A new approximation for effective Hamiltonians for homogenization of a class of Hamilton-Jacobi equations

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Abstract

We propose a new formulation to compute effective Hamiltonians for homogenization of a class of Hamilton-Jacobi equations. Our formulation utilizes an observation made by Barron-Jensen [3] about viscosity supersolutions of Hamilton-Jacobi equations. The key idea is how to link the effective Hamiltonian to a suitable effective equation. The main advantage of our formulation is that only one auxiliary equation needs to be solved in order to compute the effective Hamiltonian $\bar{H}(p)$ for all p. Error estimates and stability are proved and numerical examples are presented to show very encouraging results.

1 Introduction

Let $\mathbb{T}^n$ be the n dimensional flat torus. Assume that $u^\epsilon \in C(\mathbb{R}^n \times [0, +\infty))$ is the viscosity solution of

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}) = 0 \\ u^\epsilon(x, 0) = g(x) \end{cases} \tag{1}$$

where $g \in C(\mathbb{R}^n)$ and $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is $\mathbb{T}^n$ periodic in the second variable. Under suitable assumptions, it was proved by Lions, Papanicolaou and Varadhan in [19] that as $\epsilon \to 0$, u^ϵ uniformly converges to the unique viscosity solution u of the effective equation

$$\begin{cases} u_t + \bar{H}(Du) = 0 \\ u(x, 0) = g(x). \end{cases} \tag{2}$$

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The function $\bar{H} : \mathbb{R}^n \to \mathbb{R}$ is called the effective Hamiltonian, which is defined through the following cell problem [19].

Definition 1 (Theorem) *For each $p \in \mathbb{R}^n$, there exists a unique real number $\bar{H}(p)$ such that the partial differential equation (PDE)*

$$H(p + Dv(y), y) = \bar{H}(p) \tag{3}$$

has a $\mathbb{T}$-periodic viscosity solution v.

There has been a lot of interest in developing efficient algorithms to compute $\bar{H}$. Let us mention two motivations.

• The homogenization result in [19] is not only interesting, but also can be used to compute the homogenized solution from the x-independent effective equation (2) without the need to resolve the small scale ϵ if $\bar{H}$ is known. We refer to [12] and [1] for instance.

• Recently, a program has been launched to use nonlinear PDEs to investigate some integrable structures within a dynamical system. People believe that the effective Hamiltonian encodes a lot of interesting dynamical information. We refer to [14, 13] for more backgrounds.

Although the cell problem (3) gives an elegant mathematical description of the effective Hamiltonian it is not easy to write out an explicit formula in general, except for one dimension [19]. Numerically it is quite challenging to compute the effective Hamiltonian based on the cell problem even though there is no small scale involved. First each cell problem is difficult to solve due to the periodic boundary condition, which means that there is no way to find out where a characteristic starts and to follow it. Moreover, it requires to solve a cell problem for each single p.

Two well-known algorithms to compute the effective Hamiltonian based on the cell problem are the small-δ method and the large-T method, e.g. in [24]. They are essentially the same. The small-δ method uses the fact that under appropriate assumptions, the viscosity solution of an approximate cell problem

$$\delta u^\delta(y) + H(p + Du^\delta(y), y) = 0$$

has the following property:

$$\delta u^\delta \to -\bar{H}(p) \quad \text{as } \delta \to 0 \quad \text{uniformly in } \mathbb{R}^n.$$

The Large-T method uses the result that under appropriate assumptions, the viscosity solution of an evolution problem

$$\begin{cases} u_t + H(p + Du, y) = 0 & \text{in } \mathbb{R}^n \times (0,\ \infty) \\ u = g & \text{in } \mathbb{R}^n \times (t = 0) \end{cases}$$

has the following property:

$$\bar{H}(p) = -\lim_{t\to\infty} \frac{u(y,t)}{t}.$$

When the Hamiltonian is convex in the gradient variable, Gomes and Oberman [17] proposed an approach to calculate $\bar{H}$ based on the inf-max formula (see Contreras-Iturriaga-Paternain-Paternain [7], Gomes [16], etc.)

$$\bar{H}(p) = \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{\mathbb{T}^n} H(p + D\phi, x). \tag{4}$$

In order to compute $\bar{H}(p)$, all the above methods require the solution of the cell problem or a variational problem for each p.

Recently, Oberman, Takei and Vladimirsky [21] proposed an interesting idea to approximate $\bar{H}$ when the Hamiltonian is convex and homogeneous of degree one in the gradient variable. Their basic idea is to recover the effective Hamiltonian from a suitable effective equation. Precisely speaking, for such an H, owing to the inf-max formula, the associated $\bar{H}$ is also convex and homogeneous of degree one. Therefore we may write

$$\bar{H}(p) = \max_{|\alpha|=1} \{(p \cdot \alpha)\bar{c}(\alpha)\}$$

where $\bar{c}(\alpha) \in \mathbb{R}$. Hence to compute $\bar{H}$ it suffices to determine $\bar{c}(\alpha)$. Suppose that u is a viscosity solution of the effective equation

$$\begin{cases} \bar{H}(Du) = \max_{|\alpha|=1} \{(Du \cdot \alpha)\bar{c}(\alpha)\} = 1 \\ \qquad\qquad\qquad\qquad\qquad u(0) = 0 \end{cases}$$

The Hopf-Lax formula implies that

$$\bar{c}(\alpha) = \frac{1}{u(\alpha)}. \tag{5}$$

And u can be approximated by solutions of the oscillatory equation

$$\begin{cases} H(Du^\epsilon, \dfrac{x}{\epsilon}) = \max_{|\alpha|=1} \{(Du^\epsilon \cdot \alpha)c(\dfrac{x}{\epsilon}, \alpha)\} = 1 \\ \qquad\qquad\qquad\qquad\qquad u^\epsilon(0) = 0 \end{cases}$$

which can be numerically computed by well-established schemes. It is easy to see that by taking proper powers this method actually works for homogeneous Hamiltonian of any degree. The main advantage of this method is that only one auxiliary equation needs to be solved to approximate the effective Hamiltonian for all $p \in \mathbb{R}^n$. However, the assumptions on the Hamiltonian H is too restricted to include many interesting cases.

In this paper, we propose a general formulation along this line to include more important Hamiltonians. The main novelty is how to link the effective Hamiltonian to a suitable effective equation. In general, there does not exist an elegant relation like (5). Instead, we use an interesting observation made in [3] of viscosity supersolutions for convex Hamilton-Jacobi equations. To make our

presentation clear, we will focus on Hamilton-Jacobi equations with Hamiltonian $H(p,x)$ in the kinetic form

$$H(p,x) = a_{i,j}(x)p_i p_j + V(x)$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz continuous, $\mathbb{T}$-periodic function. And $a_{i,j}(x)$ satisfies the uniformly strict convexity condition

$$\Lambda|\xi|^2 \geq a_{i,j}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n \tag{6}$$

with $\Lambda > \lambda > 0$. Such Hamiltonians are very important in the dynamical system and the classical mechanics. Due to the presence of the potential V, the associated effective Hamiltonian is more complicated than those in [21]. For example, it is not homogeneous of any degree and its graph might contain a flat part near the origin which is related to trapping of trajectories (see [6] and computational examples in section 3). Technically speaking, our formulation works for all H which is convex and coercive in the gradient variable as long as $\min \bar{H}$ is known (see step I in section 2).

Outline. In Section 2, we provide theoretical justifications for our approach and error estimate. In Section 3, we first present the numerical algorithm to compute effective Hamiltonians. Then numerical results are presented to show both efficiency and accuracy of our method. Complexity of our method and comparisons with other methods are also discussed. In section 4, we give conclusion remarks and discuss some future projects.

2 Theoretical results

The viscosity solutions, under appropriate assumptions, for Hamilton-Jacobi equations

$$H(Du(x), u, x) = 0 \quad x \in \Omega \subset \mathbb{R}^n$$

are defined as follows [8].

Definition 2 (viscosity solution) *A function $u(x) \in C(\Omega)$ is a viscosity subsolution (resp. supersolution) of $H(Du(x), u, x) = 0$ if for any $\phi(x) \in C^\infty(\Omega)$, when $u - \phi$ attains a local maximum (resp. minimum) at point $x_0 \in \Omega$,*

$$H(D\phi(x_0), u(x_0), x_0) \leq 0 \quad (resp. \geq 0).$$

A viscosity solution of $H(Du(x), u, x) = 0$ is a viscosity subsolution and supersolution.

Our approach is strongly motivated by the following theorem in [3] .

Theorem 1 (Barron-Jensen) *If the Hamiltonian $H(p, u, x)$ is convex in p, the inequality in the definition of viscosity supersolutions is actually an equality, that is, for any $\phi(x) \in C^\infty(\Omega)$, when $u - \phi$ attains a local minimum at point $x_0 \in \Omega$,*

$$H(D\phi(x_0), u(x_0), x_0) = 0 \tag{7}$$

For readers' convenience, we sketch the idea of proof. For $K \subset \mathbb{R}^n$, denote $Co(K)$ as the convex hull of K, i.e, the smallest closed convex set containing K. We denote $C_r = Co(\{Du(x)|\ x \in B_r(x_0),\ Du(x) \text{ exists}\})$ for $r > 0$. By mollifying u, it is not hard to show that

$$D\phi(x_0) \in \cap_{r>0} C_r.$$

Then the convexity of H in the p variable implies

$$H(D\phi(x_0), u(x_0), x_0) \leq 0.$$

Combining with the definition of supersolution, the equality (7) holds.

Barron and Jensen's observation allows us to recover $\bar{H}(p)$ from a suitable effective equation. The following is our approach.

Step I: Consider the effective equation

$$(HJa) \qquad \begin{cases} \bar{H}(Du) = f(x) & \text{in } \Omega\backslash\{0\} \in \mathbb{R}^n \\ u(0) = 0 \end{cases} \tag{8}$$

For each $p \in \mathbb{R}^n$, let $w(x) = u - p \cdot x$. If w attains a local minimum at x_0, then $\bar{H}(p) = f(x_0)$ according to equality (7). A tricky point here is that we should choose $f(0) = \min \bar{H}$. Then 0 is a removable source point (see Lemma 1 below).

Step II: It might be impossible to find a local minimum point x_0. To overcome this issue, we prove a stability result (Theorem 4) which states that if $w(x_1) \leq \min_{\mathbb{R}^n} w + \delta$ for some x_1, then

$$|f(x_1) - \bar{H}(P)| = O(\sqrt{\delta}).$$

Step III: The next task is to find at least one such x_1. Let us turn to the oscillatory equation

$$(HJa)^\epsilon \qquad \begin{cases} H(Du^\epsilon, \dfrac{x}{\epsilon}) = f(x) & \text{in } \Omega\backslash\{0\} \subset \mathbb{R}^n \\ u^\epsilon(0) = 0 \end{cases} \tag{9}$$

Since $\lim_{\epsilon \to 0} u^\epsilon = u$, we may consider $w_\epsilon = u^\epsilon - p \cdot x$. If

$$w_\epsilon(x_2) \leq \min_{\mathbb{R}^n} w_\epsilon + \delta_2,$$

Step II implies

$$|f(x_2) - \bar{H}(p)| \leq O(\sqrt{\max_{B_{R(|p|)}} |u^\epsilon - u| + \delta_2}),$$

where $R(|p|)$ is a constant that only depends on $|p|$. Formally, $|u^\epsilon - u| = O(\epsilon)$. This suggests that we may expect that

$$|f(x_2) - \bar{H}(p)| \leq O(\sqrt{\epsilon + \delta_2}).$$

This will be rigorously established in Theorem 5 below and it is the main theoretical result of our paper. Although Theorem 5 is not really based on step I and II, it is strongly motivated by them and Jensen-Barron's Theorem and shares a lot of similarities in proofs.

Let's start with (HJa) and $(HJa)^\epsilon$. With appropriate $f(x)$, for any $p \in \mathbb{R}^n$, $u^\epsilon - p \cdot x$ and $u - p \cdot x$ attain at least one global minimum in $\mathbb{R}^n$.

In our discussion, $f(x) = 4\Lambda|x|^2 + \max_x V(x)$ in $(HJa)^\epsilon$ and (HJa). Note that for the kinetic Hamiltonian, it is easy to deduce from the inf-max formula that

$$\min_{p\in\mathbb{R}^n} \bar{H}(p) = \bar{H}(0) = \max_{\mathbb{T}^n} V.$$

Theorem 2 *The viscosity solution u^ϵ of $(HJa)^\epsilon$ has the following properties*

1. *$u^\epsilon(x) \geq |x|^2$ for any $x \in \mathbb{R}^n$.*
2. *$|Du^\epsilon|^2 \leq \frac{4\Lambda}{\lambda}|x|^2 + \frac{2}{\lambda}\max_x |V|$.*

Therefore, for any $p \in \mathbb{R}^n$, $u^\epsilon - p \cdot x$ attains a local minimum at some point in $\mathbb{R}^n$. And if $u^\epsilon(x) - p \cdot x$ for a fixed $p \in \mathbb{R}^n$ satisfies

$$u^\epsilon(x_0) - p \cdot x_0 = \min_x\{u^\epsilon(x) - p \cdot x\},$$

we have

$$|x_0| \leq |p|.$$

Proof:

First we prove $v(x) = |x|^2$ is a viscosity subsolution. Plug $v(x)$ into $(HJa)^\epsilon$,

$$a_{i,j}(x)2x_i2x_j + V(x) \leq 4\Lambda|x|^2 + \max_x V = f(x)$$

So $v(x) = |x|^2$ is a smooth viscosity subsolution, according to comparison principle, $u^\epsilon(x) \geq |x|^2$.

Next we prove part 2.

$$\begin{aligned}
&a_{i,j}(x)u^\epsilon_{x_i}u^\epsilon_{x_j} + V(\frac{x}{\epsilon}) = 4\Lambda|x|^2 + \max_x V \\
&\Rightarrow \\
&a_{i,j}(x)u^\epsilon_{x_i}u^\epsilon_{x_j} \leq 4\Lambda|x|^2 + 2\max_x |V| \\
&\Rightarrow \\
&\lambda|Du^\epsilon|^2 \leq 4\Lambda|x|^2 + 2\max_x |V| \\
&\Rightarrow \\
&|Du^\epsilon|^2 \leq \frac{4\Lambda}{\lambda}|x|^2 + \frac{2}{\lambda}\max_x |V|.
\end{aligned}$$

Lastly, we see that

$$|x_0|^2 - |p||x_0| \le x_0 \cdot x_0 - p \cdot x_0 \le u^\epsilon(x_0) - p \cdot x_0 = \min_x \{u^\epsilon(x) - p \cdot x\} \le 0,$$

which implies

$$|x_0| \le |p|.$$

This completes the proof. □

Owing to the inf-max formula, it is easy to deduce that

$$\lambda|p|^2 + \min_{\mathbb{T}^n} V \le \bar{H}(p) \le \Lambda|p|^2 + \max_{\mathbb{T}^n} V.$$

Following the same proof as in Theorem 2, we have the following result.

Theorem 3 *The viscosity solution u of (HJa) has the following properties*

1. $u(x) \ge |x|^2$ *for any* $x \in \mathbb{R}^n$.
2. $|Du|^2 \le \frac{4\Lambda}{\lambda}|x|^2 + \frac{2}{\lambda}\max_x |V|$.

Therefore, for any $p \in \mathbb{R}^n$, $u - p \cdot x$ attains a global minimum at some point in $\mathbb{R}^n$. And if $u(x) - p \cdot x$ for a fixed $p \in \mathbb{R}^n$ satisfies

$$u(x_0) - p \cdot x_0 = \min_x \{u(x) - p \cdot x\},$$

we have

$$|x_0| \le |p|.$$

We can further prove that the solution u is actually a viscosity solution in the whole domain including the source point $0 \in \Omega$.

Lemma 1 *Assume u is the viscosity solution of (HJa), then u is the viscosity solution in the whole domain Ω. Therefore, for any $p \in \mathbb{R}^n$, if $u - p \cdot x$ attains a local minimum at $0 \in \Omega$, then $\bar{H}(p) = f(0) = \bar{H}(0) = \max_x V$.*

Proof: First we show that u is a viscosity supersolution. We only need to consider the source point 0. For any $\phi \in C^1(\Omega)$ and $u - \phi$ attains a local minimum at $0 \in \Omega$, we need to show that $\bar{H}(D\phi(0)) \ge \max_x V$. This is automatically true according to the fact that $\max_x V = \min_p \bar{H}(p)$, which implies

$$\bar{H}(D\phi(0)) \ge \min_p \bar{H}(p) = \max_x V.$$

Next we prove that u is a viscosity subsolution. This follows from a result in Barron-Jensen [3] which says that if H is convex in the gradient variable, then u is a viscosity subsolution of $H(Du, x) = 0$ in Ω if and only if that

$$H(Du, x) \le 0 \quad \text{for a.e } x \in \Omega.$$

This completes the proof. □

The following is a stability result.

Theorem 4 *Denote $w(x) := u(x) - p \cdot x$ where u is the viscosity solution of (HJa). If for some point $x_1 \in \Omega$ such that*

$$u(x_1) - p \cdot x_1 = w(x_1) \leq \min_x w(x) + \delta = \min_x \{u(x) - p \cdot x\} + \delta,$$

then,

$$\bar{H}(p) = f(x_1) + O(\sqrt{\delta})$$

Proof: If u is the viscosity solution of (HJa), denote

$$g(x) = u(x) - p \cdot x - (\delta - 2|x - x_1|^2).$$

We show that $g(x)$ attains a local minimum in $B(x_1, \sqrt{\delta})$.

We have

$$g(x_1) = u(x_1) - p \cdot x_1 - \delta \leq \min_x w(x)$$

and

$$g(x) = u(x) - p \cdot x + \delta > \min_x w(x) \quad \text{for any} \quad x \in \partial B(x_1, \sqrt{\delta}).$$

Therefore $g(x)$ attains a local minimum at some point $x_0 \in B(x_1, \sqrt{\delta})$, which implies $u(x) - (p \cdot x + (\delta - 2|x - x_1|^2))$ attains a local minimum at $x_0 \in B(x_1, \sqrt{\delta})$. According to Barron-Jensen's observation and lemma above,

$$\bar{H}(p - 4(x_0 - x_1)) = f(x_0),$$

which is

$$\bar{H}(p) + O(|x_0 - x_1|) = f(x_1) + O(|x_0 - x_1|),$$

which implies

$$\bar{H}(p) = f(x_1) + O(|x_0 - x_1|) = f(x_1) + O(\sqrt{\delta}).$$

This completes the proof. □

Suppose $|p| \leq M$. Theorem 2 allows us to restrict the discussion within the cube $\Omega = [-M,\ M]^n$. Denote

$$e_M = \max_\Omega |u^\epsilon - u|.$$

Assume that u^ϵ is the viscosity solutions of $(HJa)^\epsilon$. According to Theorem 4, if

$$u^\epsilon(x_0) - p \cdot x_0 \leq \min_x \{u^\epsilon(x) - p \cdot x\} + \delta$$

for some x_0 in Ω, then

$$\bar{H}(p) = f(x_0) + O(\sqrt{e_M + \delta}).$$

It is a subtle question that how fast the error $e_M \to 0$ as $\epsilon \to 0$. Heuristically, $\epsilon_M = O(\epsilon)$ due to the formal asymptotic expansion $u^\epsilon = u + \epsilon v(x, \frac{x}{\epsilon})$, where $v = v(x, y)$ satisfies

$$H(Du(x), D_y v(x, y)) = \bar{H}(Du(x)).$$

However, it is usually hard to derive this rigorously except in some very special situations (for example, when u is a constant as in [11]). The main reason is that the above v might not be regular enough to justify the above expansion. Nevertheless, through a different approach, we are able to establish the following result which is our main theorem to compute $\bar{H}$.

Theorem 5 *Assume u^ϵ is the viscosity solution of $(HJa)^\epsilon$. Suppose that $|p| \leq M$ and*

$$u^\epsilon(x_0) - p \cdot x_0 \leq \min_x \{u^\epsilon(x) - p \cdot x\} + \delta$$

for some x_0 in Ω, then

$$\bar{H}(p) = f(x_0) + O(\sqrt{\epsilon + \delta}).$$

Proof: For convenience, we assume that $a_{ij}(x)$ and $V(x)$ are all smooth. Such assumptions can be easily removed by mollification and approximation. Let $w \in W^{1,\infty}(\mathbb{T}^n)$ be a viscosity solution of

$$-H(p + Dw(y), y) = -\bar{H}(p) \quad \text{in } \mathbb{T}^n$$

and satisfy that $w(0) = 0$. The existence of w is similar to Definition 1 (the cell problem). See also [15]. It is clear that $||w||_{W^{1,\infty}(\mathbb{T}^n)} \leq C$ for a constant $C \geq 1$ which only depends on M and H. Denote $\tau = \delta + \epsilon$ and

$$v_\epsilon(x) = p \cdot x + \epsilon w(\frac{x}{\epsilon}) - (2C + 1)|x - x_0|^2.$$

Note that

$$u^\epsilon(x_0) - v_\epsilon(x_0) \leq \min_{\mathbb{R}^n}(u^\epsilon - p \cdot x) + C\tau$$

and

$$u^\epsilon(x) - v_\epsilon(x) \geq \min_{\mathbb{R}^n}(u^\epsilon - p \cdot x) + (C + 1)\tau \quad \text{if } |x - x_0| = \sqrt{\tau}.$$

Hence there must exist $\bar{x} \in B_{\sqrt{\tau}}(x_0)$ such that

$$u^\epsilon(\bar{x}) - v_\epsilon(\bar{x}) = \min_{\bar{B}_{\sqrt{\tau}}}(u^\epsilon - v_\epsilon).$$

Since a_{ij} and V are all smooth, according to [18] and [15], u^ϵ is semiconcave in $\mathbb{R}^n \backslash \{0\}$ and w is semiconvex in $\mathbb{R}^n$.

Case 1: If $\bar{x} \neq 0$, then both u^ϵ and v_ϵ are differentiable at $\bar{x}$. Hence

$$H(p + Dw(\frac{\bar{x}}{\epsilon}) - 2(2C + 1)(\bar{x} - x_0), \frac{\bar{x}}{\epsilon}) = f(\bar{x}).$$

Since $|\bar{x} - x_0| \leq \sqrt{\tau}$ and

$$H(p + Dw(\frac{\bar{x}}{\epsilon}), \frac{\bar{x}}{\epsilon}) = \bar{H}(p),$$

we derive that

$$|f(x_0) - \bar{H}(p)| = O(\sqrt{\tau}).$$

Case 2: If $\bar{x} = 0$, as in the proof of Barron-Jensen's theorem, we first mollify u^ϵ. Then by comparing the mollification with the semiconvex function v_ϵ, it is not hard to show that there exists a sequence $x_m \to 0$ as $m \to +\infty$ such that v_ϵ is differentiable at x_m and

$$H(p + Dw(\frac{x_m}{\epsilon}) - 2(2C+1)(x_m - x_0), \frac{x_m}{\epsilon}) \leq f(x_m) + o(1),$$

where $o(1) \to 0^+$ as $m \to +\infty$. Again, since

$$H(p + Dw(\frac{x_m}{\epsilon}), \frac{x_m}{\epsilon}) = \bar{H}(p),$$

we have that

$$\bar{H}(p) \leq f(0) + O(\sqrt{\tau}) = \min_{\mathbb{R}^n} \bar{H} + O(\sqrt{\tau}).$$

Accordingly,

$$|\bar{H}(p) - f(0)| \leq O(\sqrt{\tau}).$$

Note that $|x_0| = |x_0 - \bar{x}| \leq \sqrt{\tau}$. So

$$|\bar{H}(p) - f(x_0)| \leq O(\sqrt{\tau}).$$

□

Remark 1 *The δ in the above Theorem 5 could be the numerical error in computing u^ϵ. So a good guideline for the balance between numerical error, denoted by $\delta(h)$ for a mesh size h, and the asymptotic error is $\delta(h) \sim \epsilon$, which will be used in our numerical examples in Section 3.*

When $\delta = 0$, the error estimate from the above theorem is $\sqrt{\epsilon}$. However, computational examples suggest that the optimal error estimate might be $O(\epsilon)$. When $n = 1$, we are able to rigorously derive that. For simplicity, let us look at the mechanical Hamiltonian $H(p, x) = \frac{1}{2}|p|^2 + V(x)$. According to [19, 15], the effective Hamiltonian $\bar{H}$ is explicitly given by

$$\begin{cases} \bar{H}(p) = \max_{\mathbb{T}^1} V & \text{if } |p| \leq p_0 \\ |p| = \int_0^1 \sqrt{2\bar{H}(p) - 2V(y)}\, dy & \text{otherwise,} \end{cases}$$

where

$$p_0 = \int_0^1 \sqrt{2(\max_{\mathbb{T}^1} V - V(y))}\, dy.$$

Without loss of generality, we may assume that $\max_{\mathbb{T}^1} V = 0$. Then the following theorem holds.

Theorem 6 *Suppose that $n = 1$, $\bar{H}(p) > \max_{\mathbb{T}^1} V$ and u^ϵ is the viscosity solution of $(HJa)^\epsilon$. Suppose that for some $x_0 \in \mathbb{R}^1$*

$$u_\epsilon(x_0) - p \cdot x_0 = \min_{\mathbb{R}^1}\{u_\epsilon - p \cdot x\}.$$

Then

$$|f(x_0) - \bar{H}(p)| \leq O(\epsilon).$$

Proof: Without loss of generality, we assume that $p > 0$. Then for $x \geq 0$,

$$u^\epsilon(x) = \int_0^x \sqrt{2f(y) - 2V(\frac{y}{\epsilon})}\, dy.$$

Since $u^\epsilon - px$ attains minimum at x_0, for any $\Delta x > 0$, we have that

$$\int_{x_0}^{x_0+\Delta x} \sqrt{2f(y) - 2V(\frac{y}{\epsilon})}\, dy \geq p\Delta x.$$

Choose $\Delta x = \epsilon$ and $x = \frac{y}{\epsilon}$. We derive that

$$\int_{x_\epsilon}^{x_\epsilon+1} \sqrt{2f(\epsilon x) - 2V(x)}\, dx \geq p,$$

where $x_\epsilon = \frac{x_0}{\epsilon}$. According to Theorem 5, $f(x_0) > \max_{\mathbb{T}^1} V$ when ϵ is small. Hence it is clear that

$$\int_{x_\epsilon}^{x_\epsilon+1} \sqrt{2f(\epsilon x) - 2V(x)}\, dx = \int_{x_\epsilon}^{x_\epsilon+1} \sqrt{2f(x_0) - 2V(x)}\, dx + O(\epsilon).$$

Since V is periodic,

$$p = \int_0^1 \sqrt{2\bar{H}(p) - 2V(x)}\, dx = \int_{x_\epsilon}^{x_\epsilon+1} \sqrt{2\bar{H}(p) - 2V(x)}\, dx.$$

Then we have that

$$\int_{x_\epsilon}^{x_\epsilon+1} \sqrt{2f(x_0) - 2V(x)}\, dx \geq \int_{x_\epsilon}^{x_\epsilon+1} \sqrt{2\bar{H}(p) - 2V(x)}\, dx - O(\epsilon).$$

This implies that

$$f(x_0) \geq \bar{H}(p) - O(\epsilon).$$

Similarly, we can deduce that

$$f(x_0) \leq \bar{H}(p) + O(\epsilon)$$

by considering the inequality

$$\int_{x_0-\Delta x}^{x_0} \sqrt{2f(y) - 2V(\frac{y}{\epsilon})}\, dy \leq p\Delta x.$$

for $\Delta x = \epsilon$. □

Remark 2 *The above theorem can be easily generalized to convex Hamiltonian in 1D using control interpretation. We believe that the above theorem should also hold for $n > 1$ in proper sense. However, when $n > 1$, the situation becomes much more complicated due to the lack of explicit relation between the $\bar{H}(p)$ and the potential V. So far, we do not have any clue about how to approach it.*

2.1 A scaling property of $\bar{H}$

Here we point out a simple scaling property of $\bar{H}$ which will be useful for computing $\bar{H}(p)$ with large p. This scaling property allows us to avoid solving $(HJa)^\epsilon$ on a large domain $\Omega = [-M,\ M]^n$ with $|p| < M$.

The cell problems

$$H(Du + p, x) = a_{i,j}(x)(u_{x_i} + p_i)(u_{x_j} + p_j) + V(x) = \bar{H}(p)$$

and

$$H(\frac{Du + p}{M}, x) = a_{i,j}(x)\frac{(u_{x_i} + p_i)}{M}\frac{(u_{x_j} + p_j)}{M} + \frac{V(x)}{M^2} = \frac{\bar{H}(p)}{M^2}$$

are equivalent.

Denote

$$H_M(Dv, x) = a_{i,j}(x)v_{x_i}v_{x_j} + \frac{V(x)}{M^2}, \quad v(x) = \frac{u}{M},$$

we have

$$H_M(Dv + \frac{p}{M}, x) = \bar{H}_M(\frac{p}{M}) = \frac{\bar{H}(p)}{M^2},$$

which implies

$$\bar{H}(p) = M^2\bar{H}_M(\frac{p}{M}).$$

Therefore, we only need to compute $\bar{H}_M(p)$ with small p, which requires to solve $(HJa)^\epsilon$ only on a small domain.

3 Numerical Method and Examples

In this section we first present our numerical method to compute the effective Hamiltonians. Then we test a few computational examples and use careful error analysis to verify our method. For notational simplicity, the method is illustrated with examples in one or two dimensions, but generalization to higher dimension is straightforward. Complexity and accuracy for computing the effective Hamiltonian will be discussed.

The first and key step in our numerical method is to solve the fast oscillatory Hamilton-Jacobi equation $(HJa)^\epsilon$ to obtain a numerical approximation u_h^ϵ, where h is the mesh size. Then for each $p \in \mathbb{R}^N$, we find the minimum of $u_h^\epsilon - p \cdot x$ approximately on a mesh point x_0. Finally, $\bar{H}(p)$ is approximated by Theorem 5.

3.1 Numerical Solution of Hamilton-Jacobi equations

Although the equation $(HJa)^\epsilon$ contains fast oscillations it is a convex Hamilton-Jacobi equation with a given source point, for which several fast algorithms with optimal complexity are available. In this work we use the fast sweeping method [4, 28] to solve $(HJa)^\epsilon$. The fast sweeping method is an efficient iterative method using Gauss-Seidel iteration with monotone upwind scheme and alternating orderings. The method is very easy to implement, especially for general convex Hamilton-Jacobi equation [22] and has the optimal complexity $O(N)$, where N is the number of grid points. Although the constant in the complexity estimate for the original fast sweeping method depends on how fast characteristics change direction [23], which implies that the number of iteration for $(HJa)^\epsilon$ will depend on ϵ. Here we adopt the improvement proposed in [2] using locking and queuing techniques, which dramatically reduces the computational cost for $(HJa)^\epsilon$ in which fast variation is present. As shown in [2], the CPU time is almost always linear in N with a fixed constant no matter how fast the oscillation is. We point out that other fast algorithms such as fast marching method [27, 25] for isotropic Eikonal equation and ordered upwind method [26] for anisotropic Eikonal equation can also be used here. Our main goal here is to show that if one can solve $(HJa)^\epsilon$ efficiently then one can approximate the effective Hamiltonians $\bar{H}(p)$ for all p easily using our formulation.

3.2 Numerical error

The total approximation error to the effective Hamiltonian is composed of asymptotic error, which depends on ϵ, and numerical error, which depends on the grid size h. Theoretically, the total error is $O(\sqrt{\epsilon + \delta(h)})$ according to Theorem 5, where $\delta(h) = |u^\epsilon - u^\epsilon_h|$ is the numerical error. So a good balance is to have numerical error $\delta(h)$ comparable to ϵ. However, the above theoretical estimate usually provides a quite conservative lower bound in practice. Numerical results suggest that the total error is $O(\epsilon)$, which is proved for one dimensional case in Theorem 6. In high dimension, we suspect that it is related to the integrability of the Hamiltonian system.

Numerically, although we are solving the u^ϵ from $(HJa)^\epsilon$, which contains fast oscillation, the key point is that we only need to approximate $u(x)$, which is the solution to the effective equation (HJa) without fast oscillation, to the order of ϵ and approximate the cell solution $v(\frac{x}{\epsilon})$ to $O(1)$ according to the asymptotic expansion

$$u^\epsilon(x, \frac{x}{\epsilon}) = u(x) + \epsilon v(x, \frac{x}{\epsilon}) + \dots \tag{10}$$

If the potential V is regular, the numerical error in maximum norm for monotone upwind scheme is typical $O(|h \log h|)$ when there are isolated source points [28, 23, 22, 20], where the constant is proportional to the size of the computation domain. Hence we may choose $h \sim O(\epsilon)$.

3.3 Computational Examples

In this part, we present a few 1-d and 2-d examples to demonstrate our new method. In particular we show computational study of convergence as $\epsilon \to 0$. As discussed above, we choose our grid size $h = \frac{\epsilon}{W}$ for some number W independent of ϵ. For $|p| \le M$, Theorem 2 allows us to restrict the computation within the cube $\Omega = [-M,\ M]^n$. If $|p|$ is large we show an example using the scaling argument presented in section 2.1.

3.3.1 1-d Example

The Hamiltonian is

$$H(p,x) = \frac{1}{2}p^2 + \cos(2\pi x), \quad f(x) = 2x^2 + 1.$$

$\bar{H}(p)$ is exactly known [19]. We will compute $\bar{H}(2) = 2.0637954$.

First we show the accuracy for the fast sweeping method for solving $(HJa)^\epsilon$. The computational domain is $[0,\ 1]$. The grid size h is chosen to resolve ϵ with $h = \frac{\epsilon}{20}$. Table 1 shows the maximum error of the fast sweeping method. The numerical solution is converging in $O(\epsilon)$.

Error of fast sweeping method for $(HJa)^\epsilon$ in domain $[0,\ 1]$						
Mesh (ϵ, $h = \frac{\epsilon}{20}$)	$\epsilon = \frac{1}{2}$	$\epsilon = \frac{1}{4}$	$\epsilon = \frac{1}{8}$	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$
Error $\lvert u^\epsilon - u^\epsilon_h \rvert$	0.0307068	0.0162982	0.0084036	0.0042489	0.0021060	0.0010217

Table 1: 1-d example: the exact solution $\int_0^1 \sqrt{2(f(x) - \cos(\frac{2\pi x}{\epsilon}))}dx$ is approximated with a much smaller $h = 10^{-5}$.

Next we compute $\bar{H}(2)$ with our new method. The computational domain for solving $(HJa)^\epsilon$ is $[-2,\ 2]$. We choose $h = \frac{\epsilon}{20}$. Table 2 shows the maximum error, which appears to be $O(\epsilon)$.

Error of $\bar{H}(2)$			
Mesh (ϵ, $h = \frac{\epsilon}{20}$)	$\epsilon = \frac{1}{10}$	$\epsilon = \frac{1}{1000}$	$\epsilon = \frac{1}{100000}$
Error of $\bar{H}(2)$	0.0269954	0.00033012	0.000003

Table 2: 1-d example: accuracy of numerically computing $\bar{H}(2)$.

3.3.2 2-d Examples

We present a few 2-d examples to illustrate our method.

Example 1: The Hamiltonian is

$$H(p,q,x,y) = \frac{1}{2}(p^2+q^2) - \frac{1}{2}((2+\cos(2\pi x)\sin(2\pi y))^2 + (\sin(2\pi x)\cos(2\pi y))^2),$$

$$f(x,y) = 2(x^2+y^2) - \frac{1}{2}.$$

Especially, $\bar{H}(2,0) = 0$. We verify our method by computing $\bar{H}(2,0)$. The computational domain is set on $[-2,\ \ 2] \times [-2,\ \ 2]$. We choose $h = \dfrac{\epsilon}{10}$. Table 3 shows the maximum error, which appears to be $O(\epsilon)$.

Numerical error of $\bar{H}(2,0)$, exact $\bar{H}(2,0) = 0$					
Mesh (ϵ, $h = \frac{\epsilon}{10}$)	$\epsilon = \frac{1}{2}$	$\epsilon = \frac{1}{4}$	$\epsilon = \frac{1}{8}$	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$
Error	-0.235000	-0.133750	-0.070937	-0.036484	-0.018496

Table 3: 2-d example 1: Numerical error for $\bar{H}(2,0)$.

Example 2: The Hamiltonian is

$$H(p,q,x,y) = \frac{1}{2}(p^2+q^2) - \cos(2\pi x) - \cos(2\pi y),\ \ f(x,y) = 2(x^2+y^2)+2.$$

Especially, $\bar{H}(1.4006, 1.1205) = 2.1241$. Table 4 shows the maximum error for computing $\bar{H}(1.4006, 1.1205)$ with our new method. We choose computational domain to be $[-2,\ \ 2] \times [-2,\ \ 2]$ and $h = \dfrac{\epsilon}{10}$. The numerical solution appears to converge by $O(\epsilon)$. We would like to mention that this example was also calculated in [1] using the Large-T method, with a semi-implicit scheme. By choosing $T = 500$, $\Delta t = 1$ and $h = \frac{1}{200}$, the error was 0.0014 which is almost the same as what we get when $\epsilon = \frac{1}{128}$. Our methods takes 8 iterations to converge for computing u_h^ϵ for the standard fast sweeping method. So the overall complexity is of $8 \times O(5120^2) + O(5120^2)$, although the second part, which is the complexity for finding the global minimum, is very cheap compared to the first part. The Large-T methods in [1] takes $500 \times O(200^2)$ operations to get $\bar{H}(1.4006, 1.1205)$. Although CFL condition for time step is relaxed by using semi-implicit scheme, a large system of linear equations has to be solved at each time step. So the coefficent depends on the method used to solve the large system of linear equations. More importantly, our method can compute various p easily once the $(HJa)^\epsilon$ is solved.

Numerical error of $\bar{H}(1.4006, 1.1205)$, exact $\bar{H}(1.4006, 1.1205) = 2.1241$					
Mesh (ϵ, $h = \frac{\epsilon}{10}$)	$\epsilon = \frac{1}{8}$	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$
Error	0.022537	0.014334	0.007596	0.003568	0.001389

Table 4: 2-d example 2: Numerical error for $\bar{H}(1.4006, 1.1205)$.

Example 3: The Hamiltonian is

$$H(p,q,x,y) = \frac{1}{2}(p^2+q^2)+\cos(2\pi x)+\cos(2\pi y)+\cos(2\pi(x-y)),\ f(x,y) = 2(x^2+y^2)+3.$$

Figure 1 shows the plots of $\bar{H}(p)$. We choose $\epsilon = \dfrac{1}{20}$ and $h = \dfrac{\epsilon}{10}$. The computational domain is $[-4,\ \ 4] \times [-4,\ \ 4]$. The domain of $\bar{H}(p)$ is $[-4,\ \ 4] \times [-4,\ \ 4]$ with a resolution of 81×81 points.

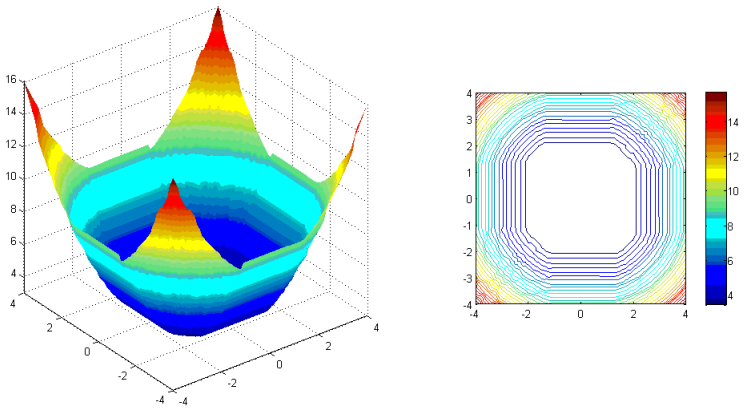

Figure 1: 2-d example 3: $\bar{H}(p)$–surf plot and contour plot.

The flat region in Figure 1 (blank region in contour plot) corresponds to $\bar{H}(p) = \max_{x,y} V(x,y) = \max_{x,y} \cos(2\pi x) + \cos(2\pi y) + \cos(2\pi(x-y)) = 3$. Simple computation with Large-T method in [24] by forward Euler first order Godunov scheme gives $\bar{H}(-4,4) = 16.0400$ with spatial discretization $h = 0.1$, time step $\triangle t = 0.001$ and terminal time $T_f = 100$. It is close to the one computed with our method, which gives $\bar{H}(-4,4) = 16.0682$.

Example 4: Double pendulum. The Hamiltonian for the double pendulum is

$$H(p,q,x,y) = \frac{1}{2}\frac{p^2 - 2pq\cos(2\pi(x-y)) + 2q^2}{2 - \cos^2(2\pi(x-y))} + 2\cos(2\pi x) + \cos(2\pi y).$$

with

$$f(x) = 6(x^2 + y^2) + 3.0.$$

Figure 2 shows the plots of $\bar{H}(p)$. We choose $\epsilon = \frac{1}{20}$ and $h = \frac{\epsilon}{10}$. The computational domain is $[-4,\ 4] \times [-4,\ 4]$. The domain of $\bar{H}(p)$ is $[-4,\ 4] \times [-4,\ 4]$ with a resolution of 81×81 points.

The flat region in Figure 2 (blank region in contour plot) corresponds to $\bar{H}(p) = \max_{x,y} V(x,y) = \max_{x,y} 2\cos(2\pi x) + \cos(2\pi y) = 3$. Simple computation with Large-T method in [24] by forward Euler first order Godunov scheme gives $\bar{H}(4,4) = 18.0970$ with spatial discretization $h = 0.1$, time step $\triangle t = 0.001$ and terminal time $T_f = 100$. It is close to the one computed with our method, which gives $\bar{H}(4,4) = 17.9532$.

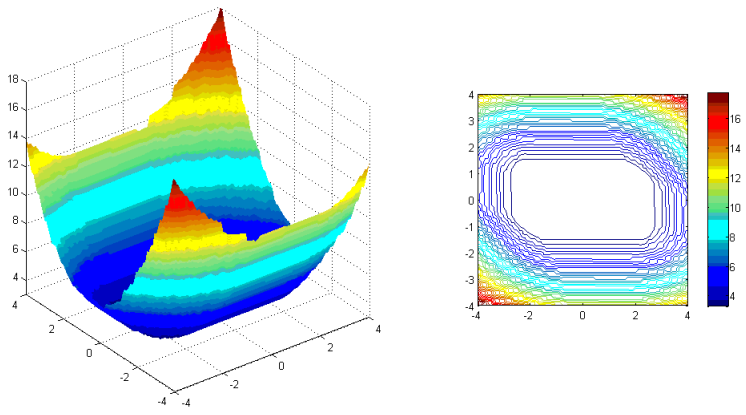

Figure 2: 2-d example 3: $\bar{H}(p)$–surf plot and contour plot

3.3.3 Examples of $\bar{H}(p)$ with large p

In this part, we compute $\bar{H}(100)$ for the above 1-d example and $\bar{H}(100, 100)$ for the above 2-d Example 2. We rescale the problem as discussed above by choosing $M = 100$. Therefore we only need to compute $\bar{H}_M(1)$ and $\bar{H}_M(1, 1)$ respectively.

1-d Example: $H(p, x) = \frac{1}{2}|p|^2 + \cos(2\pi x)$, and $H_M(p, x) = \frac{1}{2}|p|^2 + \frac{\cos(2\pi x)}{M^2}$. We choose $f(x) = 2x^2 + \frac{1.0}{M^2}$, compute $\bar{H}_M(1)$, and get $\bar{H}(100) = M^2\bar{H}_M(1)$. Table 5 shows the results.

$\bar{H}(100)$			
Mesh (ϵ, $h = \frac{\epsilon}{20}$)	$\epsilon = \frac{1}{4}$	$\epsilon = \frac{1}{8}$	$\epsilon = \frac{1}{16}$
$\bar{H}(100)$	5001.00	4876.78	4938.70

Table 5: 1-d example: computing $\bar{H}(100)$ with rescaling.

2-d Example: $H(p, q, x, y) = \frac{1}{2}(p^2+q^2) - \cos(2\pi x) - \cos(2\pi y)$, and $H_M(p, q, x, y) = \frac{1}{2}(p^2 + q^2) - \dfrac{\cos(2\pi x) + \cos(2\pi y)}{M^2}$.

We choose $f(x) = 2x^2 + \frac{2.0}{M^2}$, compute $\bar{H}_M(1, 1)$, and get $\bar{H}(100, 100) = M^2\bar{H}_M(1, 1)$. Table 6 shows the results.

For the above two examples, theoretically, $\bar{H}(100, 100)$ (2-d) is the sum of $\bar{H}(100) + \bar{H}(100)$ (1-d) dimension by dimension, which is verified as shown in Table 5 and 6.

$\bar{H}(100,100)$			
Mesh (ϵ, $h=\frac{\epsilon}{20}$)	$\epsilon=\frac{1}{4}$	$\epsilon=\frac{1}{8}$	$\epsilon=\frac{1}{16}$
$\bar{H}(100,100)$	9508.25	9753.56	9877.39

Table 6: 2-d example: computing $\bar{H}(100,100)$ with rescaling.

4 Conclusion and future problems

We present a new approximation for the effective Hamiltonians, which has accuracy and efficiency verified by numerical examples with encouraging results. The key point is that only one auxiliary PDE needs to be solved for all p. Unfortunately, our method can not deal with nonconvex Hamiltonians, since the "equality" for the definition of viscosity supersolution is no longer valid. Considering that most interesting Hamiltonians are convex, such a restriction is acceptable. Besides, different formulations with the same idea can be used, which we remark here.

For example, instead of using the Eikonal type equation $(HJa)^\epsilon$ and (HJa), we can also employ the following PDEs $(HJb)^\epsilon$ and (HJb), e.g. in infinite horizon problems,

$$(HJb) \qquad \begin{cases} \bar{H}(Du)+u=f(x) & \text{in } \Omega\backslash\{0\}\in\mathbb{R}^n \\ u(0)=0 \end{cases} \tag{11}$$

and the corresponding oscillatory equation

$$(HJb)^\epsilon \qquad \begin{cases} H(Du^\epsilon,\dfrac{x}{\epsilon})+u^\epsilon(x)=f(x) & \text{in } \Omega\backslash\{0\}\subset\mathbb{R}^n \\ u^\epsilon(0)=0 \end{cases} \tag{12}$$

All procedures are similar except that we need to use $f(x_0)-u(x_0)$ instead of $f(x_0)$ to calculate $\bar{H}(p)$ if $u-p\cdot x$ attains local minimum at x_0. One advantage of using (HJb) and $(HJb)^\epsilon$ is the possibility to derive some rigorous error estimates between u^ϵ and u_h^ϵ as in [5, 9, 10]. Meanwhile, we would like to point out that theoretical error estimates are usually too conservative compared to the real numerical errors.

As we mentioned in the introduction, one of the motivations to compute $\bar{H}$ is to provide a more efficient way to solve equation for general initial data g. An interesting future project is to find a good approach to solve equation based on $\bar{H}$ derived with our method. Similar work has been done in [1] using the large-T method.

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