{\mu(1/\overline{z})} \frac{\partial \tilde{f}(z,\overline{z})}{\partial z} = \tilde{\mu}(z) \frac{\partial \tilde{f}(z,\overline{z})}{\partial z}. \end{split}$$

THEOREM 4.3. The normalized solution f^{μ} depends holomorphically on μ . Let $\{\mu(t)\}$ be a family of Beltrami coefficients depending on a real or complex parameter t. Suppose also that $\mu(t)$ is differentiable at t = 0, that is, $\mu(t)$ can be written in the form

$$\mu(t)(z) = \mu(z) + t\nu(z) + t\epsilon(t)(z)$$
(4.7)

for $z \in \mathbb{D}$, with suitable μ in the unit ball of $L^{\infty}(\mathbb{D})$, $\nu, \epsilon(t) \in L^{\infty}(\mathbb{D})$ such that $\| (\epsilon(t) \|_{\infty} \to 0 \text{ as } t \to 0$. Then:

$$f^{\mu(t)}(w) = f^{\mu}(w) + t\dot{f}^{\mu}[\nu](w) + o(|t|)$$
(4.8)

locally uniformly on \mathbb{C} as $t \to 0$, for $w \in \mathbb{C}$, and where

$$\dot{f}[\nu](w) = -\frac{f^{\mu}(w)(f^{\mu}(w) - 1)}{\pi} \\
\left(\int_{\mathbb{D}} \frac{\nu(z)((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z) - 1)(f^{\mu}(z) - f^{\mu}(w))} \, dx \, dy + \int_{\mathbb{D}} \frac{\overline{\nu(z)}(\overline{(f^{\mu})_{z}(z)})^{2}}{\overline{f^{\mu}(z)}(1 - \overline{f^{\mu}(z)})(1 - \overline{f^{\mu}(z)})} \, dx \, dy. \right).$$
(4.9)

Proof. According to Quasiconformal Teichmuller Theory, there is an one-to-one correspondence between the set of quasiconformal homeomorphisms of $\overline{\mathbb{C}}$ fixing 3 points and the set of measurable complex-valued functions μ on $\overline{\mathbb{D}}$ for which $\sup |\mu| = k < 1$. If a diffeomorphism f on $\overline{\mathbb{C}}$ satisfies equation (4.5), then $1/\overline{f(1/\overline{z})}$ also satisfies the same equation. By the uniqueness of the solution according to Theorem 4.1, we must have $f(z) = 1/\overline{f(1/\overline{z})}$. On $\partial \mathbb{D}$, $z = 1/\overline{z}$. This implies $f(z) = 1/\overline{f(z)}$, and hence |f(z)| = 1 on $\partial \mathbb{D}$. Therefore, by restricting the solution of equation (4.5) on $\overline{\mathbb{C}}$ fixing 0, 1 and ∞ to \mathbb{D} , we can get a diffeomorphism of \mathbb{D} fixing 0 and 1. Equation (4.3) can then be applied on \mathbb{D} to get diffeomorphisms of \mathbb{D} fixing 0 and 1 that satisfy different Beltrami coefficients. It can also be used to adjust the conformality of diffeomorphisms on \mathbb{D} as we do on $\overline{\mathbb{C}}$ in Section 4.1. To get the corresponding flow on \mathbb{D} , we evaluate the integral in equation (4.3). For simplicity, we consider $\tilde{f} = f$

$$\int_{\mathbb{C}} \frac{\nu(z)((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))} \, dx \, dy$$
$$= \int_{\mathbb{D}} \frac{\nu(z)((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))} \, dx \, dy + \int_{\mathbb{C}\setminus\mathbb{D}} \frac{\nu(z)((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))} \, dx \, dy$$

Now, outside the disk \mathbb{D} ,

$$\nu(z) = \frac{z^2}{\overline{z^2}} \overline{\nu(1/\overline{z})} \quad \text{and} \quad \frac{\partial \tilde{f}(z)}{\partial z} = z^{-2} \overline{f(1/\overline{z}, 1/z)}^{-2} \overline{f_z(1/\overline{z}, 1/z)}$$

We have:

$$\begin{split} &\int_{\mathbb{C}} \frac{\nu(z)((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))} \, dx \, dy \\ &= \int_{\mathbb{D}} \frac{\nu(z)((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))} \, dx \, dy + \int_{\mathbb{C}\setminus\mathbb{D}} \frac{\nu(z)((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))} \, dx \, dy \\ &= \int_{\mathbb{D}} \frac{\nu(z)((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))} \, dx \, dy + \int_{\mathbb{C}\setminus\mathbb{D}} \frac{(z^{2}/\overline{z})\overline{\nu(1/\overline{z})}((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))} \, dx \, dy \\ &= \int_{\mathbb{D}} \frac{\nu(z)((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))} \, dx \, dy + \int_{\mathbb{D}} \frac{(z^{2}/\overline{z})\overline{\nu(z)}((f^{\mu})_{z}(1/\overline{z}))^{2}}{f^{\mu}(1/\overline{z})^{-1} - 1)(\overline{f^{\mu}(1/\overline{z})^{-1}} - f^{\mu}(w))} \frac{1}{|z|^{4}} \, dx \, dy \\ &= \int_{\mathbb{D}} \frac{\nu(z)((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))} \, dx \, dy + \int_{\mathbb{D}} \frac{\overline{\nu(z)}((\overline{f^{\mu})_{z}(z))^{2}}}{\overline{f^{\mu}(z)}(1-\overline{f^{\mu}(z)})f^{\mu}(w))} \, dx \, dy \end{split}$$

Substituting Equation 22 into Equation 4.3, we get an integral flow equation on $\mathbb D$ which is given by

$$\begin{split} \dot{f}[\nu](w) &= -\frac{f^{\mu}(w)(f^{\mu}(w)-1)}{\pi} \\ \left(\int_{\mathbb{D}} \frac{\nu(z)((f^{\mu})_{z}(z))^{2}}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))} \, dx \, dy + \int_{\mathbb{D}} \frac{\overline{\nu(z)}(\overline{(f^{\mu})_{z}(z)})^{2}}{\overline{f^{\mu}(z)}(\overline{1-f^{\mu}(z)})(1-\overline{f^{\mu}(z)}f^{\mu}(w))} \, dx \, dy. \right) \\ \Box \end{split}$$

This theorem gives us an extension of the Beltrami holomorphic flow to solve the Beltrami equation on $\overline{\mathbb{D}}$ with an arbitrary Beltrami coefficient. More generally, we can use equation (4.3) to find the quasiconformal diffeomorphism between two open surfaces with boundaries, that is associated with a prescribed Beltrami coefficient. Specifically, given two open surfaces S_1 and S_2 respectively. We first map them conformally onto the unit disk \mathbb{D} , using, for example, the discrete Yamabe flow method. To reconstruct the diffeomorphism associated with the prescribed Beltrami coefficient, we start with the identity map from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$. Using equation (4.3), we can adjust the Beltrami coefficient of the map slight each iteration to get a diffeomorphism of $\overline{\mathbb{C}}$ fixing 0, 1 and ∞ . We continue to iterate until the desired Beltrami coefficient is satisfied. Finally, the required diffeomorphism from S_1 to S_2 can also be obtained via the composition maps.

5. Applications. In this section, we show that the theory behind quasiconformal diffeomorphisms of the Riemann sphere and unit disk can be used to adjust surface maps in a fairly general setting. First, we show that our algorithm can be used to reconstruct surface diffeomorphisms from the Beltrami coefficient. Using the same formula, we illustrate how we can flow a surface map to satisfy desired Beltrami coefficients. This includes applications in restoring local conformality. In particular, we derive a new method of constructing the generalized Riemann from a surface patch to a disk. Such map is defined as a conformal map from a surface patch to a disk fixing three points, which is guaranteed to exist by the uniformization theorem. Furthermore, using the same flow, we are able to restore non-diffeomorphic surface maps to obtain diffeomorphisms. This is very useful because non-diffeomorphic maps commonly occur in the construction of surface maps satisfying certain criteria, and it is often a complicated process to avoid their occurrence. Our method can restore surface diffeomorphisms without creating a large distortion to the surface map. Indeed, we are able to distribute distortions evenly in our method. Hence the restored diffeomorphisms still retain most of the desired properties of the original maps.

5.1. Reconstruction of Genus Zero Surface Diffeomorphisms with Beltrami Coefficients. To describe a diffeomorphism of surfaces S_1 and S_2 using Beltrami equation for further adjustments, where S_1 and S_2 are two genus zero closed surface. We first map them conformally onto the Riemann sphere. This can be done using various approaches such as the Yamabe flow method [10]. With this map, the Beltrami coefficient can be easily computed. To reconstruct the diffeomorphism, we start with the identity map from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$. Using Equation (4.3), we can adjust the Beltrami coefficient of the map slight each iteration to get a diffeomorphism of $\mathbb C$ fixing 0, 1 and ∞ . We continue to iterate until the desired Beltrami coefficient is satisfied. In this way, the required diffeomorphism from S_1 to S_2 is also obtained. In Figure 5.1, the diffeomorphism on $\overline{\mathbb{C}}$ satisfying a required Beltrami coefficient is visualized as a diffeomorphism on the unit sphere via the stereographic projection. Figure 5.1(a) shows the value of the mesh of the sphere under the identity map, which is a regular mesh. In this example, we set $\mu(x+yi) = (0.5+0.5i)e^{-(x^2+y^2)}$, and the modulus of the Beltrami coefficient is visualized in Figure 5.1(b). After that, Equation (4.3) is applied repetitively to get the desired diffeomorphism. The mapping of the original mesh is shown in Figure 5.1(c). In all the figures, we visualize the area of the sphere corresponding to the area around origin in $\overline{\mathbb{C}}$. The distortion of the diffeomorphism is clearly seen in Figure 5.1(c).

5.2. Reconstruction of Diffeomorphisms on Surface Patches with Beltrami Coefficients. Using the flow for the unit disk (Equation (4.9)), we are also able to construct diffeomorphisms with desired Beltrami coefficients on simply connected surface patches. As shown in Figure 5.2, an inscribed square in a unit circle is adjusted using the Beltrami holomorphic flow. The map is distorted by the flow but its diffeomorphic property is maintained. This can be seen as interior of the unit disk is moved to the interior, and boundary is moved to boundary.

5.3. Preservation of Local Conformality of Surface Maps. Let $f: S_1 \to S_2$ be a diffeomorphism between two surfaces with Beltrami coefficient μ . In order to study two corresponding regions $\Omega \subset S_1$ and $f(\Omega) \subset S_2$ better, it is sometimes desirable to make f as conformal as possible on Ω . Indeed, the conformality of the map can be completely restored using the Beltrami holomorphic flow method. This is done by first constructing a new Beltrami coefficient μ' with $\mu' = 0$ on Ω and $\mu' = \mu$ outside Σ , where Σ is a compact set containing Ω . Then we can flow the diffeomorphism using the Beltrami holomorphic flow until the desired Beltrami coefficient is satisfied.

In Figure 5.3, we map a unit square nonconformally to a region on the plane under a diffeomorphism. The nonconformality can be seen from its Beltrami coefficient, which shows some variations on the whole surface. To restore the conformality in a central region of the domain, we construct a new Beltrami coefficient which vanishes in the central region according to the above discussion. After flowing the diffeomorphism using the Beltrami holomorphic flow on $\overline{\mathbb{C}}$ fixing 0, 1 and ∞ , it is found that the red grids in the central region are less distorted and looks closer to squares than the original grids in the same region. This shows that our method effectively restores





(a) The image under the identity map with $\mu = 0$.

(b) The image under the distorted map with nonzero μ .



(c) Comparison of the images under both maps.

FIG. 5.1. The image of an inscribed square in a unit disk under the identity map and the map after the Beltrami holomorphic flow.

conformality in the region we want.

5.4. Construction of Riemann Map of Surface Patches. It is a widely accepted method to find conformal maps of cortical surfaces to study medical imaging problems, as they are free of angular distortions and represent local geometry well. Given a simply connected surface patch \mathcal{K} , by the uniformization theorem, it is conformally equivalent to the unit disk. Therefore, there exists a unique conformal diffeomorphism φ from \mathcal{K} to \mathbb{D} fixing three points. In this section, we propose a method to construct the Riemann map of surface patches using the Beltrami holomorphic flow.

To compute the Riemann map of a simply connected domain S, we firstly find an arbitrary diffeomorphism $f: \mathbb{D} \to S$. We then look for a map $G: \mathbb{D} \to \mathbb{D}$ such that the composition map $f \circ G: \mathbb{D} \to S$ is a conformal diffeomorphism. Suppose the Beltrami coefficient of f is μ_f . We compute g as the quasiconformal map with Beltrami coefficient μ_f using the Beltrami holomorphic flow. It can be proven that $f \circ g^{-1}: \mathbb{D} \to S$ is conformal and hence $f \circ g^{-1}: \mathbb{D} \to S$ is a Riemann map of S. The UCLA CAM Report



with $\mu = 0$.



(b) The image under the distorted map with nonzero μ .



FIG. 5.2. The image of an inscribed square in a unit disk under the identity map and the map after the Beltrami holomorphic flow. Note that the points on the boundary circle slides along the boundary, instead of flowing across the boundary.

detail of the idea can be explained by the following proposition and theorem.

PROPOSITION 5.1. Let μ , σ and τ be Beltrami coefficients of quasiconformal maps f^{μ} , f^{σ} and f^{τ} with $f^{\tau} = f^{\sigma} \circ (f^{\mu})^{-1}$. Then

$$\tau = \left(\frac{\sigma - \mu}{1 - \overline{\mu}\sigma}\frac{1}{\theta}\right) \circ (f^{\mu})^{-1}, \tag{5.1}$$

where $\theta = \frac{\overline{p}}{p}$ and $p = \frac{\partial}{\partial z} f^{\mu}(z)$. In particular, if f^{σ} is the identity, that is, if $\sigma = 0$, then

$$\tau = -(\mu \frac{p}{\bar{p}}) \circ (f^{\mu})^{-1}.$$
(5.2)

Proof. Let T and T_1 be two complex linear maps defined by the equations

$$w = T(z) = A(z + \mu\overline{z}) = Az + B\overline{z}$$
$$w_1 = T_1(z) = A_1(z + \mu_1\overline{z}) = A_1z + B_1\overline{z}.$$





(b) The original diffeomorphism

(a) The Beltrami coefficient of the original diffeomorphism



FIG. 5.3. A non-conformal diffeomorphic map of the unit square and the new map with restored conformality in the central region

We are interested in the Beltrami coefficient of $w = T \circ (T_1)^{-1}(w_1)$ as a function of w_1 . From (5.3), we have

$$\overline{A_1}w_1 - B_1\overline{w_1} = z(|A_1|^2 - |B_1|^2)$$

Therefore we can write w as

$$w = T \circ (T_1)^{-1}(w_1) = \left(\frac{A\overline{A_1} - B\overline{B_1}}{|A_1|^2 - |B_1|^2}\right) \left\{w_1 + \frac{BA_1 - AB_1}{A\overline{A_1} - B\overline{B_1}}\overline{w_1}\right\}.$$

Using the equations $\mu = B/A$ and $\mu_1 = B_1/A_1$, the Beltrami coefficient of $T \circ (T_1)^{-1}$ is equal to

$$\left(\frac{\mu-\mu_1}{1-\overline{\mu_1}\mu}\right)\frac{1}{\theta_1}$$

where $\theta_1 = \overline{A_1}/A_1$. \Box

THEOREM 5.2. Let S be an open surface with boundary. Suppose $f : \mathbb{D} \to S$ be any arbitrary diffeomorphism from the unit disk to the surface S. Let μ be the Beltrami coefficient of f. Let $g : \mathbb{D} \to \mathbb{D}$ be the quasiconformal map with Beltrami

coefficient μ . Then the map $f \circ g^{-1} : \mathbb{D} \to S$ is a Riemann (conformal) map from \mathbb{D} to the surface S.

Proof. Suppose $f_t = f^{\mu(t)} \circ (f^{\mu})^{-1}$, where $f^{\mu(t)}$ and f^{μ} are the quasiconformal map with Beltrami coefficient $\mu(t)$ and μ respectively. Then the Beltrami coefficient $\lambda(t)$ of f_t is given by:

$$\lambda_t = \left(\frac{\mu(t) - \mu}{1 - \overline{\mu}\mu} \frac{(f^{\mu})_z}{(f^{\mu})_z}\right) \circ (f^{\mu})^{-1}$$
(5.3)

Let $f^{\mu(t)} = g$ and so $\mu(t) = \mu$. We have $\lambda_t = \left(\frac{\mu - \mu}{1 - \overline{\mu}\mu} \frac{(f^{\mu})_z}{(f^{\mu})_z}\right) \circ (f^{\mu})^{-1} = 0$ Since $\lambda_t = 0$, it means the composition map $f_t = f^{\mu(t)} \circ (f^{\mu})^{-1} = f \circ g^{-1}$ is conformal.

As an application, we test our method on finding the Riemann map from a surface patch of a human face to the unit disk. Figure 5.4(a) and 5.4(c) shows an arbitrary diffeomorphism constructed from the unit disk to the face. For easy identification of conformality, we mark the unit disk with perpendicular grid lines. It is clearly seen that the grid lines lose their perpendicularity in Figure 5.4(c) as the map is not conformal. Then the above discussed method is used to find the Beltrami coefficient of the map. Using the Beltrami holomorphic flow method, we find a diffeomorphism of \mathbb{D} with the same Beltrami coefficient and the map is shown in Figure 5.4(b) using the perpendicular grid lines in Figure 5.4(a). At this point, the Riemann map of the face can be obtained and is illustrated by the corresponding grid lines in Figure 5.4(b) and 5.4(c). For a clearer illustration of the conformality of the Riemann map, we use the grid lines in Figure 5.4(a) and the same map is shown in Figure 5.4(d). The conformality is clearly seen from the perpendicular grid lines on the face. This shows that our method is also useful on 3D surface patches.

5.5. Local Adjustment of Surface Diffeomorphisms. When adjusting the surface diffeomorphism, it is often preferable to tune properties of the surface map locally while retaining the original information outside the local region. With Beltrami holomorphic flow, we can locally adjust the Beltrami coefficient while keeping the Beltrami coefficient unchanged outside the local region. Since the Beltrami coefficient is unchanged outside the local region, it can be proven that we can reconstruct the original map at that region exactly. The detail can be explained with the following theorem.

THEOREM 5.3. Let S be a genus zero closed surface. Suppose $f^{\mu}: S \to \mathbb{S}^2 \cong \overline{\mathbb{C}}$ with Beltrami coefficient μ . Let $\Omega \subseteq S$ be a simply-connected domain of S such that $f^{\mu(\Omega)} = \mathbb{D} \subseteq \overline{\mathbb{C}}$. Suppose $\tilde{\mu}: S \to \mathbb{C}$ is another Beltrami coefficient satisfying $\tilde{\mu}|_{\Omega} = \mu|_{\Omega}$. That is, $\tilde{\mu}$ is a local adjustment of μ fixing Ω . Then:

$$f^{\mu}|_{\Omega} = \phi^{\Omega} \circ f^{\widetilde{\mu}}|_{\Omega}$$

where $\phi^{\Omega} : f^{\tilde{\mu}}(\Omega) \to \mathbb{D}$ is the Riemann map of $f^{\mu}(\Omega)$. In other words, by adjusting the Beltrami coefficient while fixing it on Ω , one can reconstruct the original surface map on Ω exactly.

Proof. Denote $f^{\tilde{\mu}}(\Omega) = \Omega'$. Since $f^{\mu} : \Omega \to \mathbb{D}$ and $f^{\tilde{\mu}} : \Omega \to \Omega'$ have the same Beltrami coefficient, $f^{\tilde{\mu}} \circ f^{\mu-1} : \mathbb{D} \to \Omega'$ is conformal according to Theorem 5.2. Let





(a) The canonical domain for our face surface.



(b) The mapping from the unit disk to the face.



(c) The diffeomorphism of the unit disk with the same Beltrami coefficient as the map from the disk to the face.

(d) The representation of the final diffeomorphism using the original grid on the unit disk

FIG. 5.4. Finding the Riemann map from a face to the unit disk

 a_1, a_2, a_3 be three points in Ω and let $\phi^{\Omega} : \Omega' \to \mathbb{D}$ be a conformal map such that $\phi^{\Omega}(f^{\tilde{\mu}}(a_i)) = f^{\mu}(a_i)$. Then, $\phi^{\Omega} \circ f^{\tilde{\mu}} \circ f^{\mu-1} : \mathbb{D} \to \mathbb{D}$ is a conformal map fixing three points. And so it must be the identity map. As a result, $f^{\mu}|_{\Omega} = \phi^{\Omega} \circ f^{\tilde{\mu}}|_{\Omega}$. \Box

5.6. Restoration of Diffeomorphisms of Surface Maps. In shape analysis, it is a common approach to compare shapes by computing diffeomorphisms between two surfaces satisfying certain properties, such as landmark-matching and harmonic energy minimizing. However, it is not true that every such map is diffeomorphic. It can also be complicated to implement measures to prevent surface overlapping. Moreover, there is no canonical way to restore non-diffeomorphic surface maps for each specific problem. To overcome this problem, we propose a natural and canonical approach to restore general surface diffeomorphisms using the Beltrami holomorphic flow.

Given a surjective map $f: S_1 \to S_2$ of two simply connected closed surfaces or surface patches and its corresponding complex map $\tilde{f}: K \to K$, where K is \mathbb{C} or \mathbb{D} . Let μ be the Beltrami coefficient of \tilde{f} . If $|\mu| < 1$ everywhere on D, then it can be shown that f must be an orientation-preserving diffeomorphism. If f is not diffeomorphic, then around some region, $|\mu|$ must be bigger than or equal to 1. To restore the diffeomorphic property of \tilde{f} , we propose to damp the value of μ at regions where $|\mu| > 1$, while at the same time keep μ smooth. By constructing a new Beltrami coefficient satisfying these properties, we reconstruct the map using the Beltrami holomorphic flow to get a diffeomorphism.

Specifically, we have:

THEOREM 5.4. Suppose $f: S_1 \to S_2$ be a surface map between two Riemann surfaces, which is not diffeomorphic on $\Omega \subseteq S_1$. Then the coefficient $\mu_f = \frac{f_{\overline{z}}}{f_z}$ on Ω has a supreme norm $\sup ||\mu_f||_{\infty} \geq 1$. Also, let $\tilde{\mu}$ be a smooth approximation of μ_f such that $\sup ||\tilde{\mu}|_{\Omega}||_{\infty} < 1$. Then $f^{\tilde{\mu}}$ is a smooth diffeomorphic approximation of f.

Proof. It can be verified easily that the Jacobian $J = |f_z|^2 - |f_{\overline{z}}|^2 = (1 - |\mu_f|^2)|f_z|^2$. Suppose $sup||\mu_f||_{\infty} < 1$. Then, J > 0 everywhere. By inverse function theorem, $f: S_1 \to S_2$ is a diffeomorphism which is a contradiction. Similarly, if we damp μ_f to $\tilde{\mu}$ such that $sup||\tilde{\mu}|_{\Omega}||_{\infty} < 1$. Then, the quasiconformal map $f^{\tilde{\mu}}$ has Jacobian J > 0 everywhere and so it is a diffeomorphism by inverse function theorem. \Box

Figure 5.5 shows how we can apply this method to restore surface diffeomorphism. In Figure 5.5(A), a non-diffeomorphic map from a grid to a face surface is visualized. It corresponds to a unit square portion of a map from \mathbb{C} to \mathbb{C} , which is shown in Figure 5.5(B). Its non-diffeomorphic property is clearly shown in the plot of its Beltrami coefficient in Figure 5.5(C), where the modulus of the Beltrami coefficient can be as large as 7. To correct this map, we construct a new Beltrami coefficient using the method described above, which is shown in Figure 5.5(D). Using Beltrami holomorphic flow, we obtain a diffeomorphic map from \mathbb{C} to \mathbb{C} , where the value it takes on a unit square is shown in Figure 5.5(E). Mapping the map back on the face surface, we get a diffeomorphic surface map as shown in Figure 5.5(F).

5.7. Construction of Landmark-Matching Surface Diffeomorphisms. As discussed above, the construction of meaningful surface diffeomorphisms that preserve landmarks is not a trivial task. Often, complicated measures have to be implemented to avoid overlapping of surface maps while satisfying several desired properties. To solve this problem, the Beltrami holomorphic flow provides a method of constructing landmark-matching surface maps by directly specifying appropriate Beltrami coefficients. Since the map is diffeomorphic and orientation-preserving if and only if the modulus of the Beltrami coefficient is everywhere strictly less than 1, this approach automatically makes the resulting maps diffeomorphic, thus avoiding the need to correct the surface maps afterwards.

The first question one needs to ask is how to find the correct Beltrami coefficient that represents the landmark matching surface diffeomorphism. After the Beltrami coefficient μ that represents the landmark matching diffeomorphism is defined, we can further adjust μ so that the diffeomorphism may satisfy other properties such as conformality, etc. A smooth diffeomorphism can then be obtained by Beltrami holomorphic flow. The appropriate Beltrami coefficient can be constructed according to the landmark curves. The correctness of this method is seen in the following theorem.

THEOREM 5.5. Let $\alpha : [0, s_{\alpha}] \to \mathbb{C}$ be a vertical line in \mathbb{C} and $\beta = (\beta_1 + i\beta_2) : [0, s_{\beta}] \to \mathbb{C}$ be another landmark curve such that: $\alpha(t)$ corresponds to $\beta(t), \alpha(0) =$



FIG. 5.5. Restoration of surfaces diffeomorphisms by damp the Beltrami coefficient.

 $\beta(0)$ and $\alpha(s_{\alpha}) = \beta(s_{\beta})$. Let $\varepsilon > 0$ and $\Omega_{\varepsilon} = \{x \in \mathbb{C} \mid x = \alpha(t) \pm \delta, \ \delta \in \mathbb{R}, \ \delta < \varepsilon\}$. Consider the parameterization of Ω_{ε} defined by: $\phi(s,t) = \alpha(t) + s$. Let Ω be a domain enclosing Ω_{ε} . Define $\mu(\phi(s,t))$ by:

$$\mu \circ \phi = \frac{A + iB}{C + iB}$$

where:

$$A = \left[-\frac{(\varepsilon-s)^3}{2\varepsilon^3}(2+\beta'_2(t)) - \frac{3(\varepsilon-s)^2}{\varepsilon^3}(\beta_1(t)-s)\right];$$

$$B = \left[\frac{(\varepsilon-s)^3}{2\varepsilon^3}(1+\beta'_1(t)) - \frac{3(\varepsilon-s)^2}{\varepsilon^3}(\beta_2(t)-s)\right] and;$$

$$C = \left[1 + \frac{(\varepsilon-s)^3}{2\varepsilon^3}(\beta'_2(t)-2) - \frac{3(\varepsilon-s)^2}{\varepsilon^3}(\beta_1(t)-s)\right].$$

Suppose $\tilde{\mu}$ is a smooth extension such that $\tilde{\mu}|_{\mathbb{C}\setminus\Omega_{\varepsilon}} = \vec{0}$. Then the quasiconformal map $f^{\tilde{\mu}}$ with Beltrami coefficient $\tilde{\mu}$ has properties that: (i) $f^{\tilde{\mu}}(\alpha(t)) = \beta(t)$.

Proof. Consider the parameterization of Ω_{ε} defined by:

$$\phi(s,t) = \alpha(t) + s$$

[Here, we identify (s, t) with $\phi(s, t)$.] Clearly, f maps $\alpha(t)$ to $\beta(t)$. Also,

$$\begin{split} \frac{\partial f}{\partial s} &= (1,0) - (\frac{(\varepsilon - s)^3}{\varepsilon^3} + \frac{3(\beta_1(t) - s)(\varepsilon - s)^2}{\varepsilon^3}, \frac{3(\beta_2(t) - s)(\varepsilon - s)^2}{\varepsilon^3})\\ \frac{\partial f}{\partial t} &= (0,1) + (\frac{\beta_1'(t)(\varepsilon - s)^3}{\varepsilon^3}, \frac{(\beta_2'(t) - 1)(\varepsilon - s)^3}{\varepsilon^3}) \end{split}$$

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Thus,

$$\begin{split} \frac{\partial f}{\partial z} &= \frac{1}{2} \left(\frac{\partial f}{\partial s} - i \frac{\partial f}{\partial t} \right) \\ &= \left[1 + \frac{(\varepsilon - s)^3}{2\varepsilon^3} (\beta_2'(t) - 2) + \frac{3(\varepsilon - s)^2}{\varepsilon^3} (\beta_1(t) - s) \right] \\ &- i \left[\frac{(\varepsilon - s)^3 (\beta_1'(t) + 1)}{2\varepsilon^3} + \frac{3(\varepsilon - s)^2 (\beta_2(t) - s)}{\varepsilon^3} \right] \end{split}$$

$$\begin{split} \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2} \big(\frac{\partial f}{\partial s} + i \frac{\partial f}{\partial t} \big) \\ &= [-\frac{(\varepsilon - s)^3}{2\varepsilon^3} (\beta_2'(t) + 2) + \frac{3(\varepsilon - s)^2}{\varepsilon^3} (\beta_1(t) - s)] \\ &+ i [\frac{(\varepsilon - s)^3 (\beta_1'(t) + 1)}{2\varepsilon^3} + \frac{3(\varepsilon - s)^2 (\beta_2(t) - s)}{\varepsilon^3}] \end{split}$$

Define $\mu: \Omega_{\varepsilon} \to C$ by:

$$\begin{split} \mu(\phi(s,t)) &= \frac{\partial f}{\partial z} / \frac{\partial f}{\partial z} = \\ &\frac{\left[-\frac{(\varepsilon-s)^3}{2\varepsilon^3} (2+\beta_2'(t)) - \frac{3(\varepsilon-s)^2}{\varepsilon^3} (\beta_1(t)-s) \right] + i \left[\frac{(\varepsilon-s)^3}{2\varepsilon^3} (1+\beta_1'(t)) - \frac{3(\varepsilon-s)^2}{\varepsilon^3} (\beta_2(t)-s) \right]}{\left[1 + \frac{(\varepsilon-s)^3}{2\varepsilon^3} (\beta_2'(t)-2) - \frac{3(\varepsilon-s)^2}{\varepsilon^3} (\beta_1(t)-s) \right] + i \left[\frac{(\varepsilon-s)^3}{2\varepsilon^3} (\beta_1'(t)+1) - \frac{3(\varepsilon-s)^2}{\varepsilon^3} (\beta_2(t)-s) \right]}{\left[1 + \frac{(\varepsilon-s)^3}{2\varepsilon^3} (\beta_2'(t)-2) - \frac{3(\varepsilon-s)^2}{\varepsilon^3} (\beta_1(t)-s) \right] + i \left[\frac{(\varepsilon-s)^3}{2\varepsilon^3} (\beta_1'(t)+1) - \frac{3(\varepsilon-s)^2}{\varepsilon^3} (\beta_2(t)-s) \right]}{\left[1 + \frac{(\varepsilon-s)^3}{2\varepsilon^3} (\beta_2'(t)-2) - \frac{3(\varepsilon-s)^2}{\varepsilon^3} (\beta_1(t)-s) \right] + i \left[\frac{(\varepsilon-s)^3}{2\varepsilon^3} (\beta_1'(t)+1) - \frac{3(\varepsilon-s)^2}{\varepsilon^3} (\beta_2(t)-s) \right]}{\left[1 + \frac{(\varepsilon-s)^3}{2\varepsilon^3} (\beta_1'(t)-2) - \frac{3(\varepsilon-s)^2}{\varepsilon^3} (\beta_2'(t)-s) \right]} \\ \end{split}$$

Note that $\mu = 0$ on $\partial \Omega_{\varepsilon}$. We can smoothly extend μ to $\tilde{\mu} : \mathbb{C} \to \mathbb{C}$ by setting $\tilde{\mu} = 0$ outside Ω_{ε} . The quasiconformal map $f^{\tilde{\mu}}$ with Beltrami coefficient $\tilde{\mu}$ can be computed by Beltrami holomorphic flow and has the property that $f^{\tilde{\mu}}(\alpha(t)) = \beta(t)$.

Note that the landmark curve α is not necessarily a vertical line. For a general curve, we can initially map the parameter domain with an arbitrary diffeomorphism such that it maps the curve to a vertical line. Alternatively, by considering a narrow band of the general curve, we can obtain the expression of μ as in the above proof. Figure 5.6 shows the process in a which landmark-matching diffeomorphism is constructed on the plane. Figure 5.6(A) and Figure 5.6(B) show two surfaces S_1 and S_2 respectively. They are parameterized onto the complex plane and the surface diffeomorphism between them can be constructed using the composition map. However, the landmarks on both figures are not being mapped consistently (See blue curve on S_1 and red curve on S_2). With a suitable Beltrami coefficient (as shown in Figure 5.6(E)), a landmark-matching diffeomorphism can be constructed using Beltrami holomorphic flow as shown in Figure 5.6(D), where landmark curves are mapped exactly (see blue curve on S_2). In Figure 5.7, we show that our method can be applied to match multiple landmarks accurately as well.

6. Conclusion. In this work, we propose the novel ideal of representing surface diffeomorphisms by Beltrami coefficients. We demonstrate how we can reconstruct surface diffeomorphisms fixing three points using the Beltrami holomorphic flow on simply connected closed surfaces and surface patches. This allows us to make fine adjustments to surface maps, including the restoration of the conformality of surface maps, the construction of Riemann map from arbitrary simply connected closed surfaces, the restoration of surface diffeomorphisms, and the construction of landmark-matching diffeomorphisms. This shows that the Beltrami holomorphic flow method can provide us with various powerful applications, and is general enough to handle a large class of surface maps, including non-conformal maps and maps that are not even diffeomorphic.



FIG. 5.6. The procedure for constructing a landmark-matching diffeomorphism on the plane. (A) and (B) show two surfaces S_1 and S_2 respectively. They are parameterized onto the complex plane and surface diffeomorphism between them can be constructed using the composition map. Landmarks are not mapped consistently (See blue curve on S_1 and red curve on S_2). With a suitable Beltrami coefficient (as shown in (E)), a landmark matching diffeomorphism can be constructed using Beltrami holomorphic flow as shown in (D). Landmark curves are mapped exactly (see blue curve on S_2)



FIG. 5.7. The procedure for constructing a diffeomorphism on the plane matching multiple landmarks. (A) and (B) show two surfaces S_1 and S_2 respectively. They are parameterized onto the complex plane and surface diffeomorphism between them can be constructed using the composition map. Landmarks are not mapped consistently (See blue curve on S_1 and red curve on S_2). With a suitable Beltrami coefficient (as shown in (D)), a landmark matching diffeomorphism can be constructed using Beltrami holomorphic flow as shown in (E). Landmark curves are mapped exactly (see blue curve on S_2)

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