

Some Variational Problems from Image Processing ^{*†}

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Abstract

We consider in this paper a class of variational models introduced for image decomposition into cartoon and texture in [17] (see also [10]) of the form $\inf_u \left\{ \|u\|_{BV} + \lambda \|K * (f - u)\|_{L^p}^q \right\}$ where K is a real analytic integration kernel. We analyse and characterize the extremals of these functionals and list some of their properties.

1 Introduction and Motivations

A variational model for decomposing a given image-function f into $u + v$ can be given by

$$\inf_{(u,v) \in X_1 \times X_2} \left\{ F_1(u) + \lambda F_2(v) : f = u + v \right\},$$

where $F_1, F_2 \geq 0$ are functionals and X_1, X_2 are function spaces such that $F_1(u) < \infty$, and $F_2(v) < \infty$, if and only if $(u, v) \in X_1 \times X_2$. The constant $\lambda > 0$ is a tuning (scale) parameter. A good model is given by a choice of X_1 and X_2 so that with the given desired properties of u and v , we have: $F_1(u) \ll F_1(v)$ and $F_2(u) \gg F_2(v)$. The decomposition model is equivalent with:

$$\inf_{u \in X_1} \left\{ F_1(u) + \lambda F_2(f - u) \right\}$$

In this work we are interested in the analysis of a class of variational BV models arising in the decomposition of an image function f into cartoon or BV component, and a texture or oscillatory component. This topic has been of much interest in the recent years. We first recall the definition of BV functions.

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Definition 1. Let $u \in L^1_{\text{loc}}(\mathbb{R}^d)$ be real. We say $u \in BV$ if

$$\sup \left\{ \int u \operatorname{div} \varphi dx : \varphi \in C_0^1(\mathbb{R}^d), \sup |\varphi(x)| \leq 1 \right\} = \|u\|_{BV} < \infty.$$

If $u \in BV$ there is an \mathbb{R}^d valued measure $\vec{\mu}$ such that $\frac{\partial u}{\partial x_j} = (\vec{\mu})_j$ as distributions, a positive measure μ , and a Borel function $\vec{\rho}: \mathbb{R}^d \rightarrow S^{d-1}$ such that

$$Du = \vec{\mu} = \vec{\rho} \mu$$

and

$$\|u\|_{BV} = \int d\mu.$$

(see Evans-Gariepy [16], for example).

1.1 History

Assume $f \in L^2(\mathbb{R}^d)$, f real. We list here several variational BV models that have been proposed as image decomposition models.

Rudin-Osher-Fatemi [23] (1992) proposed the minimization

$$\inf_{u \in BV} \left\{ \|u\|_{BV} + \lambda \int |f - u|^2 dx \right\}.$$

In this model, we call u a “cartoon” component, and $f - u$ a “noise+texture” component of f , with $f = u + v$. Note that there exists a unique minimizer u by the strict convexity of the functional. A limitation of this model is illustrated by the following example [21, 13]: let $f = \alpha \chi_D$, $d = 2$, with D a disk centered at the origin and of radius R ; if $\lambda R \geq 1/\alpha$, then $u = (\alpha - (\lambda R)^{-1}) \chi_D$ and $v = f - u = (\lambda R)^{-1} \chi_D$; if $\lambda R \leq 1/\alpha$, then $u = 0$. Thus, although $f \in BV$ is without texture or noise, we do not have $u = f$. The work by Tadmor et al. [27], [28] aims to overcome this limitation by computing hierarchical (BV, L^2) decompositions $u \approx \sum_k u_k$, where u_k is a minimizer of a specific ROF model at a dyadic scale λ_k ; in the particular case when $f = \alpha \chi_D$, it was shown that $\sum_k u_k \rightarrow f$ as $k \rightarrow \infty$, thus the intensity loss is diminished. A multiscale image representation using novel integro-differential equation is proposed as an alternative to [27], [28] by Tadmor and Athavale in recent work [26].

Chan-Esedoglu [12] (2005) considered and analyzed the minimization (see also Alliney [5] for the one-dimensional discrete case)

$$\inf_{u \in BV} \left\{ \|u\|_{BV} + \lambda \int |f - u| dx \right\}.$$

The minimizers of this problem exist, but they may not be unique. If $d = 2$, $f = \chi_{B(0,R)}$, then $u = f$ if $R > \frac{2}{\lambda}$ and $u = 0$ if $R < \frac{2}{\lambda}$.

W. Allard [2, 3, 4] (2007) analyzed extremals of

$$\inf_{u \in BV} \left\{ \|u\|_{BV} + \lambda \int \gamma(u - f) dx \right\}$$

where $\gamma(0) = 0$, $\gamma \geq 0$, γ locally Lipschitz. Then there exist minimizers u , perhaps not unique, and

$$\partial^*(\{u > t\}) \in C^{1+\alpha}, \quad \alpha \in (0, 1)$$

where ∂^* denotes “measure theoretic boundary”. Also, Allard gave mean curvature estimates on $\partial^*(\{u > t\})$.

Y. Meyer [21] (2001) in his book *Oscillatory Patterns in Image Processing* analysed further the R-O-F minimization and refined these models proposing

$$\inf_{u \in BV} \left\{ \|u\|_{BV} + \lambda \|u - f\|_X \right\}$$

where

$$X = (W^{1,1})^* = \left\{ \operatorname{div} \vec{g} : \vec{g} \in L^\infty \right\} = G, \quad X = \left\{ \operatorname{div} \vec{g} : \vec{g} \in BMO \right\} = F,$$

or

$$X = \left\{ \Delta g : g \text{ Zygmund} \right\} = E.$$

Inspired by the proposals of Y. Meyer, recently a rich literature of models has been developed and analyzed theoretically and computationally. We list the more relevant ones.

Osher-Vese [30] (2002) proposed

$$\inf_{u, \vec{g}} \left\{ \|u\|_{BV} + \mu \|f - (u + \operatorname{div} \vec{g})\|_2^2 + \lambda \|\vec{g}\|_p \right\}, \quad p \rightarrow \infty$$

to approximate the (BV, G) Meyer’s model and make it computationally amenable. Osher-Solé-Vese [22] proposed the minimization

$$\inf_u \left\{ \|u\|_{BV} + \lambda \|f - u\|_{H^{-1}} \right\}$$

and later Linh-Lieu [20] generalized it to

$$\inf_u \left\{ \|u\|_{BV} + \lambda \|f - u\|_{H^{-s}} \right\}, \quad s > 0.$$

Similarly, Le-Vese [19] (2005) approximated (BV, F) Meyer’s model by

$$\inf_{u, \vec{g}} \left\{ \|u\|_{BV} + \mu \|f - (u + \operatorname{div} \vec{g})\|_2^2 + \lambda \|\vec{g}\|_{BMO} \right\}.$$

Aujol et al. [7, 8] addressed the original (BV, G) Meyer’s problem and proposed an alternate method to minimize

$$\inf_u \left\{ \|u\|_{BV} + \lambda \|f - u - v\|_2 \right\},$$

subject to the constraint $\|v\|_G \leq \mu$.

Garnett-Le-Meyer-Vese [17] (2007) proposed reformulations and generalizations of Meyer’s (BV, E) model (see also Aujol-Chambolle [10]), given by

$$\inf_{u, \vec{g}} \left\{ \|u\|_{BV} + \mu \|f - (u + \Delta \vec{g})\|_2^2 + \lambda \|\vec{g}\|_{\dot{B}_{p,q}^\alpha} \right\}$$

where $1 \leq p, q \leq \infty$, $0 < \alpha < 2$, and the exact decompositions are given by

$$\inf_u \left\{ \|u\|_{BV} + \lambda \|f - u\|_{\dot{B}_{p,q}^{\alpha-2}} \right\}.$$

In a subsequent work, Garnett-Jones-Le-Meyer [18] proposed different formulations,

$$\inf_{u, \vec{g}} \left\{ \|u\|_{BV} + \mu \|f - (u + \Delta \vec{g})\|_2^2 + \lambda \|\vec{g}\|_{BMO^\alpha} \right\},$$

with $BMO^\alpha = I_\alpha(BMO)$, $\|v\|_{BMO^\alpha} = \|I_\alpha v\|_{BMO}$, and

$$\inf_{u, \vec{g}} \left\{ \|u\|_{BV} + \mu \|f - (u + \Delta \vec{g})\|_2^2 + \lambda \|\vec{g}\|_{\dot{W}^{\alpha,p}} \right\},$$

with $\|v\|_{\dot{W}^{\alpha,p}} = \|I_\alpha v\|_p$, $0 < \alpha < 2$.

Generalizing (BV, H^{-s}) , $(BV, \dot{B}_{p,q}^\alpha)$, and the *TV-Hilbert* model [9], an easier cartoon+texture decomposition model can be defined using a smoothing convolution kernel K (previously introduced in [17]):

$$\inf_{u \in BV} \left\{ \|u\|_{BV} + \lambda \|K * (f - u)\|_{L^p}^q \right\}. \quad (1)$$

This can be seen as a simplified version of all the previous models.

2 The Variational Problems

In this paper we assume K is a positive, even, bounded and real analytic kernel on \mathbb{R}^d such that $\int K dx = 1$ and such that the map $L^p \ni u \rightarrow K * u$ is injective. For example we may take K to be a Gaussian or a Poisson kernel. We fix $\lambda > 0$, $1 \leq p < \infty$ and $1 \leq q < \infty$. For real $f(x) \in L^1$ we consider the extremal problems

$$m_{p,q,\lambda} = \inf \{ \|u\|_{BV} + \mathcal{F}_{p,q,\lambda}(f - u) : u \in BV \} \quad (2)$$

where

$$\mathcal{F}_{p,q,\lambda}(h) = \lambda \|K * h\|_{L^p}^q. \quad (3)$$

Since $BV \subset L^{\frac{d}{d-1}}$ and $K \in L^\infty$, a weak-star compactness argument shows that (2) has at least one minimizer u (see Section 3 below for a more detailed argument). Our objective is to describe, given f , the set $\mathcal{M}_{p,q,\lambda}(f)$ of minimizers u of (2).

The papers of Chan-Esedoglu [12] and Allard [2, 3, 4] give very precise results about the minimizers for variations like (2) but without the real analytic kernel K , and this paper is intended to complement those works.

Remark 1. According to the definition of *admissibility* given in [2], the functional $\mathcal{F}_{p,q,\lambda}$ is admissible for an appropriate choice of K , for instance take K to be bounded (i.e. heat kernel K_t or Poisson kernel P_t for some $t > 0$). Thus the regularity results from section 1.5 in [2] holds for minimizers in $\mathcal{M}_{p,q,\lambda}(f)$. On the other hand, If K is not a Dirac delta function, then $\mathcal{F}_{p,q,\lambda}$ is not *local* as defined in [2].

2.1 Convexity

Since the functional in (2) is convex, the set of minimizers $\mathcal{M}_{p,q,\lambda}(f)$ is a convex subset of BV . If $p > 1$ or if $q > 1$, then the functional (3) is strictly convex and the problem (2) has a unique minimizer because $K * u$ determines u .

Lemma 1. *If $p = q = 1$ and if $u_1 \in \mathcal{M}_{p,q,\lambda}$ and $u_2 \in \mathcal{M}_{p,q,\lambda}$, then*

$$\frac{K * (f - u_1)}{|K * (f - u_1)|} = \frac{K * (f - u_2)}{|K * (f - u_2)|} \quad \text{almost everywhere,} \quad (4)$$

and

$$\vec{\rho}_k \cdot \frac{d\vec{\mu}_j}{d\mu_k} = \left| \frac{d\vec{\mu}_j}{d\mu_k} \right|, \quad j \neq k, \quad (5)$$

where for $j = 1, 2$,

$$Du_j = \vec{\mu}_j = \vec{\rho}_j \mu_j$$

with $|\vec{\rho}_j| = 1$ and $\mu_j \geq 0$.

Proof: Since $\mathcal{M}_{p,q,\lambda}(f)$ is a convex subset of BV , $\frac{u_1 + u_2}{2}$ is also a minimizer. This implies,

$$\begin{aligned} \left\| \frac{u_1 + u_2}{2} \right\|_{BV} + \lambda \left\| K * \left(f - \frac{u_1 + u_2}{2} \right) \right\|_1 &= \frac{1}{2} [\|u_1\|_{BV} + \|u_2\|_{BV}] \\ &+ \frac{\lambda}{2} [\|K * (f - u_1)\|_1 + \|K * (f - u_2)\|_1]. \end{aligned} \quad (6)$$

On the other hand, using convexity of $\|\cdot\|_{BV}$ and $\|\cdot\|_{L^1}$, we have

$$\begin{aligned} \left\| \frac{u_1 + u_2}{2} \right\|_{BV} &\leq \frac{1}{2} [\|u_1\|_{BV} + \|u_2\|_{BV}], \quad \text{and} \\ \left\| K * \left(f - \frac{u_1 + u_2}{2} \right) \right\|_1 &\leq \frac{1}{2} [\|K * (f - u_1)\|_1 + \|K * (f - u_2)\|_1] \end{aligned} \quad (7)$$

Combining (6) and (7), we obtain

$$\left\| K * \left(f - \frac{u_1 + u_2}{2} \right) \right\|_1 = \frac{1}{2} (\|K * (f - u_1)\|_1 + \|K * (f - u_2)\|_1),$$

which implies (4). Moreover,

$$\|u_1 + u_2\|_{BV} = \|u_1\|_{BV} + \|u_2\|_{BV}. \quad (8)$$

For $j = 1, 2$, let

$$Du_j = \vec{\mu}_j = \vec{\rho}_j \mu_j, \quad \text{with } |\vec{\rho}_j| = 1 \text{ and } \mu_j \geq 0.$$

Then for $k = 1, 2$, $k \neq j$, equation (8) implies

$$\int \left| \vec{\rho}_k + \frac{d\vec{\mu}_j}{\mu_k} \right| d\mu_k = \int d\mu_k + \int \left| \frac{d\vec{\mu}_j}{\mu_k} \right| d\mu_k,$$

which implies (5). □

2.2 Properties of $u \in \mathcal{M}_{p,q,\lambda}(f)$

Lemma 2. *Given an $f \in L^1$. Suppose u is a minimizer of (2) such that $u \neq f$. Let*

$$Du = \vec{\mu} = \vec{\rho} \cdot \mu.$$

For each real-valued $h \in BV$, write $Dh = \vec{v}$ and $\vec{v} = \frac{d\vec{v}}{d\mu}\mu + \vec{v}_s$ as the Lebesgue decomposition of \vec{v} with respect to μ . Then

$$\left| \int \vec{\rho} \cdot \frac{d\vec{v}}{d\mu} d\mu - \lambda \int h(K * J_{p,q}) dx \right| \leq \|\vec{v}_s\|, \quad (9)$$

where

$$J_{p,q} = q \frac{F|F|^{p-2}}{\|F\|_p^{p-q}} \text{ with } F = K * (f - u) \quad (10)$$

and $\|\vec{v}_s\|$ denotes the norm of the vector measure \vec{v}_s . Conversely, if $u \in BV$, $u \neq f$ and (9) and (10) hold, then $u \in \mathcal{M}_{p,q,\lambda}(f)$.

Note that since $u \neq f$ and $K * (f - u)$ is real analytic, $J_{p,q}$ is defined almost everywhere.

Proof: Let $|\epsilon|$ be sufficiently small. Since u is extremal, we have

$$\|u + \epsilon h\|_{BV} - \|u\|_{BV} + \mathcal{F}_{p,q,\lambda}(f - u - \epsilon h) - \mathcal{F}_{p,q,\lambda}(f - u) \geq 0. \quad (11)$$

On the other hand, we have

$$\left| \vec{\rho} + \epsilon \frac{d\vec{v}}{d\mu} \right| = \left(1 + 2\epsilon \vec{\rho} \cdot \frac{d\vec{v}}{d\mu} + \epsilon^2 \left\| \frac{d\vec{v}}{d\mu} \right\|^2 \right)^{1/2} = \left(1 + \epsilon \vec{\rho} \cdot \frac{d\vec{v}}{d\mu} + o(|\epsilon|) \right),$$

where in the last equality, we use the estimate $(1 + \alpha)^{1/2} = 1 + \frac{\alpha}{2} + o(|\alpha|)$. This implies,

$$\|u + \epsilon h\|_{BV} - \|u\|_{BV} = |\epsilon| \|\vec{v}_s\| + \int \left(\left| \vec{\rho} + \epsilon \frac{d\vec{v}}{d\mu} \right| - 1 \right) d\mu = |\epsilon| \|\vec{v}_s\| + \epsilon \int \vec{\rho} \cdot \frac{d\vec{v}}{d\mu} d\mu + o(|\epsilon|).$$

Moreover,

$$\begin{aligned} \mathcal{F}_{p,q,\lambda}(f - u - \epsilon h) - \mathcal{F}_{p,q,\lambda}(f - u) &= -\lambda \epsilon \int (K * h) J_{p,q} dx + o(|\epsilon|) \\ &= -\lambda \epsilon \int h(K * J_{p,q}) dx + o(|\epsilon|) \end{aligned}$$

since K is even (symmetric). By (11), we have

$$-\epsilon \left[\int \vec{\rho} \cdot \frac{d\vec{v}}{d\mu} d\mu - \lambda \int h(K * J_{p,q}) dx \right] \leq |\epsilon| \|\vec{v}_s\| + o(|\epsilon|)$$

Taking $\pm\epsilon$ and since the right hand side of the above equation does not depend on the sign of ϵ , we see that (9) holds.

The converse holds because the functional (3) is convex. \square

Following Meyer [21], we define

$$\|v\|_* = \inf \left\{ \|u\|_\infty : v = \sum_{j=1}^d \frac{\partial u_j}{\partial x_j}, |u|^2 = \sum_{i=1}^d |u_i|^2 \right\}$$

and note that $\|v\|_*$ is the norm of the dual of $W^{1,1} \subset BV$, when $W^{1,1}$ is given the norm of BV . By the weak-star density of $W^{1,1}$ in BV ,

$$\left| \int h v dx \right| \leq \|h\|_{BV} \|v\|_* \quad (12)$$

whenever $v \in L^2$.

Remark 2. Taking $h \in BV$ in Lemma 2 such that $\vec{v}_s = 0$, i.e. Dh is absolutely continuous with respect to Du , then (9) implies

$$\int \vec{\rho} \cdot \frac{d\vec{v}}{d\mu} d\mu - \lambda \int h(K * J_{p,q}) dx = 0. \quad (13)$$

In particular, for any $h \in W^{1,1}$, the above equation holds. I.e.

$$\int h(K * J_{p,q}) dx = \frac{1}{\lambda} \int \vec{\rho} \cdot \frac{d\vec{v}}{d\mu} d\mu. \quad (14)$$

We have the following characterization of minimizers in terms of $\|\cdot\|_*$ (following Meyer [21]).

Lemma 3. *Let $u \in BV$ such that $u \neq f$, and let $J_{p,q}$ be defined as in Lemma 2. Then u is a minimizer for the problem (2) if and only if*

$$\|K * J_{p,q}\|_* = \frac{1}{\lambda} \quad (15)$$

and

$$\int u(K * J_{p,q}) dx = \frac{1}{\lambda} \|u\|_{BV}. \quad (16)$$

Proof: If u is a minimizer, we use Lemma 2. For any $h \in W^{1,1}$, (14) yields

$$\|K * J_{p,q}\|_* \leq \frac{1}{\lambda}.$$

By (12)

$$\left| \int u(K * J_{p,q}) dx \right| \leq \|u\|_{BV} \|K * J_{p,q}\|_*,$$

and by setting $h = u$ in (13), we obtain

$$\lambda \int u(K * J_{p,q}) dx = \|u\|_{BV}.$$

Therefore (15) and (16) hold.

Conversely, assume $u \in BV$ satisfies (15) and (16) and note that u determines $J_{p,q}$. Still following Meyer [21], we let $h \in BV$ be real. Then for small $\epsilon > 0$, (12), (15) and (16) give

$$\begin{aligned}
\|u + \epsilon h\|_{BV} + \lambda \|K * (f - u - \epsilon h)\|_1 &\geq \lambda \int (u + \epsilon h)(K * J_{p,q}) dx + \lambda \|K * (f - u)\|_1 \\
&\quad - \epsilon \lambda \int h(K * J_{p,q}) dx + o(\epsilon) \\
&= \|u\|_{BV} + \epsilon \lambda \int h(K * J_{p,q}) dx - \epsilon \lambda \int h(K * J_{p,q}) dx + o(\epsilon) \\
&\geq 0.
\end{aligned}$$

Therefore u is a local minimizer for the functional (2), and by convexity that means u is a global minimizer.

2.3 Radial Functions

Assume K is radial, $K(x) = K(|x|)$. Also assume f is radial and $f \notin \mathcal{M}_{p,q,\lambda}(f)$. Then averaging over rotations shows that each $u \in \mathcal{M}_{p,q,\lambda}(f)$ is radial, so that

$$Du = \rho(|x|) \frac{x}{|x|} \mu$$

where μ is invariant under rotations and where $\rho(|x|) = \pm 1$ a.e. $d\mu$. Let $H \in L^1(\mu)$ be radial and satisfy $\int H d\mu = 0$ and $H = 0$ on $|x| < \epsilon$, and define

$$h(x) = \int_{B(0,|x|)} H(|y|) \frac{1}{|y|^{d-1}} d\mu.$$

Then $h \in BV$ is radial and

$$Dh = \vec{\nu} = H(|x|) \frac{x}{|x|} \mu.$$

Consequently $\vec{\nu}_s = 0$ and (9) gives

$$\int \rho H d\mu = \lambda \int K * J_{p,q}(x) \int_{B(0,|x|)} \frac{H(y)}{|y|^{d-1}} d\mu(y) dx = \lambda \int \left(\int_{|x|>|y|} K * J_{p,q}(x) dx \right) \frac{H(|y|)}{|y|^{d-1}} d\mu(y),$$

so that a.e. $d\mu$,

$$\rho(|y|) = \frac{\lambda}{|y|^{d-1}} \int_{|x|>|y|} K * J_{p,q}(x) dx. \quad (17)$$

But the right side of (17) is real analytic in $|y|$, with a possible pole at $|y| = 0$, and $\rho(|y|) = \pm 1$ almost everywhere μ . Therefore there is a finite set

$$\{r_1 < r_2 < \dots < r_n\} \quad (18)$$

of radii such that

$$Du = \frac{x}{|x|} \sum_{j=1}^n c_j \Lambda_{d-1} | \{ |x| = r_j \} |$$

for real constants c_1, \dots, c_n , where Λ_{d-1} denotes $d-1$ dimensional Hausdorff measure. By Lemma 1, $J_{p,q}$ is uniquely determined by f , and hence the set (18) is also unique. Moreover, it follows from Lemma 1 that for each j , either $c_j \geq 0$ for all $u \in \mathcal{M}_{p,1,\lambda}(f)$ or $c_j \leq 0$ for all $u \in \mathcal{M}_{p,1,\lambda}(f)$. We have proved:

Theorem 1. *Suppose K and f are both radial. If $f \notin \mathcal{M}_{p,q,\lambda}(f)$, then there is a finite set (18) such that all $u \in \mathcal{M}_{p,q,\lambda}(f)$ have the form*

$$\sum_{j=1}^n c_j \chi_{B(0,r_j)}. \quad (19)$$

Moreover, there is $X^+ \subset \{1, 2, \dots, n\}$ such that $c_j \geq 0$ if $j \in X^+$ while $c_j \leq 0$ if $j \notin X^+$.

Note that by convexity $\mathcal{M}_{p,q,\lambda}(f)$ consists of a single function unless $p = q = 1$. In Section 2.6 we will say more about the solutions of the form (19).

2.4 Example

Unfortunately, Theorem 1 does not hold more generally. The reason is that when u is not radial it is difficult to produce BV functions satisfying $\bar{\nu} \ll \mu$. For simplicity we take $d = 2$ and $p = q = 1$. Let $J = J_{1,1} = \chi_{0 < x \leq 1} - \chi_{-1 < x \leq 0}$ and $J(x+2, y) = J(x, y)$. Choose $\lambda > 0$ so that $U = \lambda K * J$ satisfies $\|U\|_* = 1$, and note that $\frac{U}{|U|} = J$. Notice that $u \in C^2$ solves the curvature equation

$$\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) = U \quad (20)$$

if and only if the level sets $\{u = a\}$ are curves $y = y(x)$ that satisfy the simple ODE $y'' = U(x, 0)(1 + (y')^2)^{3/2}$ on the line. Consequently (20) has infinitely many solutions u and both u and J satisfy (15) and (16). Hence by Lemma 3 u is a minimizer for f provided that

$$J = \frac{K * (f - u)}{|K * (f - u)|} \quad (21)$$

and there are many f that satisfy (21). For example, one can choose u and f so that $f - u = J$. Note that in this example u can be real analytic except on $U^{-1}(0)$ and not piecewise constant. Similar examples can be made when $(p, q) \neq (1, 1)$.

2.5 Properties of Minimizers when $q = 1$

Here we follow the paper of Strang [25].

Lemma 4. *If $q = 1$ and $u \in \mathcal{M}_{p,1,\lambda}(f)$, then $u \in \mathcal{M}_{p,1,\lambda}(u)$.*

Proof: If

$$\|h\|_{BV} + \lambda \|K * (u - h)\|_p < \|u\|_{BV},$$

then by the triangle inequality

$$\|h\|_{BV} + \lambda \|K * (f - h)\|_p < \|u\|_{BV} + \lambda \|K * (f - u)\|_p$$

so that u is not a minimizer for f . □

We write

$$\mathcal{M} = \mathcal{M}_{p,1,\lambda} = \bigcup_f \mathcal{M}_{p,1,\lambda}(f).$$

Lemma 5. *Let $u \in BV$. Then $u \in \mathcal{M}$ if and only if*

$$\left| \int \rho \cdot \frac{d\vec{v}}{d\mu} d\mu \right| \leq \|(\vec{v})_s\| + \lambda \|K * h\|_p \quad (22)$$

for all $h \in BV$, where $Dh = \vec{v}$.

Proof: This follows like the proof of Lemma 2. Let $a < b$ be such that

$$\mu(\{u = a\} \cup \{u = b\}) = 0. \quad (23)$$

Then $u_{a,b} = \text{Min}\{(u - a)^+, (b - a)\} \in BV$ and $D(u_{a,b}) = \chi_{a < u < b} \vec{\rho} \mu$. □

Lemma 6. *Assume $q = 1$.*

(a) *If $u \in \mathcal{M}$, then $u_{a,b} \in \mathcal{M}$.*

(b) *More generally, if $u \in \mathcal{M}$ and if $v \in BV$ satisfies $\mu_v \ll \mu_u$ and $\rho_v = \rho_u$ a.e. $d\mu_v$, then $v \in \mathcal{M}$.*

Proof: To prove (a) we verify (22). Write $\mu_{a,b} = \chi_{(a,b)} \mu$ so that $D(u_{a,b}) = \vec{\rho} \mu_{a,b}$. Let $h \in BV$ and write $Dh = \vec{v}$. Then by (23)

$$\vec{v} = \chi_{a < u < b} \frac{d\vec{v}}{d\mu} \mu + ((\vec{v})_s + \chi_{u(x) \notin [a,b]} \frac{d\vec{v}}{d\mu} \mu)$$

is the Lebesgue decomposition of \vec{v} with respect to $\mu_{a,b}$, and

$$\int \vec{\rho} \cdot \frac{d\vec{v}}{d\mu_{a,b}} d\mu_{a,b} = \int \vec{\rho} \cdot \frac{d\vec{v}}{d\mu} d\mu - \int_{g(x) \notin [a,b]} \vec{\rho} \cdot \frac{d\vec{v}}{d\mu} d\mu.$$

Then (22) for ν and $\mu_{a,b}$ follows from (22) for μ and ν . The proof of (b) is similar. □

For simplicity we assume $u \geq 0$. Write $E_t = \{x : u(x) > t\}$. Then by Evans-Gariepy [16], E_t has finite perimeter for almost every t ,

$$\|u\|_{BV} = \int_0^\infty \|\chi_{E_t}\|_{BV} dt, \quad (24)$$

and

$$u(x) = \int_0^\infty \chi_{E_t}(x) dt. \quad (25)$$

Moreover, almost every set E_t has a *measure theoretic boundary* $\partial_* E_t$ such that

$$\Lambda_{d-1}(\partial_* E_t) = \|\chi_{E_t}\|_{BV} \quad (26)$$

and a *measure theoretic outer normal* $\vec{n}_t : \partial_* E_t \rightarrow S^{d-1}$ so that

$$D(\chi_{E_t}) = \vec{n}_t \Lambda_{d-1} \lfloor \partial_* E_t. \quad (27)$$

Theorem 2. *Assume $q = 1$.*

(a) *If $u \in \mathcal{M}$, then for almost every t , $\chi_{E_t} \in \mathcal{M}$.*

(b) *If $u \in \mathcal{M}$ and $u \geq 0$, then for all nonnegative c_1, \dots, c_n and for almost all $t_1 < \dots < t_n$, $\sum c_j \chi_{E_{t_j}} \in \mathcal{M}$.*

Proof: Suppose (a) is false. Then there is $\beta < 1$, and a compact set $A \subset (0, \infty)$ with $|A| > 0$ such that for all $t \in A$ (26) and (27) hold and there exists $h_t \in BV$ such that

$$\|\chi_{E_t} - h_t\|_{BV} + \lambda \|K * h_t\|_p \leq \beta \|\chi_{E_t}\|_{BV}. \quad (28)$$

Choose an interval $I = (a, b)$ such that (23) holds and $|I \cap A| \geq \frac{|I|}{2}$. Define $h_t = 0$ for $t \in I \setminus A$, and take finite sums such that

$$\sum_{j=1}^{N_n} \chi_{E_{t_j^{(n)}}} \Delta t_j^{(n)} \rightarrow u_{a,b} \quad (n \rightarrow \infty), \quad (29)$$

$$\sum_{j=1}^{N_n} \|\chi_{E_{t_j^{(n)}}}\|_{BV} \Delta t_j^{(n)} \rightarrow \|u_{a,b}\| \quad (n \rightarrow \infty), \quad (30)$$

and $t_j^{(n)} \in A$ whenever possible. Write $h^{(n)} = \sum_{j=1}^{N_n} h_{t_j^{(n)}} \Delta t_j^{(n)}$. Then by (25) and (28) $\{h^{(n)}\}$ has a weak-star limit $h \in BV$, and by (28), (29) and (30),

$$\|u_{a,b} - h\|_{BV} + \lambda \|K * h\|_p \leq \frac{1 + \beta}{2} \|u_{a,b}\|_{BV},$$

contradicting Lemma 6. The proof of (b) is similar. \square

We believe that the converse of Theorem 2 is false, but we have no counterexample.

2.6 Radial Minimizers

In this section we assume $q = 1$ and $p = 1$. For convenience we assume the kernel $K = K_t$ is Gaussian, so that K has the form

$$K_t(x) = t^{-d} K\left(\frac{x}{t}\right) \quad (31)$$

and

$$K_s * K_t = K_{\sqrt{s^2 + t^2}}. \quad (32)$$

Note that (31) and (32) imply that

$$\|K_t * f\|_1 \text{ decreases in } t \quad (33)$$

and for $f \in L^1$ with compact support

$$\lim_{t \rightarrow \infty} \|K_t * f\|_1 = \left| \int f dx \right|. \quad (34)$$

For fixed λ and t we set

$$R(\lambda, t) = \{r > 0 : \chi_{B(0,r)} \in \mathcal{M}\}.$$

By Theorem 1 and Theorem 2 we have $R(\lambda, t) \neq \emptyset$. For $t = 0$ and $K = I$ our problem (2) becomes the problem

$$\inf\{\|u\|_{BV} + \lambda\|f - u\|_{L^1}\}$$

studied by Chan and Esedoglu in [12], and in that case Chan and Esedoglu showed $R(\lambda, 0) = [\frac{2}{\lambda}, \infty)$.

Theorem 3. *There exists $r_0 = r_0(\lambda, t)$ such that*

$$R(\lambda, t) = [r_0, \infty). \quad (35)$$

Moreover

$$[0, \infty) \ni t \rightarrow r_0(t) \text{ is nondecreasing} \quad (36)$$

and

$$\lim_{t \rightarrow \infty} r_0(t) = \infty. \quad (37)$$

Proof: Assume $r \notin R(\lambda, t)$ and $0 < s < r$. Write $\alpha = \frac{r}{s} > 1$ and $f = \chi_{B(0,r)}$. By hypothesis there is $g \in BV$ such that

$$\|g\|_{BV} + \lambda\|K_t * (f - g)\|_1 < \|f\|_{BV}. \quad (38)$$

We write $\tilde{g}(x) = g(\alpha x)$, $\tilde{f}(x) = f(\alpha x) = \chi_{B(0,s)}(x)$, and change variables carefully in (38) to get

$$\alpha\|\tilde{g}\|_{BV} + \lambda\left\|\frac{1}{t^d} \int K\left(\frac{x-y}{t}\right)(\tilde{f} - \tilde{g})\left(\frac{y}{\alpha}\right)dy\right\|_{L^1(x)} < \alpha\|\tilde{f}\|_{BV}$$

so that

$$\alpha\|\tilde{g}\|_{BV} + \lambda\left\|\frac{\alpha^d}{t^d} \int K\left(\frac{\alpha x' - \alpha y'}{t}\right)(\tilde{f} - \tilde{g})(y')dy'\right\|_{L^1(\alpha x')} < \alpha\|\tilde{f}\|_{BV}$$

and

$$\alpha\|\tilde{g}\|_{BV} + \lambda\alpha^d \int \left|K_{\frac{t}{\alpha}} * (\tilde{f} - \tilde{g})(x')\right| dx' < \alpha\|\tilde{f}\|_{BV}.$$

Since $\alpha > 1$, this and (33) show

$$\|\tilde{g}\|_{BV} + \lambda\|K_t * (\tilde{f} - \tilde{g})\|_1 < \|\tilde{f}\|_{BV}$$

so that $s \notin R(\lambda, t)$. That proves (35), and (36) now follows easily from (33). To prove (37) take $g = \frac{r^d}{s^d}\chi_{B(0,s)}$, $s > r$ and use (34). \square

We note that not all radial minimizers have the form $\chi_{B(0,r)}$. This is seen by considering separately, for large fixed t and λ , the function $\chi_{B(0,r_2)} + \chi_{B(0,r_1)}$ with r_1 and $r_2 - r_1$ large.

2.7 Characteristic Functions

Still assuming $q = 1$ we let E be such that $\chi_E \in \mathcal{M}$. Then by Evans-Gariepy [16] $\partial_* E = N \cup \bigcup K_j$, where $D(\chi_E)(N) = \Lambda_{n-1}(N) = 0$, K_j is compact and $K_j \subset S_j$, where S_j is a C^1 -hypersurface with continuous unit normal $\vec{n}_j(x)$, $x \in S_j$, and \vec{n}_j is the measure theoretic outer normal of E . After a coordinate change write $S_j = \{x_d = f_j(y)\}$, $y = (x_1, \dots, x_{d-1})$ with ∇f_j continuous and $\vec{n}_j(y, f_j(y)) \perp (\nabla f_j, 1)$. Assume $y = 0$ is a point of Lebesgue density of $(f_j, 1)^{-1}(K_j)$, let $V \subset \mathbb{R}^{d-1}$

be a neighborhood of $y = 0$, let $g \in C_0^\infty(V)$ with $g \geq 0$, and consider the variation $u_\epsilon = \chi_{E_\epsilon}$ where $\epsilon > 0$ and

$$E_\epsilon = E \cup \{0 \leq x_d \leq \epsilon u(y), y \in V\}.$$

Then $E \subset E_\epsilon$, and writing $u_0 = \chi_E$, we have

$$\|u_\epsilon\|_{BV} - \|u_0\|_{BV} = \int_V \sqrt{(1 + |\nabla(f_j + \epsilon g)|^2)} - \sqrt{(1 + |\nabla f_j|^2)} dy = o(\epsilon) \quad (39)$$

because by [16]

$$\Lambda_{d-1}((\partial_* E) \cup (E_\epsilon \setminus E)) = o(\epsilon)$$

Λ_{d-1} a.e. on K_j . Also, for a similar reason

$$\lambda \|K * (u_\epsilon - u_0)\|_p = \lambda |\epsilon| \int_V u dy + o(\epsilon). \quad (40)$$

Together (39) and (39) show

$$\int_V \nabla u \cdot \left(\frac{\nabla f_j}{\sqrt{1 + |\nabla f_j|^2}} \right) dy + \lambda \int_V u dy \geq 0. \quad (7.3)$$

Repeating this argument with $\epsilon < 0$, we obtain:

Theorem 4. *At Λ_{d-1} almost every $x \in \partial_* E$,*

$$\left| \operatorname{div} \left(\frac{\nabla f_j}{\sqrt{1 + |\nabla f_j|^2}} \right) \right| \leq \lambda. \quad (41)$$

as a distribution on \mathbb{R}^{d-1} .

2.8 Smooth Extremals

For convenience we assume $d = 2$ and we take $p = q = 1$.

Theorem 5. *Let $u \in C^2 \cap \mathcal{M}_{1,1,\lambda}(f)$ and assume $u \neq f$. Set $E_t = \{u > t\}$ and $J = \frac{K*(f-u)}{|K*(f-u)|}$. Then*

- (i) $\Lambda_1(\partial_* E_t) = \lambda \iint_{E_t} K * J dx dy$, (ii) *the level curve $\{u(z) = c\}$ has curvature $\lambda(K * J)(z)$, and*
- (iii) *if $|\nabla u| \neq 0$, then*

$$\frac{d}{dt} \Lambda_1(\partial_* E_t) = - \int_{\partial E_t} \frac{\lambda(K * J)(z)}{|\nabla u(z)|} ds.$$

Theorem 5 is proved using the variation $u \rightarrow u + \epsilon h, h \in C_0^2$. It should be true in greater generality, but we have no proof at this time.

3 Existence of minimizers

Although the proof of the existence of minimizers of our problem can be seen as a generalization and application of more classical techniques [1], [11], [29], we include it here for completeness in several cases. We consider the cases of bounded domain Q and of the whole domain \mathbb{R}^d , with various kernel operators $Ku = K * u$. We recall that here, for $u \in BV(Q)$, $\|u\|_{BV(Q)}$ denotes the semi-norm

$$\|u\|_{BV(Q)} = \sup \left\{ \int u \operatorname{div} \varphi dx : \varphi \in C_0^1(Q, \mathbb{R}^d), \sup |\varphi(x)| \leq 1, x \in Q \right\}.$$

3.1 Bounded domain, general operator K and general case $p \geq 1, 1 \leq q < \infty$

We recall that $K(x)$ is non-negative and even on \mathbb{R}^d with $\int K(x)dx = 1$, thus $K \in L^1(\mathbb{R}^d)$, $\|K\|_{L^1} = 1$, with $K1 = 1 \neq 0$. The linear and continuous operator $u \mapsto Ku = K * u$ is well defined on $L^1(\mathbb{R}^d)$. There are several ways to adapt linear and continuous convolution operators Ku to the case of bounded domains Q , e.g. as shown in [17].

Theorem 6. *Assume $p \geq 1, 1 \leq q < \infty, \lambda > 0, Q$ open, bounded and connected subset of \mathbb{R}^d , with Lipschitz boundary ∂Q . If $f \in L^p(Q)$ and $K : L^1(Q) \rightarrow L^p(Q)$ is linear and continuous, such that $\|K\chi_Q\|_{L^1(Q)} > 0$, then the minimization problem*

$$\inf_{u \in BV(Q)} \|u\|_{BV(Q)} + \lambda \|K(f - u)\|_{L^p(Q)}^q \quad (42)$$

has an extremal $u \in BV(Q)$.

Proof: Let $E(u) = \|u\|_{BV} + \lambda \|K(f - u)\|_p^q$. Infimum of E is finite since $E(u) \geq 0$, and $E(0) = \lambda \|Kf\|_p^q < \infty$. Let u_n be a minimizing sequence, thus $\inf_v E(v) = \lim_{n \rightarrow \infty} E(u_n)$. Then $E(u_n) \leq C < \infty, \forall n \geq 1$. Poincaré-Wirtinger inequality implies that there is a constant $C' = C'(d, Q) > 0$ such that for all $n \geq 1$, we have $\|u_n - u_{n,Q}\|_1 \leq C' \|u_n\|_{BV}$, where $u_{n,Q}$ is the mean of u_n over Q . Let $v_n = u_n - u_{n,Q}$, thus $v_{n,Q} = 0$ and $Dv_n = Du_n$. Similarly, we have $\|v_n\|_1 \leq C' \|v_n\|_{BV}$.

Since Q is bounded, we have for some constant $C_1 > 0$,

$$\begin{aligned} (C/\lambda)^{2/q} &\geq \|K(f - u_n)\|_p^2 \geq C_1 \|K(f - u_n)\|_1^2 \\ &= C_1 \|Ku_n - Kf\|_1^2 = C_1 \|(Kv_n - Kf) + Ku_{n,Q}\|_1^2 \\ &\geq C_1 \left| \|Kv_n - Kf\|_1 - \|Ku_{n,Q}\|_1 \right|^2 \\ &\geq C_1 \|Ku_{n,Q}\|_1 (\|Ku_{n,Q}\|_1 - 2\|Kv_n - Kf\|_1) \\ &\geq C_1 \|Ku_{n,Q}\|_1 \left(\|Ku_{n,Q}\|_1 - 2\|K\|(\|v_n\|_1 + \|f\|_1) \right). \end{aligned}$$

Let $x_n = \|Ku_{n,Q}\|_1$ and $a_n = \|K\|(\|v_n\|_1 + \|f\|_1)$. Then $x_n(x_n - 2a_n) \leq \frac{(C/\lambda)^{2/q}}{C_1} = c$, with $0 \leq a_n \leq \|K\|(C' + \|f\|_1)$, thus we obtain $0 \leq x_n \leq a_n + \sqrt{a_n^2 + c^2} \leq C_2$ for some constant $C_2 > 0$, which implies

$$\|Ku_{n,Q}\|_1 = \frac{|\int_Q u_n dx|}{|Q|} \|K\chi_Q\|_1 \leq C_2.$$

Thanks to assumptions on K , we deduce that the sequence $|u_{n,Q}|$ is uniformly bounded. By Poincaré-Wirtinger inequality we obtain $\|u_n\|_1 \leq \text{constant}$. Thus, $\|u_n\|_{BV(Q)} + \|u_n\|_{L^1(Q)}$ is uniformly bounded. Following e.g. [16], we deduce that there is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$, and $u \in BV(Q)$, such that u_{n_j} converges to u strongly in $L^1(Q)$. Then we also have $\|u\|_{BV(Q)} \leq \liminf_{n_j \rightarrow \infty} \|u_{n_j}\|_{BV(Q)}$. Since $(u_{n_j} - f) \rightarrow (u - f)$ in $L^1(Q)$, and K is continuous from $L^1(Q)$ to $L^p(Q)$, we deduce that $\|K(u_{n_j} - f)\|_p \rightarrow \|K(u - f)\|_p$ as $n_j \rightarrow \infty$. We conclude that

$$E(u) \leq \liminf_{n_j \rightarrow \infty} E(u_{n_j}) = \inf_v E(v),$$

thus u is extremal. □

3.2 Convolution operator K and particular case $p = q = 1$

In this section, we study the existence of minimizers for different choices of convolution kernels K , in the particular case $p = q = 1$.

3.2.1 Smooth Kernels

Suppose $Kv = K_t * v$, where for example K_t is the Poisson kernel of scale $t > 0$. We have $\hat{K}_t(\xi) = e^{-2\pi t|\xi|}$. Let f be a distribution such that $\|K_t * f\|_{L^1} < \infty$. We recall our minimization problem,

$$\inf_{u \in BV} \{ \mathcal{J}(u) = \|u\|_{BV} + \lambda \|K_t * (f - u)\|_{L^1} \}. \quad (43)$$

To motivate the proposed minimization model (43) with $t > 0$ for the decomposition of an image f into a BV component u and an oscillatory component $f - u$ (rather than taking $t = 0$), we consider the following two examples of functions or distributions f with $\|K_t * f\|_{L^1}$ small while $\|f\|_{L^1}$ is large.

Example 1. Suppose $f(x) = \sin(2\pi nx)$, $x \in \mathbb{R}$, is an oscillatory function. Then $K_t * f = 2 \sin(2\pi nx) e^{-2\pi tn}$. For $Q = [-m/n, m/n]$, we have

$$\|K_t * f\|_{L^1(Q)} = \frac{8m}{\pi n} e^{-2\pi tn}.$$

On the other hand, $\|f\|_{L^1(Q)} = \frac{4m}{\pi n}$. Clearly, $\|K_t * f\|_{L^1} \ll \|f\|_{L^1}$ when n is large.

Example 2. Suppose we are in \mathbb{R} and $f = \sum_{i=0}^{\infty} a_i \delta_{x_i}$ with $\sum_{i=0}^{\infty} |a_i| < \infty$ can also be seen as a (generalized) oscillatory distribution. Note that $f \notin L^1(\mathbb{R})$. However,

$$\|K_t * f\|_{L^1} \leq \sum_{i=0}^{\infty} |a_i| < \infty.$$

Recall that by using the standard property of convolution (Young's inequality), we have for all $v \in L^p$, $1 \leq p \leq \infty$,

$$\|K_t * v\|_{L^p} \leq \|K_t\|_{L^1} \|v\|_{L^p} = \|v\|_{L^p}.$$

Also, using the same arguments as the ones from Lemma 3.24 in Chapter 3 of [6], one obtains the following result

Lemma 7. *Let $u \in BV(\mathbb{R}^d)$. Then*

$$\|K_t * u - u\|_{L^1} \leq t\|u\|_{BV}. \quad (44)$$

Theorem 7. *Let $Q = (0, 1)^d$ or $Q = \mathbb{R}^d$, $\lambda > 0$. For each distribution f such that $\|K_t * f\|_{L^1} < \infty$, the variational problem (43) has a minimizer.*

Proof. Let $\{u_n\}$ be a minimizing sequence for (43). This minimizing sequence exists because $\mathcal{J}(u) \geq 0$ for all $u \in BV(Q)$ and $\mathcal{J}(0) = \|K_t * f\|_{L^1(Q)} < \infty$. We have the following uniform bounds,

$$\|u_n\|_{BV(Q)} \leq C, \quad (45)$$

$$\|K_t * (f - u_n)\|_{L^1(Q)} \leq C. \quad (46)$$

Suppose $Q = \mathbb{R}^d$, then

$$\|u_n\|_{L^1(Q)} \leq \|u_n - K_t * u_n\|_{L^1(Q)} + \|K_t * u_n\|_{L^1(Q)} \leq t\|u_n\|_{BV(Q)} + \|K_t * u_n\|_{L^1(Q)}. \quad (47)$$

This shows that $\|u_n\|_{L^1(Q)}$ is uniformly bounded. On the other hand, if $Q = (0, 1)^d$, then (45) and (46) imply that $\|u_n\|_{L^1(Q)}$ is uniformly bound. Indeed, suppose (45) and (46) hold. Let $w_n = u_n - u_{n,Q}$ as before, then

$$\|w_n\|_{BV(Q)} \leq C.$$

By Poincare's inequality, we have

$$\|w_n\|_{L^1(Q)} = \|w_n - w_{n,Q}\|_{L^1(Q)} \leq C_Q\|w_n\|_{BV} \leq C.$$

But,

$$|u_{n,Q}| = \|K_t * u_{n,Q}\|_{L^1} \leq \|K_t * u_n\|_{L^1(Q)} + \|K_t * w_n\|_{L^1(Q)} \leq \|K_t * u_n\|_{L^1(Q)} + \|w_n\|_{L^1(Q)} \leq C,$$

thus $u_{n,Q}$ is uniformly bounded. Moreover, by applying Poincare's inequality to u_n , we have

$$\|u_n\|_{L^1} \leq |Q|\|u_{n,Q}\| + \|u_n - u_{n,Q}\|_{L^1(Q)} \leq |Q|\|u_{n,Q}\| + C_Q\|u_n\|_{BV(Q)} \leq C.$$

Therefore, $\|u_n\|_{L^1(Q)}$ is uniformly bounded.

Now, using the compactness property in BV and the lower semicontinuity property of the map $u \rightarrow \|u\|_{BV(Q)}$ [16, 6], there exists $u \in BV(Q)$ such that, up to a subsequence (which we still denote by u_n), $u_n \rightarrow u$ in $L^1(Q)$ and

$$\|u\|_{BV(Q)} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{BV(Q)}. \quad (48)$$

Moreover,

$$\|K_t * (u_n - u)\|_{L^1(Q)} \leq \|u_n - u\|_{L^1(Q)} \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (49)$$

This together with the assumption that $K_t * f \in L^1(Q)$, we have

$$\|K_t * (f - u)\|_{L^1(Q)} \leq \lim_{n \rightarrow \infty} \|K_t * (f - u_n)\|_{L^1(Q)}. \quad (50)$$

Combining (48) and (50), one obtains

$$\mathcal{J}(u) \leq \liminf_{n \rightarrow \infty} \mathcal{J}(u_n),$$

which shows that u is a minimizer. □

3.2.2 Riesz Potential

Recall the Riesz potential I_α , $0 < \alpha < d$, defined as [24]

$$I_\alpha f = (-\Delta)^{-\alpha/2} f = K_\alpha * f,$$

where $\hat{K}_\alpha(\xi) = (2\pi|\xi|)^{-\alpha}$. For each $\alpha \in (0, d)$, the homogeneous Sobolev potential space $\dot{W}^{-\alpha,1}$ is defined as

$$\dot{W}^{-\alpha,1} = \{f : \|I_\alpha f\|_{L^1} < \infty\}.$$

Equipped with the norm $\|f\|_{\dot{W}^{-\alpha,1}} = \|I_\alpha f\|_{L^1}$, $\dot{W}^{-\alpha,1}$ becomes a Banach space. From Stein [24] (Chapter V, Section 1.2), if $1 < p < \infty$ and $1/q = 1/p - \alpha/d$, then

$$\|I_\alpha f\|_{L^q(\mathbb{R}^d)} \leq A_{p,q} \|f\|_{L^p(\mathbb{R}^d)}. \quad (51)$$

Here we would like to model the oscillatory component using I_α , $0 < \alpha < d$. Thus the variational problem (43) can be rewritten as

$$\inf_{u \in BV} \{\mathcal{J}(u) = \|u\|_{BV} + \lambda \|K_\alpha * (f - u)\|_{L^1}\}. \quad (52)$$

Theorem 8. *Let $Q = (0, 1)^d$. For each $0 < \alpha < d$ and a distribution f such that $\|K_\alpha * f\|_{L^1(\Omega)} < \infty$, the above variational problem (52) has a minimizer.*

Proof. Again, as before, let $\{u_n\}$ be a minimizing sequence for (52). We have

$$\|u_n\|_{BV(Q)} \leq C, \quad (53)$$

$$\|K_\alpha * (f - u_n)\|_{L^1(Q)} \leq C. \quad (54)$$

As in the proof of Thm. 7, condition (54) implies that $u_{n,\Omega}$ is uniformly bounded, and so by Poincaré's inequality, $\|u_n\|_{L^1} \leq C$, for all n . This implies that the BV -norm of u_n is uniformly bounded. Thus, there exists $u \in BV$ such that, up to a subsequence, $u_n \rightarrow u$ in L^1 and

$$\|u\|_{BV} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{BV}.$$

By the compactness of BV in L^p , $1 \leq p < d/(d-1)$, we have up to a subsequence, $u_n \rightarrow u$ in L^p , $1 \leq p < d/(d-1)$. Now for a fixed $p \in (1, d/(d-1))$, we have

$$\|K_\alpha * (u_n - u)\|_{L^1} \leq C_q \|K_\alpha * (u_n - u)\|_{L^q} \leq C_{p,q} \|u_n - u\|_{L^p} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This implies, up to a subsequence,

$$\|K_\alpha * (f - u)\|_{L^1} = \lim_{n \rightarrow \infty} \|K_\alpha * (f - u_n)\|_{L^1}.$$

Therefore, u is a minimizer. □

4 Characterization of Minimizers 2

In this section, we apply the general duality techniques of Ekeland-Temam [15] and in particular of Demengel-Temam [14] to our minimization problem. We note that these results may be seen as related with the other characterization of minimizers from Lemma 2 and Lemma 3, but expressed and proven here in a different language.

4.1 Dual problem and optimality conditions $p = q = 1$

Let $f : L^1(Q)$ be the given data, with $Q \subset \mathbb{R}^d$ open, bounded, connected, and K a smoothing (analytic) convolution kernel, such as the Gaussian kernel or the Poisson kernel. The minimization problem for $p = 1, q = 1$ is

$$(P_1) \quad \inf_{u \in BV(Q)} E(u) = \|u\|_{BV(Q)} + \lambda \|K * (u - f)\|_{L^1(Q)},$$

using the notation $\|u\|_{BV(Q)} = \int_Q |Du|$ for the semi-norm of u in $BV(Q)$. As we have seen, this problem has a solution $u \in BV(Q) \subset L^2(Q)$. For $u \in L^1(Q)$, we will also use the operator notation $Ku = K * u$ to be the corresponding linear and continuous operator from L^1 to L^1 , with adjoint K^* (with radially symmetric kernel K , then the operator K is self-adjoint). We wish to characterize the solution u of (P_1) using duality techniques.

We have

$$\inf_{u \in BV(Q)} E(u) = \inf_{u \in W^{1,1}(Q)} F(u),$$

since for any $u \in BV(Q)$, we can find $u_n \in W^{1,1}(Q)$ such that $u_n \rightarrow u$ strongly in $L^1(Q)$ and $\|u_n\|_{BV(Q)} \rightarrow \|u\|_{BV(Q)}$. Thus let's first consider the simpler problem

$$(P_2) \quad \inf_{u \in W^{1,1}(Q)} F(u) = \int_Q |\nabla u| dx + \lambda \|K * (u - f)\|_{L^1(Q)}.$$

We now write (P_2^*) , the dual of (P_2) , in the sense of Ekeland-Temam [15]. We first recall the definition of the Legendre transform (or polar) of a function: let V and V^* be two normed vector spaces in duality by a bilinear pairing denoted $\langle \cdot, \cdot \rangle$. Let $\phi : V \rightarrow \bar{\mathbb{R}}$ be a function. Then the Legendre transform $\phi^* : V^* \rightarrow \bar{\mathbb{R}}$ is defined by

$$\phi^*(u^*) = \sup_{u \in V} \left\{ \langle u^*, u \rangle - \phi(u) \right\}.$$

We let $G_1(w_0) = \lambda \int_Q |w_0 - K * f| dx$ and $G_2(\bar{w}) = \int_Q |\bar{w}| dx$, with $G_1 : L^1(Q) \rightarrow \mathbb{R}$, $G_2 : L^1(Q)^d \rightarrow \mathbb{R}$, and using $w = (w_0, w_1, w_2, \dots, w_d) \in L^1(Q)^{d+1}$, we define $G(w) = G_1(w_0) + G_2(\bar{w})$.

Let $\Lambda = (Ku, \nabla u) : W^{1,1}(Q) \rightarrow L^1(Q)^{d+1}$, and Λ^* be its adjoint. Then $E(u) = F(u) + G(\Lambda u)$, with $F(u) \equiv 0$.

Then (P_2^*) is ([15], Chapter III, Section 4):

$$(P_2^*) \quad \sup_{p^* \in L^\infty(Q)^{d+1}} -F^*(\Lambda^* p^*) - G^*(-p^*).$$

We have $F^*(\Lambda^* p^*) = 0$ if $\Lambda^* p^* = 0$, and $F^*(\Lambda^* p^*) = +\infty$ otherwise. It is easy to see that

$$G^*(p^*) = G_1^*(p_0^*) + G_2^*(\bar{p}^*), \text{ for } p^* = (p_0^*, \bar{p}^*).$$

We have that,

$$G_1^*(p_0^*) = \int_{\Omega} p_0^*(K * f) dx$$

if $|p_0^*| \leq \lambda$ a.e., $G_1^*(p_0^*) = +\infty$ otherwise, and

$$G_2^*(\bar{p}^*) = 0$$

if $|\bar{p}^*| \leq 1$ a.e., $G_2^*(\bar{p}^*) = +\infty$ otherwise.

Thus we have

$$(P_2^*) \quad \sup_{p^* \in X} - \int_{\Omega} (-p_0^*)(K * f) dx,$$

where $X = \{(p_0^*, p_1^*, \dots, p_d^*) = (p_0^*, \bar{p}^*) \in L^\infty(\Omega)^{d+1}, |p_0^*| \leq \lambda, |\bar{p}^*| \leq 1, \Lambda^* p^* = 0\}$.

Under the satisfied assumptions, we have that $\inf(P_1) = \inf(P_2) = \sup(P_2)^*$ and (P_2^*) has at least one solution p^* .

Using the definition of Λ , we can show that [29]

$$X = \{(p_0^*, p_1^*, \dots, p_d^*) = (p_0^*, \bar{p}^*) \in L^\infty(\Omega)^{d+1}, |p_0^*| \leq \lambda, |\bar{p}^*| \leq 1, K^* p_0^* - \operatorname{div} \bar{p}^* = 0, \bar{p}^* \cdot \nu = 0 \text{ on } \partial Q\}.$$

Now let $u \in BV(Q)$ be the solution of (P_1) and $p = (p_0, \bar{p}) \in X$ be the solution of (P_2^*) . We must have the extremality relation

$$\|u\|_{BV(Q)} + \|K * u - K * f\|_{L^1(Q)} = \int_Q p_0(K * f) dx.$$

We have that $Du \cdot \bar{p}$ is an unsigned measure, satisfying a Generalized Green's formula

$$\int_Q Du \cdot \bar{p} = - \int_Q u \operatorname{div} \bar{p} dx + \int_{\partial Q} u(\bar{p} \cdot \nu) ds.$$

Since $\bar{p} \cdot \nu = 0$ $\partial\Omega$ a.e., we have

$$\int_Q |Du| + \int_Q |K * u - K * f| dx + \int_Q p_0 K u dx + \int_Q Du \cdot \bar{p} - \int_Q p_0(K * f) dx = 0,$$

or using the decomposition $Du = \nabla u dx + D_s u = \nabla u dx + C_u + J_u = \nabla u dx + C_u + (u^+ - u^-) \nu d\mathcal{H}^{d-1}|_{S_u}$ [16], with S_u the support of the jump measure J_u , we get

$$\begin{aligned} & \int_Q |\nabla u| dx + \int_{Q \setminus S_u} |C_u| + \int_{S_u} (u^+ - u^-) d\mathcal{H}^1 + \int_Q \nabla u \cdot \bar{p} dx + \int_{Q \setminus S_u} \bar{p} \cdot C_u \\ & + \int_{S_u} (u^+ - u^-) \bar{p} \cdot \nu d\mathcal{H}^{d-1} + \int_Q |K * u - K * f| dx + \int_Q p_0 K u dx - \int_Q p_0(K * f) dx = 0. \end{aligned}$$

Since for any function ϕ and its polar ϕ^* we must have $\phi^*(u^*) - \langle u^*, u \rangle + \phi(u) \geq 0$ for any $u \in V$ and $u^* \in V^*$, we obtain:

1. $|K * u - K * f| - (-p_0)(K * u) + (-p_0)(K * f) \geq 0$ for dx a.e. in Ω
2. $|\nabla u| - \nabla u \cdot (-\bar{p}) + 0 \geq 0$ for dx a.e. in Ω where $\nabla u(x)$ is defined
3. $0 - (-\bar{p}) \cdot C_u + |C_u| = (1 + \bar{p} \cdot h)|C_u| \geq 0$, since $|\bar{p}| \leq 1$ (letting $C_u = h \cdot |C_u|$, $h \in L^1(|C_u|)^d$, $|h| = 1$)

4. $0 - (-\bar{p} \cdot \nu)(u^+ - u^-) + (u^+ - u^-) = (u^+ - u^-)(1 + \bar{p} \cdot \nu) \geq 0$ for $d\mathcal{H}^{d-1}$ a.e. in S_u (again since $|\bar{p}| \leq 1$).

Therefore, each expression in 1-4 must be exactly 0 and we obtain another characterization of extremals u :

Theorem 9. *u is a minimizer of (P_1) if and only if there is $(p_0, p_1, \dots, p_d) = (p_0, \bar{p}) \in (L^\infty)^{d+1}$ such that*

$$\begin{aligned} |p_0| &\leq \lambda, \quad |\bar{p}| \leq 1, \\ \bar{p} \cdot \nu &= 0 \text{ on } \partial Q, \\ K^* p_0 - \operatorname{div} \bar{p} &= 0, \\ |K * (u - f)| + p_0(K * u - K * f) &= 0, \\ |\nabla u| + \nabla u \cdot \bar{p} &= 0, \\ 1 + \bar{p} \cdot \nu &= 0 \text{ on } S_u \text{ and } |\bar{p}| = 1 \text{ on } S_u, \end{aligned} \tag{55}$$

and

$$\operatorname{supp}|C_u| \subset \{x \in \Omega \setminus S_u, 1 + \bar{p}(x) \cdot h(x) = 0, h \in L^1(|C_u|)^d, |h| = 1, C_u = h|C_u|\}.$$

4.2 Case $1 \leq p, q < \infty$

A similar statement as Thm. 9 can be shown for the general case $1 \leq p, q < \infty$. The main change is in the definition of \mathcal{G}_1 , which becomes $\mathcal{G}_1(w_0) = \lambda \|w_0 - K * f\|_p^q = \lambda \left(\int_Q |w_0 - K * f|^p dx \right)^{q/p}$ for $w_0 \in L^p(Q)$. For example, if $1 < q < \infty$, then $\mathcal{G}_1^*(p_0^*) = \lambda q \left[\frac{1}{q'} \left\| \frac{p_0^*}{\lambda q} \right\|_{p'}^{q'} + \int_Q (K * f) \frac{p_0^*}{\lambda q} dx \right]$ and (55) changes accordingly, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ (similarly in the case $1 \leq p < \infty, q = 1$).

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