

A Variational Multiphase Model based on the Piecewise Constant Level Set Method and Phase Transitions *

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Abstract. A novel interfacial energy for variational multiphase problems based on the piecewise constant level set method (PCLSM) and the gradient theory of phase transitions is proposed. We identify multiple regions using one labeling function ϕ and formulate our free energy that acts on its associated characteristic functions. Our proposed model can be applied to multiphase motion problems, image classification, segmentation and restoration. In contrast to some multiphase models in phase field literature, our variational multiphase model does not weigh the length of the object boundaries. This is shown via Γ -convergence of our proposed model. The numerical schemes for the proposed model are included and applied to multiphase motion problem and some synthetic images for demonstration purpose.

Keywords. multiphase, triple junction, PCLSM, phase transition, Γ -convergence, perimeter, image segmentation and classification.

1 Introduction

In this paper we consider multiphase problems involving N different phases, where N is an integer greater than or equal to 2. They include, but not exhaustively, multiphase motion problem, multiphase image segmentation and classification problem. These are the problems which we are mainly concerned with in this paper. We begin with reviewing previous works relevant to our model.

There exist different approaches for multiphase flow, such as front tracking methods, MBO-type projection method, variational level set approach, phase field methods, and diffusion-generated approach [43], [68], [52], [53]. For the model that they consider, the normal velocity v_{ij} of the interface is a positive multiple of the curvature of the interface plus the difference of the bulk energies:

$$v_{ij} = f_{ij}\kappa_{ij} + (e_i - e_j). \quad (1)$$

where f_{ij} is the surface tension of Γ_{ij} , κ_{ij} is the curvature of Γ_{ij} , and e_{ij} is the bulk energy density. Such motion arises from the total energy E of this system, which is given by

$$E = \sum_{1 \leq i < j \leq N} f_{ij} \text{Length}(\Gamma_{ij}) + \sum_{1 \leq i \leq N} e_i \text{Area}(\Omega_i). \quad (2)$$

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In Zhao et al. [68], each phase is represented as the zero level set of a function. Therefore, for N different phases, N different level set functions are needed. One drawback associated with their approach is that vacuum or overlaps between different phases may occur. To prevent such possibilities occurring, they introduced a Lagrange multiplier associated with a constraint. Most recently, the authors in [36] proposed to apply the PCLSM to multiphase motion problems. They formulated a variational approach for mean curvature motion problem using one level set function and associating an energy functional which consists of surface tension.

The next problem that this paper is concerned with is multiphase image segmentation problem. It consists of partitioning an image into regions which correspond to meaningful parts of objects in a visual scene. The image intensity has to be as uniform as possible inside each region, while sharp transitions take place across the boundaries. Let Ω be a bounded and open domain in \mathbb{R}^n , where $n = 2$ or 3 . Let $f : \Omega \rightarrow \mathbb{R}$ be a given image. The Mumford-Shah functional was introduced in [47] and [48] for image segmentation problems within a variational framework. Their energy functional is given by

$$F^{MS}(u, \Gamma) = \beta \int_{\Omega} |u - f|^2 dx + \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \mu \mathcal{H}^{n-1}(\Gamma), \quad (3)$$

where β and μ are positive parameters, Γ is the boundary set, and \mathcal{H}^{n-1} is the $n-1$ dimensional Hausdorff measure (counting measure if $n = 1$, length measure for $n = 2$ and area measure for $n = 3$). Any minimizing pair (u, Γ) is such that u is close to f and slowly varying in $\Omega \setminus \Gamma$ and Γ has to be as small as possible. The solution image u is formed by smooth regions, separated by sharp boundaries.

The existence of a minimizer for (3) was conjectured in [48] and has been established in [23],[24] by using the semicontinuity and the compactness result of Ambrosio [2] related to functionals defined in *SBV* spaces.

A common simplification to (3) is to assume that u is piecewise constant on each region of the segmentation. Thus the term $\int_{\Omega \setminus \Gamma} |\nabla u|^2 dx$ is dropped. The energy function becomes

$$F(u, \Gamma) = \sum_i \int_{\Omega_i} |f - c_i|^2 dx + \mu \mathcal{H}^{n-1}(\Gamma), \quad (4)$$

where $u = c_i$ in each Ω_i . In image segmentation literature, there exist different techniques to solve (4), including curve evolution, level set method and the PCLSM.

Curve evolution techniques and implicit representations [49] have been proposed to solve particular cases of the minimal partition problem, where the number of regions Ω_i or an upper bound are assumed to be known. Thus, in [16], [17], [64], [63], restrictions of the energy (3) to piecewise-constant functions taking a finite number of regions and intensities, in a variational level set approach, have been considered.

The variational level set approach from [68] has been used, together with the level set method [49]. The boundaries were represented by zero level lines of some level set functions.

In [20],[19], they continue the approaches from [16], [17], [64] and show that one can use even fewer number of level set functions to represent disjoint regions making up Ω . The main idea is to use more than one level set of the Lipschitz continuous function ϕ to represent the discontinuity set of u .

Recently some piecewise constant level set methods (PCLSM) were proposed [37], [38], [39] for image segmentation and other interface problems. The methods need to minimize a smooth cost functional under some special constraints and need to use one level set function to identify arbitrary number of regions. In [60], they proposed fast algorithms for Mumford-Shah image segmentation using the PCLSM. They use the MBO type of projection to deal with the constraint and AOS (additive operator splitting) schemes or multiplicative operator splitting (MOS) schemes to solve the Euler-Lagrange equations for the minimization of the energy functional.

Besides all the above-mentioned models, there are variational image classification and segmentation models based on phase transition from fluid mechanics and material sciences. Samson et al. [56] proposed a variational model for image classification and edge preserving restoration. The theoretical soundness of their variational model can be shown via Γ -convergence of the sequence of functionals. Jung et al. [35] proposed the sine-sinc model for multiphase image segmentation via Modica-Mortola phase transition. Similar to those considered in [64],[20],[19], [38], their objective is to identify multiphases by piecewise constant values. Their model is built upon the phase transition model of Modica-Mortola. We note that the previous works based on phase transitions [56] ,[8], [35] yield weighed length with the weight depending on phase labeling.

In this paper, we propose a novel variational interfacial energy for multiphase problems: multiphase motion, multiphase piecewise constant image segmentation and classification. Based on our observation, we formulate an interfacial energy for multiphase problems using the Van der Waals-Cahn-Hilliard energy and the PCLSM. Our main goal in this paper is to identify multiple regions and to compute the length of each interface using one labeling function. The paper is organized as follows. In Section 2 we will review some previous works relevant to our new model. In Section 3 we give our new interfacial energy and show its Γ -convergence. In Section 4 we give numerical implementation and experiments for multiphase motion. In Section 5 we give a new variational multiphase piecewise constant image segmentation and classification model, and numerical implementation, and show some numerical experiments.

2 Interface problems: sharp interface, diffuse interface

There exist different methods for solving interface problems. Among many of them, two most relevant methods to our proposed model are sharp interface and diffuse interface. Before showing our proposed model, first we review the PCLSM, phase transitions and Γ -convergence.

2.1 The Piecewise Constant Level Set Method (PCLSM)

A variant of level set method called the piecewise constant level set method (PCLSM) [38], [39] was first proposed in the year 2003. Instead of representing the interfaces by the zero level set of the level set function, they are represented by discontinuities of a piecewise constant level set function ϕ . Assume that one need to partition the domain Ω into subdomains Ω_i , $i = 1, 2, \dots, N$ and the number of subdomains is a priori known. In order to identify the subdomains, they try to identify a piecewise constant level set function ϕ such that

$$\phi = i \text{ in } \Omega_i, \quad i = 1, 2, \dots, N. \quad (5)$$

Thus for any given partition $\{\Omega_j\}_{j=1}^N$ of the domain Ω , it corresponds to a unique PCLSM function ϕ which takes the values $1, 2, \dots, N$. They associate with such a level set function ϕ , the characteristic functions to the subdomains, which are defined as

$$\psi_i = \frac{1}{\alpha_i} \prod_{j=1, j \neq i}^N (\phi - j) \text{ and } \alpha_i = \prod_{k=1, k \neq i}^N (i - k), \quad \text{for } i = 1, 2, \dots, N. \quad (6)$$

If ϕ is given as in (5), then they have $\psi_i(x) = 1$ for $x \in \Omega_i$, and $\psi_i(x) = 0$ elsewhere. They use the characteristic functions to extract geometric information for the subdomains and the interface between the subdomains.

$$\text{Length}(\partial\Omega_i) = \int_{\Omega} |\nabla \psi_i| dx, \quad \text{Area}(\Omega_i) = \int_{\Omega} \psi_i dx. \quad (7)$$

The PCLSM is an alternative interface tracking method and one can avoid the reinitialization to a signed distance function and discretization of Heaviside and Dirac delta functions as the case for the traditional level set method. The constraint and energy functionals are all smooth functionals which do not involve Heaviside and Dirac delta functions.

2.2 Γ -convergence

The notion of Γ -convergence was introduced in a paper by E. De Giorgi and T. Franzoni in 1975 [26], and was since then much developed especially in connection with applications to problems in the Calculus of Variations. We recall the definition and some properties of Γ -convergence. We refer to [13],[22] for further details on the subject.

Definition 2.1. (Γ -convergence) *Let $X = (X, d)$ be a metric space and let $\mathcal{E}_\varepsilon : X \rightarrow [0, +\infty]$ be a function defined on X for every $\varepsilon > 0$. We say that \mathcal{E}_ε Γ -converges to \mathcal{E} on X as $\varepsilon \rightarrow 0$, and write $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}$ if the following two conditions hold for all $u \in X$:*

(i) *for every sequence (u_ε) converging to $u \in X$,*

$$\mathcal{E}(u) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) \quad (\text{liminf estimate}),$$

(ii) *there exists (u_ε) converging to u such that*

$$\mathcal{E}(u) \geq \limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) \quad (\text{limsup estimate}).$$

The main properties of Γ -convergence are stated in the following theorem.

Theorem 2.1. (*main properties of the Γ -limit*)

(i) *the Γ -limit \mathcal{E} , if exists, is always lower semicontinuous on X .*

(ii) *if $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}$ and u_ε minimizes \mathcal{E}_ε over X , then every cluster point of (u_ε) minimizes \mathcal{E} over X .*

(iii) *if $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}$ and \mathcal{G} is continuous, then $\mathcal{E}_\varepsilon + \mathcal{G} \xrightarrow{\Gamma} \mathcal{E} + \mathcal{G}$.*

We note that the property (ii) can be used only after one verifies that the minimizing sequence (u_ε) is pre-compact in X . This notion of Γ -convergence has been used for modeling phase transition and thin films in material sciences, and approximating the Mumford-Shah model in free-discontinuity problems [4].

2.3 The Gradient Theory of Phase Transitions

Consider a system consisting of two unstable components (or phases), and every configuration of the system is described by a function u . We can expect that at equilibrium they separate into two phases with a minimal interface area. The energy is given by

$$F(u) = \sigma \mathcal{H}^{n-1}(S_u), \quad (8)$$

where σ is called the surface tension between the two phases and \mathcal{H}^{n-1} is the $n - 1$ dimensional Hausdorff measure and S_u is the singular set of u or the interface between the two phases. An alternative way to studying such system is to assume that the transition is not given by a separating interface, but is rather a continuous phenomenon occurring in a thin layer which we identify with the interface. Van der Waals and Cahn-Hilliard [62],[15] proposed to use a thin layer of continuous interface to model this separation. The energy that they minimize is

$$F_\varepsilon^{VWCH}(u) := \varepsilon^2 \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} W(u) dx, \quad (9)$$

where ε is a small positive parameter and $W(s) = s^2(s-1)^2$ is a double-well potential which vanishes at 0 and 1. When we try to minimize F_ε^{VWCH} , the term $\int_\Omega W(u)$ favors those configurations which take values close to 0 and 1 (phase separation), while the term $\varepsilon^2 \int_\Omega |\nabla u|^2$ penalizes the spatial inhomogeneity of u . A connection between the classical model (8) and the Cahn-Hilliard model was established by Modica [44] who proved that the minimizers of suitable rescalings of the functionals F_ε^{VWCH} Γ -converges to F .

The following theorem is one of the main results from the phase transition literature and justifies sharp discontinuities between two phases as limits of smooth interfaces. For further analysis, we refer to [45], [58], [44].

Theorem 2.2. *Set: $\sigma = 2 \int_0^1 \sqrt{W(u)} du$, and for every $\varepsilon > 0$ let*

$$F_\varepsilon(u) = \begin{cases} \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

and

$$F(u) = \begin{cases} \sigma \text{Per}_\Omega \{u = 1\} & \text{if } u \in BV(\Omega), W(u(x)) = 0 \text{ a.e.}, \\ +\infty & \text{otherwise.} \end{cases}$$

Then the functionals F_ε Γ -converges to F .

All the previous works suggest that there is a strong connection between sharp interface approach and diffuse interface approach. Our objectives in this paper is to reveal the connection between the PCLSM and phase field, to identify N - phases and to compute the length of $\partial\Omega_i$, $i = 1, \dots, N$, by one labeling function. In the following section, we give our proposed model based on the PCLSM and Theorem 2.2.

3 Proposed model

In order to identify N - phases and compute the length of $\partial\Omega_i$ for $i = 1, \dots, N$, we propose to minimize the following free energy that acts on ψ_i 's associated with a labeling function ϕ :

$$E_\varepsilon(\phi) = \sum_{i=1}^N \int_\Omega \varepsilon |\nabla \psi_i(\phi)|^2 dx + \sum_{i=1}^N \int_\Omega \frac{1}{\varepsilon} W(\psi_i(\phi)) dx, \quad (10)$$

where W and ψ_i 's are the functions defined as in the previous sections.

3.1 Γ -convergence of the proposed model

We first show the following proposition which will be followed by the main theorem of the paper.

Proposition 3.1. *Consider the functionals*

$$\tilde{E}_\varepsilon(\psi_i(\phi)) = \begin{cases} \int_\Omega \varepsilon |\nabla \psi_i(\phi)|^2 dx + \int_\Omega \frac{1}{\varepsilon} W(\psi_i(\phi)) dx & \text{if } \phi \in H^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

and let

$$\tilde{E}(\psi_i(\phi)) = \begin{cases} \sigma \text{Per}_\Omega \{\psi_i(\phi) = 1\} & \text{if } \phi \in BV(\Omega), W(\psi_i(\phi(x))) = 0 \text{ a.e.}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\sigma = 2 \int_0^1 \sqrt{W(s)} ds$. *Then the functionals \tilde{E}_ε Γ -converges to \tilde{E} in $X = L^1(\Omega)$.*

Proof. We shall adapt the proof in [14]. We need to show

(i) For each $\phi \in L^1(\Omega)$ and every sequence $\{\phi^\varepsilon\}$ converging to ϕ in $L^1(\Omega)$,

$$\tilde{E}(\psi_i(\phi)) \leq \liminf_{\varepsilon \rightarrow 0} \tilde{E}_\varepsilon(\psi_i(\phi^\varepsilon)).$$

(ii) There exists a sequence $\{\phi^\varepsilon\}$ converging to ϕ in $L^1(\Omega)$,

$$\tilde{E}(\psi_i(\phi)) \geq \limsup_{\varepsilon \rightarrow 0} \tilde{E}_\varepsilon(\psi_i(\phi^\varepsilon)).$$

First we prove (i). We note that if $W(\psi_i(\phi(x))) \neq 0$ on a set $\Omega_0 \subset \Omega$ of positive measure, then $\tilde{E}(\psi_i(\phi)) = +\infty$. But we have

$$\liminf_{\varepsilon \rightarrow 0} \tilde{E}_\varepsilon(\psi_i(\phi^\varepsilon)) \geq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} W(\psi_i(\phi^\varepsilon)) dx = +\infty.$$

So (i) is immediate. Therefore, we consider only $\phi \in L^1(\Omega)$ satisfying $W(\psi_i(\phi(x))) = 0$ for almost all $x \in \Omega$, $i = 1, \dots, N$. Assume that the sequence $\{\psi_i(\phi^\varepsilon)\}$ satisfies $0 \leq \psi_i(\phi^\varepsilon) \leq 1$. Define $h : \mathbb{R} \rightarrow \mathbb{R}$ by

$$h(t) = 2 \int_0^t \sqrt{W(s)} ds. \quad (11)$$

Since $\psi_i(\phi(x)) \in \{0, 1\}$ for almost all $x \in \Omega$,

$$\int_{\Omega} |\nabla h(\psi_i(\phi))| dx = \left(2 \int_0^1 \sqrt{W(s)} ds \right) \text{Per}_{\Omega} \{ \psi_i(\phi) = 1 \} = \tilde{E}(\psi_i(\phi)).$$

Applying the inequality $a^2 + b^2 \geq 2ab$ to \tilde{E}_ε , we have

$$\tilde{E}_\varepsilon(\psi_i(\phi^\varepsilon)) \geq 2 \int_{\Omega} \sqrt{W(\psi_i(\phi^\varepsilon))} |\psi_i'(\phi^\varepsilon)| |\nabla \phi^\varepsilon| dx = \int_{\Omega} |\nabla h(\psi_i(\phi^\varepsilon))| dx.$$

By the lower semicontinuity of the total variation functionals and $h(\psi_i(\phi^\varepsilon)) \rightarrow h(\psi_i(\phi))$ in $L^1(\Omega)$, we obtain

$$\liminf_{\varepsilon \rightarrow 0} \tilde{E}_\varepsilon(\psi_i(\phi^\varepsilon)) \geq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla h(\psi_i(\phi^\varepsilon))| dx \geq \int_{\Omega} |\nabla h(\psi_i(\phi))| dx = \tilde{E}(\psi_i(\phi)).$$

In order to prove (ii), we need to construct a sequence of functions which interpolates the values 0 and 1. We prove for functions $\psi_i(\phi)$ such that $\psi_i(\phi(x)) = 1$ for $x \in \Omega_i$ and $\psi_i(\phi(x)) = 0$ for $x \in \Omega \setminus \Omega_i$. More general case can be deduced from this particular case by density argument. For any $0 \leq t \leq 1$ define

$$\eta_\varepsilon(t) = \int_0^t \frac{\varepsilon}{\sqrt{\varepsilon + W(s)}} ds.$$

Let $\rho_\varepsilon : [0, \eta_\varepsilon(1)] \rightarrow [0, 1]$ be the inverse of η_ε . Then we have

$$\rho_\varepsilon' = \frac{1}{\eta_\varepsilon'(\eta_\varepsilon^{-1})} = \frac{\sqrt{\varepsilon + W(\rho_\varepsilon)}}{\varepsilon}.$$

Extend ρ_ε to the entire \mathbb{R} by

$$\rho_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \eta_\varepsilon^{-1}(t) & \text{if } 0 \leq t \leq \eta_\varepsilon(1) \\ 1 & \text{if } t \geq \eta_\varepsilon(1) \end{cases}$$

Define $d : \mathbb{R} \rightarrow \mathbb{R}$ by

$$d(x) = \begin{cases} \text{dist}(x, \partial\Omega_i), & \text{if } x \in \Omega_i \\ -\text{dist}(x, \partial\Omega_i), & \text{if } x \notin \Omega_i \end{cases}$$

Then d is lipschitz continuous, $|\nabla d(x)| = 1$ for a.e. $x \in \mathbb{R}$ and if $\Delta_t = \{x \in \mathbb{R} : d(x) = t\}$,

$$\lim_{t \rightarrow 0} \mathcal{H}^{n-1}(\Delta_t \cap \Omega) = \mathcal{H}^{n-1}(\partial\Omega_i \cap \Omega).$$

Let $\gamma_\varepsilon = \sup_{|t| \leq \eta_\varepsilon(1)} \mathcal{H}^{n-1}(\Delta_t \cap \Omega)$. Define $\psi_i(\phi^\varepsilon(x)) = \rho_\varepsilon(d(x))$. We have $\psi_i(\phi^\varepsilon) \rightarrow \psi_i(\phi)$ in $L^1(\Omega)$.

$$\begin{aligned} \tilde{E}_\varepsilon(\psi_i(\phi^\varepsilon)) &= \int_\Omega \left(\varepsilon |\rho'_\varepsilon(d(x))|^2 + \frac{1}{\varepsilon} W(\rho_\varepsilon(d(x))) \right) |\nabla d| dx \\ &= \int_{\mathbb{R}} \left(\varepsilon |\rho'_\varepsilon(t)|^2 + \frac{1}{\varepsilon} W(\rho_\varepsilon(t)) \right) \mathcal{H}^{n-1}(\Delta_t \cap \Omega) dt \\ &\leq \gamma_\varepsilon \int_0^{\eta_\varepsilon(1)} \left(\varepsilon |\rho'_\varepsilon(t)|^2 + \frac{1}{\varepsilon} W(\rho_\varepsilon(t)) \right) dt. \end{aligned}$$

Since

$$\rho'_\varepsilon = \frac{1}{\eta'_\varepsilon(\eta_\varepsilon^{-1})} = \frac{\sqrt{\varepsilon + W(\eta_\varepsilon^{-1})}}{\varepsilon} = \frac{\sqrt{\varepsilon + W(\rho_\varepsilon)}}{\varepsilon},$$

we have

$$\begin{aligned} \tilde{E}_\varepsilon(\psi_i(\phi^\varepsilon)) &\leq \gamma_\varepsilon \int_0^{\eta_\varepsilon(1)} \left(\frac{\varepsilon + W(\rho_\varepsilon(t))}{\varepsilon} + \frac{W(\rho_\varepsilon(t))}{\varepsilon} \right) dt \\ &\leq \frac{2\gamma_\varepsilon}{\varepsilon} \int_0^{\eta_\varepsilon(1)} (\varepsilon + W(\rho_\varepsilon(t))) dt \\ &= 2\gamma_\varepsilon \int_0^1 \sqrt{\varepsilon + W(s)} ds. \end{aligned}$$

Since $\lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon = \mathcal{H}^{n-1}(\partial\Omega_i \cap \Omega) = \text{Per}_\Omega\{\psi_i(\phi) = 1\}$, $\limsup_{\varepsilon \rightarrow 0} \tilde{E}_\varepsilon(\psi_i(\phi^\varepsilon)) \leq \sigma \text{Per}_\Omega\{\psi_i(\phi) = 1\}$.

□

From the above proposition and Theorem 2.1, a subsequence of ϕ_ε of the minimizer of \tilde{E}_ε will converge to a ϕ satisfying $W(\psi_i(\phi)) = 0, \forall i$. The following lemma shows that this ϕ must take integer values.

Lemma 3.1. *For a given $x \in \Omega$, assume $\psi_i(\phi(x)) = 1, i = 1, \dots, N$, then we have $\phi(x) = i$ for $x \in \Omega_i, i = 1, \dots, N$.*

Proof. Let $K(\phi) = \prod_{i=1}^N (\phi - i)$. Suppose $\psi_j = 1$ for some fixed $j \in \{1, \dots, N\}$. It is easy to see that

$$\psi_j = \frac{K(\phi)}{(\phi - j)\alpha_j} = 1 \Leftrightarrow K(\phi) = (\phi - j)\alpha_j.$$

We shall prove by contradiction. Suppose $\phi(x) \neq j$. Then we have two cases to consider:

case 1. $\phi(x) \in \{1, \dots, j-1, j+1, \dots, N\}$.

Then we have $K(\phi) = 0 = (\phi - j)\alpha_j$, which is a contradiction since $\alpha_j \neq 0 \quad \forall j$.

case 2. $\phi(x) = \phi_0 \notin \{1, \dots, j-1, j+1, \dots, N\}$. It is easy to see that $\prod_{i=1}^N (\phi_0 - i) = (\phi_0 - j)\alpha_j$ and thus

$$\prod_{i=1, i \neq j}^N (\phi_0 - i) = \alpha_j.$$

The above equation is satisfied by the roots of a polynomial which only has $N - 1$ roots. However, ϕ_0 is none of them and this is a contradiction. □

The following theorem, which follows from the previous proposition, states that our proposed interfacial energy $E_\varepsilon(\phi)$ Γ -converges to σ times the sum of the length of $\partial\Omega_i$, $i = 1, \dots, N$.

Theorem 3.1. *Consider the functionals*

$$E_\varepsilon(\phi) = \begin{cases} \sum_{i=1}^N \int_{\Omega} \varepsilon |\nabla \psi_i(\phi)|^2 dx + \sum_{i=1}^N \int_{\Omega} \frac{1}{\varepsilon} W(\psi_i(\phi)) dx & \text{if } \phi \in H^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

and let

$$E(\phi) = \begin{cases} \sigma \sum_{i=1}^N \text{Per}_{\Omega} \{\psi_i(\phi) = 1\} & \text{if } \phi \in BV(\Omega), W(\psi_i(\phi(x))) = 0 \text{ a.e. } \forall i, \\ +\infty & \text{otherwise.} \end{cases}$$

Then the functionals E_ε Γ -converge to E in $X = L^1(\Omega)$.

Remarks: There are several remarks we would like to make to highlight the differences among our proposed model and the other models that have previously been proposed.

1. We use only one function ϕ in order to identify any number of phases. In [10] and [64], they proposed to introduce vector-valued density functions and vector-valued level set function, respectively, in order to handle multiphases.
2. Our proposed free energy E_ε Γ -converges to the constant σ times the sum of the length of $\partial\Omega_i$, $i = 1, \dots, N$. For many applications, it is crucial to regularize the length instead of a weighted length. In [35], they adopted the result by Modica-Mortola [44] where they proved that

$$F_\varepsilon(z) = \int_{\Omega} \left(\varepsilon |\nabla z|^2 + \frac{1}{\varepsilon} \sin^2 \pi z \right) dx \quad \Gamma\text{-converges to} \quad \frac{4}{\pi} \int_{\Omega} |\nabla z| dx,$$

for phase field z that takes almost only integer values. One can see that the constant $\frac{4}{\pi}$ corresponds to the constant σ in the Γ -limit of our proposed interfacial energy. So the weighing nature of their model can be seen from their Γ -limit $\int_{\Omega} |\nabla z| dx$.

3. We do not need to reinitialize our function ϕ to a signed distance function during the iterations.
4. Our constraints and energy functionals are all smooth and do not involve Heaviside and Dirac delta functions. In [35], they chose to use the sinc function in place of Heaviside function.

4 Multiphase motions

In this section, we use our proposed interfacial energy functional E_ε for multiphase motion. We consider the problem (2) with $e_i = 0$, $f_{ij} = 1$ for simplicity. We then solve the following problem:

$$\min_{\Gamma_{ij}} \sum_{1 \leq i < j \leq N} \text{Length}(\Gamma_{ij}).$$

Having shown that our proposed interfacial energy Γ -converge to $\sum_{1 \leq i < j \leq N} \text{Length}(\Gamma_{ij})$, we solve the following minimization problem:

$$\min E_\varepsilon(\phi) = \sum_{i=1}^N \int_{\Omega} \varepsilon |\nabla \psi_i(\phi)|^2 dx + \sum_{i=1}^N \int_{\Omega} \frac{1}{\varepsilon} W(\psi_i(\phi)) dx. \quad (12)$$

We shall impose Dirichlet boundary conditions $\phi|_{\partial\Omega} = g$.

4.1 Numerical discretization and schemes

To minimize (12) using gradient descent method, we first compute the differential $\frac{\partial E_\varepsilon}{\partial \phi}$:

$$\frac{\partial E_\varepsilon}{\partial \phi} = \sum_{i=1}^N \left(-2\varepsilon \nabla \cdot (\psi'_i(\phi) \nabla \phi) \psi'_i(\phi) + \frac{1}{\varepsilon} W'(\psi_i(\phi)) \psi'_i(\phi) \right).$$

Parameterizing the descent direction by an artificial time $t \geq 0$, we solve the following equation to steady state:

$$\frac{\partial \phi}{\partial t} = \sum_{i=1}^N \left(2\varepsilon \nabla \cdot (\psi'_i(\phi) \nabla \phi) \psi'_i(\phi) - \frac{1}{\varepsilon} W'(\psi_i(\phi)) \psi'_i(\phi) \right). \quad (13)$$

We use operator splitting schemes to solve (13). We solve the following equations alternatively on $t \in [t_j, t_{j+1}]$, where $t_j = j\tau$ and τ is the time step size:

$$\tilde{\phi}_t + \sum_{i=1}^N (-2\varepsilon \nabla \cdot (\psi'_i(\tilde{\phi}) \nabla \tilde{\phi}) \psi'_i(\tilde{\phi})) = 0, \quad t \in [t_j, t_{j+1}], \quad (14)$$

$$\begin{cases} \phi_t + \sum_{i=1}^N \frac{1}{\varepsilon} W'(\psi_i(\phi)) \psi'_i(\phi) = 0, & t \in [t_j, t_{j+1}], \\ \phi(t_j) = \tilde{\phi}(t_{j+1}) \end{cases} \quad (15)$$

The equation (14) is the diffusion step and the equation (15) tries to enforce that the minimizer take the values $1, \dots, N$. We employ a fixed point finite differences scheme to discretize the equation (14) associated with the minimization of our proposed variational model. The divergence operator can be discretized

$$\begin{aligned} \nabla \cdot (\psi' \nabla \phi)_{i,j} &\approx \frac{1}{h^2} \left(\psi'_{+0} \phi_{i+1,j} + \psi'_{-0} \phi_{i-1,j} + \psi'_{0+} \phi_{i,j+1} + \psi'_{0-} \phi_{i,j-1} \right. \\ &\quad \left. - (\psi'_{+0} + \psi'_{-0} + \psi'_{0+} + \psi'_{0-}) \phi_{i,j} \right), \end{aligned}$$

where $\Psi'_{\pm 0} = \Psi'_{i \pm \frac{1}{2}, j}$ and $\Psi'_{0\pm} = \Psi'_{i, j \pm \frac{1}{2}}$. We use a fixed point Gauss-Seidel iteration method to solve for $\phi_{i,j}^{k+1/2}$

$$\begin{aligned} \phi_{i,j}^{k+1/2} &= \phi_{i,j}^k + 2 \frac{\tau \varepsilon}{h^2} \sum_{i=1}^N \left((\Psi'_i)_{+0} \phi_{i+1,j}^k + (\Psi'_i)_{-0} \phi_{i-1,j}^k + (\Psi'_i)_{0+} \phi_{i,j+1}^k + (\Psi'_i)_{0-} \phi_{i,j-1}^k \right. \\ &\quad \left. - ((\Psi'_i)_{+0} + (\Psi'_i)_{-0} + (\Psi'_i)_{0+} + (\Psi'_i)_{0-}) \phi_{i,j}^{k+1/2} \right). \end{aligned} \quad (16)$$

To solve the equation (15), we employ the following scheme to solve ϕ^{k+1} from

$$\frac{\phi^{k+1} - \phi^{k+1/2}}{\tau} = - \sum_{i=1}^N \left(\frac{1}{\varepsilon} W'(\Psi_i(\phi^{k+1})) \Psi'_i(\phi^{k+1}) \right), \quad (17)$$

where $\phi^{k+1/2}$ is the solution given by (16). To solve the above equation (17) is equivalent to finding the roots of a polynomial of degree $4N - 5$. In [36], they give an upper bound for $\frac{\tau}{\varepsilon}$ in order to guarantee that the equation (17) has the unique solution. In our case, we can not find an upper bound that guarantees the unique solution in general. Therefore, in practice, we take a very small time step $\tau = h^2 \varepsilon$. In order to find a root, we use Newton's method. Define

$$G(\phi) = \phi - \phi^{k+1/2} + \tau \sum_{i=1}^N \left(\frac{1}{\varepsilon} W'(\Psi_i(\phi)) \Psi'_i(\phi) \right). \quad (18)$$

So then we have

$$G'(\phi) = 1 + \tau \sum_{i=1}^N \left(\frac{1}{\varepsilon} (W''(\Psi_i(\phi)) \Psi_i'^2(\phi) + W'(\Psi_i(\phi)) \Psi_i''(\phi)) \right).$$

Then we iteratively solve for ϕ^{k+1}

$$\phi^{k+1} = \phi^k - \frac{G(\phi^k)}{G'(\phi^k)}.$$

4.2 Numerical experiments

In order to test if our proposed model computes the true length of each interface, we consider a three-phase problem for which the optimal solution is given by a triple junction of 120 degree angle. In our example, we set up the boundary condition such that the triple junction is located at $(41, 11)$. $\phi(1, [12, 21]) = \phi([1, 46], 1) = 1$, $\phi(1, [1, 11]) = \phi([1, 47], 1) = 3$, $\phi(47, [2, 21]) = 2$. See Figure 1(a). The initial ϕ is set to be 2.2 initially. From Figure 1 (b)-(f), we can observe that diffuse interfaces form and evolve close to the true location. The mesh size $h = 1/119$ and the total number of iteration is 1,200,000. We reduce ε gradually, starting from $0.85 * h$, to $0.65 * h$. We mention that numerical calculations for triple junctions has also been tested in [36] showing that PCLSM is able to produce 120 degree for the junctions.

5 Multiphase image restoration, classification and segmentation

In this section, we present a new variational multiphase piecewise constant image segmentation and classification model based on our proposed interfacial energy functional (10). We first state several assumptions

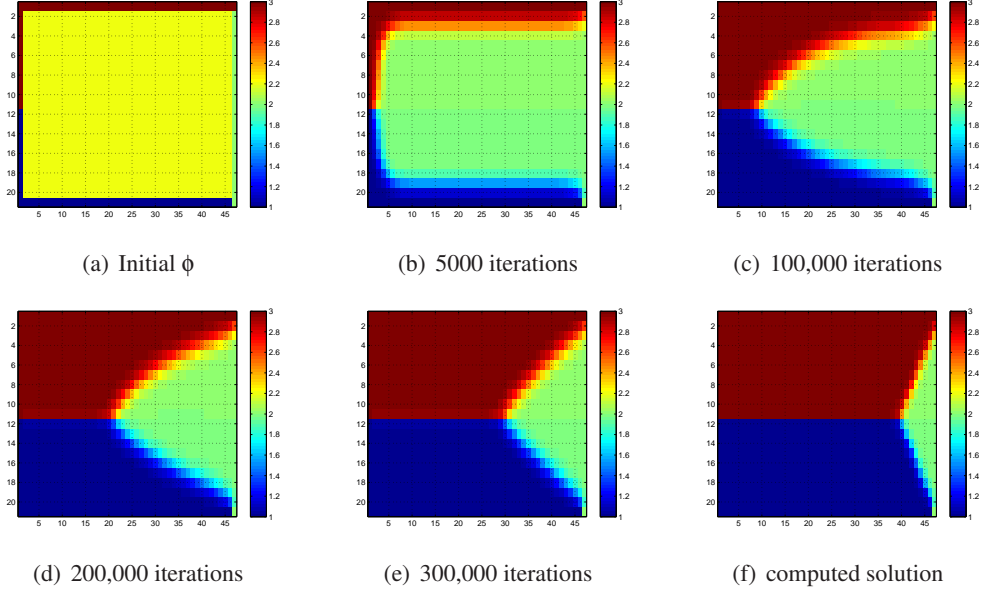


Figure 1: A three-phase problem whose optimal solution is given by a triple junction of 120 degree.

that we make; We assume that we know the number of phases in a given image a priori. In medical imaging applications, this assumption is generally accepted. In image processing, we deal with images whose intensity values are bounded. So we shall assume that the image $u_0 \in L^\infty(\Omega)$. For N -phases, we propose to minimize the following energy functional:

$$J_\varepsilon(\phi) = \sum_{i=1}^N \int_{\Omega} \varepsilon |\nabla \psi_i(\phi)|^2 dx + \sum_{i=1}^N \int_{\Omega} \frac{1}{\varepsilon} W(\psi_i(\phi)) dx + \sum_{i=1}^N \mu \int_{\Omega} |C_i - u_0|^2 \psi_i^2(\phi) dx, \quad (19)$$

where $\mu > 0$, and C_i is the average of a degraded image u_0 inside of Ω_i , $i = 1, \dots, N$:

$$C_i = \frac{\int_{\Omega} u_0 \psi_i^2(\phi) dx}{\int_{\Omega} \psi_i^2(\phi) dx}. \quad (20)$$

The first two terms correspond to the length regularization term, and the third term is the data fitting term. We note that we have smooth functions ψ_i acting as the heaviside function for Ω_i , $i = 1, \dots, N$.

5.1 Existence, Γ -convergence and compactness of minimizing sequences

We first show the existence of minimizers for the energy functional J_ε (19).

Proposition 5.1. (existence of minimizers). *Suppose $u_0 \in L^2(\Omega)$. Then for every $\varepsilon > 0$, there exists a minimizer of J_ε in $H^1(\Omega)$.*

Proof. In order to establish the existence, we shall follow the direct method of the calculus of variations [9], [21]. Take $\phi = 1$. Then $C_1 = \frac{1}{|\Omega|} \int_{\Omega} u_0 dx < \infty$,

$$J_\varepsilon(\phi = 1) = \int_{\Omega} |C_1 - u_0|^2 dx < \infty. \quad (21)$$

And since $J_\varepsilon(\phi) \geq 0$, the infimum is finite. Let $\{\phi_n^\varepsilon\}$ be a minimizing sequence for $J_\varepsilon(\phi)$. We note that the growth condition satisfies the coercivity of the energy $J_\varepsilon(\phi)$. Thus the minimizing sequences $\{\phi_n^\varepsilon\}$ is uniformly bounded by a constant $C > 0$. By reflexivity of $H^1(\Omega)$, there exists a subsequence of $\{\phi_n^\varepsilon\}$, which is still denoted by $\{\phi_n^\varepsilon\}$, converging weakly to $\phi_0^\varepsilon \in H^1(\Omega)$. To prove that ϕ_0^ε is a minimizer of $J_\varepsilon(\phi)$, it suffices to show the inequality $J_\varepsilon(\phi_0^\varepsilon) \leq \liminf_{n \rightarrow \infty} J_\varepsilon(\phi_n^\varepsilon)$, which implies that $J_\varepsilon(\phi_0^\varepsilon) = \min J_\varepsilon(\phi)$.

$$\begin{aligned} |\nabla \psi_i(\phi_n^\varepsilon)|^2 &= |\psi_i'|^2 |\nabla \phi_n^\varepsilon|^2 = |\psi_i'|^2 [|\nabla \phi_0^\varepsilon|^2 + 2\nabla \phi_0^\varepsilon \cdot (\nabla \phi_n^\varepsilon - \nabla \phi_0^\varepsilon) + |\nabla \phi_n^\varepsilon - \nabla \phi_0^\varepsilon|^2] \\ &\geq |\psi_i'|^2 [|\nabla \phi_0^\varepsilon|^2 + 2\nabla \phi_0^\varepsilon \cdot (\nabla \phi_n^\varepsilon - \nabla \phi_0^\varepsilon)] \\ \int_\Omega |\nabla \psi_i(\phi_n^\varepsilon)|^2 dx &\geq \int_\Omega |\psi_i'|^2 |\nabla \phi_0^\varepsilon|^2 dx + 2 \int_\Omega |\psi_i'|^2 \nabla \phi_0^\varepsilon \cdot (\nabla \phi_n^\varepsilon - \nabla \phi_0^\varepsilon) dx. \end{aligned}$$

Since $\nabla \phi_0^\varepsilon \in L^2(\Omega)$, $\nabla \phi_n^\varepsilon - \nabla \phi_0^\varepsilon \rightharpoonup 0$ in L^2 , by weak convergence in L^2 we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_\Omega |\psi_i'|^2 \nabla \phi_0^\varepsilon \cdot (\nabla \phi_n^\varepsilon - \nabla \phi_0^\varepsilon) dx &= 0 \\ \liminf_{n \rightarrow \infty} \int_\Omega |\nabla \psi_i(\phi_n^\varepsilon)|^2 dx &\geq \int_\Omega |\psi_i'|^2 |\nabla \phi_0^\varepsilon|^2 dx = \int_\Omega |\nabla \psi_i(\phi_0^\varepsilon)|^2 dx \\ \liminf_{n \rightarrow \infty} \sum_{i=1} \int_\Omega |\nabla \psi_i(\phi_n^\varepsilon)|^2 dx &\geq \sum_{i=1} \int_\Omega |\nabla \psi_i(\phi_0^\varepsilon)|^2 dx. \end{aligned}$$

And the other terms can pass to the limit on these sequences thanks to Fatou's lemma. Hence $J_\varepsilon(\phi_n^\varepsilon)$ is lower semicontinuous. Therefore, we have

$$\inf J_\varepsilon(\psi_i(\phi^\varepsilon)) \geq \liminf_{n \rightarrow \infty} J_\varepsilon(\psi_i(\phi_n^\varepsilon)) \geq \inf J_\varepsilon(\psi_i(\phi_0^\varepsilon)).$$

□

We state the following Γ -convergence of our proposed energy functional for multiphase image segmentation and classification.

Theorem 5.1. *Consider the functionals*

$$\bar{J}_\varepsilon(\phi) = \begin{cases} J_\varepsilon(\phi) & \text{if } \phi \in H^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

and define

$$J(\phi) = \begin{cases} \sigma \sum_{i=1}^N \text{Per}_\Omega \{\psi_i(\phi) = 1\} + \sum_{i=1}^N \mu \int_\Omega |C_i - u_0|^2 \chi_{\{\psi_i(\phi)=1\}} dx & \text{if } \phi \in BV(\Omega), W(\psi_i(\phi(x))) = 0 \text{ a.e. } \forall i, \\ +\infty & \text{otherwise,} \end{cases}$$

Then \bar{J}_ε Γ -converges to J in $X = L^1(\Omega)$.

Proof. This is a direct consequence of Theorem 3.1, Theorem 2.1 (iii), and the continuity of the fidelity term in $L^1(\Omega)$.

□

In the following, we show the compactness of the minimizing sequence of J_ε .

Theorem 5.2. (Compactness of the minimizers of J_ε). Let ϕ^ε be minimizer of $J_\varepsilon(\phi^\varepsilon)$ such that $J_\varepsilon(\phi^\varepsilon) \leq C < \infty$ for all $\varepsilon > 0$. Then $\{\phi^\varepsilon\}_{\varepsilon>0}$, $i = 1, \dots, N$ are compact in $L^1(\Omega)$

Proof. Define $\tilde{h}(t) = 2 \sum_{i=1}^N \int_0^t \sqrt{W(\psi_i(s))} ds$. Let $v^\varepsilon = \tilde{h}(\phi^\varepsilon)$. Because of our assumption on W , there exist positive numbers c_1, c_2, t_0 and a number $p \geq 2$ such that

$$c_1|t|^p \leq W(\psi_i(t)) \leq c_2|t|^p \quad \text{for } |t| \geq t_0, i = 1, \dots, N.$$

Then we have that

$$\tilde{h}(t) \leq 2 \sum_{i=1}^N \int_0^{t_0} \sqrt{W(\psi_i(s))} ds + \frac{4\sqrt{c_2}}{p+2} t^{\frac{p}{2}+1}, \quad t \geq t_0.$$

Since $p \geq 2$, $\frac{p}{2} + 1 \leq p$,

$$\tilde{h}(t) \leq c_3 + c_4 \sum_{i=1}^N W(\psi_i(t)), \quad t \geq t_0,$$

for some positive constants c_3 and c_4 . Hence,

$$\begin{aligned} \int_{\Omega} v^\varepsilon dx &\leq c_3|\Omega| + c_4 \sum_{i=1}^N \int_{\Omega} W(\psi_i(\phi^\varepsilon(x))) dx \\ &\leq c_3|\Omega| + c_4 \varepsilon J_\varepsilon(\phi^\varepsilon). \end{aligned}$$

So (v^ε) is bounded in $L^1(\Omega)$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_{\Omega} |\nabla v^\varepsilon| dx &= 2 \sum_{i=1}^N \int_{\Omega} \sqrt{W(\psi_i(\phi^\varepsilon))} |\nabla \psi_i(\phi^\varepsilon)| dx \leq \sum_{i=1}^N \int_{\Omega} \varepsilon |\nabla \psi_i(\phi^\varepsilon)|^2 dx \\ &+ \sum_{i=1}^N \int_{\Omega} \frac{1}{\varepsilon} W(\psi_i(\phi^\varepsilon)) dx \leq J_\varepsilon(\phi^\varepsilon) \leq C \text{ for all } \varepsilon. \end{aligned}$$

So $|v^\varepsilon|_{BV(\Omega)}$ is uniformly bounded in ε , and then pre-compact in $L^1(\Omega)$. Thus there exists a subsequence $\tilde{h}(\phi^{\varepsilon_j}) := v^{\varepsilon_j}$ converging to $\tilde{h}(\phi^0) = v^0$ in $L^1(\Omega)$. From the assumption on $W(\psi_i(t))$, we have

$$\tilde{h}'(t) = 2 \sum_{i=1}^N \sqrt{W(\psi_i(t))} \geq 2N\sqrt{c_1}|t|^{\frac{p}{2}} \quad \text{for } |t| \geq t_0.$$

Thus \tilde{h}^{-1} exists and is uniformly continuous on compact sets. By the uniform continuity of \tilde{h}^{-1} , $\phi^{\varepsilon_j} = \tilde{h}^{-1}(v^{\varepsilon_j})$ converges in measure to ϕ^0 . Since the assumption on $W(\psi_i(t))$ implies that ϕ^{ε_j} are uniformly bounded in L^p , $p \geq 2$, it converges in $L^1(\Omega)$. □

Remark: Combining Theorem 2.1 (ii) and Theorem 5.2, any limit point of a minimizing sequence of J_ε is a minimizer of J , and solves the Euler-Lagrange Equation associated with J .

5.2 Numerical discretization and schemes

In this section, we present the numerical schemes for our proposed multiphase image segmentation model. To minimize (19) using gradient descent method, we first compute the differential $\frac{\partial J_\varepsilon}{\partial \phi}$:

$$\frac{\partial J_\varepsilon}{\partial \phi} = \sum_{i=1}^N \left(-2\varepsilon \nabla \cdot (\psi'_i(\phi) \nabla \phi) \psi'_i(\phi) + \frac{1}{\varepsilon} W'(\psi_i(\phi)) \psi'_i(\phi) + \mu |C_i - u_0|^2 \psi'_i(\phi) \psi_i(\phi) \right). \quad (22)$$

Parameterizing the descent direction by an artificial time $t \geq 0$, we solve the following equation to steady state:

$$\frac{\partial \phi}{\partial t} = \sum_{i=1}^N \left(2\varepsilon \nabla \cdot (\psi'_i(\phi) \nabla \phi) \psi'_i(\phi) - \frac{1}{\varepsilon} W'(\psi_i(\phi)) \psi'_i(\phi) - \mu |C_i - u_0|^2 \psi'_i(\phi) \psi_i(\phi) \right). \quad (23)$$

5.2.1 Two-phase case

For the case of two phases, we have $\phi = 1$ or 2 and

$$\psi_1 = 2 - \phi, \quad \psi_2 = \phi - 1.$$

Thus

$$\psi'_1 = -1, \quad \psi'_2 = 1.$$

Hence, the time evolution equation (23) becomes

$$\frac{\partial \phi}{\partial t} = 4\varepsilon \Delta \phi - \sum_{i=1}^2 \left(\frac{1}{\varepsilon} W'(\psi_i(\phi)) \psi'_i(\phi) + \mu |C_i - u_0|^2 \psi'_i(\phi) \psi_i(\phi) \right). \quad (24)$$

To get better stability, we can try to treat the fidelity term implicitly and solve

$$\frac{\phi^{k+1} - \phi^k}{\tau} = 4\varepsilon \Delta \phi^{k+1} - \sum_{i=1}^2 \left(\frac{1}{\varepsilon} W'(\psi_i(\phi^k)) \psi'_i(\phi^k) + \mu |C_i - u_0|^2 \psi'_i(\phi^k) \psi_i(\phi^{k+1}) \right). \quad (25)$$

5.2.2 Multi-phase case

For multi-phase case, we shall use operator-splitting schemes to solve (23). We solve the following equations alternatively on $t \in [t_j, t_{j+1}]$ where $t_j = j\tau$ and τ is the time step size:

$$\tilde{\phi}_t + \sum_{i=1}^N (-2\varepsilon \nabla \cdot (\psi'_i(\tilde{\phi}) \nabla \tilde{\phi}) \psi'_i(\tilde{\phi})) = 0, \quad t \in [t_j, t_{j+1}], \quad (26)$$

$$\begin{cases} \phi_t + \sum_{i=1}^N \left(\frac{1}{\varepsilon} W'(\psi_i(\phi)) \psi'_i(\phi) + \mu |C_i - u_0|^2 \psi'_i(\phi) \psi_i(\phi) \right) = 0, & t \in [t_j, t_{j+1}], \\ \phi(t_j) = \tilde{\phi}(t_{j+1}) \end{cases} \quad (27)$$

We employ a fixed point finite differences scheme to discretize the equation (26) associated with the minimization of our proposed variational model.

To solve the equation (26), we use a fixed point Gauss-Seidel iteration method to solve for $\phi_{i,j}^{k+1/2}$:

$$\begin{aligned} \phi_{i,j}^{k+1/2} &= \phi_{i,j}^k + 2 \frac{\tau \varepsilon}{h^2} \sum_{i=1}^N \left((\psi'_i)_{+0} \phi_{i+1,j}^k + (\psi'_i)_{-0} \phi_{i-1,j}^k + (\psi'_i)_{0+} \phi_{i,j+1}^k + (\psi'_i)_{0-} \phi_{i,j-1}^k \right. \\ &\quad \left. - ((\psi'_i)_{+0} + (\psi'_i)_{-0} + (\psi'_i)_{0+} + (\psi'_i)_{0-}) \phi_{i,j}^{k+1/2} \right). \end{aligned} \quad (28)$$

To solve the equation (27), we employ the following scheme to solve ϕ^{k+1} from

$$\frac{\phi^{k+1} - \phi^{k+1/2}}{\tau} = -\sum_{i=1}^N \left(\frac{1}{\varepsilon} W'(\psi_i(\phi^{k+1})) \psi_i'(\phi^{k+1}) + \mu |C_i - u_0|^2 \psi_i'(\phi^{k+1}) \psi_i(\phi^{k+1}) \right), \quad (29)$$

where $\phi^{k+1/2}$ is the solution given by (28). To solve the above equation (29) is equivalent to finding the roots of a polynomial of degree $4N - 5$. As we already discussed in the previous section, in general, we can not give an upper bound on $\frac{\tau}{\varepsilon}$ that guarantees the unique solution. Therefore, in practice, we take a very small time step $\tau = h^2 \varepsilon$. In order to find a root, we use Newton's method. Define

$$G(\phi) = \phi - \phi^{k+1/2} + \tau \sum_{i=1}^N \left(\frac{1}{\varepsilon} W'(\psi_i(\phi)) \psi_i'(\phi) + \mu |C_i - u_0|^2 \psi_i'(\phi) \right). \quad (30)$$

So then we have

$$G'(\phi) = 1 + \tau \sum_{i=1}^N \left(\frac{1}{\varepsilon} (W''(\psi_i(\phi)) \psi_i'^2(\phi) + W'(\psi_i(\phi)) \psi_i''(\phi)) + \mu |C_i - u_0|^2 \psi_i''(\phi) \right).$$

Then we iteratively solve for ϕ^{k+1} :

$$\phi^{k+1} = \phi^k - \frac{G(\phi^k)}{G'(\phi^k)}.$$

Once we found a new ϕ^{k+1} , we update the constant C_i inside Ω_i , $i = 1, \dots, N$ using the equation (20).

5.3 Numerical experiments

We demonstrate some numerical experiments according to the proposed model and numerical schemes. In all our experiments, all test images are re-scaled so that their intensity values are in $[0, 1]$. We employ a fixed rectangular grid with uniform mesh size $h = 1$. We impose Neumann boundary condition in all our numerical experiments.

5.3.1 Two-phase case

In Figure 2, we show our test result for two-phase case. The original noisy image is of dimension 100×100 . It consists of one background and three disconnected objects of the same phase. We initialize ϕ to be 0.5. The total number of iteration is 70. We notice that in contrast to the multiphase motion experiment, the number of iteration is much smaller because the fidelity term, acting as a forcing term, quickly drives it to steady state. We start with $\varepsilon = 6$ and gradually decrease ε . The final ε is 0.5. By reducing ε , our segmentation result yields much sharper boundaries. In [35], they discuss the effect of ε .

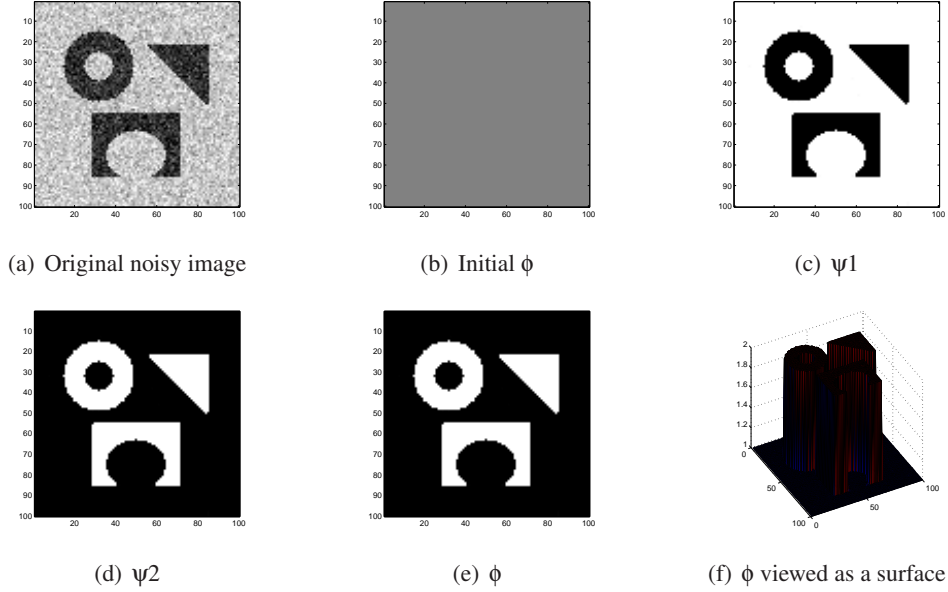


Figure 2: Two-phase example: $\mu = 20$, $\tau = 0.05$, 70 iterations, $C_1 = 0.7803$, $C_2 = 0.204$.

5.3.2 Four-phase case

In Figure 3, we show the test result for four-phase case. We use a noisy synthetic image containing three objects and one background. We use a very noisy data for the initial ϕ , as can be seen from Figure 3 (b). To speed up convergence and to ensure that ϕ takes values 1, 2, 3 or 4 only at convergence, we initially set $\varepsilon = 10$, gradually decreasing it to 0.01. i.e. we encourage image restoration initially and then we gradually reduce ε to enforce the classification constraints. The selection of the tuning parameter μ depends on the noise level: for very noisy images, smaller μ is chosen. In this experiment we fix $\mu = 50$. The number of iteration is 140. The computed solution shown in Figure 3 (g)-(h) is quite satisfactory albeit the initial ϕ is very noisy. We observe that by reducing ε below the mesh size, we are able to get sharp boundaries. In Figure 3 (c)-(f), we show the characteristic functions ψ_i 's for the corresponding phases. In Figure 4, we show another test result for four-phase case. We use a noisy synthetic image containing three stars on four different backgrounds, as shown in Figure 4 (a). We re-scale the given noisy data between -0.3 and 4.7 and use the re-scaled noisy data as the initial ϕ . We initially set $\varepsilon = 10$, gradually decreasing it to 0.01. In this experiment, we fix $\mu = 50$. The number of iteration is 140. In Figure 4 (c)-(f), we show the characteristic functions ψ_i 's for the corresponding phases. All the examples above indicate that by reducing ε , diffuse interfaces approach sharp interfaces, and reveal the connection between phase transitions, multiphase image segmentation and classification problems.

6 Conclusion

We proposed a new variational interfacial energy for multiphase problems based on the gradient theory of phase transitions and the PCLSM. We showed that our interfacial energy Γ -converges to σ times the sum of the length of $\partial\Omega_i$, $i = 1, \dots, N$, with σ being the same constant regardless of the phase labeling. Therefore our proposed model treats jumps equally and does not weigh the object boundaries. We also revealed the close connection between sharp and diffusive interfaces for interface problems from the two perspectives:

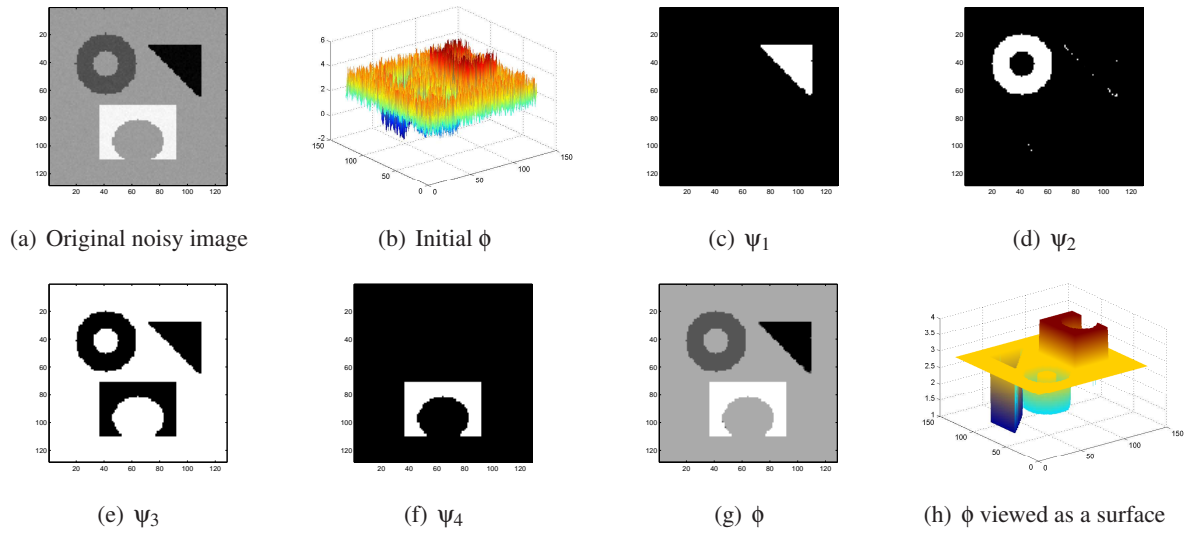


Figure 3: Four-phase example: $\mu = 50$, $\tau = 0.001$, initial $\varepsilon = 10$, $C_1 = 0.029$, $C_2 = 0.32$, $C_3 = 0.61$, $C_4 = 0.96$.

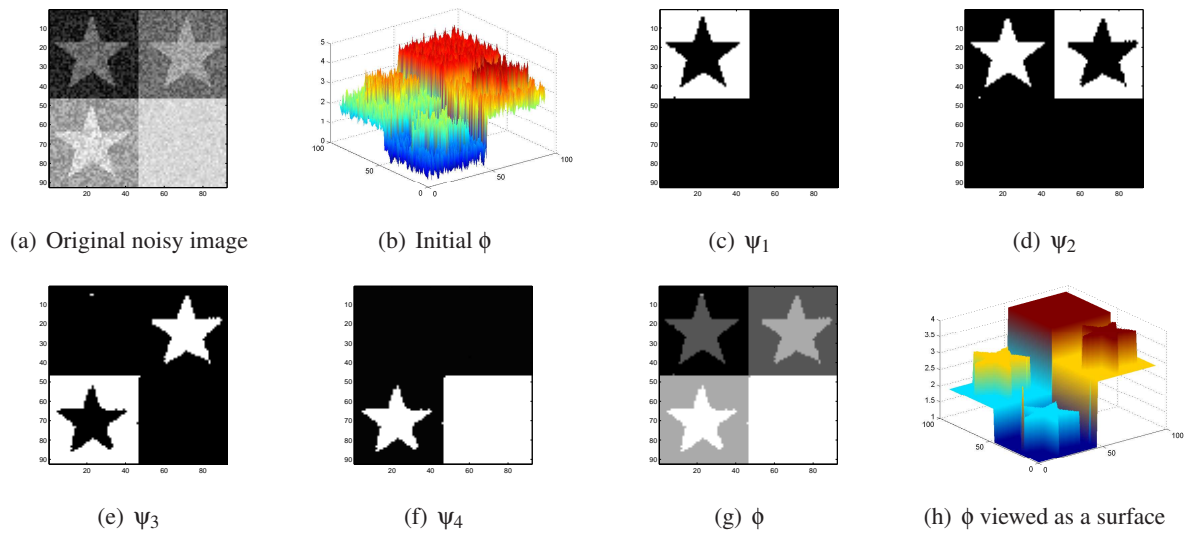


Figure 4: Four-phase example: $\mu = 50$, $\tau = 0.001$, initial $\varepsilon = 10$, $C_1 = 0.14$, $C_2 = 0.426$, $C_3 = 0.634$, $C_4 = 0.84$.

the PCLSM and phase fields.

In forthcoming papers, we plan to include an elliptic approximating functional to curvature because there have been much interest in studying limit interface of

$$\int_{\Omega} \frac{1}{\varepsilon} \left(\varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) \right)^2 dx.$$

in dimension $n \leq 3$ as $\varepsilon \rightarrow 0$ [11], [46], [28], [29], [27]. Recently a modified De Giorgi conjecture has been proved in space dimensions $n = 2$ and 3 : in [51] they proved that

$$\int_{\Omega} \left(\varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx + \int_{\Omega} \frac{1}{\varepsilon} \left(\varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) \right)^2 dx. \quad (31)$$

Γ -converges to the sum of the perimeter functional and the Willmore functional. It would be interesting to see if we can extend their two-phase result to multiphase case.

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