THE DIRECT DISCONTINUOUS GALERKIN (DDG) METHOD FOR DIFFUSION WITH INTERFACE CORRECTIONS

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Abstract. In [24] we developed a direct discontinuous Gallerkin (DDG) method for diffusion problems based on a general numerical flux formula for the solution gradient only. In this work, we introduce a refined DDG method by using interface corrections in place of involving higher order (≥ 4) derivatives in the numerical flux. This way the optimal $(k+1)$th order of accuracy is obtained for any p^k elements with a simple form of numerical fluxes. The admissible parameters in the flux formulation is estimated, and the leading coefficient of the solution jump is shown to be necessarily large when jump of $2nd$ order derivatives is not included in the numerical flux. The refined DDG method is then extended to solve convection diffusion problems in both one and two dimensional settings. A series of numerical tests are presented to demonstrate the high order accuracy of the method.

Contents

1. Introduction

In this paper, we continue our study on the direct discontinuous Galerkin (DDG) method for solving nonlinear diffusion equations of the form

$$\partial_t U - \nabla \cdot (A(U)\nabla U) = 0, \quad \Omega \times (0, T), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, the matrix $A(U) = (a_{ij}(U))$ is symmetric and positive definite, and U is an unknown function of (x, t). The method will also be applied to convection-diffusion problems by using some well known flux for the convection.

The Discontinuous Galerkin (DG) method is a finite element method using a completely discontinuous piecewise polynomial space for the numerical solution and the test functions. A key ingredient of this method is the suitable design of the inter-element boundary treatments (the so-called numerical fluxes) to obtain high order accurate and stable schemes. The DG method

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has been vigorously developed for hyperbolic problems since it was first introduced in 1973 by Reed and Hill [26] for neutron transport equations. A major development of the DG method is carried out by Cockburn, Shu, and collaborators in a series of papers [18, 17, 13, 20] for nonlinear hyperbolic conservation laws. While it is being actively developed, the DG method has found rapid applications in many areas; we refer to [11, 16, 21] for further references.

However, the DG method when applied to diffusion problems encounters subtle difficulties, see e.g. [28]. There have been various DG methods suggested in the literature to solve the problem, including the method originally proposed by Bassi and Rebay [4] for compressible Navier-Stokes equations, its generalization called the local discontinuous Galerkin (LDG) methods introduced in [19] by Cockburn and Shu and further studied in [7, 12, 15, 8]; as well as the method introduced by Baumann-Oden [5, 25]. The Baumann-Oden method once written into a primal formulation is similar to a class of interior penalty methods, independently proposed and studied for elliptic and parabolic problems in the 1970s; see, e.g., [1, 3, 32]. Considering the similarities among the recently introduced DG methods, Arnold et al. [2] have set the existing DG methods into a unified framework with a systematic analysis of these methods via linear elliptic problems. Another framework using both the equation in each element and continuity relations across interfaces was recently analyzed in [6].

More recent works such as those by Van Leer and Nomura in [31], Gassner et al. in [22], and Cheng and Shu in [9] attempt to develop some direct DG methods for diffusion. We note that these prior works exploit repeated integration by parts for the diffusion term, this brings a possibility of choosing both solution and its derivatives to contribute to the interface flux.

Our approach beginning in [24] is to propose a path which sticks to the direct weak formulation of the underlying parabolic equation in each cell and let cells communicate via the numerical flux $\widehat{u_x}$ only. To illustrate the idea, we simply take the one-dimensional heat equation, $u_t = u_{xx}$, as an example, and formulate the DDG method as follows,

$$\int_{I_j} u_t v dx - \widehat{(u_x)} v|_{j-\frac{1}{2}}^{j+\frac{1}{2}} + \int_{I_j} u_x v_x dx = 0,$$

where I_j is the $j-th$ computational cell, and v is the test function. The numerical flux involves the average $\overline{u_x}$ and the jumps of even order derivatives of u:

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] + \cdots . \tag{1.2}$$

It was shown in [24] that there is a large class of numerical fluxes of the form (1.2) being admissible in the sense that the resulting scheme is stable. For piecewise p^k polynomial approximations, kth order of accuracy is ensured if the numerical flux is admissible.

Numerical experiments in [24] show that in order to achieve the optimal $(k+1)$th order of accuracy, the term $(\Delta x)^{k-1}[\partial_x^k u]$ for $k = even$ is necessary. We note that the scheme is sensitive in term of the choice of $\beta_{k/2}$. For example, for $k = 2$, only $\beta_1 = \frac{1}{12}$ with a range of β_0 gives the optimal accuracy. The situation for $k = odd$ is easier in the sense that the optimal order of accuracy of the DDG scheme can be achieved even with

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} \tag{1.3}$$

provided β_0 is suitably large.

However, the DDG method turns to be less practical when $k = even \geq 4$ due to involving $4-$th (or higher) order solution derivatives in the numerical flux. This problem is amplified when solving high-dimensional equations. Our goal in this work is to refine the DDG method by using interface corrections so that an optimal accuracy can be achieved for all k using the numerical flux (1.2) or just (1.3) with a proper large β_0.

The refined DDG scheme takes the following form

$$\int_{I_j} u_t v dx - \widehat{(u_x)} v|_{j-\frac{1}{2}}^{j+\frac{1}{2}} + \int_{I_j} u_x v_x dx + \frac{1}{2}[u](v_x)^-_{j+\frac{1}{2}} + \frac{1}{2}[u](v_x)^+_{j-\frac{1}{2}} = 0. \tag{1.4}$$

Upon summation of (1.4) with numerical flux (1.3) over all j, we find the symmetric Internal Penalty method originally introduced by Arnold in [1] for elliptic problems, see also [3, 32] (in the format of so-called primal formulation). It is well known that the penalty parameter (β_0 in (1.3)) is a coefficient depending on the order of the polynomial p^k, see e.g. [27]. It has to be large enough to stabilize the scheme, especially for high order polynomials. For the DDG method with interface corrections, we obtain a sharp estimate on β_0 with the assistance of the admissibility condition. Furthermore, our numerical results show that including $[u_{xx}]$ term in the numerical flux $\widehat{u_x}$ (1.2) will relieve the dependency of β_0 on k. Namely, we obtain $(k+1)$th order of accuracy for all p^k polynomial approximations with fixed small β_0, e.g. $\beta_0 = 2$ for p^k up to $k = 9$.

This paper is organized as follows. In §2, we introduce the DDG method with interface corrections for one-dimensional problems. For this model problem, the main idea of devising the method and the scheme formulation are presented. We then prove stability of the DDG scheme for any admissible numerical fluxes. In §3, we extend the DDG method to nonlinear convection diffusion equations in both one and two-dimensional problems in which U is a scalar and $A = (a_{ij})_{d\times d}$ is a positive and semi-definite matrix. Finally in §4, we present a series of numerical results to validate the refined DDG method.

2. One dimensional case

Our new DDG algorithm for diffusion consists of an addition of interface corrections, upon the one proposed in [24], and hence allows a wider choice of numerical fluxes for obtaining the optimal accuracy. Discretization in time with a matching accuracy is obtained by an appropriate Runge-Kutta solver.

2.1. **Scheme formulation.**

We begin with the one-dimensional linear diffusion

$$U_t - U_{xx} = 0 \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

subject to initial data

$$U(x, 0) = U_0(x) \quad \text{on } \Omega, \tag{2.2}$$

and periodic boundary conditions.

First we partition the domain Ω into computational cells $\Omega = \cup_{j=1}^N I_j$, with mesh $\{I_j = [x_{j-1/2}, x_{j+1/2}]\}$ of uniform size $\Delta x = x_{j+1/2} - x_{j-1/2}$. We seek an approximation u to U such that for any time $t \in [0, T]$, $u \in \mathbb{V}_{\Delta x}$,

$$\mathbb{V}_{\Delta x} := \{v \in L^2(\Omega) : \quad v|_{I_j} \in P^k(I_j), \quad j = 1, \cdots, N\},$$

where $P^k(I_j)$ denotes the space of polynomials on I_j with degree at most k.

Set

$$\widehat{u_x} v\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} := (\widehat{u_x})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (\widehat{u_x})_{j-\frac{1}{2}} v^-_{j-\frac{1}{2}}.$$

The DDG method introduced in [24] can be written as

$$\int_{I_j} u_t v dx - \widehat{u_x} v\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} + \int_{I_j} u_x v_x dx = 0, \tag{2.3}$$

with numerical flux $\widehat{u_x}$ evaluated at x by

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \sum_{m=1}^{\lfloor k/2 \rfloor} \beta_m (\Delta x)^{2m-1} [\partial_x^{2m} u], \tag{2.4}$$

where $\beta_0, \beta_1, \cdots, \beta_{[k]/2}$ are coefficients to be chosen to ensure the stability of the scheme, and the following notations have been used:

$$u^{\pm} = u(x \pm 0, t), \quad [u] = u^+ - u^-, \quad \overline{u} = \frac{u^+ + u^-}{2}.$$

The admissibility of β_i's and their effects on the numerical accuracy were studied in [24], in which a numerical flux of the following form was tested

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \frac{1}{12}\Delta x[u_{xx}]. \tag{2.5}$$

This scheme with $\beta_0 = 1$ was numerically shown to produce optimal accuracy of $(k+1)th$ order for $k \leq 3$, as well as for $k = odd(> 3)$ with a slightly larger β_0.

For $k \geq 4$, instead of relying on high order terms such as $\beta_2[\Delta x]^3[u_{xxxx}]$, in this paper we turn to a refined DDG method with inclusion of interface corrections:

$$\int_{I_j} u_t v dx - \widehat{u_x} v\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} + \int_{I_j} u_x v_x dx + \frac{1}{2}[u](v_x)^-_{j+1/2} + \frac{1}{2}[u](v_x)^+_{j-1/2} = 0, \tag{2.6}$$

$$\int_{I_j} u(x,0)v(x)dx = \int_{I_j} U_0(x)v(x)dx, \tag{2.7}$$

where the numerical flux $\widehat{u_x}$ is still (2.5) or more general

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x[u_{xx}]. \tag{2.8}$$

We recall that $\beta_1 = 1/12$ was identified through a procedure suggested in [24] for $k = 2, 3$: we used the Stirling interpolation formula based on four symmetric points

$$x_{j+\frac{1}{2}} \pm \frac{1}{2}\Delta x, \quad x_{j+\frac{1}{2}} \pm \Delta x,$$

leading to a unique 3rd order polynomial, whose derivative when evaluated at the cell interface $x_{j+1/2}$ gives

$$D_x u = \frac{7}{6}\frac{[u]}{\Delta x} + \overline{u_x} + \frac{\Delta x}{12}[u_{xx}]. \tag{2.9}$$

Therefore in this work we shall use the flux (2.5) for the refined DDG scheme. For non-uniform mesh, Δx needs to be understood as $(\Delta x_j + \Delta x_{j+1})/2$. The 1D scheme is now well defined.

We prove in next section that a large class of β_i' exists for the stability of the DDG method. Note that the scheme (2.6) with (2.8) when $\beta_1 = 0$ becomes the classical symmetric DG method, and in such a case sufficiently large β_0 is indeed needed to penalize the interface jumps [1], see Lemma 2.2 below.

2.2. **Admissibility and stability.** As usual for the DG method the guiding principle for the choice of numerical flux is the stability requirement. Following [24] we adopt the following admissibility criterion:

Definition 2.1. (Admissibility) We call a numerical flux $\widehat{u}_x$ of the form (2.8) *admissible* if there exists a $\gamma \in (0,1)$ and $0 < \alpha \leq 1$ such that

$$\gamma \sum_{j=1}^N \int_{I_j} u_x^2(x,t)dx + \sum_{j=1}^N \widehat{u_x}_{j+1/2}[u]_{j+1/2} + \sum_{j=1}^N [u]_{j+1/2}\overline{u_x}_{j+1/2} \geq \alpha \sum_{j=1}^N \frac{[u]^2_{j+1/2}}{\Delta x} \tag{2.10}$$

holds for any piecewise polynomials of degree k, i.e. $u \in \mathbb{V}_{\Delta x}$.

This admissibility ensures the stability of the DDG method.

Theorem 2.1. *(Energy stability) Consider the DDG scheme (2.6)-(2.7). If the numerical flux (2.8) is admissible as described in (2.10), then we have*

$$\frac{1}{2}\int_0^1 u^2(x,T)dx + (1-\gamma)\int_0^T \sum_{j=1}^N \int_{I_j} u_x^2(x,t)dxdt + \alpha\int_0^T \sum_{j=1}^N \frac{[u]^2}{\Delta x}dt \le \frac{1}{2}\int_0^1 U_0^2(x)dx. \quad (2.11)$$

This can be proved by summation of (2.6) with $v = u$ over $j \in \{1, \cdots N\}$, and using the admissibility condition (2.10).

In the following two lemmas we show that there is indeed a large set of β_i', making admissible fluxes (2.8) for polynomial approximations of any given degree k.

Lemma 2.1. *Consider the one-dimensional linear diffusion (2.1). The numerical flux (2.8) is admissible for any piecewise polynomial of degree $k \ge 0$ provided $(\beta_0, \beta_1) = (1, 0)$ when $k = 0$, and for any $k > 0$*

$$\beta_0 \ge \alpha + \frac{1}{\gamma}M(k, \beta_1),$$

where

$$M(k,\beta_1) = \max_{u\in P^k(I_j)} \frac{\Delta x \sum_j (\overline{u}_x + \frac{\beta_1}{2}\Delta x[u_{xx}])^2}{\sum_j \int_{I_j} u_x^2 dx}.$$

Proof. Note for $k = 0$, $(\beta_0, \beta_1) = (1, 0)$ is admissible since $\beta_0 \ge \alpha$. For admissibility condition to hold when $k \ge 1$, it suffices to select (β_0, β_1) so that the underlying flux (2.8) is admissible locally around each cell, i.e.,

$$\gamma\Delta x\int_{I_j} u_x^2 dx + (2\overline{u_x} + \beta_1\Delta x[u_{xx}])[u]\Delta x + (\beta_0 - \alpha)[u]^2 \ge 0, \quad k \ge 1.$$

This is ensured to hold for all $u|_{I_j} \in P^k(I_j)$ provided

$$(2\overline{u_x} + \beta_1\Delta x[u_{xx}])^2(\Delta x)^2 - 4(\beta_0 - \alpha)\gamma\Delta x\int_{I_j} u_x^2 dx \le 0.$$

Summation of this inequality over all index $j \in \{1, \cdots N\}$ yields

$$\beta_0 \ge \alpha + \frac{1}{\gamma}\frac{\Delta x \sum_j (\overline{u}_x + \frac{\beta_1}{2}\Delta x[u_{xx}])^2}{\sum_j \int_{I_j} u_x^2 dx}.$$

Maximization of the right hand side over all $u|_{I_j} \in P^k(I_j)$ gives

$$\beta_0 \ge \alpha + \frac{M(k,\beta_1)}{\gamma}.$$

□

In the numerical flux (2.8) the β_1 term provides a leverage to compensate the β_0 term. Indeed we show below that when $\beta_1 = 0$, a larger β_0 in terms of k is needed for the flux to be admissible.

Lemma 2.2. *For a given $k \ge 1$, (2.8) with $\beta_1 = 0$ is admissible if*

$$\beta_0 \ge \alpha + \frac{1}{4\gamma}\lambda_{\max}(H^{-1/2}OH^{-1/2}),$$

where H is the Hilbert matrix $H = \left(\frac{1}{m+l-1}\right)$ of size k and O is a $k \times k$ matrix with each entry to be 1.

Proof. From the proof in Lemma 2.1 we have

$$M(k,0) = \max_{u\in P^k(I_j)} \frac{\Delta x \sum_j \overline{u}_x^2}{\sum_j \int_{I_j} u_x^2 dx} = \max_{v\in P^{k-1}(I_j)} \frac{\Delta x \sum_j \overline{v}^2}{\sum_j \int_{I_j} v^2 dx}.$$

k	1	2	3	4	5	6	7	8	9
β_0	2	4	6	10	14	20	26	34	42

TABLE 1. analytical estimation on $\beta_0(k)$.

Set $v|_{I_j} = \sum_{m=1}^k a_m^j \xi_j^{m-1}$ with base functions chosen to be $\xi_j = \frac{x - x_{j-1/2}}{\Delta x}$ for $x \in I_j$. Here a_m^j is the coefficient of the base function in cell I_j. Thus the average of v at the interface $x = x_{j+1/2}$ is

$$\bar{v}\Big|_{x_{j+1/2}} = \frac{1}{2}\left(\sum_{m=1}^k a_m^j \cdot 1^{m-1} + \sum_{m=1}^k a_m^{j+1} \cdot 0^{m-1}\right) = \frac{1}{2}\sum_{m=1}^k a_m^j.$$

On the other hand we have

$$\int_{I_j} v^2 dx = \Delta x \int_0^1 \left(\sum_{m=1}^k a_m^j \cdot \xi^{m-1}\right)^2 d\xi = \Delta x \sum_{m,l=1}^k a_m^j a_l^j \frac{1}{m+l-1}.$$

These together lead to

$$M(k,0) = \frac{1}{4} \max_{a_m^j, m=1\cdots k} \frac{\sum_{j=1}^N \sum_{m,l=1}^k a_m^j a_l^j}{\sum_{j=1}^N \sum_{m,l=1}^k a_m^j a_l^j \frac{1}{m+l-1}} = \frac{1}{4} \max_{a^j \in \mathbb{R}^k} \frac{\sum_{j=1}^N a^j \cdot O a^j}{\sum_{j=1}^N a^j \cdot H a^j},$$

where

$$a^j = (a_1^j, \cdots, a_k^j)^\top, \quad O_{m,l} = 1, \quad H_{m,l} = \frac{1}{m+l-1}.$$

This is clearly bounded from above by

$$\frac{1}{4} \max_{a \in \mathbb{R}^k} \frac{a \cdot Oa}{a \cdot Ha} = \frac{1}{4} \max_{y \in \mathbb{R}^k} \frac{y \cdot \left(H^{-1/2} O H^{-1/2}\right) y}{\|y\|_2^2},$$

where we have used the fact that the Hilbert matrix H is symmetric and positive definite to transform a via $y = H^{1/2}a$. Due to symmetry of the matrix $H^{-1/2}OH^{-1/2}$, the induced spectral norm is just the largest eigenvalue of this matrix, as claimed. □

From this result we now specify the choice of β_0 for each fixed k, and compare with our numerical results. For instance, we take $\alpha = 1$ and $\gamma = 1/2$, and let β_0 to be an integer as

$$\beta_0 = \left[\frac{1}{2}\lambda_{\max}\left(H^{-1/2} O H^{-1/2}\right)\right] + 2, \tag{2.12}$$

where $[\cdot]$ denotes the integer part. Some calculation for k up to 9 shows that $\beta_0 = [k^2/2] + 2$, which is summarized in Table 1.

The numerical results for $\beta_1 = 0$ is consistent with those given in Table 1. With β_1 non-vanishing we numerically show that $(k+1)$th order of accuracy is obtained for p^k polynomials with a fixed β_0. For instance, optimal accuracy is observed for all k up to 9 when taking $(\beta_0, \beta_1) = (2, \frac{1}{12})$ in our numerical tests.

Remark 2.1. A similar calculation for $M(k, \beta_1)$ yields an estimate about as (2.12) but with O replaced by

$$O_{ml} = \left(1 - \frac{\beta_1}{2}(m-1)\right)\left(1 - \frac{\beta_1}{2}(l-1)\right).$$

For some $\beta_1 > 0$ this indeed leads to a smaller β_0 than (2.12).

3. Extensions

In this section we extend the refined DDG method to nonlinear convection diffusion problems and multi-dimensional problems.

3.1. **One dimensional convection diffusion equations.** We consider the nonlinear convection diffusion equation,

$$U_t + f(U)_x - (a(U)U_x)_x = 0 \quad \text{in } \Omega \times (0,T), \tag{3.1}$$

subject to initial data $U(x,0) = U_0(x)$ and periodic boundary conditions. The diffusion coefficient $a(U) > 0$ is non-negative.

Take $v \in \mathbb{V}_{\Delta x}$ as the test function, the DDG scheme with interface correction is defined as the following,

$$\int_{I_j} u_t v dx + (\widehat{f(u)} - \widehat{a(u)u_x})v\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} - \int_{I_j} (f(u) - a(u)u_x)v_x dx + \frac{1}{2}[b(u)](v_x)^-_{j+1/2} + \frac{1}{2}[b(u)](v_x)^+_{j-1/2} = 0, \tag{3.2}$$

where $b(u) = \int_0^u a(s)ds$, $\widehat{f(u)} - \widehat{a(u)u_x}$ is the numerical flux to be chosen.

For the convection part we may choose any entropy satisfying numerical flux, for example, the Lax-Friedrichs flux,

$$\widehat{f(u)} = \widehat{f}(u^-, u^+) = \frac{1}{2}\left(f(u^-) + f(u^+) - \theta(u^+ - u^-)\right), \tag{3.3}$$

where $\theta = \max_{u \in [u^-, u^+]} |f'(u)|$.

For the diffusion part, the numerical flux is chosen as

$$\widehat{a(u)u_x} = \beta_0 \frac{[b(u)]}{\Delta x} + \overline{b(u)_x} + \beta_1 \Delta x [b(u)_{xx}]. \tag{3.4}$$

Here β_0 and β_1 are taken the same as those for the linear case.

3.2. **Multi-dimensional extensions.** We now formulate the refined DDG method for multi-D problems. Since numerical flux for convection can be treated via dimension-wise extension, we present schemes only for nonlinear diffusion problems, for simplicity, in two-dimensional setting. Let $J_1 = a_{11}(U)U_x + a_{12}(U)U_y$ and $J_2 = a_{21}(U)U_x + a_{22}(U)U_y$. Then two-dimensional problem can be written as

$$U_t - \partial_x J_1 - \partial_y J_2 = 0, \quad \text{in } \Omega = [0,1]^2 \subset \mathbb{R}^2, \tag{3.5}$$

subject to initial data $U(x,y,0) = U_0(x,y)$ and periodic boundary conditions. The diffusion coefficient matrix (a_{ij}) is assumed to be symmetric and positive definite.

Let a partition of Ω be denoted by rectangular meshes

$$\Omega = \sum_{j,k}^{N,M} I_{j,k}, \quad I_{j,k} = I_j \times I_k = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [y_{k-\frac{1}{2}}, y_{k+\frac{1}{2}}]$$

of uniform mesh sizes $\Delta = \max(\Delta x, \Delta y)$. We denote the finite element space by

$$\mathbb{V}_\Delta = \left\{ v : \quad v|_{I_{j,k}} \in P^k, \quad \forall I_{j,k} \subset \Omega \right\}, \tag{3.6}$$

where P^k is a polynomial of degree at most k.

Set $b_{ij}(u) = \int_0^u a_{ij}(s)ds$. Then the DDG scheme on each computational cell can be written as

$$\iint_{I_{j,k}} u_t v dx dy - \int_{I_k} \widehat{J_1 v}\Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} dy - \int_{I_j} \widehat{J_2 v}\Big|_{y_{k-\frac{1}{2}}}^{y_{k+\frac{1}{2}}} dx + \iint_{I_{j,k}} (J_1 v_x + J_2 v_y) dx dy + B = 0, \tag{3.7}$$

$$\iint_{I_{j,k}} u(x,y,0)v(x,y)dxdy = \iint_{I_{j,k}} U_0(x,y)v(x,y)dxdy, \tag{3.8}$$

where the boundary correction is

$$B = \frac{1}{2}\left\{ \int_{I_k} ([b_{11}(u)]v_x^-)_{x_{j+\frac{1}{2}}} + ([b_{11}(u)]v_x^+)_{x_{j-\frac{1}{2}}} dy + \int_{I_j} ([b_{22}(u)]v_y^-)_{y_{k+\frac{1}{2}}} + ([b_{22}(u)]v_y^+)_{y_{k-\frac{1}{2}}} dx \right\}.$$

Here and below $[u]_{x_{j+\frac{1}{2}}} := u(x^+_{j+\frac{1}{2}}, y, t) - u(x^-_{j+\frac{1}{2}}, y, t)$ and $[u]_{y_{k+\frac{1}{2}}} = u(x, y^+_{k+\frac{1}{2}}, t) - u(x, y^-_{k+\frac{1}{2}}, t)$. The numerical flux $\widehat{J}_i$ is defined by

$$\widehat{J}_1\Big|_{x_{j+\frac{1}{2}}} = \beta_0 \frac{[b_{11}(u)]}{\Delta x} + \overline{b_{11}(u)_x + b_{12}(u)_y} + \beta_1 \Delta x [b_{11}(u)_{xx}], \tag{3.9}$$

$$\widehat{J}_2\Big|_{y_{k+\frac{1}{2}}} = \beta_0 \frac{[b_{22}(u)]}{\Delta y} + \overline{b_{21}(u)_x + b_{22}(u)_y} + \beta_1 \Delta y [b_{22}(u)_{yy}], \tag{3.10}$$

where we take (β_0, β_1) as obtained for one-dimensional case. The 2D algorithm is now well defined.

An appropriate choice of (β_0, β_1) is to ensure the L^2-stability of the method.

Theorem 3.1. *(Energy stability) Assume that for $p \in \mathbb{R}$, $\exists \gamma$ and γ^* such that the eigenvalues of matrix $(a_{ij}(p))$ lie between $[\gamma, \gamma^*]$. Consider the refined DDG scheme with numerical flux (3.9)-(3.10). Then the numerical solution satisfies*

$$\iint_\Omega u^2(x,y,T)dxdy + \int_0^T \sum_{I_{j,k}} \iint_{I_{j,k}} (J_1 u_x + J_2 u_y) dxdydt + \gamma\beta_0 \int_0^T \left(\sum_{I_j} \int_{I_j} \sum_{k=1}^M \frac{[u]^2_{y_{k+1/2}}}{\Delta y} dx + \sum_{I_k} \int_{I_k} \sum_{j=1}^N \frac{[u]^2_{x_{j+1/2}}}{\Delta x} dy \right) dt \le \iint_\Omega U_0^2(x,y) dxdy, \tag{3.11}$$

provided β_0 is suitably large.

This stability result can be proved by following the similar argument as that explored in [24]. Details are omitted.

Up to now, we have taken the method of lines approach and have left t continuous. For time discretization we use the explicit third order TVD Runge-Kutta method [30, 29] to match accuracy in space.

4. Numerical examples

In this section we provide a few numerical examples to illustrate the accuracy and capacity of the DDG method with interface corrections. We would like to illustrate the high order accuracy of the method through these numerical examples from one-dimensional to two-dimensional linear and nonlinear problems.

***Example* 4.1. lD heat equation**.

$$U_t - U_{xx} = 0, \qquad x \in (0, 2\pi) \tag{4.1}$$

with initial condition $U(x, 0) = \sin(x)$ and periodic boundary conditions. In this example we will use this model equation to test the performance of the new DDG method. Two different numerical fluxes are investigated, one is to take $\beta_1 = 0$ and the other is to take $\beta_1 \neq 0$ in (2.8).

The numerical flux (2.8) with $\beta_1 = 0$ reduces to the following,

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x}. \tag{4.2}$$

In light of the admissibility studies in §2 we know that β_0 is a parameter depending on the degree of the approximation polynomial. We need to choose β_0 big enough to stabilize the scheme for high order approximations. We refer to Table 1 for suitable choices of β_0 with different k. The DDG method based on p^k polynomial approximations with $k = 0, 1, 2, 3, 4$ are tested and $(k+1)$th order of accuracy is obtained. L^2 and L^∞ errors are listed in Table 2. Note that in this and the remaining examples, L^∞ error is obtained by evaluating on 200 sample points per cell.

k		N=10	N=20		N=40		N=80	
		error	error	order	error	order	error	order
0	L^2	4.86e-02	2.38e-02	1.03	1.19e-02	1.00	5.90e-03	1.00
	L^∞	1.17e-01	5.80e-02	1.02	2.89e-02	1.00	1.45e-02	1.00
1	L^2	5.36E-03	1.41e-03	1.93	3.57e-04	1.98	8.96e-05	1.99
	L^∞	1.03e-02	2.87e-03	1.84	7.46e-04	1.94	1.88e-04	1.98
2	L^2	3.21e-04	3.73e-05	3.10	4.56e-06	3.02	5.68e-07	3.00
	L^∞	1.31e-03	1.60e-04	3.03	1.98e-05	3.00	2.48e-06	3.00
3	L^2	2.48e-05	1.56e-06	3.98	9.78e-08	3.99	6.11e-09	3.99
	L^∞	5.47e-05	3.60e-06	3.92	2.31e-07	3.96	1.45e-08	3.99
4	L^2	4.03e-07	1.03e-08	5.29	3.01e-10	5.09	9.24e-12	5.02
	L^∞	1.83e-06	5.62e-08	5.02	1.75e-09	5.00	5.46e-11	5.00

TABLE 2. 1D heat equation with numerical flux (4.2). L^2 and L^∞ errors at $t = 1.0$. p^k polynomial approximations with $k = 0, 1, 2, 3, 4$.

The second test is to use numerical flux (2.8) with $\beta_1 \neq 0$. As suggested in §2 we use the following numerical flux in the DDG scheme (2.6),

$$\widehat{u_x} = 2\frac{[u]}{\Delta x} + \overline{u_x} + \frac{\Delta x}{12}[u_{xx}]. \tag{4.3}$$

p^k polynomial approximations with $k = 2, \cdots, 9$ are tested and again optimal $(k+1)$th order of accuracy is obtained. To save the space here we only list the errors and orders for $k = 2, 3, 4, 5, 6, 7$ in Table 3. Note including term $[u_{xx}]$ in the numerical flux does relieve the dependence of β_0 on k. We use fixed $\beta_0 = 2$ for all p^k polynomial approximations up tp $k = 9$. We also investigate this scheme on nonuniform mesh and still $(k+1)$th order of accuracy is obtained. Errors and orders are listed in Table 4. For the nonuniform mesh, the partition of the domain $[0, 2\pi]$ consists of repeated pattern of $1.1\Delta x$ and $0.9\Delta x$ for odd and even number of index $i = 1, ...N$, where $\Delta x = 2\pi/N$ with even number N.

The above comparison indicates that the term $[u_{xx}]$ is important for the scheme to remain accurate for high order approximations with a fixed β_0. Hence in the rest examples we use numerical flux (4.3) only.

***Example* 4.2. 1D fully nonlinear equation**.

$$U_t + UU_x - \frac{1}{2}(UU_x)_x = 0 \qquad x \in (0, 1) \tag{4.4}$$

with initial condition $U(x, 0) = e^x$ and boundary condition $U(0, t) = 1$ and $U(1, t) = e$. The exact solution is given as $U(x, t) = e^x$.

For the nonlinear diffusion term $(a(u)u_x)_x = b(u)_{xx}$ we use the following numerical flux,

$$\widehat{b(u)_x} = 2\frac{[b(u)]}{\Delta x} + \overline{b(u)_x} + \frac{\Delta x}{12}[b(u)_{xx}].$$

In this example $b(u) = \frac{1}{2}u^2$. We conduct the DDG scheme on this fully nonlinear equation. We obtain $(k+1)$th order of accuracy for p^k approximations. Errors and orders are listed in Table 5.

***Example* 4.3. 2D linear convection diffusion equation**.

$$U_t + c(U_x + U_y) - \mu(U_{xx} + U_{yy}) = 0 \qquad (x, y) \in (0, 2\pi) \times (0, 2\pi) \tag{4.5}$$

with initial condition $U(x, y, 0) = \sin(x+y)$ and periodic boundary condition. The exact solution is $U(x, y, t) = e^{-2\mu t}\sin(x + y - 2ct)$. In this example we take $c = 1$ and $\mu = 1$. Accuracy test is performed on a $N \times N$ rectangular mesh. DDG scheme with interface correction is implemented

k		**N=10**	**N=20**		**N=40**		**N=80**	
		error	error	order	error	order	error	order
2	L^2	3.73e-04	4.65e-05	3.00	5.80e-06	3.00	7.25e-07	3.00
	L^∞	7.21e-04	9.11e-05	2.98	1.14e-05	2.99	1.43e-06	2.99
3	L^2	2.59e-05	1.58e-06	4.03	9.80e-08	4.00	6.12e-09	4.00
	L^∞	6.17e-05	3.72e-06	4.05	2.33e-07	3.99	1.46e-08	3.99
4	L^2	2.13e-06	7.60e-08	4.80	2.47e-09	4.94	7.79e-11	4.98
	L^∞	6.74e-06	2.46e-07	4.77	8.03e-09	4.93	2.54e-10	4.98
		N=4	**N=8**		**N=12**		**N=16**	
5	L^2	1.13e-05	1.88e-07	5.91	1.67e-08	5.96	3.00e-09	5.98
	L^∞	1.97e-05	3.39e-07	5.86	3.03e-08	5.96	5.42e-09	5.98
6	L^2	7.30e-07	4.22e-09	7.43	2.34e-10	7.14	3.07e-11	7.05
	L^∞	1.23e-06	1.09e-08	6.82	7.25e-10	6.67	1.02e-10	6.83
7	L^2	8.43e-08	3.87e-10	7.77	1.57e-11	7.91	1.75e-12	7.63
	L^∞	1.28e-07	5.92e-10	7.76	2.40e-11	7.91	2.65e-12	7.65

TABLE 3. 1D heat equation with numerical flux (4.3). p^k polynomial approximations with $k = 2, 3, 4, 5, 6, 7$.

k		**N=10**	**N=20**		**N=40**		**N=80**	
		error	error	order	error	order	error	order
2	L^2	4.07e-04	5.04e-05	3.01	6.29e-06	3.00	7.86e-07	3.00
	L^∞	9.37e-04	1.19e-04	2.97	1.50e-05	2.99	1.88e-06	3.00
3	L^2	3.10e-05	1.91e-06	4.02	1.19e-07	4.00	7.43e-09	4.00
	L^∞	8.84e-05	5.46e-06	4.01	3.44e-07	3.99	2.16e-08	3.99
4	L^2	4.07e-06	1.69e-07	4.59	5.76e-09	4.87	1.84e-10	4.97
	L^∞	1.39e-05	5.89e-07	4.56	2.02e-08	4.87	6.46e-10	4.96

TABLE 4. 1D heat equation with numerical flux (4.3) on none uniform mesh. p^k approximations with $k = 2, 3, 4$.

k		**N=10**	**N=20**		**N=30**		**N=40**	
		error	error	order	error	order	error	order
0	L^2	1.08e-01	5.54e-02	0.97	3.72e-02	0.98	2.81e-02	0.98
	L^∞	2.11e-01	1.08e-01	0.97	7.24e-02	0.98	5.45e-02	0.99
1	L^2	1.19e-03	2.99e-04	1.99	1.33e-04	2.00	7.49e-05	2.00
	L^∞	3.13e-03	8.40e-04	1.90	3.83e-04	1.94	2.18e-04	1.96
2	L^2	1.02e-05	1.28e-06	2.99	3.80e-07	3.00	1.60e-07	3.00
	L^∞	2.15e-05	2.72e-06	2.99	8.06e-07	3.00	3.40e-07	3.00
3	L^2	5.95e-08	3.72e-09	4.00	7.35e-10	3.98	2.34e-10	3.99
	L^∞	1.39e-07	8.88e-09	3.97	1.77e-09	3.92	5.74e-10	3.95

TABLE 5. 1D fully nonlinear equation (4.4). L^2 and L^∞ errors at $t = 0.5$. p^k polynomial approximation with $k = 0, 1, 2, 3, 4$.

k		N=10	N=20		N=40		N=80	
		error	error	order	error	order	error	order
0	L^2	8.58e-02	4.74e-02	0.86	2.50e-02	0.92	1.28e-02	0.96
	L^∞	2.23e-01	1.22e-01	0.88	6.26e-02	0.96	3.17e-02	0.98
1	L^2	1.11e-02	2.73e-03	2.02	6.71e-04	2.02	1.66e-04	2.01
	L^∞	5.49e-02	1.39e-02	1.98	3.49e-03	1.99	8.84e-04	1.98
2	L^2	1.13e-03	1.40e-04	3.01	1.74e-05	3.00	2.18e-06	3.00
	L^∞	7.64e-03	9.58e-04	2.99	1.17e-04	3.03	1.46e-05	3.00
3	L^2	1.33e-04	8.16e-06	4.02	5.06e-07	4.01	3.15e-08	4.00
	L^∞	7.03e-04	4.49e-05	3.97	2.87e-06	3.97	1.82e-07	3.98

TABLE 6. 2D linear convection diffusion equation (4.5). L^2 and L^∞ errors at $t = 0.5$. p^k polynomial approximation with $k = 0, 1, 2, 3$.

k		N=10	N=20		N=40		N=80	
		error	error	order	error	order	error	order
0	L^2	2.03e-01	1.07e-01	0.92	5.56e-02	0.95	2.88e-02	0.95
	L^∞	6.10e-01	3.12e-01	0.97	1.60e-01	0.96	8.09e-02	0.98
1	L^2	3.62e-02	9.44e-03	1.94	2.39e-03	1.98	5.93e-04	2.00
	L^∞	2.26e-01	6.11e-02	1.88	1.54e-02	1.99	3.78e-03	2.02
2	L^2	3.90e-03	4.48e-04	3.12	5.50e-05	3.02	6.69e-06	3.03
	L^∞	3.44e-02	4.29e-03	3.00	5.02e-04	3.09	5.75e-05	3.12
3	L^2	4.44e-04	2.51e-05	4.14	1.55e-06	4.02	9.51e-08	4.02
	L^∞	3.49e-03	2.72e-04	3.68	1.46e-05	4.21	7.86e-07	4.20

TABLE 7. 2D anisotropic case (4.6). L^2 and L^∞ errors at $t = 0.3$. p^k polynomial approximations with $k = 0, 1, 2, 3$.

at $t = 0.5$. Similar to 1D problems, we choose fixed $\beta_0 = 2$ in the numerical flux for all p^k polynomials. In the x-direction, we take

$$\widehat{u_x}|_{x_{j+1/2}} = \left(2\frac{[u]}{\Delta x} + \overline{u_x} + \frac{\Delta x}{12}[u_{xx}]\right)|_{x_{j+1/2}}.$$

Similar formula in the y-direction is applied. Again $(k+1)$th order of accuracy is obtained with piecewise p^k polynomial approximations. L^2 and L^∞ errors and orders are listed in Table 6.

***Example* 4.4. 2D anisotropic diffusion equation**.

$$U_t + c(U_x + U_y) - \mu(U_{xx} + U_{yy} + U_{xy}) = 0 \qquad (x, y) \in (0, 2\pi) \times (0, 2\pi) \tag{4.6}$$

with initial condition $U(x, y, 0) = \sin(x+y)$ and periodic boundary condition. The exact solution is $U(x, y, t) = e^{-3\mu t}\sin(x + y - 2ct)$. We use this example to test the DDG scheme for diffusion problem with non-isotropic term. For the mixed term u_{xy} we use numerical flux (3.9) on rectangular mesh. In this example we take $c = 1$ and $\mu = 0.01$. p^k polynomial approximations with $k = 0, 1, 2, 3$ are tested and $(k + 1)$th order of accuracy is obtained. Errors and orders are listed in Table 7.

***Example* 4.5. 2D incompressible Navier-Stokes equation in vorticity formulation**. In this example we consider two-dimensional incompressible Navier-Stokes equation in vorticity based formulation.

$$\omega_t + \nabla \cdot (\mathbf{u}\omega) = \frac{1}{Re}\triangle\omega \qquad (x, y) \in (0, 2\pi) \times (0, 2\pi) \tag{4.7}$$

k		N=16	N=32		N=64		N=128	
		error	error	order	error	order	error	order
1	L^2	2.19e-02	5.13e-03	2.09	1.25e-03	2.04	3.06e-04	2.02
	L^∞	1.16e-01	2.59e-02	2.16	6.25e-03	2.05	1.47e-03	2.09
2	L^2	1.34e-03	1.55e-04	3.11	1.85e-05	3.06	2.19e-06	3.07
	L^∞	1.10e-02	1.19e-03	3.20	1.32e-04	3.17	1.49e-05	3.14
3	L^2	3.23e-04	1.07e-05	4.91	4.62e-07	4.52	2.33e-08	4.31
	L^∞	2.09e-03	5.73e-05	5.18	2.30e-06	4.63	1.18e-07	4.28

TABLE 8. 2D Navier-Stokes equation in vorticity formulation (4.7). L^2 and L^∞ errors at $t = 1.0$. p^k approximations with $k = 1, 2, 3$.

Again we use this example to check the high order accuracy of the DDG method. To simplify the computation, we take incompressible velocity field $\mathbf{u} = (u, v)$ as a given function. Here $(u(x, y, t), v(x, y, t)) = e^{-\frac{2t}{Re}}(-\cos x \sin y, \sin x \cos y)$ and the exact solution is known as $\omega(x, y, t) = 2e^{-\frac{2t}{Re}} \cos x \sin y$, see [10]. Periodic boundary conditions are applied and we take the Reynolds number $Re = 100$. We compute the solution at time $t = 1$. L^2 and L^∞ errors are listed in Table 8 and we obtain $(k + 1)$th order of accuracy with p^k polynomial approximations.

***Example* 4.6. 2D Buckley-Leverett equation**.
Finally, we consider the two dimensional convection diffusion equation [23]

$$u_t + f(u)_x + g(u)_y = \epsilon(u_{xx} + u_{yy}) \qquad (x, y) \in (-1.5, 1.5) \times (-1.5, 1.5). \tag{4.8}$$

The nonlinear convection terms are given as

$$f(u) = \frac{u^2}{u^2 + (1 - u)^2},$$

$$g(u) = f(u)(1 - 5(1 - u)^2),$$

and the initial condition is taken as

$$u(x, y, 0) = \begin{cases} 1, & x^2 + y^2 < 0.5, \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

This is the two-dimensional Buckley-Leverett equation with small diffusion. Essentially it is a convection dominated problem with non-convex flux functions. Here we take $\epsilon = 0.01$. We compute the DDG solution with p^1 polynomial approximation up to $t = 0.5$ with mesh size $N \times N = 100 \times 100$. In Figure 1 we show the solution slice at $y = 0.75$ and the solution contours in $(0, 1)$. Figure 2 shows the $3D$ outlook of the solution.

In our simulation we observed that when ϵ is relatively large, say $\epsilon = 0.1$, the scheme is stable and accurate. When smaller ϵ is used, we observe some instability phenomena which is related to the steep shock fronts. Here we use sloper limiters as introduced in [14] to stabilize the scheme. Again, the DDG method shows its capability to obtain high resolution solutions across sharp transition areas and gives satisfactory results.

5. Concluding remarks

Built upon the DDG method introduced in [24], we have developed a refined direct discontinuous Galerkin(DDG) method for diffusion problems. We include extra interface corrections in the scheme formulation with numerical flux involving only up to second order derivatives of the numerical solution. The refined DDG scheme has the advantage of obtaining optimal accuracy of $(k + 1)$th order for all p^k elements. We prove that there exists a large class of coefficients (β_0, β_1) in the numerical flux formulation, $\widehat{u_x} = \beta_0[u]/\Delta x + \overline{u_x} + \beta_1 \Delta x[u_{xx}]$, ensuring the stability of the scheme. We also prove that when $\beta_1 = 0$, β_0 has to be big enough to guarantee

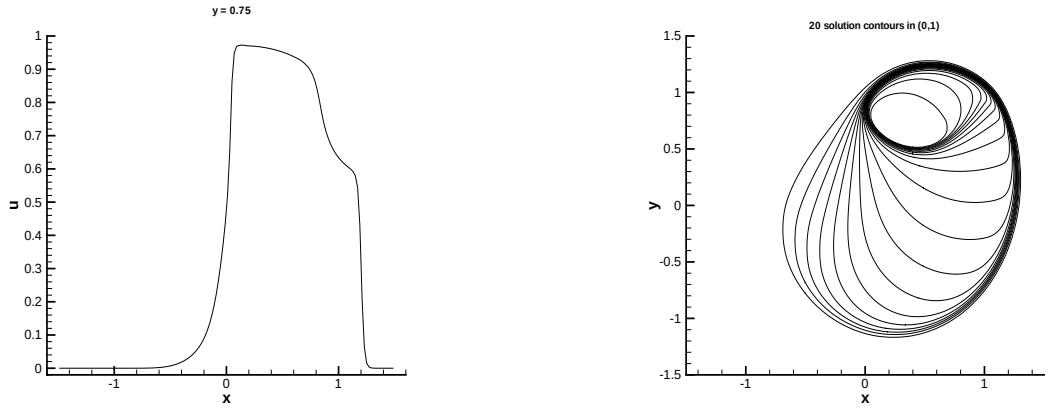

FIGURE 1. 2D Buckley-Leverett equation (4.8). Left: solution slice at $y = 0.75$. Right: solution contours.

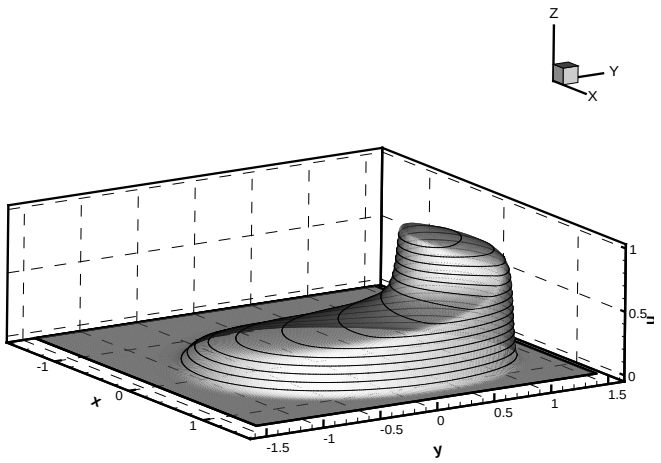

FIGURE 2. 2D Buckley-Leverett equation (4.8). solution at $t = 0.5$ on 100x100 mesh.

the scheme stability; actually we estimate the precise dependence of β_0 on the polynomial degree k. Extensions of the method to convection diffusion problems in both one and two dimensional settings are given. Finally we carry out a series of numerical tests from linear to nonlinear, one dimensional to two dimensional problems to demonstrate the high order accuracy of the method. Our numerical results show that $\beta_1 \neq 0$ does provide a leverage to compensate the β_0 term, thus a fixed β_0 can be used for all p^k polynomials.

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