GLOBAL WELL-POSEDNESS FOR THE MICROSCOPIC FENE MODEL WITH A SHARP BOUNDARY CONDITION

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Abstract. We prove global well-posedness for the microscopic FENE model under a sharp boundary requirement. The well-posedness of the FENE model that consists of the incompressible Navier-Stokes equation and the Fokker-Planck equation has been studied intensively, mostly with the natural flux boundary condition. Recently it was illustrated by C. Liu and H. Liu [2008, SIAM J. Appl. Math., 68(5):1304–1315] that any preassigned boundary value of a weighted distribution will become redundant once the non-dimensional parameter $b > 2$. In this article, we show that for the well-posedness of the microscopic FENE model ($b > 2$) the least boundary requirement is that the distribution near boundary needs to approach zero faster than the distance function. This condition is strictly weaker than the natural flux boundary condition. Under this condition it is shown that there exists a unique weak solution in a weighted Sobolev space. The sharpness of this boundary requirement is shown by a construction of infinitely many solutions when the distribution approaches zero as fast as the distance function.

Contents

1. Introduction

It is well-known that the following system coupling incompressible Navier-Stokes equation for the macroscopic velocity field $v(t, x)$ and the Fokker-Planck equation for the probability density function $f(t, x, m)$ describes diluted solutions of polymeric liquids with noninteracting polymer chains, where $x \in \mathbb{R}^n$ is the macroscopic Eulerian

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coordinate and $m \in \mathbb{R}^n$ is the microscopic molecular configuration variable

$$\partial_t v + (v \cdot \nabla)v + \nabla p = \nabla \cdot \tau_p + \nu \Delta v, \tag{1.1}$$

$$\nabla \cdot v = 0, \tag{1.2}$$

$$\partial_t f + (v \cdot \nabla) f + \nabla_m \cdot (\nabla v m f) = \frac{2}{\zeta} \nabla_m \cdot (\nabla_m \Psi(m) f) + \frac{2kT}{\zeta} \Delta_m f. \tag{1.3}$$

In this model, a polymer is idealized as an elastic dumbbell consisting of two beads joined by a spring that can be modeled by a vector m (see e.g [2]). In the Navier-Stokes equation (1.1), p is hydrostatic pressure, ν is the fluid viscosity, and τ_p is a tensor representing the polymer contribution to stress,

$$\tau_p = \lambda \int m \otimes \nabla_m \Psi(m) f dm,$$

where Ψ is the elastic spring potential, and λ is the polymer density constant. In the Fokker-Planck equation (1.3), ζ is the friction coefficient of the dumbbell beads, T is the temperature, and k is the Boltzmann constant. Notice that, the Fokker-Planck equation can be written as a stochastic differential equation (see [17]).

One of the simplest model is the Hookean model in which the potential Ψ is given by

$$\Psi(m) = \frac{H|m|^2}{2},$$

where H is the elasticity constant. For a finite $m \in B(0, \sqrt{b})$, a more realistic model is the finite extensible nonlinear elasticity (FENE) model in which case

$$\Psi(m) = -\frac{Hb}{2} \log \left(1 - \frac{|m|^2}{b} \right),$$

where $\sqrt{b}$ is the maximum dumbbell extension. In this work we shall focus our attention on the case $b > 2$, which is known to contain the parameter range of physical interest. We refer the readers to [3, 2] for a comprehensive survey of the physical background.

In past years the well-posedness of the FENE model has been studied intensively in several aspects. For local well-posedness of strong solutions we refer the readers to [7] for the FENE model (in the setting where the Fokker-Planck equation is formulated by a stochastic differential equation) with $b > 2$ or sometime $b > 6$, [5] for a polynomial force and [19] for the FENE model with $b > 76$. For a preliminary study on some related coupled PDE systems, we refer to the earlier work [18](however, the FENE model was not addressed there). Moreover, the authors [11] proved global existence of smooth solutions near equilibrium under some restrictions on the potential, which have been extended in subsequent works [12, 10]. More recently, N. Masmoudi [15] proved local and global well-posedness for the FENE dumbbell model for a general class of potentials.

Global existence of weak solutions was also proved in [13] for the co-rotational model, see also [1] for $b \geq 10$. For an earlier existence result of weak solutions, we refer to [4] for the Fokker-Planck equation alone with $b > 4$. On the other hand, the authors in [8], investigated the long-time behavior of both Hookean models and FENE models in various special flows in a bounded domain with suitable boundary conditions.

The main complexity with the FENE potential lies mainly with the singularity of the equation at the boundary. In [14], C. Liu and H. Liu closely examined the necessity of Dirichlet boundary conditions for the microscopic FENE model. By the method of the Fichera function the authors were able to conclude that $b = 2$ is a threshold in the sense that for $b > 2$ any preassigned value of a weighted distribution will become redundant, and for $b < 2$ that value has to be a priori given. For the microscopic FENE model, singularity in the potential requires at least the zero Dirichlet boundary condition

$$f|_{\partial B} = 0, \tag{1.4}$$

where $B \stackrel{\text{def.}}{=} B(0, \sqrt{b})$, the ball with center 0 and radius $\sqrt{b}$. In most of a priori works the natural flux boundary condition has been used:

$$\left(\frac{2}{\zeta}\nabla_m \Psi f + \frac{2kT}{\zeta}\nabla_m f - \nabla v m f\right) \cdot \frac{m}{|m|}\Big|_{\partial B} = 0, \tag{1.5}$$

which is stronger than (1.4).

The boundary issue for the underlying FENE model is fundamental, and our main quest in this paper is whether one can identify a sharp boundary requirement so that both existence and uniqueness of a global weak solution to the microscopic FENE model can be proved. The answer is positive, and we claim that f must have the following boundary condition

$$f d^{-1}|_{\partial B} = 0 \quad \text{for almost all } t > 0, \tag{1.6}$$

where $d \stackrel{\text{def.}}{=} d(m, \partial B)$ denotes the distance function from $m \in B$ to the boundary ∂B. Note that (1.6) is strictly weaker than (1.5) and stronger than (1.4). Our claim is supported by our main results: the global well-posedness for the Fokker-Planck equation stated in Theorem 2, and the sharpness of (1.6) stated in Proposition 3.

The importance of the Fokker-Planck equation itself as the added complexity with the FENE potential affects mostly the analysis of the Fokker-Planck equation. In this article, we focus on the underlying Fokker-Planck equation alone. Assuming f is independent of x and the fluid velocity is steady and homogeneous, we obtain the following equation from a suitable scaling ([14]).

$$\partial_t f + \nabla_m \cdot (\kappa m f) = \frac{1}{2}\nabla_m \cdot \left(\frac{bm}{\rho} f\right) + \frac{1}{2}\Delta_m f, \tag{1.7}$$

where $\rho = b - |m|^2$ and $\kappa = \nabla v$ is a constant matrix such that $\text{Tr}(\kappa) = 0$ and with the matrix max norm $|\kappa|$. Suppose that a sufficiently smooth function f solves (1.7). Then for any test function $\tilde{\varphi} \in C_c^1(B)$, i.e. a continuously differentiable function compactly supported in B, it follows that

$$\int_B \left[\partial_t f \tilde{\varphi} - f \kappa m \cdot \nabla \tilde{\varphi} + \frac{b f m \cdot \nabla \tilde{\varphi}}{2\rho} + \frac{1}{2}\nabla f \cdot \nabla \tilde{\varphi}\right] dm = 0. \tag{1.8}$$

Here, we omit m from ∇_m. Note that (1.8) is well defined for any $f(t, \cdot) \in H^1_{loc}(B)$ and $\partial_t f(t, \cdot) \in H^{-1}_{loc}(B)$. Moreover, the compact support of a test function makes f free on the boundary of the domain. From this, a weak solution of the Fokker-Planck equation (1.7) with the initial condition

$$f(0, m) = f_0(m), \quad m \in B, \tag{1.9}$$

is defined in the following manner:

Definition 1. *Suppose that*

$$f \in L^2(0,T;H^1(B')), \text{ and } \partial_t f \in L^2(0,T;H^{-1}(B')) \tag{1.10}$$

for an arbitrary subdomain B' of B such that $\overline{B'} \subset B$. We say f is a solution of (1.7), (1.9) *if* (1.8) *holds for any $\tilde{\varphi} \in C_c^1(B)$ and almost all $t \in (0,T)$ such that*

$$f(0,m) = f_0(m) \quad \text{in } L^2(B'). \tag{1.11}$$

Note that (1.11) makes sense since $f \in C([0,T];L^2(B'))$.

Consider the equilibrium f^{eq} of (1.7) for non-flow case, i.e. $\kappa = 0$. Rewrite (1.7) as

$$\partial_t f = \frac{1}{2}\nabla \cdot (\nabla(\rho^{-b/2} f)\rho^{b/2}).$$

It follows that

$$f^{eq} = \rho^{b/2}.$$

Obviously, f^{eq} has the zero trace on the boundary ∂B and satisfies (1.6) for

$$b > 2. \tag{1.12}$$

Thus, (1.8) is satisfied even for a test function $\varphi \in H^1(B)$ without the compactly supported property. Also

$$f^{eq} \in H^1_{-b/2}(B)$$

as long as (1.12) is assumed. Here, $H^1_{-b/2}(B) = \{\phi : \phi, \partial_{m_j}\phi \in L^2_{-b/2}(B)\}$ with

$$L^2_{-b/2}(B) = \left\{\phi : \int_B \phi^2 (b - |m|^2)^{-b/2} dm < \infty\right\}.$$

Our main results are summarized in Theorem 2 and Proposition 3 below.

Theorem 2. *Assume* (1.6) *and* (1.12). *For any $T > 0$, there exists at most one solution f to the Fokker-Planck equation* (1.7) *and* (1.9) *in the sense of Definition 1 for any initial data $f_0(m)$. Furthermore, if*

$$f_0(m) \in L^2_{-b/2}(B)$$

then there exists a unique solution and

$$\max_{0\le t\le T} ||f(t,\cdot)||_{L^2_{-b/2}(B)} + ||f||_{L^2(0,T;H^1_{-b/2}(B))} + ||\partial_t f||_{L^2(0,T;((H^1{}_{-b/2})^*(B))} \le C||f_0||_{L^2_{-b/2}(B)}. \tag{1.13}$$

From the equilibrium f^{eq}, the restriction on b (1.12) is essential to have the energy estimate (1.13). We, however, remark that the well-posedness of the weak solution for $b \le 2$ was discussed in [15] with a different function space.

The following proposition states that the boundary condition (1.6) is sharp for the uniqueness of the weak solution.

Proposition 3. *If the boundary condition* (1.6) *fails, that is,*

$$fd^{-1}|_{\partial B} \neq 0$$

is assumed, then the Fokker-Planck equation (1.7) *with $f_0(m) = 0$ has infinitely many non-trivial solutions for $b > 2$.*

The article is organized as follows. In Section 2, we transform the Fokker-Planck equation to a certain Cauchy-Dirichlet problem, named as W-problem and define a weak solution of W-problem in a weighted Sobolev space. The well-posedness of the W-problem is shown in Section 3 by the Galerkin method and the Banach fixed point theorem. This leads to the well-posedness of the Fokker-Planck equation, Theorem 2; details of the proof are presented in Section 4. In the last section, we construct a non-trivial solution for the Fokker-Planck equation described in Proposition 3.

2. Transformation of the microscopic FENE model

In what follows we shall call the Fokker-Planck equation (1.7) with initial condition (1.9) and boundary condition (1.6) as the Fokker-Planck-FENE (FPF) problem. We first formulate a time evolution equation from the FPF problem. Define $w(t, m)$ ([14]) as

$$f(t, m) = w(t, m)\rho^{\alpha} \tag{2.1}$$

with α to be determined. Then (1.7) is transformed to

$$\rho^{\alpha}\left[\partial_t w - \frac{1}{2}\Delta w - \frac{(b-4\alpha)m - 2\rho\kappa m}{2\rho}\cdot\nabla w - \frac{c}{\rho^2}w\right] = 0, \tag{2.2}$$

where,

$$c(m) = 2\alpha\rho m\cdot\kappa m + (n\rho + 2|m|^2 - 2|m|^2\alpha)(b/2-\alpha). \tag{2.3}$$

Setting a parameter

$$\beta = -\frac{b}{2} + 2\alpha,$$

we rewrite (2.2) as

$$\rho^{b/2-\alpha}\left[\partial_t w\rho^{\beta} - \frac{1}{2}\nabla\cdot(\nabla w\rho^{\beta}) + \kappa m\cdot\nabla w\rho^{\beta} - cw\rho^{\beta-2}\right] = 0.$$

If

$$\alpha \le 1$$

is taken, the boundary condition (1.6) implies that $w(t, \cdot)$ satisfies a homogeneous boundary condition for almost all t since the distance function d and ρ are equivalent (see (2.9)).

The FPF problem is formally transformed to the following W-problem:

$$\partial_t w\rho^{\beta} - \frac{1}{2}\nabla\cdot(\nabla w\rho^{\beta}) + \kappa m\cdot\nabla w\rho^{\beta} - cw\rho^{\beta-2} = 0, \qquad (t, m) \in (0, T] \times B, \tag{2.4}$$

$$w(0, m) = w_0(m), \qquad m \in B, \tag{2.5}$$

$$w(t, m) = 0, \qquad (t, m) \in [0, T] \times \partial B. \tag{2.6}$$

Here,

$$w_0(m) = f_0(m)\rho^{-\alpha}$$

according to the transformation (2.1).

In order to define a weak solution of W-problem we introduce a weighted Sobolev space $H^1(\Omega;\sigma)$ for a nonnegative measurable function σ as a set of measurable function ϕ such that

$$||\phi||^2_{H^1(\Omega;\sigma)} := \int_\Omega (|\nabla\phi|^2 + \phi^2)\sigma dm < \infty.$$

Similarly, a weighted $L^2(\Omega;\sigma)$ can be defined. $\overset{\circ}{H}{}^1(\Omega;\sigma)$ denotes a completion of $C_c^\infty(\Omega)$ with $||\cdot||_{H^1(\Omega;\sigma)}$. It is obvious that $H^1(\Omega;\sigma)$ and $\overset{\circ}{H}{}^1(\Omega;\sigma)$ are Hilbert spaces with the inner product $\langle\cdot,\cdot\rangle_{H^1(\Omega;\sigma)}$ defined as

$$\langle\phi_1,\phi_2\rangle_{H^1(\Omega;\sigma)} = \int_\Omega (\nabla\phi_1\cdot\nabla\phi_2 + \phi_1\phi_2)\sigma dm$$

and

$$\overset{\circ}{H}{}^1(\Omega;\sigma) \subset H^1(\Omega;\sigma).$$

For convenient notation, we use $H^1_\mu(\Omega), \overset{\circ}{H}{}^1_\mu(\Omega)$ and $L^2_\mu(\Omega)$ for $H^1(\Omega;\rho^\mu), \overset{\circ}{H}{}^1(\Omega;\rho^\mu)$ and $L^2(\Omega;\rho^\mu)$ respectively. We also omit a domain Ω if it is obvious.

Lemma 4. *Suppose that $\Omega = B$ and $\mu < 1$.*

(1) *If $\phi \in \overset{\circ}{H}{}^1_\mu$, then*

$$||\phi||_{L^2_{\mu-2}} \le C_0 ||\phi||_{H^1_\mu}. \tag{2.7}$$

(2) *If $\phi \in H^1_\mu$, then the trace map*

$$\begin{aligned} \mathcal{T}:\ H^1_\mu(\Omega) &\to L^2(\partial\Omega) \\ \phi &\mapsto \phi|_{\partial\Omega} \end{aligned}$$

is well defined, i.e. it is a bounded linear map.

In particular, for $\phi \in \overset{\circ}{H}{}^1_\mu$

$$\mathcal{T}(\phi) = 0. \tag{2.8}$$

Proof. In [16](see also [9]), it was proved that

$$\overset{\circ}{H}{}^1(\Omega;d^\mu) \hookrightarrow L^2(\Omega;d^{\mu-2})$$

provided $\partial\Omega$ is Lipschitz continuous. Recall that d denotes the distance from m to the boundary of Ω. (2.7) follows from

$$\sqrt{b}d \le \rho \le 2\sqrt{b}d. \tag{2.9}$$

It is also known that the trace map $\mathcal{T}$ is well defined for $0 \le \mu < 1$ ([16, 9]). For $\mu < 0$

$$||\phi||_{H^1} \le b^{-\mu/2}||\phi||_{H^1_\mu}$$

since $\rho^\mu \ge b^\mu$ for all $m \in B$. Therefore, $\mathcal{T}$ is well defined for $\mu < 1$. (2.8) is obvious from the definitions of the trace map and $\overset{\circ}{H}{}^1_\mu$. □

Note that (2.7) remains true for $\mu > 1$.

We now define a weak solution to W-problem in a standard manner. Multiplication by a test function $\tilde{\varphi} \in C_c^1(B)$ to the equation (2.4) and integration over B yield

$$\int_B \left[\partial_t w \tilde{\varphi} \rho^\beta + \frac{1}{2} \nabla w \cdot \nabla \tilde{\varphi} \rho^\beta + \kappa m \cdot \nabla w \tilde{\varphi} \rho^\beta - c w \tilde{\varphi} \rho^{\beta-2} \right] dm = 0.$$

This equation is well defined assuming that $\partial_t w(t,\cdot) \in (\overset{\circ}{H}{}^1_\beta)^*$, the dual space of $\overset{\circ}{H}{}^1_\beta$, and $w(t,\cdot), \tilde{\varphi} \in \overset{\circ}{H}{}^1_\beta$ due to the boundedness of c and Lemma 4. Moreover,

$$\overset{\circ}{H}{}^1_\beta \subset L^2_\beta$$

implies

$$\overset{\circ}{H}{}^1_\beta \subset L^2_\beta \subset (\overset{\circ}{H}{}^1_\beta)^*.$$

Thus

$$w(t,x) \in C([0,T]; L^2_\beta).$$

Here we identify L^2_β with its dual space.

Definition 5. *A function $w(t,m)$ such that*

$$w(t,m) \in L^2(0,T; \overset{\circ}{H}{}^1_\beta), \text{ with } \partial_t w(t,m) \in L^2(0,T; (\overset{\circ}{H}{}^1_\beta)^*)$$

is a weak solution of W-problem, (2.4)-(2.6), *provided*

(1) *For each $\varphi \in \overset{\circ}{H}{}^1_\beta$ and almost every $0 \le t \le T$,*

$$(\partial_t w(t,\cdot), \varphi)_{\overset{\circ}{H}{}^1_\beta} + \mathbf{L}[w,\varphi;t] = \int_B c w(t,m) \varphi \rho^{\beta-2} dm,$$

(2) $w(0,m) = w_0(m)$ *in L^2_β sense. i.e.*

$$\int_B |w(0,m) - w_0(m)|^2 \rho^\beta dm = 0.$$

Here, we let $(\cdot,\cdot)_H$ denote the paring of a Hilbert space H with its dual space H^* and

$$\mathbf{L}[w,\varphi;t] = \frac{1}{2} \int_B \nabla w(t,m) \cdot \nabla \varphi \rho^\beta dm + \int_B \kappa m \cdot \nabla w(t,m) \varphi \rho^\beta dm.$$

The following energy estimate for $\mathbf{L}[w,w;t]$ for fixed t can be achieved from a simple modification of energy estimates for the bilinear form in elliptic equations, see [6] for details.

Lemma 6. *There exist positive constants C_1 and C_2 depending only on b and $|\kappa|$ such that*

$$C_1 ||w(t,\cdot)||^2_{H^1_\beta} \le \mathbf{L}[w,w;t] + C_2 ||w(t,\cdot)||^2_{L^2_\beta}.$$

3. Well-posedness for the transformed problem

In this section, we show the well-posedness of the weak solution to W-problem. For this aim, we consider the following U-problem containing a non-homogeneous term $h(t,m) \in L^2(0,T;L^2_{2-\beta})$.

$$\partial_t u\rho^\beta - \frac{1}{2}\nabla\cdot(\nabla u\rho^\beta) + \kappa m\cdot\nabla u\rho^\beta - h = 0, \qquad (t,m)\in(0,T]\times B, \tag{3.1}$$

$$u(0,m) = u_0(m), \qquad m\in B, \tag{3.2}$$

$$u(t,m) = 0, \qquad (t,m)\in[0,T]\times\partial B. \tag{3.3}$$

The weak solution of U-problem is defined similarly.

Definition 7. *We say a function u such that*

$$u \in L^2(0,T;\overset{\circ}{H}{}^1_\beta),\ \text{with } \partial_t u \in L^2(0,T;(\overset{\circ}{H}{}^1_\beta)^*)$$

is a weak solution of U-problem provided

(1) *for each $\varphi\in\overset{\circ}{H}{}^1_\beta$ and almost every $0\le t\le T$*

$$(\partial_t u(t,\cdot),\varphi)_{\overset{\circ}{H}{}^1_\beta} + \mathbf{L}[u,\varphi;t] = \int_B h(t,m)\varphi dm,$$

(2) *$u(0,m) = u_0(m)$ in L^2_β.*

Recall that

$$\mathbf{L}[u,\varphi;t] = \frac{1}{2}\int_B \nabla u(t,m)\cdot\nabla\varphi\rho^\beta dm + \int_B \kappa m\cdot\nabla u(t,m)\varphi\rho^\beta dm.$$

We remark that $\int_B h(t,m)\varphi dm$ is finite for any $h(t,\cdot)\in L^2_{2-\beta}$ since $\varphi\in L^2_{\beta-2}$ from (2.7). Thus $\int_B h(t,m)\varphi dm$ can be understood as the L^2_0 inner product although $h(t,\cdot)$ may not belong to L^2_0.

The well-posedness for U-problem follows from the standard Galerkin method.

Lemma 8. *For given $h\in L^2(0,T;L^2_{2-\beta})$ and $u_0\in L^2_\beta$, U-problem has a unique weak solution. Moreover,*

$$\max_{0\le t\le T}||u(t,\cdot)||_{L^2_\beta} + ||u||_{L^2(0,T;\overset{\circ}{H}{}^1_\beta)} + ||\partial_t u||_{L^2(0,T;(\overset{\circ}{H}{}^1_\beta)^*)} \le C(||h||_{L^2(0,T;L^2_{2-\beta})} + ||u_0||_{L^2_\beta}). \tag{3.4}$$

Proof. We first construct an approximate solution in a finite-dimensional space. Let $\{\phi_i\}$ be a basis of $\overset{\circ}{H}{}^1_\beta$ and L^2_β. The existence of such a basis can be verified from the fact that $\overset{\circ}{H}{}^1_\beta$ is a dense subset of L^2_β. Consider an approximation $u_l(t,m) = \sum_{i=1}^{l} d_i^l(t)\phi_i$, where d_i^l satisfies

$$(\partial_t u_l(t,\cdot),\phi_j)_{\overset{\circ}{H}{}^1_\beta} + \mathbf{L}[u_l,\phi_j;t] = \langle h(t,\cdot),\phi_j\rangle_{L^2_0}, \quad 1\le j\le l, \tag{3.5}$$

$$\sum_{i=1}^{l} d_i^l(0)\phi_i \to u_0 \text{ in } L^2_\beta, \text{ as } l\to\infty. \tag{3.6}$$

Since (3.5) and (3.6) form a system of linear differential equations, $\{d_i^l\}$ is uniquely determined for each l. We rewrite (3.5) as

$$\langle \partial_t u_l(t,\cdot), \phi_j \rangle_{L^2_\beta} + \mathbf{L}[u_l, \phi_j; t] = \langle h(t,\cdot), \phi_j \rangle_{L^2_0}, \quad 1 \le j \le l. \tag{3.7}$$

Apply d_j^l to (3.7) and sum for $1 \le j \le l$, then for almost every t

$$\langle \partial_t u_l(t,\cdot), u_l(t,\cdot) \rangle_{L^2_\beta} + \mathbf{L}[u_l, u_l; t] = \langle h(t,\cdot), u_l(t,\cdot) \rangle_{L^2_0}.$$

From Lemma 6, it follows that

$$\frac{d}{dt}||u_l(t,\cdot)||^2_{L^2_\beta} + 2C_1 ||u_l(t,\cdot)||^2_{H^1_\beta} \le 2C_2 ||u_l(t,\cdot)||^2_{L^2_\beta} + 2\langle h(t,\cdot), u_l(t,\cdot)\rangle_{L^2_0}. \tag{3.8}$$

From (2.7), for any $\delta > 0$

$$|\langle h(t,\cdot), u_l(t,\cdot)\rangle_{L^2_0}| \le \frac{1}{2\delta}||h(t,\cdot)||^2_{L^2_{2-\beta}} + \frac{\delta}{2} C_0^2 ||u_l(t,\cdot)||^2_{H^1_\beta}.$$

With $\delta = C_1/C_0^2$, (3.8) can be rewritten as

$$\frac{d}{dt}||u_l(t,\cdot)||^2_{L^2_\beta} + C_1 ||u_l(t,\cdot)||^2_{H^1_\beta} \le 2C_2 ||u_l(t,\cdot)||^2_{L^2_\beta} + C_0^2/C_1 ||h(t,\cdot)||^2_{L^2_{2-\beta}}, \tag{3.9}$$

or

$$\frac{d}{dt}||u_l(t,\cdot)||^2_{L^2_\beta} \le 2C_2 ||u_l(t,\cdot)||^2_{L^2_\beta} + C_0^2/C_1 ||h(t,\cdot)||^2_{L^2_{2-\beta}}.$$

Use Gronwall's inequality to obtain

$$\max_{0\le t\le T} ||u_l(t,\cdot)||^2_{L^2_\beta} \le C\left(||u_0||^2_{L^2_\beta} + ||h||^2_{L^2(0,T;L^2_{2-\beta})}\right),$$

where C is an appropriate constant which depends on β, b, T and $|\kappa|$. On the other hand, integration of (3.9) from 0 to T together with above inequality yields

$$||u_l||^2_{L^2(0,T;\mathring{H}^1_\beta)} \le C\left(||u_0||^2_{L^2_\beta} + ||h||^2_{L^2(0,T;L^2_{2-\beta})}\right). \tag{3.10}$$

A similar argument to that in [6] gives us the estimate for $||\partial_t u_l||$ as

$$||\partial_t u_l||^2_{L^2(0,T;(\mathring{H}^1_\beta)^*)} \le C\left(||u_0||^2_{L^2_\beta} + ||h||^2_{L^2(0,T;L^2_{2-\beta})}\right).$$

Here we have used (3.5) with $\phi \in \mathring{H}^1_\beta$ such that $||\phi||_{H^1_\beta} \le 1$ and (3.10). By passing to the limit as $l \to \infty$ and a standard argument (e.g. see [6]), we have well-posedness for U-problem. □

Now, we introduce a linear map $\mathcal{A}$ to connect W and U-problems as

$$\begin{aligned} \mathcal{A}: \quad L^2(0,\tau;L^2_\beta) &\to L^2(0,\tau;L^2_{2-\beta}) \\ w \quad &\mapsto c w \rho^{\beta-2}. \end{aligned}$$

To have a well-defined $\mathcal{A}$, we choose

$$\alpha = 1, \tag{3.11}$$

which is crucial in this argument. With this α, we rewrite $c(m)$ defined in (2.3) as follows:

$$\begin{aligned} c(m) &= 2\rho m \cdot \kappa m + n\rho(b/2 - 1), \\ &\overset{\text{def.}}{=} \tilde{c}(m)\rho. \end{aligned}$$

Since $\tilde{c}$ is bounded,

$$\begin{aligned} ||\tilde{c}w(t,\cdot)\rho^{\beta-1}||^2_{L^2_{2-\beta}} &\le ||\tilde{c}||^2_{L^\infty}\int_B w^2(t,\cdot)\rho^{2\beta-2}\rho^{2-\beta}dm, \\ &= ||\tilde{c}||^2_{L^\infty}\int_B w^2(t,\cdot)\rho^{\beta}dm, \\ &\le C||w(t,\cdot)||^2_{L_\beta}. \end{aligned}$$

Thus, $\mathcal{A}$ is well defined and

$$||\mathcal{A}(w_1)-\mathcal{A}(w_2)||^2_{L^2(0,T;L^2_{2-\beta})} \le ||\tilde{c}||^2_{L^\infty}||w_1-w_2||^2_{L^2(0,T;L^2_\beta)}.$$

We define another map $\mathcal{F}$ such that

$$\begin{aligned} \mathcal{F}:\ \ C([0,\tau];L^2_\beta) &\to C([0,\tau];L^2_\beta) \\ w \qquad &\mapsto u. \end{aligned}$$

Here, $\mathcal{F}(w)$ is given by the weak solution of U-problem with

$$h=\mathcal{A}(w),$$

and the initial condition

$$u_0(m)=w(0,m).$$

The map $\mathcal{F}$ is well defined from Lemma 8 and the definition of $\mathcal{A}$. Now we show that $\mathcal{F}$ is a contraction mapping for sufficiently small τ. Let

$$u_1=\mathcal{F}(w_1), \quad u_2=\mathcal{F}(w_2).$$

From the energy estimate (3.4),

$$\begin{aligned} ||u_1-u_2||^2_{C([0,\tau];L^2_\beta)} &\le C||\mathcal{A}(w_1)-\mathcal{A}(w_2)||^2_{L^2(0,\tau;L^2_{2-\beta})} \\ &= C\int_0^\tau ||\mathcal{A}(w_1)(t,\cdot)-\mathcal{A}(w_2)(t,\cdot)||^2_{L^2_{2-\beta}}dt \\ &\le C\int_0^\tau ||w_1(t,\cdot)-w_2(t,\cdot)||^2_{L^2_\beta}dt \\ &= C\tau||w_1-w_2||^2_{C([0,\tau];L^2_\beta)}. \end{aligned}$$

Thus, $\mathcal{F}$ has a unique fixed point w in $C([0,\tau];L^2_\beta)$ and w solves W-problem in a weak sense in $(0,\tau]\times B$, if $C\tau<1$. We are able to continue this procedure to obtain the global well-posedness for the above constant C is independent of τ.

For the fixed point w, (3.4) and the boundedness of $\mathcal{A}$ imply that for $t\in[0,\tau']$

$$\begin{aligned} &\max_{0\le t\le\tau'}||w(t,\cdot)||_{L^2_\beta}+||w||_{L^2(0,\tau';\mathring{H}^1_\beta)}+||\partial_t w||_{L^2(0,\tau';(\mathring{H}^1_\beta)^*)} \\ &\le C||w||_{L^2(0,\tau';L^2_{2-\beta})}+C||w_0||_{L^2_\beta} \\ &\le C\tau'\max_{0\le t\le\tau'}||w(t,\cdot)||_{L^2_\beta}+C||w_0||_{L^2_\beta}. \end{aligned}$$

We select a small $\tau'<T$ such that $C\tau'<1$. Then

$$\max_{0\le t\le\tau'}||w(t,\cdot)||_{L^2_\beta}+||w||_{L^2(0,\tau';\mathring{H}^1_\beta)}+||\partial_t w||_{L^2(0,\tau';(\mathring{H}^1_\beta)^*)}\le C||w_0||_{L^2_\beta}.$$

Thus,

$$||w(\tau',\cdot)||_{L^2_\beta}\le C||w_0||_{L^2_\beta}$$

and

$$\begin{aligned}\max_{\tau'\le t\le 2\tau'}\|w(t,\cdot)\|_{L^2_\beta}+\|w\|_{L^2(\tau',2\tau';\overset{\circ}{H}{}^1_\beta)}+\|\partial_t w\|_{L^2(\tau',2\tau';(\overset{\circ}{H}{}^1_\beta)^*)} &\le C\|w(\tau',\cdot)\|_{L^2_\beta}\\ &\le C^2\|w_0\|_{L^2_\beta}.\end{aligned}$$

Continuing, after finitely many steps we obtain an energy estimation similar to (3.4). We summarize this in the following Lemma.

Lemma 9. *W-problem, (2.4)-(2.6), is uniquely solvable in weak sense for $w_0 \in L^2_\beta$. Furthermore,*

$$\max_{0\le t\le T}\|w(t,\cdot)\|_{L^2_\beta}+\|w\|_{L^2(0,T;\overset{\circ}{H}{}^1_\beta)}+\|\partial_t w\|_{L^2(0,T;(\overset{\circ}{H}{}^1_\beta)^*)}\le C\|w_0\|_{L^2_\beta}.$$

4. Well-posedness for the FPF problem

In Section 2, we transformed the FPF problem to W-problem formally, but it is not difficult to show that they are equivalent even in the weak sense if (3.11) is assumed. Indeed, one can verify that the boundary condition (1.6) for the FPF problem is equivalent to the null boundary condition for W-problem. Let

$$\mathfrak{L}_1(f)=0,\quad \mathfrak{L}_2(w)=0$$

denote the Fokker-Planck equation (1.7) and W-problem (2.4), respectively. For any test function $\tilde\varphi\in C^1_c$,

$$\int_B \mathfrak{L}_1(f)\tilde\varphi dm=0\Leftrightarrow\int_B\mathfrak{L}_2(w)\tilde\varphi\rho^{b/2-1}dm=0.$$

Since $\tilde\varphi\rho^{b/2-1}$ is in C^1_c which is dense in $\overset{\circ}{H}{}^1_\beta$, the FPF problem and W-problem are equivalent.

Now we seek the function space in which the weak solution f to the FPF problem belongs. For fixed $t\in[0,T]$, (2.7) implies

$$\int_B |f|^2\rho^{-b/2}dm = \int_B |w|^2\rho^\beta dm, \tag{4.1}$$

$$\int_B (|f|^2+|\nabla f|^2)\rho^{-b/2}dm \le C\int_B (|w|^2+|\nabla w|^2)\rho^\beta dm. \tag{4.2}$$

Also, for $\phi\in H^1_{-b/2}$ we have

$$|(\partial_t f,\phi)_{H^1_{-b/2}}|\le C|(\partial_t w,\rho^{-1}\phi)_{H^1_\beta}|. \tag{4.3}$$

Recall that $\alpha=1$ and $\beta=-b/2+2\alpha$. The estimate of the weak solution, (1.13) follows from Lemma 9 together with (4.1)-(4.3). In order to finish the proof of Theorem 2, we assume that f_1, f_2 are two weak solutions of the FPF problem with arbitrary initial data $f_0(m)$. Then f_1-f_2 solves the FPF problem with zero initial data which is in $L^2_{-b/2}$. From (1.13), $f_1=f_2$ in $L^2(0,T;H^1_\beta)$.

5. Non-uniqueness

In this section we show that (1.6) is sharp in the sense that more solutions can be constructed if a weaker condition is imposed.

We construct a non-trivial solution to the Fokker-Planck equation with $f_0(m) = 0$ and the assumption

$$fd^{-1}|_{\partial B} \neq 0, \quad t \in I \text{ for nonzero measurable set } I. \tag{5.1}$$

Rewrite the Fokker-Planck equation with side conditions as follows:

$$\partial_t f + \nabla \cdot (\kappa m f) = \frac{1}{2}\nabla \cdot \left(\frac{bm}{\rho} f\right) + \frac{1}{2}\Delta f, \qquad \text{in } (0,T] \times B, \tag{5.2}$$

$$f(0,m) = 0, \qquad m \in B, \tag{5.3}$$

$$f(t,m) = 0, \qquad \text{in } (0,T] \times \partial B. \tag{5.4}$$

Obviously, $f \equiv 0$ is a solution of (5.2)-(5.4). Let

$$\tilde{g}(t,m) = \rho g(t,m)$$

for $g(t,m) \in W^{2,\infty}((0,T) \times B)$ such that $g(0,m) = 0$ and $g(t,m)|_{\partial B} \neq 0$ for $t > 0$ (e.g. $g(t,m) = t|m|^2$). We will show the existence of a nontrivial solution f which coincides with $\tilde{g}$ at the boundary. Note that $\tilde{g}$ satisfies (5.1) and

$$\tilde{g}d^{-1}|_{\partial B} \in L^\infty(\partial B).$$

Define a function $\tilde{w}$ as

$$f = \tilde{w}\rho.$$

Then,

$$\partial_t \tilde{w}\rho^\gamma - \tfrac{1}{2}\nabla \cdot (\nabla \tilde{w}\rho^\gamma) + (\beta - \gamma) m \cdot \nabla \tilde{w}\rho^{\gamma-1} + \kappa m \cdot \nabla \tilde{w}\rho^\gamma - \tilde{c}\tilde{w}\rho^{\gamma-1} = 0,$$
$$\tilde{w}(0,m) = 0, \quad m \in B,$$
$$\tilde{w}(t,m) = g(t,m), \quad (t,m) \in [0,T] \times \partial B,$$

for a parameter γ such that

$$\max\{\beta, -1\} < \gamma < 1. \tag{5.5}$$

Note that $\beta = 2 - b/2 < 1$, we can thus take such γ. Recall that

$$\tilde{c}(m) = 2m \cdot \kappa m + n(b/2 - 1).$$

In order to have the zero boundary condition, we define

$$w = \tilde{w} - g.$$

Then w solves

$$\partial_t w\rho^\gamma - \tfrac{1}{2}\nabla \cdot (\nabla w\rho^\gamma) + (\beta - \gamma) m \cdot \nabla w\rho^{\gamma-1} + \kappa m \cdot \nabla w\rho^\gamma - \tilde{h}_0 = 0, \tag{5.6}$$

$$w(0,m) = 0, \ m \in B, \tag{5.7}$$

$$w(t,m) = 0, \ (t,m) \in [0,T] \times \partial B, \tag{5.8}$$

where

$$\tilde{h}_0(t,m) = \tilde{c}w\rho^{\gamma-1} - \partial_t g\rho^\gamma + \frac{1}{2}\nabla \cdot (\nabla g\rho^\gamma) - (\beta - \gamma) m \cdot \nabla g\rho^{\gamma-1} - \kappa m \cdot \nabla g\rho^\gamma + \tilde{c}g\rho^{\gamma-1}.$$

Let

$$\begin{aligned}\mathcal{A}_0: \quad L^2(0,\tau;L^2_\gamma) &\to L^2(0,\tau;L^2_{2-\gamma})\\ w &\mapsto h_0.\end{aligned}$$

This is well defined because of (5.5) and the assumption that $g \in W^{2,\infty}((0,T)\times B)$. From the same argument as that in Section 3, it follows that (5.6)-(5.8) has a unique solution w such that

$$w(t,m) \in L^2(0,T;\overset{\circ}{H}{}^1_\gamma), \quad \partial_t w(t,m) \in L^2(0,T;(\overset{\circ}{H}{}^1_\gamma)^*),$$

provided the corresponding U-problem

$$\partial_t u\rho^\gamma - \tfrac{1}{2}\nabla\cdot(\nabla u\rho^\gamma) + (\beta-\gamma)m\cdot\nabla u\rho^{\gamma-1} + \kappa m\cdot\nabla u\rho^\gamma - h_0 = 0, \tag{5.9}$$

$$u(0,m) = 0, \ m \in B, \tag{5.10}$$

$$u(t,m) = 0, \ (t,m) \in [0,T]\times\partial B, \tag{5.11}$$

has a solution for any $h_0 \in L^2(0,T;L^2_{2-\gamma})$. Note that $\gamma < 1$ is essential in order that the trace of w at the boundary is defined. Equation (5.9) is of the form of (3.1) except $(\beta-\gamma)m\cdot\nabla u\rho^{\gamma-1}$. We thus define

$$\mathbf{L}_0[u,\varphi;t] \overset{\text{def.}}{=} \frac{1}{2}\int_B \nabla u\cdot\nabla\varphi\rho^\gamma dm + (\beta-\gamma)\int_B m\cdot\nabla u\varphi\rho^{\gamma-1}dm + \int_B \kappa m\cdot\nabla u\varphi\rho^\gamma dm.$$

We may obtain the existence and uniqueness for (5.9)-(5.11) from the same argument of the well-posedness for U-problem (3.1)-(3.3), if there is an energy estimate of $\mathbf{L}_0[u,u;t]$ which is similar to $\mathbf{L}[u,u;t]$ in Lemma 6. Indeed, for $u \in L^2(0,T;\overset{\circ}{H}{}^1_\gamma)$

$$\frac{1}{2}\int_B |\nabla u|^2\rho^\gamma dm = \mathbf{L}_0[u,u;t] - \frac{\beta-\gamma}{2}\left(\int_B m\cdot\nabla u^2\rho^{\gamma-1}dm\right) - \int_B \kappa m\cdot\nabla uu\rho^\gamma dm.$$

We now claim that

$$\int_B m\cdot\nabla u^2\rho^{\gamma-1}dm \le 0. \tag{5.12}$$

Given this together with (5.5),

$$\begin{aligned}\frac{1}{2}\int_B |\nabla u(\cdot,t)|^2\rho^\gamma dm &\le \mathbf{L}_0[u,u;t] - \int_B \kappa m\cdot\nabla uu\rho^\gamma dm\\ &\le \mathbf{L}_0[u,u;t] + |\kappa|\sqrt{b}\left(\frac{1}{2\delta}\int_B |\nabla u|^2\rho^\gamma dm + \frac{\delta}{2}\int_B |u|^2\rho^\gamma dm\right)\end{aligned}$$

for any $\delta > 0$. By taking δ so small, we obtain

$$C_1'||u(t,\cdot)||^2_{H^1_\gamma} \le \mathbf{L}_0[u,u;t] + C_2'||u(t,\cdot)||^2_{L^2_\gamma}$$

for appropriate constants C_1' and C_2'.

To verify the claim (5.12), we define the trace operator $\mathcal{T}_0$ such that

$$\begin{aligned}\mathcal{T}_0: \quad H^1_\gamma(B) &\to L^2(\partial B)\\ u &\mapsto u\rho^{\frac{\gamma-1}{2}}|_{\partial\Omega}.\end{aligned}$$

Integration by parts on (5.12) yields

$$\int_B m\cdot\nabla u^2\rho^{\gamma-1}dm = -\int_B u^2\left(n\rho^{\gamma-1} + 2(1-\gamma)|m|^2\rho^{\gamma-2}\right)dm + \sqrt{b}\int_{\partial B} u^2\rho^{\gamma-1}dS(m),$$

or

$$\begin{aligned}\sqrt{b}\int_{\partial B} u^2\rho^{\gamma-1}dS(m) &= 2\int_B m\cdot\nabla u u\rho^{\gamma-1}dm + \int_B u^2\left(n\rho^{\gamma-1}+2(1-\gamma)|m|^2\rho^{\gamma-2}\right)dm \\ &\leq C||u||^2_{H^1_\gamma}.\end{aligned}$$

Thus $\mathcal{T}_0$ is well defined, and for $u \in \overset{\circ}{H}{}^1_\gamma$, $\mathcal{T}_0(u) = 0$. Finally we obtain

$$\int_B m\cdot\nabla u^2\rho^{\gamma-1}dm = -\int_B u^2\left(n\rho^{\gamma-1}+2(1-\gamma)|m|^2\rho^{\gamma-2}\right)dm \leq 0.$$

This shows that there is a unique weak solution u of (5.9)-(5.11), and thus $w \in L^2(0,T;\overset{\circ}{H}{}^1_\gamma)$ of (5.6)-(5.8). Finally, $f = (w+g)\rho$ is a nontrivial solution of (5.2)-(5.4) satisfying (5.1). Hence the uniqueness of (5.2)-(5.4) fails as stated in Proposition 3.

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