UNIVERSITY OF CALIFORNIA

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Computational Differential Geometry and Intrinsic Surface Processing

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Mathematics

by

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2010

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University of California, Los Angeles 2010 To my parents and my wife ... for their love and constant support

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Acknowledgments

I would like to express my deep appreciation to my advisors, Professor Tony F. Chan (committee chair) and Professor Luminita Vese for their continuously encouragement and generous support throughout my graduate studies at UCLA. My appreciation is further extended to my other committee members, Professor Mark Green and Professor Paul Thompson for their helpful suggestions on my oral exams and the dissertation.

Besides, I would like to thank Professor Yonggang Shi, Professor Ivo Dinov for their stimulating discussion in the weekly group meeting at LONI(Laboratory of Neuro Imaging) in the past three years. I would like to thank Professor Arthur Toga for his generous fellowship support in the past three summers.

I am thankful to Professor Kefeng Liu for his helpful suggestions on differential geometry. I also want to thank Professor Andrea Bertozzi for organizing VIGRE summer internships that provides me an opportunity to touch this exciting field.

I am also thankful for the valuable discussion and suggestions I received from the postdoctoral researchers and graduate students in the department of mathematics, which includes Dr. Bin Dong, Dr. David Y. Mao, Pascal Getreuer, Matthew Keegan, Dr. Xiaoqun Zhang, Dr. Ernie Esser and many others.

My gratitude further goes to my friends Dr. Jinjun Xu, Dr. Wang Yang, Dr. James Y. Jian, Wenhua Gao, Dr. YingYing Li, Yifei Lou, Yongning Zhu and many others, who made my graduate study colorful and joyful. I am also thankful to all of the people who work in the mathematics department here at UCLA for all of their help, especially, Maggie Albert, Martha Abundis and Babette Dalton.

My last, but the most heartfelt acknowledgement goes to my parents and my wife Tao Yang for their constant love and support, which encourage me to overcome all difficulties during my graduate studies.

The research in this dissertation was supported by NIH U54 RR021813, NSF IIS-0914580 and ONR N00014-09-1-0105.

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R.Lai, Y. Shi, K. Scheibel, S. Fears, R. Woods, A. W. Toga, T.F.Chan: Metric-Induced Optimal Embedding for Intrinsic 3D Shape Analysis, Accepted, CVPR2010.

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Abstract of the Dissertation

Computational Differential Geometry and Intrinsic Surface Processing

by

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Doctor of Philosophy in Mathematics University of California, Los Angeles, 2010 Professor Tony F. Chan, Chair

In this work, we focus on using the intrinsic geometric method to study variational problems and Laplace-Beltrami eigen-geometry on 3D triangulated surfaces and their applications to computational brain anatomy. Two classes of problems will be discussed in this dissertation. In the first part, we study how to tackle image processing problems on surfaces by using variational approaches. Starting from the proof for the suitability of total variation for image processing problems on surfaces, we generalize the well-known total variation related imaging models to study imaging problems on surfaces by using differential geometry techniques. As an advantage of this intrinsic method, popular algorithms for solving the total variation related problems can be adapted to solve the generalized models on surfaces. We also demonstrate that this intrinsic method provides us a robust and efficient way to solve imaging problems on surfaces. In the second part, we focus on studying surfaces' own geometry. Specifically, we will study how to detect local and global surface geometry and its applications to computational brain anatomy. The main tool we use is the Laplace-Beltrami (LB) operator and its eigen-systems, which provide us an intrinsic and robust tool to study

surface geometry. We first propose to use LB nodal count sequences as a surface signature to characterize surface and demonstrate its applications to isospectral surfaces resolving and surface classification. Then, we provide a novel approach of computing skeletons of simply connected surfaces by constructing Reeb graphs from the eigenfunctions of an anisotropic Laplace-Beltrami operator. In the last topic about the LB eigen-geometry, we propose a general framework to define a mathematically rigorous distance between surfaces by using the eigen-system of the LB operator, and then we demonstrate one of its applications to tackle the challenging sulci region identification problem in computational brain anatomy.

CHAPTER 0

Introduction

With the rapid development of 3D data acquisition technology, 3D surface processing has become more and more important in many areas, such as computer vision, computer graphics, geometry modeling, medical imaging, computational anatomy, geo-physics, 3D cartoon etc. To the best of our knowledge, there are, at least, three classes of problems about 3D surface processing, namely, analyzing functions on given surfaces, studying the geometry of given surfaces and exploring relations among different surfaces. These basic classes of problems motivate several research topics, for instance, image processing on surfaces, 3D surface recognition, surface retrieval, surface mapping etc.

As studying problems in Euclidean space, partial different equations (PDEs), variational methods and many other techniques are also very useful and powerful tools to study surface processing. However, a big challenge of studying problems on surfaces is that the background space is no longer flat Euclidean space but curved surfaces. There are several ways to tackle this challenge. For instance, implicit surface representation method or parametrization method can convert problems on surfaces to problems in Euclidean space, then standard techniques in Euclidean space can be applied. Apparently, a most natural way to solve problems on surfaces is using differential geometry techniques to handle curved surfaces. This motivation inspires us to use the intrinsic geometric approach to study surface processing. As a consequence, the intrinsic geometric approach will provide us a robust and efficient way to solve problems on surfaces.

This dissertation contains two parts. In the first part, we will focus on studying variational PDEs and image processing on surfaces by using the intrinsic geometry method [58]. Thanks to the differential geometry terminologies, we can successfully prove the suitability of studying imaging on surfaces by the total variation on surfaces. As a consequence of the intrinsic geometry method, we can easily adapt many popular variational image models in Euclidean domains to analogous models on surfaces. Moreover, many fast algorithms can also be adapted to solve the generalized variational models on surfaces. As examples, we typically implement the Rudin-Osher-Fatemi (ROF) image denoising model and the Chan-Vese segmentation model on surfaces by two popular fast algorithms, and demonstrate their related applications. This intrinsic geometry method provides us a robust and efficient approach to study image processing on surfaces.

In the second part of the dissertation, we will focus on studying global and local surface geometry. The basic problem is how to extract key features from the given surface, such that these features can truly reflect surface geometry. In other words, these key features should satisfy:

- Computability: These key features should be numerically easy to compute with reasonable computational cost;
- Robustness: These key features should reflect the intrinsic properties of surface geometry. Namely, the translation, rotation, or pose variance will not affect these key features. Moreover, surfaces with similar geometry should have similar key features.
- Applicability: These key features can provide us a global or local characterization of the given surface.

To satisfy the above conditions, some intrinsic geometric quantities, such as Gauss curvature and conformal factor might be considered as key features. However, it is not easy to obtain accurate numerical approximation of Gauss curvature and conformal factor. A better choice is the surface Laplace-Belrami (LB) eigen-system. It contains a sequence of numbers and a sequence of functions intrinsically defined on the given surface. LB eigen-systems can be efficiently computed by the finite element method on surfaces and provide us a robust key feature set to characterize surfaces. In this dissertation, we will discuss our series work of how to use surface LB eigen-system to study surface geometry and their applications to computational brain anatomy [88, 56, 57].

The organization of the dissertation is as follows:

The first part will focus on studying functions on surfaces, i.e. variational PDEs and intrinsic image processing on surfaces.

In Chapter 1, we first briefly review different geometry background related to our work. Then we discuss numerical approximation of surface differential operators and their sparse matrix representations.

In Chapter 2, we propose to use differential geometry techniques to directly study image problems on surfaces. By using our approach, all plane image variation models and their algorithms can be naturally adapted to study image problems on surfaces. As examples, we show how to generalize Rudin-Osher-Fatemi (ROF) denoising model [82] and Chan-Vese (CV)[20] segmentation model on closed surfaces, and then demonstrate how to implement popular algorithms to solve the total variation related problems on surfaces. This intrinsic approach provides us a robust and efficient method to directly study image processing, in particular, total variational problems on surfaces without requiring any preprocessing [58]. The second part will focus on the surface Laplace-Beltrami eigen-geometry, which includes how to extract global and local surface geometric information from the Laplace-Beltrami eigen-systems and their applications to medical imaging.

In Chapter 3, we first briefly discuss the mathematical background of the Laplace-Beltrami eigen-system and its relation to surface functional analysis and surface geometry. After that, we describe how to use the finite element method on triangulated surfaces to approximate the numerical solution of a LB eigen-system for a given surface.

In Chapter 4, we propose a new signature based on nodal counts of the eigenfunctions. This signature provides a compact representation of the geometric information that is missing in the eigenvalues. In our experiments, we demonstrate that the proposed signature can successfully classify anatomical shapes with similar eigenvalues, and then we show a promising surface classification result by using this new signature [56].

In Chapter 5, we propose a novel approach of computing skeletons of robust topology for simply connected surfaces with boundary by constructing Reeb graphs from the eigenfunctions of an anisotropic Laplace-Beltrami operator. Our work brings together the idea of Reeb graphs and skeletons by incorporating a flux-based weight function into the Laplace-Beltrami operator. Based on the intrinsic geometry of the surface, the resulting Reeb graph is pose independent and captures the global profile of surface geometry [88].

In Chapter 6, we propose a novel and intrinsic metric, the spectral l^2 -distance, to find the optimal embedding induced by Laplace-Beltrami eigen-systems. We prove mathematically that this new distance satisfies the conditions of a rigorous metric. Using the resulting optimal embedding determined by the spectral l^2 distance, we can perform both local and global shape analysis intrinsically in the embedding space. We demonstrate this by developing a template matching approach in the optimal embedding space to solve the challenging problem of identifying major sulci on vervet cortical surfaces [57]. Part I

Variational PDEs on Surfaces

CHAPTER 1

Background of Differential Geometry

In this chapter, we would like to briefly review the background materials about differential geometry used in this dissertation. We first describe the mathematical definition of Riemannian surfaces and several important differential operators, then the discretization of these differential operators are given for triangulated surfaces. In addition, to simplify further computation, sparse matrix representations of these linear operators are given in the end.

1.1 Background of Riemannian Surfaces

A surface M is called a two dimensional smooth manifold, if M is a subset of \mathbb{R}^n such that for each point $p \in M$, there is a smooth function

$$\varphi: \qquad V \longrightarrow \mathbb{R}^n$$
$$x = (x^1, x^2) \longmapsto (\varphi_1, \cdots, \varphi_n)$$

defined on an open set $V \subset \mathbb{R}^2$ satisfying:

- 1. φ maps V homeomorphically onto an open neighbourhood U of p in M;
- 2. For each $x \in V$, the $2 \times n$ matrix $\begin{pmatrix} \frac{\partial \varphi_1(x)}{\partial x^1} & \cdots & \frac{\partial \varphi_n(x)}{\partial x^1} \\ \frac{\partial \varphi_1(x)}{\partial x^2} & \cdots & \frac{\partial \varphi_n(x)}{\partial x^2} \end{pmatrix}$ has rank 2. In other words, if we write $\frac{\partial \varphi(x)}{\partial x^i} = (\frac{\partial \varphi_1(x)}{\partial x^i}, \cdots, \frac{\partial \varphi_n(x)}{\partial x^i}), \quad i = 1, 2$, then the two vectors $\frac{\partial \varphi(x)}{\partial x^1}, \frac{\partial \varphi(x)}{\partial x^2} \in \mathbb{R}^n$ are linearly independent.



Figure 1.1: A local coordinate system at a point p on M

Here, $\varphi^{-1}: U \to V$ is called a local coordinate system at point p (Fig. 1.1). Mis called closed or open, if its topology is closed or open respectively. A function $f: M \to \mathbb{R}$ is called smooth at $p \in M$, if for any given local coordinate system $\varphi^{-1}: U \to V$ at point $p, f \circ \varphi: V \to \mathbb{R}$ is smooth at $\varphi^{-1}(p)$. f is called a smooth function on M if f is smooth at each point on M.

We say a vector $v \in \mathbb{R}^n$ is tangent to M at p if v can be expressed as the velocity vector of certain smooth path through $p \in M$. For instance, $\partial_{x_1} = \frac{\partial \varphi(x)}{\partial x^1}$, $\partial_{x_2} = \frac{\partial \varphi(x)}{\partial x^2}$ are two tangent vectors at p. In fact, any tangent vector at p can be written as a linear combination of ∂_{x_1} and ∂_{x_2} . Given a tangent vector $v = \eta_1 \partial_{x_1} + \eta_2 \partial_{x_2}$ and a smooth function $f : M \to \mathbb{R}$, the derivative of f along the direction v at $p \in M$ is given by :

$$v(f)|_{p} = \eta_{1} \frac{\partial}{\partial x^{1}}(f)|_{p} + \eta_{2} \frac{\partial}{\partial x^{2}}(f)|_{p}$$
(1.1)

where $\frac{\partial}{\partial x^i} f$ stands for $\frac{\partial}{\partial x^i} (f \circ \varphi(x_1, x_2))$, i = 1, 2. The set of all tangent vectors at p:

$$T_p M = span_{\mathbb{R}} \left\{ \frac{\partial \varphi(x)}{\partial x^1}, \frac{\partial \varphi(x)}{\partial x^2} \right\} \subset \mathbb{R}^n$$
 (1.2)

is called the tangent space of M at p. The tangent bundle TM of M is a subset of $M \times \mathbb{R}^n$ satisfying: $TM = \{(p, v) \in M \times \mathbb{R}^n \mid v \in T_pM\}$. A tangent vector field is a smooth cross section of the tangent bundle TM, which is defined by a continuous map $s: M \to TM$ such that $\Pi \circ s(p) = p$, where $\Pi: TM \to M: (p, v) \longmapsto p$.

A Riemannian surface (M, g) is a two dimensional smooth manifold with a metric g smoothly defined on each tangent space T_pM of M as an inner product:

$$g_p: T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto g_p(v, w)$$

In a local coordinate chart $\varphi^{-1} : U \to V$ at p, if we write $\partial_{x^1} = \frac{\partial \varphi(x)}{\partial x^1}, \partial_{x^2} = \frac{\partial \varphi(x)}{\partial x^2}$, then g can be locally represented as a 2 × 2 symmetric positive definite matrix $g(x) = \begin{pmatrix} g_{11}(x) & g_{12}(x) \\ g_{21}(x) & g_{22}(x) \end{pmatrix}$, where $g_{ij} = g_p(\partial_{x^i}, \partial_{x^j}), i, j = 1, 2$.

Typically, since M is a subset of \mathbb{R}^n , there is a natural metric g^{Ind} on M induced from \mathbb{R}^n given by:

$$g_p^{Ind}(\partial_{x^i}, \partial_{x^j}) = (\partial_{x^i}, \partial_{x^j})_{\mathbb{R}^n} = \sum_{\alpha=1}^n \frac{\partial \varphi^\alpha(x)}{\partial x^1} \frac{\partial \varphi^\alpha(x)}{\partial x^2}$$
(1.3)

where $(\cdot, \cdot)_{\mathbb{R}^n}$ is the standard inner product in \mathbb{R}^n . Since $\partial_{x^1}, \partial_{x^2}$ are linearly independent vectors at each point p, g_p^{ind} is a symmetric positive definite matrix. For most of visible surfaces in the physical world, they are subsets in \mathbb{R}^3 with induced metrics. In the rest of this dissertation, all Riemannian surfaces are two dimensional manifolds in \mathbb{R}^3 associated with induced metric. Let (M, g) be a two dimensional closed Riemannian manifold. For any point $p \in M$ and its local coordinate system $\{\varphi_{\alpha} : V_{\alpha} \to U_{\alpha}\}$, the coordinatesindependent surface gradient, divergence, and Laplacian-Beltrami operators are given by:

$$\nabla_M f = \sum_{i,j=1}^2 g^{ij} \frac{\partial f}{\partial x^i} \partial_{x^j}$$
(1.4)

$$\operatorname{div}_{M} \mathbb{V} = \frac{1}{\sqrt{G}} \sum_{i=1}^{2} \frac{\partial}{\partial x^{i}} (\sqrt{G} v^{i}), \text{ for } \mathbb{V} = \sum_{i=1}^{2} v^{i} \partial_{x^{i}}$$
(1.5)

$$\Delta_M f = \operatorname{div}_M(\nabla_M f) = \frac{1}{\sqrt{G}} \sum_{i=1}^2 \frac{\partial}{\partial x_i} (\sqrt{G} \sum_{j=1}^2 g^{ij} \frac{\partial f}{\partial x_j})$$
(1.6)

where (g^{ij}) is the inverse matrix of (g_{ij}) and $G = \det(g_{ij})$.

In particular, if we consider a simple Riemannian surface \mathbb{R}^2 with Euclidean metric, then $\partial_{x_1} = (1,0), \partial_{x_2} = (0,1)$ and $g_{ij} = g^{ij} = \delta_{ij}, i, j = 1, 2$. Thus $\nabla_{\mathbb{R}^2} f = (\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}), \operatorname{div}_{\mathbb{R}^2} f = \frac{\partial f}{\partial x^1} + \frac{\partial f}{\partial x^2}$ and $\triangle_{\mathbb{R}^2} f = \frac{\partial^2 f}{\partial \partial x^1} + \frac{\partial^2 f}{\partial \partial x^2}$, which are exactly the same as the gradient, divergence and Laplace operators in \mathbb{R}^2 . In this point of view, the above differential operators are natural generalizations of the standard gradient, divergence and Laplace operators in Euclidean space.

Given any two functions $f, g : M \to \mathbb{R}$, and any tangent vector field $\mathbb{V} = \sum_{i=1}^{2} v^{i} \frac{\partial}{\partial x^{i}}$, we can also have the divergence theorem similar as in the \mathbb{R}^{n} cases:

$$\int_{M} (\operatorname{div}_{M} \mathbb{V}) f \mathrm{d}s = -\int_{M} \mathbb{V} \cdot \nabla_{M} f \mathrm{d}s \qquad (1.7)$$

$$\int_{M} (\Delta_{M} f) g \mathrm{d}s = -\int_{M} \nabla_{M} f \cdot \nabla_{M} g \mathrm{d}s \qquad (1.8)$$

Moreover, one can define the l^1 and l^2 norm, respectively, as follows:

$$||f||_{1} = \int_{M} |f| \mathrm{d}s, \quad ||f||_{2} = (\int_{M} f^{2} \mathrm{d}s)^{1/2}, \quad \langle f, g \rangle = \int_{M} fg \mathrm{d}s \tag{1.9}$$

$$|\mathbb{V}| = (\sum_{i,j=1}^{2} g_{ij} v^{i} v^{j})^{1/2}, \quad ||\mathbb{V}||_{1} = \int_{M} |\mathbb{V}| \mathrm{d}s, \quad ||\mathbb{V}||_{2} = \int_{M} |\mathbb{V}|^{2} \mathrm{d}s, \qquad (1.10)$$

For details about the above differential geometry concepts, one can find these in many textbooks on differential geometry, for example [70, 59].

1.2 Discretization of Differential Operators

In the discrete case, all surfaces we considered are represented by triangle meshes. Namely, for any given surface M in \mathbb{R}^3 , we represent M as a triangle mesh $M = \{P = \{p_i\}_{i=1}^N, T = \{T_l\}_{l=1}^L\}$, where $p_i = (p_i^1, p_i^2, p_i^3) \in \mathbb{R}^3$ is the i-th vertex and $T_l = (l_1, l_2, l_3) \in \mathbb{N}^3$ represents indices of three vertices of the l-th triangle.



Figure 1.2: A surface with triangle mesh representation

Since the definition of surface gradient, divergence operator are pointwise given, we can consider the pointwise first order numerical approximation of them in the first ring of each vertex. The idea of our approximation can be realized in two steps. We first compute the discretizations on each triangle by their definition given in (1.4),(1.5), then take a weighted average in the first ring of each vertex in terms of the triangle area.

We first show the operators discretizations on a given triangle T_l with vertices $\{p_0, p_1, p_2\}$. Assuming a function $f = \{f(p_0), f(p_1), f(p_2)\}$ and a vector field $\mathbb{V} = \{\mathbb{V}(p_0), \mathbb{V}(p_1), \mathbb{V}(p_2)\}$ defined on each vertex respectively, any point $p \in T_l$,

the linear interpolation of f, \mathbb{V} in T_l can be given by

$$\begin{cases} p = x^{1}(p_{1} - p_{0}) + x^{2}(p_{2} - p_{0}) + p_{0} \\ f(p) = x^{1}(f(p_{1}) - f(p_{0})) + x^{2}(f(p_{2}) - f(p_{0})) + f(p_{0}) \\ \mathbb{V}(p) = x^{1}(\mathbb{V}(p_{1}) - \mathbb{V}(p_{0})) + x^{2}(\mathbb{V}(p_{2}) - \mathbb{V}(p_{0})) + \mathbb{V}(p_{0}) \end{cases}$$
(1.11)

where $\{(x^1, x^2, 1-x^1-x^2) \mid 0 \leq x^1, x^2, x^1+x^2 \leq 1\}$ are the barycentric coordinates of T_l .

Then we have $\partial_{x^1} = p_1 - p_0$, $\partial_{x^2} = p_2 - p_0$, and the metric matrix of T_l would be:

$$g = (g_{i,j})_{i,j=1,2} = \begin{pmatrix} \partial_{x^1} \cdot \partial_{x^1} & \partial_{x^1} \cdot \partial_{x^2} \\ \partial_{x^2} \cdot \partial_{x^1} & \partial_{x^2} \cdot \partial_{x^2} \end{pmatrix} \text{ and } (g^{i,j})_{i,j=1,2} = g^{-1} \qquad (1.12)$$

where \cdot is the dot product in \mathbb{R}^3 . We can have the following discretization:

$$\nabla_{T_{l}}^{d} f(p_{0}) = \sum_{i,j=1}^{2} g^{ij} \frac{\partial f}{\partial x^{j}} \partial_{x^{i}} = (f(p_{1}) - f(p_{0}), f(p_{2}) - f(p_{0}))g^{-1} \begin{pmatrix} \partial_{x^{1}} \\ \partial_{x^{2}} \end{pmatrix}$$

$$= (f(p_{1}) - f(p_{0}), f(p_{2}) - f(p_{0})) \begin{pmatrix} \partial_{x^{1}} \cdot \partial_{x^{1}} & \partial_{x^{1}} \cdot \partial_{x^{2}} \\ \partial_{x^{2}} \cdot \partial_{x^{1}} & \partial_{x^{2}} \cdot \partial_{x^{2}} \end{pmatrix}^{-1} \begin{pmatrix} p_{1} - p_{0} \\ p_{2} - p_{0} \end{pmatrix}$$
(1.13)

We next discretize the divergence operator on the triangle T_l . Assume $\mathbb{V} = v_1\partial_{x^1} + v_2\partial_{x^2}$ is a vector field on T_l , then the coefficients v^1, v^2 can be given by:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = g^{-1} \begin{pmatrix} \mathbb{V} \cdot \partial_{x^1} \\ \mathbb{V} \cdot \partial_{x^2} \end{pmatrix}$$
(1.14)

and differentiate both sides of above equality, we can obtain:

$$\begin{pmatrix} \frac{\partial}{\partial x^1} v_1\\ \frac{\partial}{\partial x^2} v_2 \end{pmatrix} = \begin{pmatrix} g^{11}(\mathbb{V}(p_1) - \mathbb{V}(p_0)) \cdot \partial_{x^1} + g^{12}(\mathbb{V}(p_1) - \mathbb{V}(p_0)) \cdot \partial_{x^2}\\ g^{21}(\mathbb{V}(p_2) - \mathbb{V}(p_0)) \cdot \partial_{x^1} + g^{22}(\mathbb{V}(p_2) - \mathbb{V}(p_0)) \cdot \partial_{x^2} \end{pmatrix}$$
(1.15)

Since \sqrt{G} is constant on each triangle, we have the discretization of the divergence operator on triangle T_l by directly using its definition in (1.5)

$$\operatorname{div}_{T_l}^d \mathbb{V}(p_0) = \frac{1}{\sqrt{G}} \sum_{i=1}^2 \frac{\partial}{\partial x^i} (\sqrt{G} v^i) = \frac{\partial}{\partial x^1} (v_1) + \frac{\partial}{\partial x^2} (v_2)$$
(1.16)

Now, we can give the discretization of the gradient and divergence operators on each vertex by taking a weighted average in the first ring of the vertex in terms of the triangle area. Namely, for any function f and vector field \mathbb{V} defined on a given triangle mesh $\{V = \{p_i\}_{i=1}^N, T = \{T_l\}_{l=1}^L\}$, we use the following discretization of gradient and divergence operators:



Figure 1.3: The first ring

where l goes through all triangles in the first ring of p_i as showed in Figure 1.3.

For the Laplace operator, we would like to use the discretization given by Desbrun et al. [31]:

$$\overset{\alpha_{ij}}{\underset{j \in N_i}{\overset{\beta_{ij}}{\longrightarrow}}} \mathbf{j} \qquad \Delta_M^d f(p_i) = \frac{3}{\sum_l Area(T_l)} \sum_{j \in N_i} \omega_{ij}(p_i)(f(p_j) - f(p_i))$$
(1.19)

where $\omega_{ij}(p_i) = \frac{\cot \alpha_{ij}(p_i) + \cot \beta_{ij}(p_i)}{2}$ and α_{ij}, β_{ij} are the two angles opposite to the edge $\overline{p_i p_j}$ as showed in above figure. This is also a first order discretization.

In the rest of the dissertation, we will also use ∇_M , div_M and Δ_M to denote their discretization operators respectively.

Remark 1.2.1 One thing we need to point out is that we are not the first to consider the discretization of surface gradient and divergence. Other different discretization methods and their approximation analysis can be found in a series works of M. Meyer, M. Desbrun, P. Schroder, A. Barr and G.Xu et.al [68, 31, 102]. Our contribution is using above discrete differential geometry technique to solve variational problems on surfaces, especially, total variation models, their related image processing problems and eigen-problems on surfaces. We will use finite element to compute surface Laplacian to avoid approximating the second order operators. Therefore, we just need to approximate gradient and divergence operators.

1.3 Sparse Matrix Representations of Differential Operators

With the above discretization of differential operators, the gradient and divergence on any surface can be computed. In addition, we observe that one can write down the sparse matrix representations of surface gradient and divergence. In other words, we can implement the actions of these two surface differential operators as matrix multiplications. The computation speed can be improved a lot with this sparse matrix representation of differential operators. In principle, all actions of linear operators on surface can be written as the matrix multiplication with the similar technique.

Before we give the sparse representations of differential operators, we would like to introduce several notations as follows:

We write $\overrightarrow{C} = (\overrightarrow{c_{ij}}), \overrightarrow{D} = (\overrightarrow{d_{ij}})$ as a vector matrix, i.e. each entry of $\overrightarrow{C}, \overrightarrow{D}$ is a vector in \mathbb{R}^3 instead of a number, let $A = (a_{ij})$ be a number matrix as usual and λ be a real number. We define the following multiplications:

$$(\lambda \vec{D})_{ij} = \lambda \vec{d}_{ij}, \quad (\vec{D}A)_{ij} = \sum_{k} \vec{d}_{ik} a_{kj}$$
 (1.20)

$$(\overrightarrow{C} \bullet \overrightarrow{D})_{ij} = \sum_{k} \overrightarrow{c_{ik}} \cdot \overrightarrow{d}_{kj}, \quad (\overrightarrow{C} \times \overrightarrow{D})_{ij} = \sum_{k} \overrightarrow{c_{ik}} \times \overrightarrow{d}_{kj}$$
(1.21)

where \cdot, \times in the right hand side is the dot, cross product in \mathbb{R}^3 respectively.

We first write down the matrix representation of gradient and divegence on each triangle. From (1.13) and (1.16), we have:

$$\nabla_{T_l} f(p_0) = g^{11}(f(p_1) - f(p_0))\partial_{x^1} + g^{12}(f(p_1) - f(p_0))\partial_{x^2} + g^{21}(f(p_2) - f(p_0))\partial_{x^1} + g^{22}(f(p_2) - f(p_0))\partial_{x^2}$$
(1.22)

$$\operatorname{Div}_{T_{l}} \mathbb{V}(p_{0}) = g^{11}(\mathbb{V}(p_{1}) - \mathbb{V}(p_{0})) \cdot \partial_{x^{1}} + g^{12}(\mathbb{V}(p_{1}) - \mathbb{V}(p_{0})) \cdot \partial_{x^{2}} + g^{21}(\mathbb{V}(p_{2}) - \mathbb{V}(p_{0})) \cdot \partial_{x^{1}} + g^{22}(\mathbb{V}(p_{2}) - \mathbb{V}(p_{0})) \cdot \partial_{x^{2}}$$
(1.23)

Remember $\partial_{x^1} = p_1 - p_0, \partial_{x^2} = p_2 - p_0$. If we write

$$\begin{aligned} \overrightarrow{w}_{T_{l}}^{p_{0}} &= -(g^{11} + g^{21})(p_{1} - p_{0}) - (g^{12} + g^{22})(p_{2} - p_{0}) \\ \overrightarrow{w}_{T_{l}}^{p_{1}} &= g^{11}(p_{1} - p_{0}) + g^{12}(p_{2} - p_{0}) \\ \overrightarrow{w}_{T_{l}}^{p_{2}} &= g^{21}(p_{1} - p_{0}) + g^{22}(p_{2} - p_{0}) \\ \overrightarrow{W}_{T_{l}} &= (\overrightarrow{w}_{T_{l}}^{p_{0}}, \overrightarrow{w}_{T_{l}}^{p_{1}}, \overrightarrow{w}_{T_{l}}^{p_{2}}) \\ f_{T_{l}} &= (f(p_{0}), f(p_{1}), f(p_{2}))^{t} \\ \nabla_{T_{l}} &= (\mathbb{V}(p_{0}), \mathbb{V}(p_{1}), \mathbb{V}(p_{2}))^{t} \end{aligned}$$
(1.24)

then we have,

$$\begin{cases} \nabla_{T_l} f(p_0) = \overrightarrow{W}_{T_l} f_{T_l} \\ \operatorname{div}_{T_l} \mathbb{V}(p_0) = \overrightarrow{W}_{T_l} \bullet \mathbb{V}_{T_l} \end{cases}$$
(1.25)

By plugging the above formula (1.25) in (1.17) and (1.18), we can obtain the sparse matrix representation of surface gradient and divergence. Namely, for a given vector field $\mathbb{V} = (\mathbb{V}(p_1), \cdots, \mathbb{V}(p_N))^t$ and a given function f =
$(f(p_1), \cdots, f(p_N))^t$ on the triangulated surface $M = \{P = \{p_i\}_{i=1}^N, T = \{T_l\}_{l=1}^L\},\$ the sparse differentiation matrix \overrightarrow{W} is as follows:

$$\vec{W}(p_i, p_j) = 0 \quad \text{if } p_i, p_j \text{ are not two vertex of a triangle}
\vec{W}(p_i, p_i) = \frac{1}{\sum_l Area(T_l)} \sum_l Area(T_l) \vec{w}_{T_l}^{p_i}, \quad i = 1, \cdots, N
\vec{W}(p_i, p_j) = \frac{1}{\sum_l Area(T_l)} (Area(T_{l_1}) \vec{w}_{T_{l_1}}^{p_j} + Area(T_{l_2}) \vec{w}_{T_{l_2}}^{p_j}),$$
(1.26)

if T_{l_1} and T_{l_2} are two common triangles of p_i and p_j

where l goes through the first ring of p_i . Then we have:

$$\begin{cases} \nabla_M f = \overrightarrow{W} f \\ \operatorname{div}_M \mathbb{V} = \overrightarrow{W} \bullet \mathbb{V} \end{cases}$$
(1.27)

Remark 1.3.1 To the best of our knowledge, we are the first to write down surface gradient and divergence as matrix product in the above compact form. The biggest advantage of this sparse matrix representation is to speed up the computation. In this work, our main concern is solving the total variation related imaging models. To solve total variational related problems, iterative methods are commonly used. With the above surface differential operator matrix representation, we only need to compute a series of sparse matrix products instead of computing gradient and divergence directly by their definition in each iteration. It is clear that this sparse matrix representation can decrease the computation cost.

CHAPTER 2

Variational Problems and Image Processing on Surfaces

2.1 Introduction

The variational method in image processing is quite an important approach. After decades of development, many beautiful results are explored, such as variational models of image denosing, image segmentation [82, 20, 18, 21] etc. However, most results focus on image processing in Euclidean space, in particular, image processing on the 2D plane. With the development of 3D data acquisition technology and the requirement of various applications, there has been increasing interest in studying image processing and variational problems on surfaces or general manifolds. For instance, in fields like computer vision, computer graphics, geometry modeling, medical imaging, computational anatomy, geo-physics and 3D cartoon, it is critical to consider images on 3D surfaces instead of images only on 2D planes.

Several approaches are explored to study image processing on surfaces by using the variational PDE method. To the best of our knowledge, there are, roughly speaking, two classes of approaches to study surfaces imaging, which reflect two different surface representations. One class is using implicit representation of surfaces. S. Osher, G. Sapiro, M. Bertalmio, L. T. Cheng et al. [8, 55, 7, 66, 15] view a closed surface as a zero level set of a signed distance function on a Euclidean domain or a narrow band of the given surface. They approximate differential operators on surfaces by combining the standard Euclidean differential operators with projection along the normal direction. The biggest advantage of implicit representation of surfaces is that one can easily handle topological change under surface evolution. However, it has its own limitations. For instance, fast algorithms in Euclidean cases can not be easily adapted to surface cases by implicit method; For open surfaces, or surfaces with complicated structures, like human's cortical surfaces with many close and deep folding parts, it is not easy to obtain their implicit representations. In addition, the cost of the implicit representations is the pre-step to extend all data on the definition domain of implicit function. These additional increasing data might decrease the computation speed. Another class is using explicit representation of surfaces, namely, surfaces are represented by polygon meshes, in particular, triangle meshes. J. Stam, L. Lopez-Perez, X. Gu, L. Lui et al. [93, 62, 47, 63] introduce either standard patch-wise parametrization or conformal parametrization to the given surface, then differential operators can be computed under the corresponding parametrization. However, the computation of a parametrization is a complicated pre-processing for arbitrary given surfaces, especially for those surfaces with complicated structures or high genus. To conclude, the above methods mainly focus on converting problems on surfaces to problems in Euclidean space. They require pre-processing, either extending data to the narrow band of the given surface or finding a parametrization of the given surface.

Our strategy is different from the above methods. To avoid the need for pre-processing, we will focus on studying variational imaging models directly on the given surface instead of converting them to be problems in Euclidean spaces. Specially, we take two well-known models, namely the ROF denoising model and the CV segmentation model, as examples to explain our strategy. Related work and applications of the intrinsic geometry method can be found in a series work of M. Meyer, M. Desbrun, P. Schroder, A. Barr, G. Xu, C. Bajaj and U. Clarenz, et al. [68, 31, 3, 102, 26]. However, our contribution in this work is using intrinsic geometry method to study the total variation related image processing problems on surfaces and fast algorithms. The most natural extension of the total variation (TV) on surfaces, which is also well-defined on any n-dimensional manifold, is given by M. Ben-Artzi and P. G. LeFloch in [6]. By their natural definition of the total variation on surfaces, we prove the analogous boundary perimeter formula and co-area formula of TV on surfaces, which illustrate the suitability of using TV to study image processing on surfaces. After this, we generalize the ROF denosing model and the CV segmentation model on surfaces as two examples. To implement the above models on triangulated surfaces, we approximate surface gradient and divergence operators by using their intrinsic differential geometry definition. Furthermore, we represent the action of these operators as the multiplication of sparse matrix to simplify our computation. As a consequence of this intrinsic geometry method, we can easily adapt many well-known algorithms in the total variation related problems in Euclidean cases to the generalization total variation image models on surfaces. As examples, we discuss the split Bregman iteration method [44, 43] and Chambolle's dual method [17, 12] on surfaces. In our experience, there are at least two advantages of our intrinsic method. First, we do not need to conduct pre-processing, such as extending all data on the narrow band in the implicit representation or finding a good parametrization in explicit representation. For instance, in the implicit method, when we process high resolution data, like a cortical surface, dealing with large amount of additional data will waste too much computation time; in the parametrization method, when we process surfaces with complicated structures,

it is not easy to obtain a good parametrization. Our direct method can be expected to overcome these limitations. In addition, fast algorithms in Euclidean cases can be easily adapted to solve the total variational problems on surfaces due to the intrinsic method. Second, by this intrinsic method, it is easy to handle open surfaces and surfaces with complicated geometric or topological structures, which can not be easily processed by implicit methods or parametrization methods. A brief comparison among different methods is given in the Table 2.1. To explain everything clearly, we are here just focusing on closed surface cases. One can also study open surfaces with this general technique.

The rest of this chapter is organized as follows. In Section 2.2, we first generalize the concept of the total variation (TV) on surfaces, and demonstrate the analogous version of the boundary perimeter formula and co-area formula for TV on surfaces. Then, we introduce a general form of variational models on surfaces and take the ROF denoising model, the total variational inpainting model, the CV segmentation model as examples to show how to generalize variational models of image processing on planes to variational models of image processing on surfaces. The numerical algorithms of ROF denoising and CV segmentation on surfaces are then presented in Section 2.3. In particular, we use the split Bregman iteration and the dual method on surfaces to solve above models on surfaces. Numerical comparisons with the conformal parametrization method and the level set method are given in Section 6.2.2. Meanwhile, we demonstrate applications of surface image denoising to geometric processing and surface image segmentation to cortical surface parcellation in computational anatomy. Finally, conclusions are made in Section 2.5.

method	principle	Advantage	Disadvantage
level set represen- tation	view a surface as a zero level set of a func- tion	easy to handle topological changes when dealing with surface evolution	all data need to be ex- tended to the narrow band of surface, hard to adapt fast algorithms in Eu- clidean cases
parametrization method	parameterize patches of a surface by Eu- clidean coordinates	differential operators are easy to compute after find- ing parametrization	not easy to obtain parametrization for an arbitrary surface, hard to handle topological changes.
intrinsic geometry method	computation pro- cessed on the given surface itself by differential geometry techniques	can deal with any surface without any preprocessing, easy to adapt fast algo- rithms in Euclidean cases	hard to handle topological changes.

Table 2.1: Comparisons among different methods

2.2 The Total Variation and Image Processing Models on Surfaces

In variational models of image processing on the plane, total variation plays an important role as a regularizing term. One can expect that the analogue of total variation on surfaces and similar variational models should also be useful in image processing on surfaces. In this section, we first describe the generalization of the total variation concept on surfaces, and then we prove it has similar boundary perimeter formula and co-area formula as on the plane. After these preparations, we can consider the analogous variational models on surfaces. In particular, we explain the generalization of ROF denoising and CV segmentation as two examples. One can use the same technique to generalize other plane image models to surface image models.

2.2.1 The total variation on surfaces

For a given surface M, denote the tangent bundle of M by TM and write the set of C^1 sections of TM, i.e. the set of C^1 tangent vector fields on M, by $\Gamma(TM)$. For any function $\varphi \in L^1(M)$, the total variation (TV) of φ is given by [6]:

$$TV(\varphi) = \sup_{\mathbb{V}\in\Gamma(TM), |\mathbb{V}|\leqslant 1} \int_{M} \varphi \operatorname{div}_{M} \mathbb{V} \mathrm{d}s$$
(2.1)

Let's write BV(M) for all functions in $L^1(M)$ with finite TVs. If φ is a C^1 function, then $TV(\varphi) = \int_M |\nabla_M \varphi| ds$. Therefore, we also use $\int_M |\nabla_M \varphi| ds$ to denote the total variation of φ for convenience.

The importance of the total variation in imaging on 2D planes is that: 1. TV does not penalize edges of image due to the co-area formula; 2. TV can also control the geometry of boundary because of the boundary perimeter formula. The rigorous proof of these two formulas can be found in [37, 39]. With the similar proof as in Euclidean cases, we can still have the analogue of these two formulas on surfaces. This tells us that it is reasonable to consider the total variation when we study image processing on surfaces. Moreover, we can similarly adapt other properties of total variation in Euclidean spaces to surfaces by using differential geometry techniques and functional analysis on manifolds.

Let $E \subset M$ be a measurable subset in M and χ_E be its characteristic function. We can similarly define the perimeter Per(E) of E by $TV(\chi_E)$. Since the definition of the total variation on surfaces is a natural extension from the Euclidean case, we can similarly prove the following theorem by combining with differential geometry:

Theorem 2.2.1 (boundary perimeter formula) Let E be a connected subset

in M with C^2 boundary, then

$$Per(E) = \int_{M} |\nabla_M \chi_E| ds = length(\partial E)$$
 (2.2)

[proof]: Remember

$$Per(E) = TV(\chi_E) = \sup_{\mathbb{V}\in\Gamma(TM), |\mathbb{V}|\leqslant 1} \int_M \chi_E \operatorname{div}_M \mathbb{V} \mathrm{d}s.$$

Let \overrightarrow{n} be the unit normal vector of ∂E . Then, for any $\mathbb{V} \in \Gamma(TM)$ with $|\mathbb{V}| \leq 1$, we have

$$\begin{split} |\int_{M} \chi_{E} \mathrm{div}_{M} \mathbb{V} \mathrm{d}s| &= |\int_{E} \mathrm{div}_{M} \mathbb{V} \mathrm{d}s| &= |\int_{\partial E} \mathbb{V} \cdot \overrightarrow{n} \mathrm{d}l| \leqslant \int_{\partial E} 1 \mathrm{d}l = \mathrm{length}(\partial E) \\ & \Longrightarrow Per(E) & \leqslant \ \mathrm{length}(\partial E). \end{split}$$

On the other hand, since ∂E is C^2 smooth, one can easily construct a tangent vector field $\overrightarrow{V_0} \in \Gamma(TM)$, such that $\overrightarrow{V_0}|_{\partial E} = \overrightarrow{n}$. Then,

$$\int_{M} \chi_{E} \operatorname{div}_{M} \overrightarrow{V_{0}} \mathrm{d}s = \int_{E} \operatorname{div}_{M} \overrightarrow{V_{0}} \mathrm{d}s = \int_{\partial E} \overrightarrow{V_{0}} \cdot \overrightarrow{n} \mathrm{d}l = \int_{\partial E} 1 \mathrm{d}l = \operatorname{length}(\partial E)$$
$$\implies \operatorname{Per}(E) \geqslant \operatorname{length}(\partial E)$$

Therefore,

$$Per(E) = length(\partial E)$$

Furthermore, we also have the analogue of the co-area formula. To prove co-area formula for bounded variation functions on surfaces, we need to use its smooth version as follows. It is a standard result in differential geometry. The main idea of the proof is simply change of variables [23].

Theorem 2.2.2 (Smooth Co-area formula) Given $\varphi \in C^{\infty}(M)$, we write $E_t = \{p \in M | \varphi(p) > t \}$. Then

$$TV(\varphi) = \int_{M} |\nabla_{M}\varphi| ds = \int_{-\infty}^{+\infty} Per(E_{t}) dt$$
(2.3)

Now, we prove the co-area formula on surfaces. The idea of the proof is similar to the Euclidean case in [37]. We prove it as follows by using the result of Theorem.2.2.2

Theorem 2.2.3 (Co-area formula) Given $\varphi \in TV(M)$, we write $E_t = \{p \in M | \varphi(p) > t \}$. Then

$$TV(\varphi) = \int_{M} |\nabla_{M}\varphi| ds = \int_{-\infty}^{+\infty} Per(E_{t}) dt$$
(2.4)

[Proof]: For any real number t, we define a L^1 measurable function b_t on M by:

$$b_t(p) = \begin{cases} \chi_{E_t}(p), & \text{if } t \ge 0\\ -\chi_{E_t^c}(p) = \chi_{E_t}(p) - 1, & \text{if } t < 0 \end{cases}$$

Given an arbitrary point $p \in M$,

$$\int_{-\infty}^{+\infty} b_t(p) dt = \begin{cases} \int_0^{\varphi(p)} \chi_{E_t}(p) dt = \int_0^{\varphi(p)} 1 dt = \varphi(p), & \text{if } \varphi(p) \ge 0\\ -\int_{\varphi(p)}^0 \chi_{E_t^c}(p) dt = -\int_{\varphi(p)}^0 1 dt = \varphi(p), & \text{if } \varphi(p) < 0 \end{cases}$$

Hence, for any $\mathbb{V} \in \Gamma(TM)$ with $|\mathbb{V}| \leq 1$, by Fubini's theorem

$$\int_{M} \varphi \operatorname{div}_{M} \mathbb{V} \mathrm{d}s = \int_{M} \left(\int_{-\infty}^{+\infty} b_{t} \mathrm{d}t \right) \operatorname{div}_{M} \mathbb{V} \mathrm{d}s$$
$$= \int_{-\infty}^{+\infty} \left(\int_{M} b_{t} \mathrm{div}_{M} \mathbb{V} \mathrm{d}s \right) \mathrm{d}t$$
$$= \int_{-\infty}^{+\infty} \left(\int_{M} \chi_{E_{t}} \mathrm{div}_{M} \mathbb{V} \mathrm{d}s \right) \mathrm{d}t$$
$$\leqslant \underbrace{\int_{-\infty}^{+\infty} Per(E_{t}) \mathrm{d}t}$$

where $\underline{\int}$ and $\overline{\int}$ denote lower and upper Lebesgue integrals respectively. Then we have:

$$TV(\varphi) \leqslant \underbrace{\int_{-\infty}^{+\infty} Per(E_t) dt}.$$

On the other hand, one can find a sequence $\{\varphi_k\} \subset C^{\infty}(M)$, such that:

$$\lim_{k \to \infty} \int_{M} |\varphi - \varphi_{k}| \, \mathrm{d}s = 0 \qquad (a)$$
$$\lim_{k \to \infty} \int_{M} |\nabla_{M} \varphi_{k}| \, \mathrm{d}s = TV(\varphi)$$

Denote $E_t^k = \{x \in M \mid \varphi_k > t\}$, then from the smooth co-area formula on surfaces, we have:

$$\int_{M} |\nabla_{M} \varphi_{k}| \, \mathrm{d}s = \int_{-\infty}^{+\infty} Per(E_{t}^{k}) \mathrm{d}t, \text{ for each } k$$

From (a), it is clear that there is a zero measure subset $N \subset \mathbb{R}$, such that for any $t \in \mathbb{R} - N$,

$$\lim_{k \to \infty} \int_M |\chi_{E_t} - \chi_{E_t^k}| \, \mathrm{d}s = 0 \qquad (b)$$

Given $t \in \mathbb{R} - N$, if $TV(\chi_{E_t}) < \infty$, (b) implies

$$\lim_{k \to \infty} TV(\chi_{E_t^k}) = TV(\chi_{E_t})$$

Thus for any $\epsilon > 0$, there is a integer k_0 , such that for $k \ge k_0$,

$$TV(\chi_{E_t}) \leq TV(\chi_{E_t^k}) + \epsilon$$

This implies:

$$Per(E_t) = TV(\chi_{E_t}) \leq \lim_{k \to \infty} \inf TV(\chi_{E_t^k})$$
 (c)

If $TV(\chi_{E_t}) = \infty$, (c) is also true. Now by Fatou's lemma, we have:

$$\overline{\int}_{-\infty}^{+\infty} Per(E_t) dt \leq \int_{-\infty}^{+\infty} \liminf_{k \to \infty} TV(\chi_{E_t^k}) dt$$
$$\leq \lim_{k \to \infty} \inf_{-\infty} TV(\chi_{E_t^k}) dt$$
$$= \lim_{k \to \infty} \inf_{0} TV(\varphi_k) = TV(\varphi)$$

To conclude, we have:

$$TV(\varphi) \leq \underbrace{\int_{-\infty}^{+\infty} Per(E_t) dt}_{-\infty} \leq \overline{\int_{-\infty}^{+\infty} Per(E_t) dt} \leq TV(\varphi)$$
$$TV(\varphi) = \int_{-\infty}^{+\infty} Per(E_t) dt \qquad \Box$$

Therefore:

In the following two subsections, as two examples, we will generalize ROF denoising and CV segmentation to show how to adapt variational models on Euclidean domains to variational models on surfaces by differential geometry techniques.

2.2.2 Variational models on surfaces

Similar to variational problems in the Euclidean space \mathbb{R}^n , a general setting of variational problems on a surface M can be written as:

$$\min_{\varphi \in \mathcal{S}} \mathcal{J}(\varphi) + \mathcal{H}(\varphi) \tag{2.5}$$

where S is certain function space on M, \mathcal{J} and \mathcal{H} are two convex functions on S. In particular, let S = BV(M) and $\mathcal{J}(\varphi) = TV(\varphi)$ as we discussed in Section 2.2.1, the general variational problem (2.5) becomes the total variation related problems on the surface M. More specifically, we list the surface analogous forms of several popular total variational models as follows:

A. The ROF model on surfaces

The Rudin-Osher-Fatemi (ROF) image denoising model was first introduced by Rudin et al. [82] in plane image cases. Similarly, let $I: M \to \mathbb{R}$ be an image on a surface M. Let $\mathcal{J}(\varphi) = \int_M |\nabla_M \varphi| ds$, $\mathcal{H}(\varphi) = \frac{\mu}{2} \int_M (\varphi - I)^2 ds$ The analogous ROF image denoising model on the surface M can be represented as follows:

$$\min_{\varphi \in BV(M)} E_1(\varphi) = \int_M |\nabla_M \varphi| \mathrm{d}s + \frac{\mu}{2} \int_M (\varphi - I)^2 \mathrm{d}s$$
(2.6)

More general, let $\mathcal{H}(\varphi) = \frac{\mu}{2} \int_M (K\varphi - I)^2 ds$, the analogue of the total variational image deblurring model can be written as:

$$\min_{\varphi \in BV(M)} E_2(\varphi) = \int_M |\nabla_M \varphi| \mathrm{d}s + \frac{\mu}{2} \int_M (K\varphi - I)^2 \mathrm{d}s$$
(2.7)

where K is a linear blurring kernel operator on M.

B. The total variation inpainting model

The total variation inpainting model was first introduced by Chan and Shen [19]. Similarly, assume $I: M \to \mathbb{R}$ is an image on the surface M. Let $D \subset M$ be the inpainting domain of I and $\mathcal{H}(\varphi) = \int_{M-D} (\varphi - I)^2 ds$. The analogous total variational inpainting model on surfaces can be written as the following:

$$\min_{\varphi \in BV(M)} E_3(\varphi) = \int_M |\nabla_M \varphi| \mathrm{d}s + \frac{\mu}{2} \int_{M-D} (\varphi - I)^2 \mathrm{d}s.$$
(2.8)

C. CV segmentation and its convexified version

The CV segmentation model was first introduced by Chan and Vese [20] for segmentation of images in the Euclidean space. For the surface case, suppose $I: M \to \mathbb{R}$ is an image on surface M. We also represent a closed curve C on Mas the zero level let of a function $\varphi: M \to \mathbb{R}$. The CV segmentation model on M can be given by:

$$\min_{\varphi, c_1, c_2} \int_M |\nabla_M H(\varphi)| ds + \mu \int_M (c_1 - I)^2 H(\varphi) ds + \mu \int_M (c_2 - I)^2 (1 - H(\varphi)) ds \quad (2.9)$$

where H denotes the one dimensional Heaviside function.

However, the energy of CV model is not convex, it might get "stuck" at certain local minima. Chan et al. [18] propose another convexified CV (CCV) segmentation model based on a convex energy. It can be adapted to a segmentation model on surfaces. Namely, fix $\mu \in (0, 1)$ and let $\Omega^+(\varphi^k) = \{p \in M \mid \varphi^k(p) \ge \mu\}$ and $\Omega^-(\varphi^k) = \{p \in M \mid \varphi^k(p) < \mu\}$, the whole procedure of optimizing CCV segmentation would be iterating the following two steps until the steady state: 1. Solve $\varphi^{k+1} = \arg \min_{0 \leqslant \varphi \leqslant 1} \int_M |\nabla_M \varphi| + \mu \int_M \varphi((c_1^k - I)^2 - (c_2^k - I)^2) ds$ 2. Update $c_1^{k+1} = \int_{\Omega^+(\varphi^{k+1})} I ds, c_2^{k+1} = \int_{\Omega^-(\varphi^{k+1})} I ds$

Thanks to differential geometry, we can easily adapt the total variational image models to surface by using differential geometry terminology. With similar techniques, many other popular variational PDE models in Euclidean space can be generalized on surfaces.

2.3 Numerical Algorithms for Total Variation Related Problems on Surfaces

To solve the above minimization problems, a direct method could be used is the gradient descent method to find the minimizer. However, it has its own limitation of computation speed. As an advantage of the intrinsic method, it is easy to adapt popular fast algorithms to the above total variation related problems on surfaces. As examples, we will focus on solving the ROF denoising model and the CCV segmentation models on surfaces by adapting two fast algorithms, namely the split Bregman iteration method and Chambolle's dual projection method. Similar approaches can be used to solve other relevant models on surfaces.

2.3.1 Primal approaches: Split Bregmen iterations

Bregman iteration is first introduced by S. Osher et al. [74]. Later, Tom Goldstein et al. [44, 43] introduce the split Bregman method to compute ROF and global convex segmentation problems in plane image cases. The convergence analysis of this algorithm is given by J.F. Cai et al. in [16]. This algorithm is much faster than gradient descent. Here we can adapt their algorithm to solve the total variation related problems on surfaces as follows.

We consider a general total variation related optimization problem on surfaces:

$$\min_{\varphi} \int_{M} |\nabla_{M}\varphi| \mathrm{d}s + \mathcal{H}(\varphi) \tag{2.10}$$

where $\mathcal{H}(\cdot)$ is a convex function.

Let $\Gamma(TM)$ be the linear space of all tangent vector fields on M. We also introduce the auxiliary variable $\mathbb{V} \in \Gamma(TM)$, and consider the following equivalent optimization problem:

$$\min_{\varphi, \mathbb{V} \in \Gamma(TM)} ||\mathbb{V}||_1 + \mathcal{H}(\varphi) \quad \text{subject to} \quad \mathbb{V} = \nabla_M \varphi \tag{2.11}$$

where $||\mathbb{V}||_1$ is defined in (1.10). The corresponding unconstrained problem would be:

$$(\varphi^*, \mathbb{V}^*) = \arg\min_{\varphi, \mathbb{V} \in \Gamma(TM)} ||\mathbb{V}||_1 + \mathcal{H}(\varphi) + \frac{\lambda}{2} ||\mathbb{V} - \nabla_M \varphi||_2^2$$
(2.12)

Then, we can apply the Bregman iteration on the above problem, namely, we should solve a sequence of the following problems:

$$(\varphi^k, \mathbb{V}^k) = \arg \min_{\varphi, \mathbb{V} \in \Gamma(TM)} ||\mathbb{V}||_1 + \mathcal{H}(\varphi) + \frac{\lambda}{2} ||\mathbb{V} - \nabla_M \varphi - \overrightarrow{b}^k||_2^2 \quad (2.13)$$

$$\overrightarrow{b}^{k+1} = \overrightarrow{b}^k + \nabla_M \varphi^k - \mathbb{V}^k \tag{2.14}$$

To solve (2.13), we can iteratively minimize with respect to φ and \mathbb{V} separately:

$$\varphi^{k+1} = \arg\min_{\varphi} \mathcal{H}(\varphi) + \frac{\lambda}{2} ||\mathbb{V}^k - \nabla_M \varphi - \overrightarrow{b}^k||_2^2$$
(2.15)

$$\mathbb{V}^{k+1} = \arg \min_{\mathbb{V} \in \Gamma(TM)} ||\mathbb{V}||_1 + \frac{\lambda}{2} ||\mathbb{V} - \nabla_M \varphi^{k+1} - \overrightarrow{b}^k||_2^2 \qquad (2.16)$$

For (2.16), the solution is also similar to plane image cases, which can be obtained by the following shrinkage:

$$\mathbb{V}^{k+1} = \max\{|\nabla_M \varphi^{k+1} + \overrightarrow{b}^k| - 1/\lambda, 0\} \frac{\nabla_M \varphi^{k+1} + \overrightarrow{b}^k}{|\nabla_M \varphi^{k+1} + \overrightarrow{b}^k|}$$
(2.17)

To summarize, the whole procedure of using split Bregman iterations for the minimization problem (2.10) on the surface M is the following:

- 1. Let $\mathbb{V}^0 = \overrightarrow{b}^0 = 0$, Do
- 2. Update $\varphi^{k+1} = \arg\min_{\varphi} \mathcal{H}(\varphi) + \frac{\lambda}{2} ||\mathbb{V}^k \nabla_M \varphi \overrightarrow{b}^k||_2^2;$
- 3. Update $\mathbb{V}^{k+1} = \max\{|\nabla_M \varphi^{k+1} + \overrightarrow{b}^k| 1/\lambda, 0\} \frac{\nabla_M \varphi^{k+1} + \overrightarrow{b}^k}{|\nabla_M \varphi^{k+1} + \overrightarrow{b}^k|};$
- 4. Update $\overrightarrow{b}^{k+1} = \overrightarrow{b}^{k} + \nabla_M \varphi^{k+1} \mathbb{V}^{k+1};$
- 5. while ("not converge")

Remark 2.3.1 If the initial auxiliary variables $\overrightarrow{b}^0, \mathbb{V}^0$ are two tangent vector fields on M, then each $\overrightarrow{b}^k, \mathbb{V}^k$ will also be tangent fields on M.

[Proof]: By the definition of step 3 and 4 in above algorithm, and using induction on k, the fact is obviously true.

When we implement the above algorithms, the surface we consider is an embedding surface in \mathbb{R}^3 , thus a tangent vector on the surface can also be viewed as a vector in \mathbb{R}^3 . By Remark 2.3.1, if the initial data $\overrightarrow{b}^0, \mathbb{V}^0$ are two tangent vector fields on the surface, the results of each iteration are automatically two tangent vector fields of the given surface, even if we view the tangent vector field as a vector field in \mathbb{R}^3 .

A. Split Bregman iteration for ROF deoising model on surfaces.

In ROF denoising model (2.8), $\mathcal{H}(\varphi) = \frac{\mu}{2} ||\varphi - I||_2^2$. In this case, the solution of the minimization problem (2.15) should satisfy:

$$(\mu \mathrm{Id} - \lambda \Delta_M) \varphi^{k+1} = \mu I + \lambda \operatorname{div}_M(\overrightarrow{b}^k - \mathbb{V}^k)$$
(2.18)

Therefore, the split Bregman iteration for ROF denoising model (2.8) would be given by:

- 1. Let $\mathbb{V}^0 = \overrightarrow{b}^0 = 0$, Do
- 2. Solve $(\mu \mathrm{Id} \lambda \Delta_M) \varphi^{k+1} = \mu I + \lambda \operatorname{div}_M(\overrightarrow{b}^k \mathbb{V}^k);$
- 3. Update $\mathbb{V}^{k+1} = \max\{|\nabla_M \varphi^{k+1} + \overrightarrow{b}^k| 1/\lambda, 0\} \frac{\nabla_M \varphi^{k+1} + \overrightarrow{b}^k}{|\nabla_M \varphi^{k+1} + \overrightarrow{b}^k|};$
- 4. Update $\overrightarrow{b}^{k+1} = \overrightarrow{b}^{k} + \nabla_M \varphi^{k+1} \mathbb{V}^{k+1};$
- 5. while ("not converge")
- **B.** Split Bregman iterations for CCV segmentation model on surfaces

The key step in CCV segmentation model is solving

$$\varphi^{k+1} = \arg\min_{0 \leqslant \varphi \leqslant 1} \int_{M} |\nabla_{M}\varphi| + \mu \int_{M} \varphi r^{k} \mathrm{d}s \qquad (2.19)$$

where $r^k = (c_1^k - I)^2 - (c_2^k - I)^2$. In this case, we have $\mathcal{H}(\varphi) = \mu \int_M \varphi r^k ds$. The minimization problem (2.15) becomes:

$$\varphi^{k+1} = \arg \min_{0 \leqslant \varphi \leqslant 1} \frac{\lambda}{2} ||\mathbb{V}^k - \nabla_M \varphi - \overrightarrow{b}^k||_2^2 + \mu \int_M \varphi r^k \mathrm{d}s \tag{2.20}$$

Since the above minimization is a quadratic problem with constraint $0 \leq \varphi \leq 1$, its solution can be obtained by:

solve:
$$\Delta_M \varphi^{k+1} = \frac{\mu}{\lambda} r^k + \operatorname{div}_M(\mathbb{V}^k - \overrightarrow{b}^k)$$

update: $\varphi^{k+1}(p_i) \longleftarrow \min\{\max\{\varphi^{k+1}(p_i), 0\}, 1\}$ (2.21)

Therefore, the split Bregman iteration for CCV segmentation model would be given by:

- 1. Let $\mathbb{V}^0 = \overrightarrow{b}^0 = 0$, Do
- 2. Update $r^k = (c_1^k I)^2 (c_2^k I)^2;$
- 3. Solve $riangle_M \varphi^{k+1} = \frac{\mu}{\lambda} r^k + \operatorname{div}_M (\mathbb{V}^k \overrightarrow{b}^k),$ $\varphi^{k+1}(p_i) \longleftarrow \min\{\max\{\varphi^{k+1}(p_i), 0\}, 1\};$

- 4. Update $\mathbb{V}^{k+1} = \max\{|\nabla_M \varphi^{k+1} + \overrightarrow{b}^k| 1/\lambda, 0\} \frac{\nabla_M \varphi^{k+1} + \overrightarrow{b}^k}{|\nabla_M \varphi^{k+1} + \overrightarrow{b}^k|};$ 5. Update $\overrightarrow{b}^{k+1} = \overrightarrow{b}^{k} + \nabla_M \varphi^{k+1} - \mathbb{V}^{k+1};$ 6. Update $c_1^{k+1} = \int_{\Omega^+(\varphi^{k+1})} I \mathrm{d}s, c_2^{k+1} = \int_{\Omega^-(\varphi^{k+1})} I \mathrm{d}s;$
- 7. while ("not converge")

2.3.2Dual approaches: Chambolle's projection methods

The discussion based on the variational model (2.10) with split Bregman iteration method, can be viewed as the primal approach to solve the total variation related problems on surfaces. Meanwhile, based on the definition of the total variation, there has been increasing interests on dual approaches. One famous dual algorithm is Chambolle's projection method of ROF denoising model [17]. It offers us a fast and easy-coding algorithm to solve ROF denoising model. Later, X. Bresson et al [12] propose an algorithm based on Chambolle's dual method to solve CCV model for plane image problems. Here, we can similarly apply the Chambolle's dual methods to solve the total variation related optimization problems on surfaces. Remembering the definition of the total variation on surfaces in (2.1), we consider the following variational problem:

$$\min_{\varphi} \max_{\mathbb{V} \in \Gamma(TM), |\mathbb{V}| \leq 1} \int_{M} \varphi \operatorname{div}_{M} \mathbb{V} \mathrm{d}s + \mathcal{H}(\varphi)$$
(2.22)

By the min-max theorem in optimization theory [33], we can interchange the min and max, to obtain the following equivalent optimization problem:

$$\max_{\mathbb{V}\in\Gamma(TM),|\mathbb{V}|\leqslant 1}\min_{\varphi}\int_{M}\varphi \operatorname{div}_{M}\mathbb{V}\mathrm{d}s + \mathcal{H}(\varphi)$$
(2.23)

A. Dual method for ROF denoising model on surfaces

In ROF denoising model (2.8), $\mathcal{H}(\varphi) = \frac{\mu}{2} ||\varphi - I||_2^2$. In this case, we need to consider the following problem:

$$\max_{\mathbb{V}\in\Gamma(TM),|\mathbb{V}|\leqslant 1}\min_{\varphi}\int_{M}\varphi \operatorname{div}_{M}\mathbb{V}\mathrm{d}s + \frac{\mu}{2}||\varphi - I||_{2}^{2}$$
(2.24)

The solution of the inner minimization problem can be solved exactly as $\varphi = I - \frac{1}{\mu} \operatorname{div}_M \mathbb{V}$. Plug in this back to the above problem, we have the following maximization problem:

$$arg \max_{\mathbb{V}\in\Gamma(TM),|\mathbb{V}|\leqslant 1} \int_{M} (I - \frac{1}{\mu} \operatorname{div}_{M} \mathbb{V}) \operatorname{div}_{M} \mathbb{V} \mathrm{d}s + \frac{\mu}{2} ||\frac{1}{\mu} \operatorname{div}_{M} \mathbb{V}||_{2}^{2}$$
$$= arg \max_{\mathbb{V}\in\Gamma(TM),|\mathbb{V}|\leqslant 1} \frac{\mu}{2} (||I||_{2}^{2} - ||\frac{1}{\mu} \operatorname{div}_{M} \mathbb{V} - I||_{2}^{2})$$
$$= arg \min_{\mathbb{V}\in\Gamma(TM),|\mathbb{V}|\leqslant 1} ||\frac{1}{\mu} \operatorname{div}_{M} \mathbb{V} - I||_{2}^{2}$$
(2.25)

As indicated in Chambolle's method [17], we can also solve the last minimization problem by the iterative method as follows:

$$\mathbb{V}^{n+1} = \frac{\mathbb{V}^n + \tau \nabla_M (\operatorname{div}_M \mathbb{V}^n - \mu I)}{1 + \tau |\nabla_M (\operatorname{div}_M \mathbb{V}^n - \mu I)|}$$
(2.26)

The convergence analysis in Euclidean cases can be easily adapted to surface cases to prove the convergence.

B. Dual method for CCV segmentation model on surfaces

The key step in CCV segmentation model is solving

$$\min_{0 \leqslant \varphi \leqslant 1} TV(\varphi) + \mu \int_{M} \varphi r^{k} \mathrm{d}s$$
(2.27)

where $r^k = (c_1^k - I)^2 - (c_2^k - I)^2$. It has the same set of minimizers as the following unconstrained problem [18]:

$$\min_{\varphi} TV(\varphi) + \mu \int_{M} (\varphi r^{k} + \alpha \nu(\varphi)) \mathrm{d}s \qquad (2.28)$$

where $\nu(\xi) = \max\{0, 2|\xi - 1/2| - 1\}$, provided that $\alpha > \frac{\mu}{2}||r^k(x)||_{L^{\infty}}$. As the algorithms proposed in [1, 12], we can similarly consider convex regularization of the above minimization problem on surfaces as follows:

$$\min_{\varphi,v} TV(\varphi) + \frac{\theta}{2} ||\varphi - v||_2^2 + \mu \int_M (vr^k + \alpha\nu(v)) \mathrm{d}s$$
(2.29)

whose solution can be approached by iteratively updating φ, v by the following two steps:

$$\varphi^{l+1} = \arg\min_{\varphi} TV(\varphi) + \frac{\theta}{2} ||\varphi - v^l||_2^2$$
(2.30)

$$v^{l+1} = arg\min_{v} \frac{\theta}{2} ||\varphi^{l+1} - v||_{2}^{2} + \mu \int_{M} (vr^{k} + \alpha\nu(v)) ds \qquad (2.31)$$

By the dual algorithm of ROF model, the minimizer of (2.30) can be obtained by $\varphi^{l+1} = v^l - \frac{1}{\theta} \operatorname{div}_M \mathbb{V}$, where \mathbb{V} can be iteratively solved by

$$\mathbb{V}^{n+1} = \frac{\mathbb{V}^n + \tau \nabla_M (\operatorname{div}_M \mathbb{V}^n - \theta v^l)}{1 + \tau |\nabla_M (\operatorname{div}_M \mathbb{V}^n - \theta v^l)|}$$
(2.32)

and the solution of (2.31) is given by $v^{l+1} = min\{max\{\varphi^{l+1} - \frac{\mu}{\theta}r^k, 0\}, 1\}$. All convergence proofs of this algorithm on surface optimization problem (2.27) can be naturally adapted from the proofs in Euclidean cases in [12]. To conclude, the algorithm of dual method to solve CCV segmentation model on surfaces is given as follows:

1. Let $v^0 = 0, \mathbb{V}^0 = 0$, Do

2. Update
$$r^k = (c_1^k - I)^2 - (c_2^k - I)^2;$$

3. Do

4. Do
$$\mathbb{V}^{n+1} = \frac{\mathbb{V}^n + \tau \nabla_M(\operatorname{div}_M \mathbb{V}^n - \theta v^l)}{1 + \tau |\nabla_M(\operatorname{div}_M \mathbb{V}^n - \theta v^l)|}$$
 while $(|\mathbb{V}^{n+1} - \mathbb{V}^n| > \epsilon)$

5. Update
$$\varphi^{l+1} = v^l - \frac{1}{\theta} \operatorname{div}_M \mathbb{V}^{n+1};$$

6. Update
$$v^{l+1} = min\{max\{\varphi^{l+1} - \frac{\mu}{\theta}r^k, 0\}, 1\}$$

- 7. while $(\max\{|\varphi^{l+1}-\varphi^l|,|v^{l+1}-v^l|\}>\epsilon)$
- 8. Update $c_1^{k+1} = \int_{\Omega^+(\varphi^{k+1})} I \mathrm{d}s, c_2^{k+1} = \int_{\Omega^-(\varphi^{k+1})} I \mathrm{d}s;$
- 9. while ("not converge")

To summarize, we want to point out that the successful generalization of the image models and their related algorithms from Euclidean cases to surface cases is because of differential geometry. Due to the power of differential geometry, a natural extension of Euclidean geometry, we can generalize the concept of the total variation on surfaces, then ROF denoising model and CCV segmentation model and their related algorithms are adapted on surfaces as examples. Moreover, one can also prove similar convergence results as in the Euclidean cases. With the same technique, the generalization of fast algorithms is not necessarily limited to split Bregman iteration and dual projection method, one can similarly generalize other fast algorithms such as primal-dual methods [103] and so on.

2.3.3 Implementation

So far, we extend all formulas on surfaces. It is easy to observe that each abstract formula is quite consistent with plane image cases due to differential geometry terminologies. At the first glance, the only difference is that we replace all Euclidean gradient, divergence, Laplace operators and Euclidean integrals by their corresponding surface forms. However, since the surface metric has been involved in the above surface differential operators and surface integrals, the mathematical meaning of each term is quite different and also numerical implementation is different.

A. Surface differential operators

As we described in section 1.2, the data structure of each surface is given

by a triangle mesh $M = \{P = \{p_i\}_{i=1}^N, T = \{T_l\}_{l=1}^L\}$, where $p_i \in \mathbb{R}^3$ is the i-th vertex and T_l is the l-th triangle. Any function f defined on M can be written as $f = \{f(p_i)\}_{i=1}^N$. Since M is a embedding surface in \mathbb{R}^3 , then any tangent vector \mathbb{V} on M can be written as $\mathbb{V} = \{\mathbb{V}(p_i)\}_{i=1}^N$, where each $\mathbb{V}(p_i)$ can be viewed as a vector in \mathbb{R}^3 . Let \overrightarrow{W} be the differentiation vector matrix defined in section 1.3 associated with the triangle mesh M. Then we can easily compute the surface gradient and divergence by the matrix product as the discussion in section 1.3:

$$\nabla_M f = \overrightarrow{W} f; \quad \operatorname{div}_M \mathbb{V} = \overrightarrow{W} \bullet \mathbb{V}$$
(2.33)

Once we can compute the surface gradient and divergence, then the dual method to compute ROF denoising model and CCV segmentation model on surface can be easily implemented, since the algorithms of dual method only need surface gradient and divergence.

B. Surface PDEs

The next step is to implement the split Bregman iterations on surfaces. There are two PDEs related to the Laplace operator we need to solve on surfaces. One is to solve the equation (2.18) for ROF denoising:

$$(\mu \mathrm{Id} - \lambda \Delta_M) \varphi^{k+1} = \mu I + \lambda \operatorname{div}_M(\overrightarrow{b}^k - \mathbb{V}^k)$$
(2.34)

another one is the equation (2.21) for CCV segmentation.

$$\Delta_M \varphi^{k+1} = \frac{\mu}{\lambda} r^k + \operatorname{div}_M(\mathbb{V}^k - \overrightarrow{b}^k)$$
(2.35)

Since above two equations are defined on a triangulated surface, we can not use the fast solver, fast fourier transform (FFT), to solve it as we deal with the same type of equations in 2D regular domain. One possible method is by using the discretization of Δ_M given by Meyer, Desbrun, Xu et al.[68, 31, 102], then use Gauss-Seidel, conjugate gradient to solve them. However, the approximation of Laplace-Beltrami operator on an arbitrary triangulated surface depends on the quality of the triangle mesh. To avoid discretizing the second order differential operator, we are here proposing to use finite element methods to solve equations (2.18) and (2.21).

We choose the linear elements $\{e_i\}_{i=1}^N$ on the triangle mesh $\{V = \{p_i\}_{i=1}^N, T = \{T_i\}_{i=1}^L\}$, such that $e_i(p_j) = \delta_{i,j}$ and write $S = Span_{\mathbb{R}}\{e_i\}_{i=1}^N$. Then the discrete version of the continuous variational problem of (2.18) is to find a $\varphi^{k+1} \in S$, such that

$$\mu \sum_{l} \int_{T_{l}} \varphi^{k+1} e_{j} + \lambda \sum_{l} \int_{T_{l}} \nabla_{M} \varphi^{k+1} \nabla_{M} e_{j} = \sum_{l} \int_{T_{l}} \Theta^{k} e_{j}, \ \forall e_{j} \in S.$$
(2.36)

where $\Theta^k = \mu I + \lambda \operatorname{div}_M(\overrightarrow{b}^k - \mathbb{V}^k).$

If we write

$$\begin{cases} \varphi^{k+1} = \sum_{i}^{N} x_{i} e_{i}, \quad \Theta^{k} = \sum_{i}^{N} \theta_{i} e_{i} \\ Q = (a_{ij})_{N \times N}, a_{ij} = \sum_{l} \int_{T_{l}} \nabla_{M} e_{i} \nabla_{M} e_{j} \\ K = (b_{ij})_{N \times N}, b_{ij} = \sum_{l} \int_{T_{l}} e_{i} e_{j} \end{cases}$$
(2.37)

and we also write $\varphi^{k+1} = (x_1, \cdots, x_N)^t$ and $\Theta^{k+1} = (\theta_1, \cdots, \theta_N)^t$ with abused notations, then to solve φ^{k+1} is equivalent to solving the following linear equations:

$$(\mu K + \lambda Q)\varphi^{k+1} = K\Theta \tag{2.38}$$

One fact we would like to point out here is:

Remark 2.3.2 K is a symmetric positive definite sparse matrix and Q is a symmetric nonnegative definite sparse matrix.

[Proof]: Symmetry of Q, K is easy to see. For any $f = (f_1, \dots, f_N)^t, g = (g_1, \dots, g_N)^t$, if we also write $f = \sum f_i e_i$ and $g = \sum g_i e_i$, then $fKg^t = \int_M fg$ and $fQg^t = \int_M \nabla_M f \nabla_M g$. So K is positive definite and Q is nonnegative definite. \Box

Therefore, when μ and λ are both positive, the matrix $(\mu K + \lambda Q)$ is a symmetric positive definite sparse matrix. The solution φ^{k+1} of (2.38) can be obtained by using conjugate gradient or Gauss-Seidel.

Similarly, the discrete version of the continuous variational problem of (2.21) is to find a $\varphi^{k+1} \in S$, such that

$$\sum_{l} \int_{T_l} \nabla_M \varphi^{k+1} \nabla_M e_j = -\sum_{l} \int_{T_l} \Gamma^k e_j, \ \forall e_j \in S.$$
(2.39)

where $\Gamma^k = \frac{\mu}{\lambda} r^k + \operatorname{div}_M(\mathbb{V}^k - \overrightarrow{b}^k).$

If we write $\varphi^{k+1} = \sum_{i}^{N} x_{i} e_{i}$, $\Gamma^{k} = \sum_{i}^{N} \gamma_{i} e_{i}$, the solution φ^{k+1} of (2.21) is equivalent to solving the following linear equation:

$$Q\varphi^{k+1} = -K\Gamma^k \tag{2.40}$$

which can be solved by the Gauss-Seidel method.

2.4 Experimental Results and Applications

In this section, several examples will be given to demonstrate advantages of the intrinsic method. The intrinsic method can provide us a robust and efficient method to study image problems on surfaces. It can easily handle surfaces with different complexity, different topologies. Moreover, we will further show two applications of our intrinsic method of image processing on surfaces. All algorithms are written in C++ and all experiments are ran on a PC with a 2.0GHz CPU.

	intrinsic method	conformal parametrization	level set
	CCV by split Bregman	parametrization: $126.63s +$	implicit representation with
M1	18.61s	CCV by split Bregman in 0.55s	$65\times162\times99$ grid points in 5.92s
			+ CCV by gradient descent in 84.44s
	CCV by split Bregman	parametrization: $602.91s +$	implicit representation with
M2	18.94s	CCV by split Bregman in 0.56s	$143\times350\times227$ grid points in 29.55s
			+ CCV by gradient descent in 756.4s

Table 2.2: The computation cost of different methods.

2.4.1 Comparison with other approaches

The intrinsic method can efficiently solve variational problems directly on surfaces and does not need preprocessing. Due to the natural extension of differential operators on surfaces, the fast algorithms for variational problems in Euclidean cases can also be easily adapted on surfaces by this intrinsic method. To demonstrate these advantages of our intrinsic method, we here compare our method with level set method and conformal parametrization method. We test the CCV segmentation model on following two surfaces M1, M2 with characters by the intrinsic method, the level set method [55, 66] and the conformal parametrization method [63] respectively. In Figure 2.1, Surface M2 is a cortical surface in human's brain¹ and surface M1 is a smoothing version of surface M2. Both surfaces have 39994 vertices. The computation cost comparison is listed in table 2.2.

From Table 2.2, we can observe two facts as follows:

1. Computation cost under surface structure variance:

For the conformal parametrization method and level set method, they both

¹Meshes are provided by the public available database ADNI at LONI



Figure 2.1: CCV segmentation results on two surfaces with different complexity. The first row: (a1), (b1) two views of characters image I on surface M1, (c1) the initial curve for CCV segmentation on M1, (d1), (e1) two views of the CCV segmentation results φ . The second row: (a2),(b2) two views of characters image I on surface M2, (c2) the initial curve for CCV segmentation on M2, (d2), (e2) two views of the CCV segmentation results φ .

depend on the complexity of surfaces. More specifically, it needs more time to obtain the conformal parametrization if the surface geometry is farther from the sphere. Similarly, the level set method also needs more data to represent a surface with more complicated structure, which requires more time to solve variational problems. However, the computation of the intrinsic method is fast and stable under surface structure variance.

2. Adaptability of fast algorithms:

For the conformal parametrization method, once the parametrization is obtained, one can easily transfer the surface variational problems into 2D Euclidean cases, then several fast algorithms can be also applied. For the level set method, since the surface gradient operator is computed by projection and expression of surface divergence by level set function is very complicated, the generalization of fast algorithms, like split Bregman or dual method is not straightforward. Meanwhile, the data size of implicit surface representation will consume more computation cost. However, as we discussed in section 2.3.3, fast algorithms in Euclidean cases can be easily adapted in surface cases by the intrinsic method.

2.4.2 Further demonstration on high genus surfaces and open surfaces

In many fields such as computer graphics, geometry modeling, medical imaging, computational anatomy, 3D cartoon, it is also necessary to process high genus surfaces or open surfaces. Usually, to find a parametrization of a high genus surface is not so easy, one has to cut the surface into several patches [47, 65], then process these patches separately. This artificial cutting of patches and separately processing may introduce numerical inaccuracy on the cutting edges. In addition, the parametrization method and level set method have their own limitations to study open surfaces, specially open surfaces with topological nontrivial boundaries. However, the intrinsic method can easily handle high genus surfaces and open surfaces as surfaces with spherical topology.

Here, we demonstrate this advantage of the intrinsic method in several synthetic examples. In Figure 2.2, we take a one handled cup ² with 25075 vertices as a ground surface and show ROF denoising results of the Lena image with Gaussian noise $\sigma = 40$. As we discussed in Section 2.3, the split Bregman iterations and Chambolle's dual projection method can be applied to solve ROF denoising models on surfaces. We show denoising results obtained by split Bregman and Chambolle's dual projection method respectively in Figure 2.2. In addition, we also apply our algorithms for the CCV segmentation model on surfaces to

 $^{^2 \}mathrm{One}$ handled cup is obtained from the public available database SHARP3D



Figure 2.2: ROF denoising results of the Lena image I on a 25075 vertices cup surface with split Bregman iterations and dual method. (a) the clear lena image. (b) the noise image with Gaussian noise σ =40. (c), (d) a denoising result φ by split Bregman iterations with $\lambda = 0.05, \mu = 1000$ in 17.73 seconds and its corresponding residual $I - \varphi$. (e), (f) a denoising result φ by dual method with $\mu = 10$ in 35.46 seconds and its corresponding residual $I - \varphi$.



Figure 2.3: CCV segmentation results on a 25075 vertices cup surface with split Bregman iterations and Chambolle's projection method. (a) the cameraman image and the initial segmentation curve marked by the red contour. (b), (c) the CCV segmentation result obtained by split Bregman method with $\lambda = 8, \mu = 50$ in 14.57 seconds and the corresponding edges marked by red contours in the original image. (d), (e) the CCV segmentation result obtained by Chambolle's projection method with $\mu = 0.1, \theta = 400$ in 73.34 seconds and the corresponding edges marked by red contours in the original image.

the same one handled cup and a torus. The segmentation results are showed in Figure 2.3 and Figure 2.4. Moreover, to show the advantage of dealing with open surfaces with the intrinsic method, we show CCV segmentation results in Figure 2.5 on a human hand surface ³ and Figure 2.6 on a half torus, which both are open surfaces. To summarize, our intrinsic method can provide a robust and efficient approach to study image problems on surfaces, whenever surfaces are closed or open, with genus zero or high genus, with simple or complicated geometric structure.



Figure 2.4: CCV segmentation results on a 65536 vertices torus with split Bregman iterations and Chambolle's projection method. (a) the cameraman image and the initial segmentation curve marked by the red contour. (b1), (c1), (d1), (e1) two views of the CCV segmentation result obtained by split Bregman method with $\mu = 10, \lambda = 50$ in 42.95 seconds and two views of the corresponding edges marked by red contours in the original image. (b2), (c2), (d2), (e2) two views of the CCV segmentation result obtained by Chambolle's projection method with $\mu = 0.1, \theta = 1000$ in 162.99 seconds and the corresponding edges marked by red contours on the original image.

³This model is provided by the public available database AIM@SHAPE Shape Repository



Figure 2.5: CCV segmentation results on a 58875 vertices open hand surface with split Bregman iterations and Chambolle's projection method. (a1), (a2) two views the image and the initial segmentation curve marked by the red contour. (b1), (c1), (d1), (e1) two views of the CCV segmentation result obtained by split Bregman method with $\mu = 5, \lambda = 50$ in 29.74 seconds and two views of the corresponding edges marked by red contours in the original image. (b2), (c2), (d2), (e2) two views of the CCV segmentation result obtained by Chambolle's projection method with $\mu = 0.1, \theta = 1000$ in 184.94 seconds and the corresponding edges marked by red contours on the original image.



Figure 2.6: CCV segmentation results on a 31247 vertices half double torus with split Bregman iterations and Chambolle's projection method. (a) the characters image with Gauss noise $\sigma = 100$ and the initial segmentation curve marked by the red contour. (b), (c) the CCV segmentation result obtained by split Bregman method with $\lambda = 100, \mu = 50$ in 21.33 seconds and the corresponding edges marked by red contours in the original image. (d), (e) the CCV segmentation result obtained by Chambolle's projection method with $\mu = 0.1, \theta = 1000$ in 108.4 seconds and the corresponding edges marked by red contours in the original image.

2.4.3 Applications

A. Geometric processing

An interesting application of ROF denoising is geometric processing, namely, surface denoising or geometric processing [25, 29, 30, 3, 35, 32, 63]. Given a surface $M \subset \mathbb{R}^3$, there are three coordinate functions (f_1, f_2, f_3) on M, namely we have the embedding:

$$\overrightarrow{f} = (f_1, f_2, f_3): M \longrightarrow \mathbb{R}^3$$

$$p \longmapsto (f_1(p), f_2(p), f_3(p)) \tag{2.41}$$

It is natural to view the three coordinate functions (f_1, f_2, f_3) as three image functions on the surface M. A noisy surface is a perturbation in the geometry of the surface, namely, we can view the noisy surface $\overrightarrow{f_{noise}}$ as $\overrightarrow{f_{clean}} + \overrightarrow{noise}$ with $\overrightarrow{noise} \in \mathcal{N}(0, \sigma) \times \mathcal{N}(0, \sigma) \times \mathcal{N}(0, \sigma)$. Thus each coordinate function of the noisy surface can be viewed as a noisy image on the surface, then we can study the geometry processing via the surface coordinate functions. As an example, we naively consider surface ROF denosing model on each coordinate function as an approach to study surface denosing. Figure 2.7 shows two preliminary results of the surface denoising. The first row in Figure 2.7 is the surface denoising of a cube with Gaussian noise $\sigma = 0.1$. The edge preserving property of the total variation can be observed from the denoising result. The second row is a denoising result of a human's cortical surface⁴.

B. Cortical parcellation

Image segmentation techniques are quite useful in image analysis on the 2D plane. For 3D surface analysis, image segmentation technique can also be used to detect certain special parts of the given surface. For instance, in anatomical

⁴This model is provide by the public available database ADNI at LONI

brain structure analysis, sulci/guri detection for cortical surfaces is important [64, 90, 57]. However, in most cases, cortical surfaces have many very deep and closed folding parts. These folding parts might restrict us to easily and efficiently find parameterization or use implicit method. With our intrinsic image processing on surfaces method, we can directly process the image segmentation on cortical surfaces without any preprocessing. This will help us deal with surfaces with complex structures. For the sulci/guri detection problem, we can view the mean curvature of cortical surfaces as an image on the surfaces, which can be obtained by the algorithms given in [68], then apply CCV segmentation on the surface mean curvature. The CCV segmentation result will provide us a promising cortical parcellation. In Figure 2.8, we show the segmentation results of two type of cortical surfaces. The first row is a human's cortical surface⁵ and the second row is a vervet's cortical surface⁶. Both surfaces are with deep and narrow sulcal regions.

2.5 Conclusions and Future Work

In this work, we use differential geometric techniques to study intrinsical image processing on surfaces. We generalize the total variation concept on surfaces and show it is also a suitable regularizing term when we study image processing on surfaces. Furthermore, we take ROF denoising model and CV segmentation model as two examples to illustrate our intrinsic method. As an advantage of the intrinsic method, we show the adaptability of fast algorithms in Euclidean spaces to the total variational related problems on surfaces by using the intrinsic method. Specifically, we implement the split Bregman method and Chambolle's

⁵This model is obtained from the public available database ADNI at LONI

⁶This model is provided by Dr. Scott Fears



Figure 2.7: the first column: clean surfaces. the second column: noise surfaces with Gaussian noise $\sigma = 0.1$. the third column: denoised results obtained by split Bregman method.



Figure 2.8: Top: Two different views of CCV segmentation on the mean curvature of a human's cortical surface. Bottom: Two different views of CCV segmentation on the mean curvature of a vervet's cortical surface. Surfaces are color coded with their mean curvature and the red contours mark the boundary of the sulcal and gyral regions.

dual projection method as two examples. The intrinsic method is a very general approach to study variational problems, differential equations and image processing on surfaces. This technique can be further extended to study diffusion equations, motions of curves, and other surface PDEs or variation related problems on surfaces. In the future, we will explore along this direction and demonstrate more applications of this intrinsic geometric technique.
Part II

Laplace-Beltrami Eigen-Geometry

CHAPTER 3

Mathematical Background of Laplace-Beltrami Eigen-Geometry

In this chapter, we will give a brief introduction to eigen-systems of the Laplace-Beltrami operator on two dimensional surfaces and their numerical computation by the finite element method on triangulated surfaces. More details about eigen-systems of the Laplace-Beltrami operator can be found in [23, 50].

3.1 Theoretical Background

Let (M, g) be a two dimensional closed Riemannian manifold. For any point $p \in M$, we write a local representation of the metric $g(p) = (g_{ij}(x))_{i,j=1,2}$. Then, for any smooth function $u \in C^{\infty}(M)$, the Laplace-Beltrami (LB) operator is defined by:

$$\Delta_M \phi = \operatorname{div}_M \nabla_M(\phi) = \frac{1}{\sqrt{G}} \sum_{i=1}^2 \frac{\partial}{\partial x_i} (\sqrt{G} \sum_{j=1}^2 g^{ij} \frac{\partial \phi}{\partial x_j})$$
(3.1)

where (g^{ij}) is the inverse matrix of (g_{ij}) and $G = \det(g_{ij})$

By the knowledge of differential geometry, one can easily check that the Laplace-Beltrami operator is not dependent on the choice of local coordinates. More importantly, the Laplace-Beltrami operator is an elliptic operator, so it has discrete spectrum. We denote the set of eigenvalues as $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$

and the corresponding eigenfunctions as ϕ_1, ϕ_2, \cdots such that

$$\Delta_M \phi_n = -\lambda_n \phi_n, \ n = 1, 2, \cdots \tag{3.2}$$

 $\{\lambda_i, \phi_i\}_{i=1}^{\infty}$ is called an LB eigen-system of (M, g), typically, $\{\lambda_i\}_{i=1}^{\infty}$ is called the LB spectrum of (M, g).

An important fact of the LB eigen-system is that two eigenfunctions with different eigenvalues are perpendicular to each other. Namely:

Fact 3.1.1 Let ϕ_i, ϕ_j be two eigenfunctions of the Laplace-Beltrami operator on a surface (M, g) with eigenvalue λ_i, λ_j respectively. If $\lambda_i \neq \lambda_j$, then

$$\langle \phi_i, \phi_j \rangle_M = \int_M \phi_i(x) \phi_j(x) \mathrm{d}v(x) = 0 \tag{3.3}$$

Thus, we can choose eigenfunctions $\{\phi_i\}_{i=1}^{\infty}$ to be an orthonormal basis. From now on, we always choose $\{\phi_i\}_{i=1}^{\infty}$ in an eigen-system $\{\lambda_i, \phi_i\}_{i=1}^{\infty}$ to be an orthonormal basis. Equivalently, an LB eigen-system can be also given by:

$$\lambda_{1} = \int_{M} |\nabla_{M}\phi_{1}|^{2} dv = \inf_{\mathcal{C}^{\infty}(M)} \{ \frac{\int_{M} |\nabla_{M}\phi|^{2} dv}{\int_{M} \phi^{2} dv} \}$$
$$\lambda_{n} = \int_{M} |\nabla_{M}\phi_{n}|^{2} dv = \inf_{\mathcal{B}^{n-1}} \{ \frac{\int_{M} |\nabla_{M}\phi|^{2} dv}{\int_{M} \phi^{2} dv} \}$$
(3.4)

where $\mathcal{B}^{n-1} = \{ \phi \in \mathcal{C}^{\infty}(M) \mid \langle \phi, \phi_j \rangle_M = 0, \ j = 1, \cdots, n-1 \}.$

The Laplace-Beltrami operator defined above is actually a generalization of the standard Laplace operator in Euclidean space and on a sphere. For instance, the following two examples are two special cases of the above general defined Laplace-Beltrami operator and its eigen-system.

1. On the interval $I = [-\pi, \pi]$: the Eigensystem is given by: $\frac{d^2}{dx^2}\phi_n = -\lambda_n\phi_n$, $n = 1, 2, \cdots$ $\{\phi_n\} = \{\sin mx, \cos mx\}_{m \in \mathbb{N}}$ is a Fourier basis of $L^2([\pi, \pi])$. 2. On the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$:

Under spherical coordinates, Eigensystem is given by:

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial}{\partial\theta} \phi_l^m) + \frac{1}{\sin\theta^2} \frac{\partial^2}{\partial\phi^2} \phi_l^m = -\lambda_l^m \phi_l^m, \begin{cases} l = 0, 1, 2, \cdots \\ m = -l, \cdots, l \end{cases}$$

where $\phi_l^m(\theta, \phi) = N e^{\sqrt{-1}m\phi} P_l^m(\cos \theta)$, here ϕ_l^m is called a spherical harmonic function of degree l and order m, is an associated Legendre function, N is a normalization constant, and θ and ϕ represent colatitude and longitude, respectively. $\{\phi_l^m, l = 0, 1, 2, \cdots, m = -l, \cdots, l\}$ is a basis of $L^2(S^2)$.

Since the Laplace-Beltrami operator is a complete intrinsic geometric operator on the surface itself, it is clear that the LB eigen-system of a given surface is also completely intrinsic. In other words, the LB eigen-system of a surface is isometric invariant, in particular, it is rotation and translation invariant.

It is well known that the Fourier basis is very useful for functional analysis in \mathbb{R}^n and spherical harmonic functions are quite useful for functional analysis on the sphere. Similarly, eigen-systems of the Laplace-Beltrami operator are also a very powerful tool for function analysis on M. For instance, eigen-system is highly related to the heat equation on a surface as follows:

Let M be a surface. The heat equation on M is given by:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \Delta_M u(t,x) \\ u(x,0) = f(x) \\ u(x,t) = 0 \quad \forall x \in \partial M \end{cases}$$
(3.5)

where $u \in \mathcal{C}^{\infty}([0,\infty) \times M)$ and $f \in \mathcal{C}^{\infty}(M)$. the heat kernel K(t,x,y) is the fundamental solution of the heat equation (3.5) with the following properties:

1. K(t, x, y) is C^1 in t and C^2 in x and y;

- 2. K(t, x, y) solves the equation $\frac{\partial K(t, x, y)}{\partial t} = \Delta_{M, x} K(t, x, y)$ and K(t, x, y) = 0 $\forall x \in \partial M;$
- 3. For any compactly supported function f on M, $\lim_{t\to 0} \int_M K(t, x, y) f(y) dv(y) = f(x)$.

With the heat kernel K(t, x, y), the solution of the heat equation (3.5) can be simply given by:

$$u(t,x) = \int_M K(t,x,y)f(y)\mathrm{d}v(y) \tag{3.6}$$

Therefore, the essential step to solve the heat equation on surfaces is to construct the heat kernel. Theoretically, the heat kernel of M can be represented as follows:

Theorem 3.1.1 (Sturm–Liouville decomposition) Let's (M, g) be a Riemannian surface and $\{\lambda_n, \phi_n\}$ be an eigen-system of the Laplace-Beltrami operator of (M, g). Then the heat kernel uniquely exist and can be given as follows:

$$K(t, x, y) = \sum_{i} e^{-\lambda_i t} \phi_i(x) \phi_i(y)$$
(3.7)

where $\sum_{i} e^{-\lambda_{i}t} \phi_{i}(x) \phi_{i}(y)$ is absolute and uniform convergent when t > 0.

The Sturm-Liouville decomposition clearly illustrates the strong relation of LB eigen-systems and the heat equation on its ground surface. Moreover, besides functional analysis on surfaces, another big advantage of LB eigen-systems can be further used to detect geometry of the ground surface. The following heat trace expansion will demonstrate how LB spectrum will reflect geometric information of surfaces.

Write $Z(t) = \int_M K(t, x, x) dv$ as the heat trace of the Laplace-Beltrami operator on (M, g). Since $\{\phi_i\}$ is an orthonormal basis, the heat trace is also equal

$$Z(t) = \int_{M} K(t, x, x) \mathrm{d}s = \int_{M} \sum_{i} e^{-\lambda_{i} t} \phi_{i}(x) \phi_{i}(y) \mathrm{d}s = \sum_{i} e^{-\lambda_{i} t}$$
(3.8)

Theorem 3.1.2 (heat trace asymptotical expansion) Let (M, g) be a Riemannian surface with boundary B. When $t \to 0^+$, the heat trace Z(t) has the following asymptotical expansion

$$Z(t) = \frac{1}{4\pi t} \left(\sum_{i=0}^{\infty} c_i t^{i/2}\right)$$
(3.9)

where c_0, c_1, c_2 were first computed by H. McKean and I. Singer [67] as follows:

$$c_0 = area(M), \quad c_1 = -\frac{\sqrt{\pi}}{2} length(B), \quad c_2 = \frac{1}{3} \int_M K - 1/6 \int_B J \quad (3.10)$$

where K is the Gauss curvature of M and J is the mean curvature of B in M. Moreover, if M is a closed surface with Euler number $\chi(M)$, then $c_2 = 2\chi(M)/3$

More results about the relation between surface geometry and its eigenvalues can be found in [67, 84, 97].

From the above asymptotical formula, we can see how to use eigenvalue to study surface geometry. However, eigenvalues are only one part of the eigensystem, one can expect that more surface geometric information can be detected by using eigenvalues together with eigenfunctions. In fact, P.Bérard, G. Besson, S. Gallot ([2]) have the following theoretical result:

Given an orthonormal basis $\alpha = \{\phi_n^{\alpha}\}$ of eigenfunctions of the Laplace-Beltrami operator of (M, g), for each point $x \in M$, one can have a convergent sequence $\{e^{-\lambda_i t/2}\phi_i^{\alpha}(x)\}_{i\geq 1}$ corresponding to an element in the standard sequence Hilbert space l^2 . Now consider the following map:

$$I_{t,g}^{\alpha}: M \to l^2, \ I_{t,g}^{\alpha}(x) = \{e^{-\lambda_i t/2}\phi_i^{\alpha}(x)\}_{i \ge 1}$$
 (3.11)

to:

Here l^2 is the standard sequence Hilbert space. P.Bérard, et al have the following key theorem:

Theorem 3.1.3 For any given surface (M,g), any given basis $\alpha = \{\phi_n^{\alpha}\}$ and any t. the map $I_{t,g}^{\alpha}$ is an embedding.

By this embedding theorem, one can represent all of surfaces (it is also right for all closed manifolds) in a standard space l^2 . An important fact here is that a surface is completely determined by its LB eigen-system. This inspires us to use eigenfunctions instead of only eigenvalues of surface to analyze shapes. The essential question here is how do we introduce certain models to use LB eigenfunctions to detect the geometry of corresponding surfaces. This is a widely open problem. We would like to first discuss how to compute the LB eigen-system for a given triangulated surface in the following section. Then, based on our work in [88, 87, 56, 57], we will discuss how to use LB eigen-system to study surface global and local geometric information and their application to computational anatomy.

3.2 Numerical Computation: The Finite Element Method on Surfaces

Numerically, we use the finite element method (FEM) to compute eigen-systems of the LB operator. For any closed surface (M, g), to solve the original LB eigensystem problem, is equivalent to solve the following weak version of the original problem. Namely, for any $u \in C^{\infty}(M)$, ϕ should satisfy:

$$\int_{M} \Delta_{M} \phi u dv = -\lambda \int_{M} \phi u dv$$
$$\implies \int_{M} (\nabla_{M} \phi, \nabla_{M} u)_{M} dv = \lambda \int_{M} \phi u dv \qquad (3.12)$$

In a local coordinate chart $\{U, x = (x^1, x^2)\}$ of (M, g), we locally represent $g = (g_{ij}(x))_{i,j=1,2}$ and denote $(g_{ij}(x))_{i,j=1,2}^{-1}$ by $(g^{ij}(x))_{i,j=1,2}$. We further denote coordinate vector fields by $\partial_i = \frac{\partial}{\partial x_i}$, i = 1, 2. Then, the following representation can be obtained:

$$\begin{cases} \nabla_M \phi |_U = \sum_{i,j=1,2} g^{ij} \frac{\partial \phi}{\partial x^i} \partial_j \\ (\nabla_M \phi, \nabla_M u)_M = \sum_{i,j=1,2} g_{ij} \frac{\partial \phi}{\partial x^i} \frac{\partial u}{\partial x^j} \end{cases}$$
(3.13)

In particular, we mainly focus on studying surfaces in \mathbb{R}^3 . Therefore, these surfaces metric can be viewed as induced metric from \mathbb{R}^3 (see section 1.1), and tangent vectors $\nabla_M \phi$, $\nabla_M u$ can be viewed as vectors in \mathbb{R}^3 . Thus:

$$(\nabla_M \phi, \nabla_M u)_M = (\nabla_M \phi, \nabla_M u)_{\mathbb{R}^3} = \nabla_M \phi \cdot \nabla_M u \tag{3.14}$$

Let M be a surface represented by triangular mesh $\{V = \{p_i\}_{i=1}^N, T = \{T_l\}_{l=1}^L\}$, where p_i means the i-th vertex and T_l means the l-th triangle. To implement the finite element method on the given triangle mesh, we choose linear elements $\{\psi_i^h\}_{i=1}^N$, such that $\psi_i^h(v_j) = \delta_{i,j}$ and write $S^h = Span_{\mathbb{R}}\{\psi_i^h\}_{i=1}^N$. Then the discrete version of the continuous variational problem is to find a $\phi^h \in S^h$, such that

$$\sum_{l} \int_{T_{l}} \nabla_{M} \phi^{h} \cdot \nabla_{M} \psi^{h}_{i} = \lambda^{h} \sum_{l} \int_{T_{l}} \phi^{h} \psi^{h}_{i}, \ \forall \psi^{h}_{i} \in S^{h}, i = 1, \cdots, N$$
(3.15)

Therefore, the key of the remaining computation is to obtain numerical estimation of the above two types of integral in (3.15) on each triangle, which can be approximated as follows:

Consider a triangle $T_l = \{p_0, p_1, p_2\}$ with two functions $\phi = \{\phi_0, \phi_1, \phi_2\}$ and $\psi = \{\psi_0, \psi_1, \psi_2\}$ defined on each vertex respectively. Any point $p \in T_l$, the first order linear interpolation of ϕ and ψ in T_l can be given by

$$\begin{cases} p = x^{1}(p_{1} - p_{0}) + x^{2}(p_{2} - p_{0}) + p_{0} \\ \phi(p) = x^{1}(\phi_{1} - \phi_{0}) + x^{2}(\phi_{2} - \phi_{0}) + \phi_{0} \\ \psi(p) = x^{1}(\psi_{1} - \psi_{0}) + x^{2}(\psi_{2} - \psi_{0}) + \psi_{0} \end{cases}$$
(3.16)

where $0 \leq x^1, x^2, x^1 + x^2 \leq 1$ are the barycenter coordinates.

We first compute the integral on the right hand side of the formula in (3.15) as follows:

$$\int_{T_l} \phi(p)\psi(p) dv = \int_0^1 \int_0^{1-x^1} \phi(p)\psi(p) dx_2 dx_1$$
(3.17)

Next, we approximate another type integral of the formula in (3.15). Since we use the first order finite element method, the gradient $\nabla_{T_l}\phi$ and $\nabla_{T_l}\psi$ will be two constant vectors on the triangle T_l . Therefore:

$$\int_{T_l} \nabla_{T_l} \phi \cdot \nabla_{T_l} \psi dv = area(T_l) (\nabla_{T_l} \phi \cdot \nabla_{T_l} \psi)$$
(3.18)

The only two terms we need to estimate are $\nabla_{T_l}\phi$ and $\nabla_{T_l}\psi$, which can be given as follows (see Section 1.2):

On the triangle T_l , we have $\partial_{x^1} = p_1 - p_0$, $\partial_{x^2} = p_2 - p_0$, and the metric matrix of T_l would be:

$$g = (g_{i,j})_{i,j=1,2} = \begin{pmatrix} \partial_{x^1} \cdot \partial_{x^1} & \partial_{x^1} \cdot \partial_{x^2} \\ \partial_{x^2} \cdot \partial_{x^1} & \partial_{x^2} \cdot \partial_{x^2} \end{pmatrix}$$
(3.19)

where \cdot is the dot product in \mathbb{R}^3 . Then,

$$\nabla_{T_l} \phi = \sum_{i,j}^2 g^{ij} \frac{\partial \phi}{\partial x^j} \partial_{x^i} = \sum_{i,j}^2 g^{ij} (\phi_j - \phi_0) (p_i - p_0)$$
$$= (\phi_1 - \phi_0, \phi_2 - \phi_0) \begin{pmatrix} \partial_{x^1} \cdot \partial_{x^1} & \partial_{x^1} \cdot \partial_{x^2} \\ \partial_{x^2} \cdot \partial_{x^1} & \partial_{x^2} \cdot \partial_{x^2} \end{pmatrix}^{-1} \begin{pmatrix} p_1 - p_0 \\ p_2 - p_0 \end{pmatrix} (3.20)$$

Similarly, we have:

$$\nabla_{T_{i}}\psi = \sum_{i,j}^{2} g^{ij} \frac{\partial \psi}{\partial x^{j}} \partial_{x^{i}} = \sum_{i,j}^{2} g^{ij} (\psi_{j} - \psi_{0}) (p_{i} - p_{0})$$

$$= (\psi_{1} - \psi_{0}, \psi_{2} - \psi_{0}) \begin{pmatrix} \partial_{x^{1}} \cdot \partial_{x^{1}} & \partial_{x^{1}} \cdot \partial_{x^{2}} \\ \partial_{x^{2}} \cdot \partial_{x^{1}} & \partial_{x^{2}} \cdot \partial_{x^{2}} \end{pmatrix}^{-1} \begin{pmatrix} p_{1} - p_{0} \\ p_{2} - p_{0} \end{pmatrix} (3.21)$$

Now we can go back to solve the problem 3.15. Let h_l be the diameter of the triangle T_l and $h = \max \{h_l \mid l = 1, \dots, L\}$. If we write

$$\begin{cases} \phi^{h} = \sum_{i}^{N} x_{i} \psi_{i}^{h} \\ A^{h} = (a_{ij})_{N \times N}, a_{ij} = \sum_{l} \int_{T_{l}} \nabla_{M} \psi_{i}^{h} \nabla_{M} \psi_{j} \\ B^{h} = (b_{ij})_{N \times N}, b_{ij} = \sum_{l} \int_{T_{l}} \psi_{i}^{h} \psi_{j}^{h} \end{cases}$$
(3.22)

then the discrete variational problem is equivalent to the generalized matrix eigenproblem:

$$\begin{cases} A^h x = \lambda^h B^h x, \text{ where } x = (x_1, \cdots, x_N)^T \\ \phi^h = \sum_i^N x_i \psi_i^h \end{cases}$$
(3.23)

Therefor, the solutions of the LB eigen-system problem can be approximated by solving the above generalize matrix eigen-problem, which can be obtained by a variety of linear algebra packages. We here use matlab to solve the matrix eigen-problem in (3.23). In figure 3.1, we show several examples of surface eigenfunctions obtained by above finite element method. It is clear to see that this general approach is very robust to surfaces with boundaries, complicated topology or complicated geometry.

Theoretically, the following two estimations can be obtained by the numerical analysis of the FEM given in [94].

Theorem 3.2.1 Let (ϕ_n^h, λ_n^h) be the eigen-system computed with FEM, then we have:

$$||\phi_n - \phi_n^h|| \leqslant Ch^2 \lambda_n \tag{3.24}$$

$$\lambda_n \leqslant \lambda_n^h \leqslant \lambda_n + 2\delta h^2 \lambda_n^2 \tag{3.25}$$

where (ϕ_n, λ_n) are the true eigen-system, and C and δ are constants.







homer, 1st eigenfunction



hippo, 1st eigenfunction





a patch, 1st eigenfunction



sphere, 10th eigenfunction



homer, 10th eigenfunction



hippo, 10th eigenfunction









sphere, 20th eigenfunction



homer, 20th eigenfunction



hippo, 20th eigenfunction





a patch, 20th eigenfunction

Figure 3.1: LB Eigenfunctions obtained by FEM are color coded on surfaces.

This theorem tells us that the accuracy of eigenfunctions obtained by finite element method is decreasing with increasing the order of eigenvalues. The high order of eigenvalues and eigenfunctions are, the less accuracy they will be. There are at least two approaches to improve the numerical accuracy. First, we can upsample the triangle mesh for a given surface. It often can be done by adding the middle point of each triangle as a new vertex, thus constructing four new triangles in each original triangle. This means we decrease the triangle length h in theorem 3.2.1 to increase the numerical accuracy. Second, we can use high order finite element method to approach high accuracy solutions [94, 79].

CHAPTER 4

Laplace-Beltrami nodal counts: From "How to hear the shape of a drum?" to "How to count the shape of a drum?"

4.1 Introduction

The analysis of 3D shapes is an important problem in medical imaging. By studying shapes, we can obtain detailed information about morphometry changes of anatomical structures. Recently there has been increasing interests in using the eigenvalues of the Laplace-Beltrami operators to study shapes [79, 71]. Features based on eigenvalues, however, have limitations in resolving isospectral shapes. To overcome this difficulty, we propose in this work a new signature derived from the nodal counts of eigenfunctions and demonstrate its advantage in classifying medical shapes.

Using the eigenvalues of the Laplace-Beltrami operator, the shape DNA feature was proposed in [79] as a vector of eigenvalues ordered according to their magnitude. The shape DNA feature has been successfully applied to the classification of anatomical structures [71]. One limitation of the shape DNA feature, however, is that it cannot resolve so called isospectral shapes with the same eigenvalues. There were various examples of isospectral surfaces created by mathematician [69, 96, 46, 22, 36, 81]. In practice, we have also observed shapes with quite different geometry but very similar distribution of eigenvalues. To address this ambiguity in the shape DNA feature, we propose here a new signature derived from the eigenfunctions of the Laplace-Beltrami operator. This new feature is intrinsically defined over the surfaces and is pose and scale invariant. Using the nodal counts of the eigenfunctions, this feature provides a compact representation of the new geometric information that is not described by the eigenvalues. In our experiments, we show that it has the ability of resolving the ambiguity in the shape DNA feature. In addition, we illustrate its potential to surface classification.

4.2 Shape DNA and Isospectral Surfaces

As we discussed in section 3.1, the asymptotical expansion of the heat trace illustrates that the spectrum of the surface Laplace-beltrami operator can be used to detect geometric information of the ground surface. Recently, M. Reuter, F. Wolter, N. Peinecke [79] use the LB spectrum, which is also called shape DNA in their paper, to obtain promising results of surface classification. However, only using LB spectrum can not characterize the whole information of surface geometry. In other words, there are two surfaces sharing the same LB eigenvalues, but they have different geometry. These surfaces are called isospectral surfaces. The first example of isospectral surfaces was found by Milnor [69]. He found two flat tori in \mathbb{R}^{16} with the same spectrum but not isometric. After Milnor's work, many other pairs of isospectral, non-isometric manifolds [96]. Gordan et al. [46] and Chapman [22] used Sunada's method to construct isospectral domains in \mathbb{R}^2 . On the discrete version of Laplacian, Fisher gave a few examples of distinct graphs with the same spectrum [36], Sunda's method also be used to obtain isospectral discrete graphs by Brooks in [81]. In Figure 4.1, we show two groups of isospectral 2 dimensional manifolds. Each pair shares exactly the same LB spectrum. However, it is clearly that they have different geometries. These isospectral phenomenons do not only exist in synthetic data, but also can



Figure 4.1: two groups of isospectral surfaces. (a) isospectral shapes found by Gordon, Webb, and Wolpert; (b) isospectral shapes found by P.Bérard

be found in our physical world. In practice, we show in Figure 4.2 two types of anatomical structures, putamen and caudate surfaces, in human's brain. These two groups of surfaces have different geometries, however, their Laplace-Beltrami spectrums are quite similar. Namely, one can not expect to use Laplace-Beltrami spectrum to differentiate these two groups. Practically, we can say they are isospectral surfaces.

4.3 Laplace-Beltrami Nodal Count Sequences

A natural question is how can we resolve isospectral surfaces by using other information stored in LB eigen-systems. S. Gnutzmann, P.D. Karageorge, and U. Smilansky suggested that one can use nodal count sequences to "count" the



Figure 4.2: Isospectral anatomical structures . (a)(b)(c) Caudate surfaces. (d)(e)(f)Putamen. (g) Top: the first 300 eigenvalues of the 6 shapes. Bottom: MDS embedding results with the shape DNA. (red: caudate; blue: putamen.)

shape of drum [40], they also developed trace formulas for nodal count sequences of simple tori and surfaces of revolution [41]. Their trace formulas demonstrate that nodal count sequences do store geometric information. S. Gnutzmann, R. Band et al. [42, 4] also suggested that one can use nodal count sequences to resolve isospectral surfaces and isospectral quantum graphs. In this section, we will discuss the nodal count sequences and introduce the weight l^2 -distance to measure the difference between two surfaces by using their nodal count sequences.

Definition 4.3.1 Let (M, g) be a given two dimensional compact Riemannian manifold and ϕ be an eigenfunction of the Laplace-Beltrami operator of (M, g), then $\phi^{-1}(0)$ is called nodal lines of ϕ on (M, g). Each connected component of $M \setminus \phi^{-1}(0)$ is called a nodal domain of ϕ and the number of nodal domains is called the nodal number of ϕ .



Figure 4.3: (a): Nodal lines and nodal domains of the 2-nd eigenfunction of a putamen and the nodal number is 3. (b) Nodal lines and nodal domains of the 4-th eigenfunction of an armadillo, the nodal number is 5 (c) Nodal lines and nodal domains of the 4-th eigenfunction of a cow and the nodal number is 3.

As examples, we show nodal lines on three different surfaces in Fig.4.3. From this picture, it is clear that nodal lines are loops on surfaces and nodal numbers of the eigenfunctions in Fig.4.3 are finite. Theoretically, Courant and Cheng prove the following properties about nodal lines and nodal domains for general two dimensional manifolds in [28, 24]):

Theorem 4.3.1 Given a two dimensional compact Riemannian manifold (M, g),

- (Courant's nodal domain theorem) The number of nodal domains of the i-th eigenfunction ≤ i+1;
- The nodal lines consist of a number of C²-immersed one dimensional closed submanifolds. In other words, nodal lines are number of C²-immersed circles on M.

Since LB eigenfunctions are intrinsically defined on surfaces, the corresponding nodal lines and nodal numbers are also intrinsic geometric quantities. Namely, nodal numbers are isometric invariant. In particular, nodal numbers are rotation and translation invariant. Moreover, because the number of connected components of an eigenfunction ϕ only depends on how many times the eigenfunction ϕ crosses the zero level line, thus function ϕ and its scale $c\phi$ will have the same nodal numbers (c > 0), which means nodal numbers are also scale invariant. However, this scale invariant property can not be preserved by LB spectrum. In Fig.4.4, we show a series of pose variation of David from publicly available TOSCA datasets [13, 14, 14]. It is clear that eigenfunctions are pose invariant. In Fig.4.5, we deform a pose of David with different scales. Fig.4.5(c) shows how LB spectrum will change under different scale and Fig.4.5(d) illustrates the scale invariant property of nodal numbers.



Figure 4.4: Pose invariance of nodal numbers. Nodal lines are marked by red lines. The first row: nodal lines of the first eigenfunction. The second row: nodal lines of the fifth eigenfunction.

According to Theorem 4.3.1, for a given eigenfunction sequence $\{\phi_1, \phi_2, \cdots\}$ of (M, g), we can have a sequence of numbers $\{l_1, l_2, \cdots\}$ which correspond to



Figure 4.5: Scale variance properties of LB spectrum and nodal count sequences. (a): the nodal domains of the first eigenfunctions, (b): the nodal domains of the fifth eigenfunctions, (c) the first 100 eigenvalues of the David with 3 different scales. (d) the first 100 nodal numbers of the David with 3 different scales. red dots—big, blue + — middle, black x — small

the number of nodal domains of the eigenfunction sequence. This sequence is well-defined because each l_k is a finite number. The sequence $\{l_1, l_2, \dots\}$ is called the **nodal count sequence** of $\{\phi_1, \phi_2, \dots\}$ on (M, g). As the properties we described for nodal numbers, nodal count sequences have the following invariant properties:

- 1. Nodal count sequences are isometric invariant. In particular, they are rotation, translation and pose invariant;
- 2. Nodal count sequences are scale invariant.

A natural question is if we can expect to obtain surfaces' geometric information by using their nodal count sequence. As Kac's famous question: "Can one hear the shape of a drum?" for LB spectrum, a similar question: "Can one count the shape of a drum? " can be proposed for nodal count sequences. Recently, S. Gnutzmann, P. Karageorge, U. Smilansky [41] introduce a trace formula for the nodal count sequence, which is an analogue of the spectrum trace formula. The nodal trace formula shows the dependence of the nodal count sequences on the geometry of the surface in both the smooth(Wely-like) and the fluctuating parts. The geometric information stored in nodal count sequence is not the same as the information stored in the spectrum. Moreover, in [4, 42], it was conjectured that the nodal count sequence can resolve the isospectral surface and was partially proved for flat tori and simple metric graph cases. Intuitively, if two isospectral surfaces $(M, \{\lambda_i\}, \{\phi_i\})$ and $(\tilde{M}, \{\tilde{\lambda}_i\}, \{\tilde{\phi}_i\})$ have the same LB spectrums, i.e. $\lambda_i = \tilde{\lambda}_i, i = 1, 2, \ldots$, but they have different geometries. Then they can not have the same eigenfunctions for each order. This implies that they may have different nodal count sequences. That is the reason we expect that nodal count sequences can tell us different geometric information as a complementary part of geometric information stored in the LB spectrum.

There is one weak point of the nodal count sequences, namely, in the case of multi-dim eigenspace, the choice of basis is not unique. For a given eigenvalue λ , if the corresponding eigenspace is multi-dimension, different choice of basis can be made for that eigenspace. Then, the corresponding nodal count sequence is not unique. Generally speaking, the more symmetry an object has, the more multiplicity its eigen-system has. However, Uhlenbeck's results in [100] prove that this phenomenon is rare. For generic (mathematical sense) 2- manifolds, they have simple LB eigen-system. This is equivalent to say for most of surfaces, their eigen-system have no multi-dim phenomenons. Practically, we can view our shapes as simple eigen-system surfaces except some high symmetry shapes, like a sphere.

Numerically, given a triangulated surface M, we use the finite element method on surfaces described in Section 3.2 to compute its LB eigenvalues and eigenfunctions. To obtain the corresponding nodal count sequences, we count the number of connected components of the triangular mesh with the same sign of its LB eigenfunctions. As we discussed in Section 3.2, the following two estimations of the FEM method can be obtained as it was proved in [94]:

Theorem 4.3.2 Let (ϕ_n^h, λ_n^h) and (ϕ_n, λ_n) as above description

$$||\phi_n - \phi_n^h||_0 \leqslant Ch^2 \lambda_n \tag{4.1}$$

$$\lambda_n \leqslant \lambda_n^h \leqslant \lambda_n + 2\delta h^2 \lambda_n^2 \tag{4.2}$$

where C and δ are constant.

The nodal number of a given eigenfunction ϕ_n reflects the oscillation of ϕ_n . From the above theorem, we can see the accuracy of the eigenfunction decreases as the order n increases for a given h. As a result, the nodal counts for high order eigenfunctions are noisier than that of the low order eigenfunctions. However, the more eigenfunctions we can use, the more geometric information can be obtained. We need to find a balance between using the nodal number of high frequency eigenfunction and overcoming the numerical issue. Based on this consideration, we propose the following weighted l^2 distance between two nodal count sequences $\{l_n\}_{n=1}^{\infty}$ and $\{\tilde{l}_n\}_{n=1}^{\infty}$:

$$Dist(\{l_n\}_{n=1}^{\infty}, \{\tilde{l}_n\}_{n=1}^{\infty}) = \sqrt{\sum_{n=1}^{\infty} (\frac{1}{n^{\alpha}})^2 (l_n - \tilde{l}_n)^2}$$
(4.3)

where $\alpha > 0$. In our experiments, we demonstrate that the nodal count sequences under this weighted l^2 distance provide robust performance for shape classification.

4.4 Experimental Results and Applications

In this section, we present experimental results to demonstrate the application of the nodal count sequence in shape analysis. In particular, we show that the nodal counts of the Laplace-Beltrami operator is able to resolve isospectral shapes. After that, we also use our method shapes: a group of caudate nucleus, a group of putamen, a group of hippocampus, group of Armadillo, a cow, a male, a camel. Our method can provide a promising classification results as shape DNA did. All the experiments were run on a PC with a 2.0GHz CPU.

A. Isospectral Surfaces Resolving

The 3D shapes used in the first experiment are three putamen and three caudate surfaces shown in Fig.4.2. While the putamen and caudate are visually quite different, they share very similar distribution of eigenvalues, i.e., the shape DNA, as shown in Fig.4.6(g). For each group of surfaces, we use their nodal count sequences and the shape DNA to embed them into a 2D space with multi-dimensional scaling (MDS) technique. The details of this embedding process is summarized as follows.

- 1. For a given surface (M, g) represented by a triangle mesh, we compute the first N eigenvalues and eigenfunctions of the Laplace-Beltrami operator by the finite element method to obtain the signature.
- 2. For a group of surfaces, we compute the pairwise weighted l^2 distance of their corresponding signatures. The pairwise distances are stored in a distance matrix.
- 3. Using the distance matrix, the MDS technique is applied to embed the surfaces into the Euclidean space.

In our experiment, we choose $N = 300, \alpha = 1$, and the embedding results with the shape DNA and the nodal count sequences are shown in Fig. 4.6 (g) and (h). From the results, we can see clearly that the nodal counts provide better separation of these two groups. This demonstrates the ability of the nodal count sequences in resolving isospectral surfaces.



Figure 4.6: (a)(b)(c) Caudate surfaces. (d)(e)(f) Putamen. (g) Top: the first 300 eigenvalues of the 6 shapes. Bottom: MDS embedding results with the shape DNA. (h) Top: The first 300 nodal counts of the 6 shapes. Bottom: MDS embedding results with the nodal count sequences. (red: caudate; blue: putamen.)

In the second experiment, we demonstrate the above shape classification procedures to a larger data set. This data set includes three groups of surfaces: 20 hippocampus, 20 putamen, and 20 caudate. For the three groups, the eigenvalue sequences and nodal count sequences were computed. By applying the same MDS technique as in the first experiment to these signatures, we can embed the 60 surfaces into a 2D space and the results are shown in Fig. 4.7. Clearly these results show that the nodal count sequence provides better classification.

B. Surface classification

In [79], M. Reuter et.al use LB spectrum of shapes to classify surfaces. As we discussed in section 4.3, the nodal count sequences can provide us the geometry information of surfaces. Here, we propose to use the nodal count sequences to tackle shape classification problems as spectrum did. In Fig.4.8, we choose $N = 300, \alpha = 1$ to classify different groups of surfaces. From the picture, it is



Figure 4.7: Top: the MDS embedding of the surfaces with the shape DNA signature. Bottom: the MDS embedding of the surfaces with the nodal count sequences. The first 300 eigenvalues and eigenfunctions were used in both embeddings. (red '.': caudate; blue '+': putamen; black '*': hippocampus.)

clear that different groups cluster together. Moreover, the geometric meaning of the weight l^2 nodal counts distance is also illustrated in Fig.4.8. For instance, the David group and the Michael group are similar to each other, the geometry of lioness group and the geometry cat group are closest in all groups, the geometry of centaurs group are far way from other groups, these intuitions can all be observed in Fig.4.8. It is clear that the nodal count sequences can provide us promising results for surface classification.

4.5 Conclusions and Further Work

In this work we proposed to use the nodal count sequences of the Laplace-Beltrami eigenfunctions as a novel signature of 3D shapes. We demonstrated its ability of resolving isospectral shapes by classifying anatomical structures with very similar



Figure 4.8: MDS embedding results to classify cats, horses, lionesses, dogs, wolves, centaurs, David, Michael with the nodal count sequences.

distribution of eigenvalues. We also apply the nodal count sequence to obtain a preliminary result of surface classification. In our future work, we will apply the nodal count sequences, or also combining with LB spectrum to the task of shape retrieval from databases. We are also investigating its application in classifying hippocampal surfaces from normal controls and Alzheimer's disease.

CHAPTER 5

Anisotropic Laplace-Beltrami Eigenmaps: Bridging Reeb Graphs and Skeletons

5.1 Introduction

Skeletons are important tools in studying shapes [10] as they provide an intuitive graph representation that connects well with high level understandings. The challenge of using skeletons in group studies is to maintain a consistent topology across population. In this work, we propose a novel approach of computing skeletons with consistent topology on simply connected surface patches in 3D by constructing a Reeb graph from the eigenfunction of an anisotropic Laplace-Beltrami operator.



Figure 5.1: The Hamilton-Jacobi skeleton of four cingulate gyri.

One weakness in using skeletons to represent shapes is their sensitivity to small changes on the boundary, which makes it difficult to compare a group of shapes belonging to the same category but having subtle differences. As an example, we show the skeletons of four cingulate gyri in Fig. 5.1 that are computed with the method of Hamilton-Jacobi skeletons [92]. While the skeletons are relatively clean, they have different graph structures. To address this challenge, various approaches were proposed to enforce a consistent topology on the skeleton. A skeleton of fixed topology was computed for 2D shapes by driving a snake model to the shocks in the distance map [45]. A similar approach was also taken in studying 3D shapes with a medial axis [98]. Pruning strategies based on continuity and significance were also developed to simplify skeletons [72, 95, 11]. The other powerful approach is the M-rep that uses a generative approach to match templates designed a priori to new shapes [76]. More recently, the idea of inverse skeletonization was used to compute skeletons of simplified topology via the solution of a nonlinear optimization problem [75].

Given a function defined on a surface, its Reeb graph is intuitively a graph describing the neighboring relation of the level sets of the function. Following Morse theory, Reeb graphs [78] have been used as a powerful tool to analyze geometric information contained in various sources of imaging data. A Reeb graph was constructed to build a smooth surface interpolating a series of contour lines [91]. Contour trees were constructed to store seed information for efficient visualization of volume images [101]. The Reeb graphs were also used to study terrain imaging data [9] and the matching of topological information in a database of 3D shapes [48].

In this work, we propose to use Reeb graphs to construct a skeleton of robust topology for simply connected surface patches with the aim of studying anatomical structures such as the cingulate gyrus and corpus callosum. There are two main contributions in our work. First of all, we propose to use the spectrum of the Laplace-Beltrami operator [79, 77] to construct the Reeb graph, which ensures the Reeb graph is invariant to the pose of the shape. Our second contribution is the development of an anisotropic Laplace-Beltrami operator based on a flux measure [92]. This bridges the idea of Reeb graphs with conventional skeletons and makes the Reeb graphs follow the main body of skeletons.

The rest of this chapter is organized as follows. In section 5.2, we introduce the mathematical background of Reeb graphs and its construction on triangular meshes. We describe the spectrum of the anisotropic Laplace-Beltrami operator and its use of building Reeb graphs in section 5.3. After that, a flux-based weight function is proposed in section 5.4 to define the anisotropic Laplace-Beltrami operator. Experimental results are presented in section 5.5. Finally conclusions are made in section 5.6.

5.2 Reeb Graphs



Figure 5.2: The Reeb graph of the height function on a double torus of genus two. (a) Level sets of the height function. (b) The Reeb graph of the level sets.

Let M denote a compact surface and a feature function f defined on this surface. The Reeb graph of f on M is defined as follows.

Definition 5.2.1 Let $f : M \to R$. The Reeb graph R(f) of f is the quotient space with its topology defined through the equivalent relation $x \sim y$ if $f(x) = f(y) \forall x, y \in M$.

As a quotient topological space derived from M, the connectivity of the elements in R(f), which are the level sets of f, is determined by the topology, i.e., the collection of open sets, of M. If f is a Morse function [53], which means the critical points of f are non-degenerative, the Reeb graph R(f) encodes the topology of M and it has g loops for a manifold of genus g. To compute the Reeb graph numerically, we assume the surface M is represented as a triangular mesh M = $(\mathcal{V}, \mathcal{T})$, where \mathcal{V} and \mathcal{T} are the set of vertices and triangles, respectively. The function f is then defined on each vertex in \mathcal{V} . We sample the level sets of f at a set of K values $\xi_0 < \xi_1 < \cdots < \xi_{K-1}$ and the set of contours as

$$\Gamma = \{ \Gamma_k^l, \ 0 < k < K - 1, \ 0 < l < L_k, \}$$

where L_k denotes the number of contours at the level ξ_k , and Γ_k^l represents the *l*-th contour at this level. To build edges between contours at neighboring levels, we consider the region

$$R_{k,k+1} = \{ x \in M \mid \xi_k < f(x) < \xi_{k+1} \}$$

and a contour $\Gamma_K^{l_1}$ at the level ξ_k and a contour $\Gamma_{K+1}^{l_2}$ at the level $\xi_k + 1$ are connected if they belong to the same connected component in $R_{k,k+1}$. This completes the construction of a Reeb graph on M as an undirected graph with the level contours as the nodes and the edges representing the neighboring relation of these contours.



Figure 5.3: The Reeb graphs of two different feature functions f on a cingulate gyrus. (a) f is the z-coordinates; (b) The Reeb graph from the function f in (a); (c) f is the y-coordinates; (d) The Reeb graph from the function f in (c).

As an example, we illustrate the construction of a Reeb graph on a double torus shown in Fig. 5.2(a). The feature function f used here is the height function. We sample ten level sets of f and plot them as red contours on the surface. With these contours as its nodes, the Reeb graph is shown in Fig. 5.2(b), where the centroid of each contour is used to explicitly represent the nodes of the graph. Clearly this graph has two loops and it captures the topology of the shape. Reeb graphs can also be constructed for functions defined on surfaces with boundary. For the cingulate gyrus on a left hemispherical surface, we compute the Reeb graph for two different choices of the feature function f. In Fig. 5.3 (a) and (c), the function f is the z- and y- coordinates of vertices, respectively. For these two functions, we sample 20 level sets and the resulting Reeb graphs are shown in Fig. 5.3 (b) and (d), respectively. Since the level sets here are curve segments, we use the middle point of each curve segment to explicitly represent the node of the Reeb graph. The above results demonstrate that Reeb graphs can be constructed successfully on surface patches given a feature function f, but they also help point out the main difficulty in using Reeb graphs to compare shapes across population: the selection of an appropriate feature function f. The two feature functions used above are similar to the height function used commonly in previous work [9] and there are two drawbacks of such choices. First, their Reeb graphs are pose dependent as the coordinates will change under rotation. Second, they are sensitive to noise on the boundary. We can see in Fig. 5.3(b) and (d) that spurious branches are created in the Reeb graphs because the boundary is jaggy as is pretty common for manually segmented structures. We next propose to use the eigenfunctions of an anisotropic Laplace-Beltrami operator as the feature function, which are defined intrinsically on the surface and robust to irregularities on the boundary.

5.3 Anisotropic Laplace-Beltrami Eigenmaps

In this section, we introduce the anisotropic Laplace- Beltrami operator on a surface patch and the computation of its spectrum. We then propose to use the first nontrivial eigenfunction of this operator as the feature function in the construction of Reeb graphs. The spectrum of the Laplace-Beltrami operator has been used in several work in medical imaging [71, 77]. Here we consider the more general anisotropic Laplace-Beltrami operator $\nabla_M \cdot (w\nabla_M)$ on a simply connected surface patch M, where ∇_M is the intrinsic gradient operator on M, and



Figure 5.4: The Reeb graph of the Laplace-Beltrami eigenmap. (a) The eigenmap.(b) The level sets of the eigenmap and the Reeb graph.

 $w: M \to \mathbb{R}^+$ is the weight defined over M. If we set w = 1, we have the regular Laplace-Beltrami operator. Here we only require w to be positive to ensure the operator is elliptic, so the spectrum is discrete and can be expressed as follows. We denote the set of eigenvalues as $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots$ and the corresponding eigenfunctions as ϕ_0, ϕ_1, \cdots such that

$$\nabla_M \cdot (w \nabla_M \phi_n) = -\lambda_n \phi_n, \quad n = 0, 1, \cdots$$
(5.1)

The set of eigenfunctions form orthonormal basis functions on M and can be intuitively considered as the intrinsic Fourier basis functions on the surface. In fact, they have been used for denoising in brain imaging studies [77].

To compute the spectrum, we use the weak form of (5.1). Taking the Neumann boundary condition, we can find the eigenvalues as the critical points of the following energy

$$E(\phi) = \frac{\int_M w ||\nabla_M \phi||^2 \mathrm{d}s}{\int_M ||\phi||^2 \mathrm{d}s}$$
(5.2)

For numerical implementation, we use the finite element method on surfaces similarly to the discussion in Section 3.2. Let M be a surface represented by triangular mesh $\{V = \{p_i\}_{i=1}^N, T = \{T_l\}_{l=1}^L\}$. We choose linear elements $\{\psi_i^h\}_{i=1}^N$, such that $\psi_i^h(v_j) = \delta_{i,j}$ and write $S^h = Span_{\mathbb{R}}\{\psi_i^h\}_{i=1}^N$. We write

$$\begin{cases} \phi = \sum_{i}^{N} x_{i}\psi_{i} \\ A_{w} = (a_{ij})_{N \times N}, \quad a_{ij} = \sum_{l} \int_{T_{l}} w \nabla_{M} \psi_{i} \nabla_{M} \psi_{j} \mathrm{d}s \\ B = (b_{ij})_{N \times N}, b_{ij} = \sum_{l} \int_{T_{l}} \psi_{i} \psi_{j} \mathrm{d}s \end{cases}$$
(5.3)

then the generalized eigen-problem can be approximated by the following matrix eigen-problem:

$$A_w \phi = \lambda B \phi. \tag{5.4}$$

This problem can be solved with a variety of numerical linear algebra packages. In our implementation, we represent both A_w and B as sparse matrices and use Matlab to solve 5.4. Since the sum of each row in A_w equals zero, the first eigenvalue $\lambda_0 = 0$ and ϕ_0 is constant. As the first nontrivial eigenfunction, ϕ_1 minimizes the energy E and achieves the critical value at λ_1 :

$$\lambda_{1} = \int_{M} w ||\nabla_{M} \phi_{1}||^{2} \mathrm{d}v, \qquad (5.5)$$

s.t. $||\phi_{1}||^{2} = 1.$

Thus the eigenmap ϕ_1 provides the smoothest, non-constant map from M to \mathbb{R} . Using this eigenmap, we can capture the intrinsic structure of elongated shapes such as the cingulate gyrus and corpus callosum. The eigenmap is also invariant under isometric transformations such as bending. As an example, the eigenmap ϕ_1 with the isotropic weight w = 1 for the cingulate gyrus in Fig. 5.4 is visualized in Fig. 5.4(a). The level sets of this function are plotted as red contours in Fig. 5.4(b), where the Reeb graph is computed with each node representing the middle point of the level sets. From the level sets, we can see the eigenmap ϕ_1 projects the surface smoothly onto \mathbb{R} and is robust to the jaggy boundary of the surface. Compared with the skeleton in Fig. 5.1(a), we can see the Reeb graph of ϕ_1 has a simple chain structure and approximates the main component of the skeleton very well.

5.4 Flux-based Weight Functions



Figure 5.5: The Reeb graph of a corpus callosum using the eigenmap of the isotropic Laplace-Beltrami operator. (a) The Hamilton- Jacobi skeleton. (b) The Reeb graph.

In the cingulate gyrus example in Section 5.3, we see that the Laplace-Beltrami eigenmap provides a robust way of constructing the Reeb graph and capturing the global structure of the shape. In some cases, however, the Reeb graph built from the eigenmap of the isotropic Laplace-Beltrami operator is insufficient as an approximation of the skeleton. We show such an example in Fig. 5.5. The Hamilton-Jacobi skeleton of the corpus callosum is plotted in Fig. 5.5 (a), and the Reeb graph of the eigenmap ϕ_1 computed with the isotropic weight w = 1, together with the level sets, is shown in Fig. 5.5(b). For most parts, the Reeb graph does a good job in approximating the skeleton, but it is also not hard to notice that it fails to follow the bending of the genu at the frontal end of the corpus callosum, which is well represented in the conventional skeleton in Fig. 5.5(a). In this section, we design a weight function to incorporate information in skeletons into the construction of Reeb graphs. The weight function we choose is based on the flux measure used in the method of Hamilton-Jacobi skeleton [92] and its extension to triangular meshes [89]. For a surface patch M, let ∂M denote its boundary. We define a distance transform $D: M \to \mathbb{R}$ as:

$$D(x) = \min_{y \in \partial M} d(x, y) \quad \forall x \in M$$
(5.6)

where $d(\cdot, \cdot)$ is the geodesic distance between two points. Given this distance transform, the flux measure is defined as

$$Flux(x) = \frac{\int_{\delta R} \langle \vec{N}, \nabla_M D \rangle \mathrm{d}s}{\int_{\delta R} ds}, \quad \forall x \in M$$
(5.7)

where δR is the boundary of an infinitesimal geodesic neighborhood of x, \vec{N} is the outward normal direction of δR and $\nabla_M D$ is the intrinsic gradient of D on M. To numerically compute the flux measure for a triangular mesh, we first compute the distance transform with the fast marching algorithm on triangular meshes [54] to solve the Eikonal equation on M:

$$||\nabla_M D|| = 1 \tag{5.8}$$

We then calculate the flux measure at each vertex of M as:

$$Flux(\mathcal{V}_i) \approx \frac{1}{\sharp N(\mathcal{V}_i)} \sum_{\mathcal{V}_j \in N(\mathcal{V}_i)} \langle \frac{\overline{\mathcal{V}_i \mathcal{V}_j}}{||\overline{\mathcal{V}_i \mathcal{V}_j}||}, \nabla_M D(\mathcal{V}_j) \rangle$$
(5.9)

where $\sharp N(\mathcal{V}_i)$ is the number of vertices in the 1-ring neighborhood $N(\mathcal{V}_i)$ of \mathcal{V}_i , and $\overrightarrow{\mathcal{V}_i \mathcal{V}_j}$ is the vector from the vertex \mathcal{V}_i to \mathcal{V}_j . Based on the flux measure, we define the weight function as:

$$w(x) = e^{-sign(Flux(x))|Flux(x)|^{\alpha}} \quad \forall x \in M$$
(5.10)

Following this definition, more weight is given to points on the skeleton as the flux is more negative at these points according to (5.9). Recall that the eigenmap ϕ_1 is the smoothest projection from M to \mathbb{R} by the minimization of the energy in

(5.6). As we decrease the parameter α , as shown in Fig. 5.6, we put more weight on vertices close to the skeleton, and the shape looks more like the skeleton for the energy in (5.6). Thus intuitively the projection from M to \mathbb{R} will happen along the skeleton and the level sets of the eigenmap should be more oriented in the direction normal to the skeleton. With each of the four weight functions in



Figure 5.6: The weight function.

Fig. 5.6, we compute the eigenmap of the anisotropic Laplace-Beltrami operator $\nabla_M \cdot (\omega \nabla_M)$ and use it to construct a Reeb graph for the corpus callosum in Fig. 5.5. The weight functions and the corresponding Reeb graphs are shown in Fig. 5.7. As we decrease the parameter α from 1.0 to 0.1, we can see the level sets of the eigenmap at the frontal part turn more toward the direction pointed by the main body of the skeleton. As a result, the Reeb graph follows the bending of the genu better than simply using the isotropic LB operator.


Figure 5.7: The effects of the parameter on the Reeb graph. Top row: the weight function mapped onto the surface. Bottom row: the level sets and the Reeb graph of the anisotropic Laplace-Beltrami eigenmap.

5.5 Experimental Results

In this section, we present experimental results to demonstrate our algorithm. Reeb graphs are constructed on two anatomical structures: the cingulate gyrus and corpus callosum. We illustrate that our algorithm can be used as an efficient and robust approach of computing skeletons of consistent topology for these shapes. In the first experiment, we apply our algorithm to a group of 16 cingulate gyri as shown in Fig. 5.3 with the weight function w = 1. Each surface patch is extracted from triangulated cortical surfaces with manual labeling, and it is usually composed of around 2000 vertices and 4000 triangles. The computational time is less than 2 seconds on a PC. For each shape, we sample the eigenmap at 50 level sets and use the middle point of each level contour as the node of the Reeb graph. Intuitively we can see the Reeb graphs successfully capture the global profile of these elongated surface patches. For all the examples, the Reeb graphs have the same chain structure. In the second experiment, we compute Reeb graphs for a group of 16 corpora callosa with an anisotropic Laplace-Beltrami eigenmap by choosing the parameter $\alpha = 0.25$ in 5.10. The surface patch of each corpus callosum is constructed from manually labeled bi-



Figure 5.8: The Reeb graph of 16 cingulate gyri constructed using the Laplace-Beltrami eigenmap.



Figure 5.9: The Reeb graphs of 16 corpora callosa constructed with the anisotropic Laplace-Beltrami eigenmap.

nary masks with the software triangle [86] and also composed of around 2000 vertices and 4000 triangles. Because of the need of calculating the weight function, it takes around 3 seconds, which is slightly longer than using the isotropic Laplace-Beltrami eigenmaps, to compute the Reeb graphs on a PC. Similar to the cingulate examples, a collection of 50 level sets are sampled on the eigenmap of each corpus callosum. From the results shown in Fig. 5.9, we can see all the Reeb graphs have the chain structure and successfully capture the bending of the genu.

To measure the advantage of using the anisotropic eigenmap for analyzing the corpus callosum, we have also computed the Reeb graphs with the isotropic weight w = 1 for the 16 corpora callosa. After factoring out rotation and translation, we applied a principal component analysis (PCA) to each of the two groups of Reeb graphs [27]. The variances of the principal components for both the isotropic and anisotropic Reeb graphs are plotted in Fig. 5.10. We can see clearly the anisotropic Reeb graphs generate more compact representations. This gives a quantitative validation that anatomically meaningful features are better aligned with the use of the anisotropic Laplace-Beltrami eigenmaps.

5.6 Conclusions

In this work, we propose to use the Reeb graph of an anisotropic Laplace-Beltrami eigenmap to analyze shapes represented as simply connected surface patches. Experimental results on two neuroanatomical structures have been presented to demonstrate the use of Reeb graphs as skeletons of consistent topology. Besides shape analysis, the results from our algorithm can also be used to test local morphometry changes with our results by using the length of the level sets as a width measure and the correspondences established by the Reeb graphs. For



Figure 5.10: A comparison of the eigenvalue distribution obtained by applying a PCA to Reeb graphs of corpora callosa constructed with both isotropic and anisotropic Laplace-Beltrami eigenmaps.

future work, we will also use the level sets to construct an intrinsic parameterization for the statistical analysis of anatomical/functional features distributed over the structure.

CHAPTER 6

Metric-Induced Optimal Embedding for Intrinsic 3D Shape Analysis

6.1 Introduction

The analysis of 3D shapes is a critical problem in many computer vision problems such as shape classification and model retrieval [38], shape prior construction [27], and brain mapping [99]. For the various shape analysis problems, a rigorous and intrinsic metric is fundamental as it allows the comparison of shapes in a pose and scale invariant way. In this work, we propose a novel and rigorous metric via the Laplace-Beltrami embedding for intrinsic 3D shape analysis. We demonstrate the proposed metric in solving the challenging problem of sulcal landmark identification from convoluted cortical surfaces of vervet brains.

For intrinsic 3D shape analysis, there are generally three classes of approaches developed in the context of various applications. The first approach is featurebased and defines typically application specific intrinsic features to characterize and compare shapes in the feature space [73, 5, 60]. The second approach is based on the shape space theory and each surface is viewed as a point on a manifold [51, 61]. Natural metrics on the manifold can be used to compare shapes, but it is difficult to extract local information about shapes since each surface is treated as a point in this space. Thus this approach is especially suitable in shape classification and retrieval. The third approach embeds the shapes intrinsically into a high dimensional space and studies them in the embedding space. One of the famous examples is the isometric embedding based on the pair-wise geodesic distances of vertices [34], but it is computationally expensive. More recently there have been increasing interests in the embedding constructed with the LB eigensystem because of its generality for arbitrary shapes and ease of computation [2, 52, 83, 49, 79, 80, 56, 88]. Since surfaces are still manifolds in the embedding space, both local and global comparisons can be performed intrinsically. Using the one-to-one correspondences established by the embedding, we can map the analysis results back to the original Euclidean space easily.

There is, however, a fundamental problem in analyzing and comparing shapes in the LB embedding space: the non-uniqueness of the embedding. This is caused by the ambiguity in the sign of the eigenfunctions and possible multiplicity. While conventional metrics such as the Hausdorff distance can be used for a specific embedding, they are not truly intrinsic because they depend on heuristic choices such as the sign of the eigenfunctions. For a truly intrinsic metric, the distance between two shapes should only be determined by their intrinsic geometry. To overcome this critical challenge, we propose in this work a novel metric, which we call the spectral l^2 -distance, for the LB embedding proposed by Rustamov [83]. The spectral l^2 -distance is completely determined by the intrinsic geometry of surfaces and therefore invariant to pose and scale changes. Mathematical proofs will be presented to show that this distance is a rigorous metric on the space of surfaces. Besides a distance measure, the process of computing the spectral l^2 -distance also removes the ambiguity in embedding and thus determines the optimal embedding for shape comparison. In the resulting embedding space, we can perform both global and local analysis based on intrinsic geometry, which we demonstrate by using the new metric-induced embedding to solve the challenging problem of automated sulci detection on cortical surfaces.

The rest of this chapter is organized as follows. In section 6.2, we first give a brief review of the eigen-system of the LB operator and the eigen-embedding on general surfaces. After that we present the spectral l^2 -distance and give the rigorous mathematical proof to show that it is a metric in the shape space. In section 6.3, we develop the numerical algorithm to compute the eigen-system and a tractable approximation of the spectral l^2 -distance between two triangulated surfaces. We demonstrate the application of the spectral l^2 -distance for the extraction of sulcal landmarks for cortical surfaces in section 6.4. Experimental results are presented in section 6.5 to demonstrate automated sulcal identification on a large data of 698 vervet cortical surfaces. Finally, conclusions and future work are discussed in section 6.6.

6.2 Mathematical Background

In this section, we first review the concept of the LB eigen-system on general surfaces. The spectral l^2 -distance and mathematical proofs will be presented to show that it is a rigorous metric.

6.2.1 The LB Eigen-system

Let (M, g) denote a Riemannian surface. For any smooth function $\phi \in C^{\infty}(M)$, the LB operator is defined as:

$$\Delta_M \phi = \frac{1}{\sqrt{G}} \sum_{i=1}^2 \frac{\partial}{\partial x_i} (\sqrt{G} \sum_{j=1}^2 g^{ij} \frac{\partial \phi}{\partial x_j})$$
(6.1)

where (g^{ij}) is the inverse matrix of $g = (g_{ij})$ and $G = \det(g_{ij})$.

The LB operator is self adjoint and elliptic, so its spectrum is discrete. We

denote the eigenvalues of Δ_M as $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ and the corresponding eigenfunctions as $\phi_0, \phi_1, \phi_2, \cdots$ such that

$$\Delta_M \phi_n = -\lambda_n \phi_n, \ n = 0, 1, 2, \cdots .$$
(6.2)

Let $E_i = \{u \in C^2(M) \mid \Delta_M u = -\lambda_i u\}$ denote the eigenspace associated with the eigenvalue λ_i . If the eigenspace of all eigenvalues are one dimensional, the eigensystem $\{\lambda_i, \phi_i\}|_{i=0}^{\infty}$ of M is called *simple*. The eigen-system of the LB operator has many nice properties [85, 50]:

- The eigen-system $\{\lambda_i, \phi_i\}|_{i=0}^{\infty}$ of Δ_M is intrinsic and isometric invariant. Thus properties derived from the eigen-system of Δ_M are robust to pose variations.
- Scale formula: Let A be a positive constant, {λ_i, φ_i} be the eigensystem of (M, g), and {λ̃_i, φ̃_i} be the eigensystem of (M, A · g), then

$$\lambda_i = A \cdot \tilde{\lambda}_i \text{ and } \phi_i = \sqrt{A} \cdot \tilde{\phi}_i$$
(6.3)

From a signal processing point of view, the eigenfunctions of the LB operator are extensions of the Fourier basis functions on the Euclidean domain to general manifolds. One famous example is the spherical harmonics, which are the eigenfunctions of the LB operator on the unit sphere, and they have been used in various shape analysis tasks. The focus of our research, however, is using the LB eigen-system to investigate surface geometry.

6.2.2 Metric in the embedding space

Given a surface (M, g) and its LB eigen-system $\Phi = \{\lambda_i, \phi_i\}$, the scale-invariant embedding proposed in [83] is defined as:

$$I_M^{\Phi}: M \to l^2, \ I_M^{\Phi}(x) = \{\frac{\phi_1(x)}{\sqrt{\lambda_1}}, \frac{\phi_2(x)}{\sqrt{\lambda_2}}, \cdots, \frac{\phi_k(x)}{\sqrt{\lambda_k}}, \cdots\}$$
(6.4)

Here l^2 is the standard sequence Hilbert space. This is a well defined map since the Green's function of the LB operator on M can be represented by

$$G(x,y) = \sum_{i} \frac{\phi_i(x)\phi_i(y)}{\lambda_i}, \text{ for any } x, y \in M$$
(6.5)

Namely, for any fixed point $x \in M$, $I_M(x)$ is a l^2 -convergent sequence. In Fig. 6.1, we show the embedding results of the unit sphere and a vervet cortical surface by using the first three eigenfunctions. Generally speaking, the more eigenfunctions we use, the more detailed information of the surface we can obtain.



Figure 6.1: The first three columns are color coded by the first three eigenfunctions on each surface respectively. The last column is obtained by using the first three eigenfunctions as 3 coordinates to reconstruct the original surfaces

The following theorem from [83] shows that the map in (6.4) is an embedding.

Theorem 6.2.1 For any given surface (M, g), any given basis $\Phi = \{\phi_i\}$, the map I_M^{Φ} is an embedding and it is scale invariant by the scale formula (6.3).

The biggest advantage of this embedding is that we can put all surfaces into a common space to factor out all pose and scale variations. But the problem is that the eigen-system of a given surface M is not unique. The mapping I_M depends on the choice of an orthonormal basis of eigenfunctions. For instance, both ϕ_i and $-\phi_i$ can be an element of basis. For eigenvalues with multiplicity greater than one, there is even more freedom to choose the basis. Given a surface (M,g), one can decompose the space $L^2(M,g)$ as $L^2(M,g) = \bigoplus_{i=0}^{\infty} E_i$, where E_i 's are the eigenspaces of the LB operator Δ_M . We denote $\mathcal{B}(M) = \prod_{i=0}^{\infty} \mathcal{B}(E_i)$ the set of corresponding orthonormal bases. The space $\mathcal{B}(E_i)$ can be identified with the orthogonal group $O(\dim(E_i))$ and hence $\mathcal{B}(M)$ is a compact set with respect to the product topology. Inspired by P.Bérard et al.'s work about heat kernel embedding [2], we define the following spectral l^2 -distance for any given two surfaces (M, g), (M', g') as follows:

Definition 6.2.1 Let (M,g) and (M',g) be two surfaces. For any given LB orthonormal basis Φ of M and Φ' of M', we define

$$d_{\Phi}^{\Phi'}(x, M') = \inf_{y \in M'} ||I_{M}^{\Phi}(x) - I_{M'}^{\Phi'}(y)||_{2} , \forall x \in M$$

$$d_{\Phi}^{\Phi'}(M, y) = \inf_{x \in M} ||I_{M}^{\Phi}(x) - I_{M'}^{\Phi'}(y)||_{2} , \forall y \in M'.$$
(6.6)

The spectral pre-l²-distance between M and M' with respect to Φ, Φ' is defined by:

$$d_{\Phi}^{\Phi'}(M,M') = max \left\{ \int_{M} d_{\Phi}^{\Phi'}(x,M') \mathrm{d}vol_{M}(x) , \\ \int_{M'} d_{\Phi}^{\Phi'}(M,y) \mathrm{d}vol_{M'}(y) \right\}$$
(6.7)

where $dvol_M(x), dvol'_M(x)$ are normalized area elements, i.e.

$$\int_{M} \mathrm{d}vol_{M}(x) = 1, \quad \int_{M'} \mathrm{d}vol_{M'}(y) = 1.$$
(6.8)

The spectral l^2 -distance d(M, M') between M and M' independent of the choice of eigen-systems is then defined as:

$$d(M, M') = max \Big\{ \sup_{\Phi \in \mathcal{B}(M)} \inf_{\Phi' \in \mathcal{B}(M')} \int_{M} d_{\Phi}^{\Phi'}(x, M') \mathrm{d}vol_{M}(x) ,$$
$$\sup_{\Phi' \in \mathcal{B}(M')} \inf_{\Phi \in \mathcal{B}(M)} \int_{M'} d_{\Phi}^{\Phi'}(M, y) \mathrm{d}vol_{M'}(y) \Big\}.$$
(6.9)

Because $\mathcal{B}(M)$ and $\mathcal{B}(M')$ are compact, d(M, M') can attain the optimal value for certain $\bar{\Phi} \in \mathcal{B}(M), \bar{\Phi'} \in \mathcal{B}(M')$, which we call the optimal embedding basis of the pair $\{M, M'\}$.

Since our goal is to perform scale and pose invariant shape analysis, we formalize this notion as follows. We call two surfaces $(M, g) \sim (M', g')$ if and only if there is a diffeomorphism $f: M \to M'$ and a constant c such that $g = c \cdot f^*(g')$. Then the embedding and spectral l^2 -distance given above are defined on the quotient space $\mathfrak{D} = \{(M, g)\}/\sim$. The quotient space \mathfrak{D} is equivalent to the set of all surfaces with normalized area, namely $\mathfrak{D} \cong \mathfrak{S} = \{(M, g) \mid \operatorname{area}(M) = 1\}$. On this space, we have the following result.

Theorem 6.2.2 The spectral l^2 -distance $d(\cdot, \cdot)$ is a metric on \mathfrak{S} . Hence it is also a metric on \mathfrak{D} .

With this theorem, one can use $d(\cdot, \cdot)$ as a rigorous measure to compare surfaces intrinsically. For example, this is critical for problems such as surface registration and classification. Next we prove Theorem 6.2.2 and show that the spectral l^2 -distance is a metric. It is easy to check that $d(\cdot, \cdot)$ is symmetric. To ensure $d(\cdot, \cdot)$ is a metric on \mathfrak{D} , we also need the following two lemmas.

Lemma 6.2.1 (triangle inequality) Given any three surfaces (M, g), (M', g'), $(N, \hat{g}) \in \mathfrak{S}$, then

$$d(M, M') \le d(M, N) + d(N, M').$$
(6.10)

[Proof]: Given an arbitrary eigen-system Φ of M, Φ' of M', and Θ of N. For

any point $x \in M, y \in M'$ and $z \in N$, we have:

$$\begin{aligned} d_{\Phi}^{\Phi'}(x,y) &\leq d_{\Phi}^{\Theta}(x,z) + d_{\Theta}^{\Phi'}(z,y) \\ \stackrel{\inf_{y\in M'}}{\Longrightarrow} & d_{\Phi}^{\Phi'}(x,M') &\leq d_{\Phi}^{\Theta}(x,z) + d_{\Theta}^{\Phi'}(z,M') \\ \stackrel{\inf_{z\in N}}{\Longrightarrow} & d_{\Phi}^{\Phi'}(x,M') &\leq d_{\Phi}^{\Theta}(x,N) + \inf_{z\in N} d_{\Theta}^{\Phi'}(z,M') \\ &\leq d_{\Phi}^{\Theta}(x,N) + d_{\Theta}^{\Phi'}(z,M') \end{aligned}$$

$$\begin{array}{ll} \stackrel{\int_{M}}{\Longrightarrow} & \int_{M} d_{\Phi}^{\Phi'}(x, M') & \leq \int_{M} d_{\Phi}^{\Theta}(x, N) + \int_{M} d_{\Theta}^{\Phi'}(z, M') \\ \implies & \int_{M} d_{\Phi}^{\Phi'}(x, M') & \leq \int_{M} d_{\Phi}^{\Theta}(x, N) + d_{\Theta}^{\Phi'}(z, M') \\ \stackrel{\int_{N}}{\Longrightarrow} & \int_{M} d_{\Phi}^{\Phi'}(x, M') & \leq \int_{M} d_{\Phi}^{\Theta}(x, N) + \int_{N} d_{\Theta}^{\Phi'}(z, M') \end{array}$$

Similarly, we can have:

$$\int_{M'} d_{\Phi}^{\Phi'}(M, y) \leq \int_N d_{\Phi}^{\Theta}(M, z) + \int_{M'} d_{\Theta}^{\Phi'}(N, y).$$

By the definition of d(M, M') we then have:

$$d(M, M') \leq d(M, N) + d(N, M'). \square$$

Lemma 6.2.2 For any two surfaces M, M' with Vol(M) = Vol(M'), then d(M, M') = 0 if and only if M and M' are isometric.

[Proof]: If M and M' are isometric, we have d(M, M') = 0 because the LB eigen-system is isometric invariant.

Next we show the opposite is also true. Let (M, g), (M'g') denote two surfaces with the same surface area. By the compactness of $\mathcal{B}(M)$ and $\mathcal{B}(M')$, we can find the optimal eigen-system $\Phi = \{(\lambda_i, \phi_i\}_{i=0}^{\infty}, \Phi' = \{(\lambda'_i, \phi'_i\}_{i=0}^{\infty} \text{ of } (M, g), (M', g'), \text{ respectively, such that}\}$

$$d(M, M') = max \left\{ \int_{M} d_{\Phi}^{\Phi'}(x, M') , \int_{M'} d_{\Phi}^{\Phi'}(M, y) \right\}.$$

Suppose d(M, M') = 0, we have:

$$d_{\Phi}^{\Phi'}(x,M') = 0, \ \forall \ x \in M \quad \text{and} \quad d_{\Phi}^{\Phi'}(M,y) = 0, \ \forall \ y \in M'.$$

Therefore,

$$\forall x \in M, \exists y \in M', \text{s.t.} \quad \frac{\phi_i(x)}{\sqrt{\lambda_i}} = \frac{\phi_i'(y)}{\sqrt{\lambda_i'}}, i = 1, 2, \cdots$$
 (6.11)

We denote $f: M \to M': x \longmapsto y$, and

$$\forall y \in M', \exists x \in M, \text{s.t.} \quad \frac{\phi_i(x)}{\sqrt{\lambda_i}} = \frac{\phi_i'(y)}{\sqrt{\lambda_i'}}, i = 1, 2, \cdots$$
 (6.12)

Similarly we define $h: M' \to M: y \mapsto x$. Both f, h are injective since $I_M^{\Phi}, I_{M'}^{\Phi'}$ are one-to-one. Thus $f \circ h = Id_{M'}$ and $h \circ f = Id_M$ and one can easily show that f and h are diffeomorphisms.

Moreover, if we integrate (6.11) for all $i \ge 1$:

$$0 = \int_{M} \frac{\phi_{i}(x)}{\sqrt{\lambda_{i}}} \mathrm{d}vol_{M}(x) = \int_{M} \frac{\phi_{i}'(f(x))}{\sqrt{\lambda_{i}'}} \mathrm{d}vol_{M}(x)$$
$$= \int_{M'} \frac{\phi_{i}'(y)}{\sqrt{\lambda_{i}'}} \mathcal{J}(h) \mathrm{d}vol_{M'}(y).$$

This means $\mathcal{J}(h)$, which is the Jacobian of h, is orthogonal to all $\phi'_i, i \geq 1$. Therefore, $\mathcal{J}(h)$ must be a constant. On the other hand, we have:

$$1 = \int_{M} dvol_{M}(x) = \frac{Vol(M')}{Vol(M)} \int_{M'} \mathcal{J}(h) dvol_{M'}(y)$$
$$= \frac{Vol(M')}{Vol(M)} \mathcal{J}(h) \int_{M'} dvol_{M'}(y) = \frac{Vol(M')}{Vol(M)} \mathcal{J}(h).$$

So we have $\mathcal{J}(h) = Vol(M)/Vol(M') = 1$. Similarly, one can show $\mathcal{J}(f) = 1$.

To conclude, we have shown that both f, h are isometry. Therefore M is isometric to M'.

In summary, the first lemma proves the triangular inequality and the second lemma tells us two surfaces of equal area are the same in \mathbb{R}^3 if and only if their spectral l^2 -distance is zero. This completes our proof that the spectral l^2 -distance is a rigorous metric on \mathfrak{D} .

6.3 Numerical Implementation of Spectral l²-distance

In this section, we first describe the numerical method to compute the eigensystem of the LB operator of a triangulated surface. After that, we develop the numerical approximation of the spectral l^2 -distance.

6.3.1 Eigen-system computation

Numerically we use the finite element method (FEM) to compute the eigensystem of the LB operator [79, 77]. For any given surface M in \mathbb{R}^3 , we represent M as a triangular mesh $\{V = \{v_i\}_{i=1}^N, T = \{T_l\}_{l=1}^L\}$, where v_i is the i-th vertex and T_l is the l-th triangle. We denote h_l as the diameter of the triangle T_l and $h = max \{h_l\}$. One can choose linear elements $\{\psi_i^h\}_{i=1}^N$, such that $\phi_i^h(v_j) = \delta_{i,j}$ and write $S^h = Span_{\mathbb{R}}\{\phi_i^h\}_{i=1}^N$. Then the discrete version of the continuous variational problem is to find a $\phi^h \in S^h$, such that

$$\sum_{l} \int_{T_{l}} \nabla_{M} \phi^{h} \nabla_{M} \psi^{h}_{n} = \lambda^{h} \sum_{l} \int_{T_{l}} \phi^{h} \psi^{h}_{n}, \ \forall \psi^{h}_{n} \in S^{h}.$$
(6.13)

If we write

$$\begin{cases} \phi^{h} = \sum_{i}^{N} x_{i} \psi_{i}^{h} \\ A^{h} = (a_{ij})_{N \times N}, a_{ij} = \sum_{l} \int_{T_{l}} \nabla_{M} \psi_{i}^{h} \nabla_{M} \psi_{j} \\ B^{h} = (b_{ij})_{N \times N}, b_{ij} = \sum_{l} \int_{T_{l}} \psi_{i}^{h} \psi_{j}^{h} \end{cases}$$

$$(6.14)$$

then the discrete variational problem is equivalent to the generalized matrix eigen-problem:

$$\begin{cases} A^h x = \lambda^h B^h x, \text{ where } x = (x_1, \cdots, x_N)^T \\ \phi^h = \sum_i^N x_i \psi_i^h \end{cases}$$
(6.15)

This can be solved with existing numerical packages such as MATLAB.

6.3.2 Implementation of the spectral *l*²-distance

The implementation of the spectral l^2 -distance includes two parts, the distance itself and the optimal embedding basis. To implement the spectral l^2 -distance in general form is computationally not tractable. However, Uhlenbeck [100] proved that surfaces with simple eigen-systems are generic. In other words, most surfaces in practice have simple eigen-systems. In particular, our applications will focus on studying anatomical brain structures, which usually have simple eigen-systems in our experience. We describe here how to compute the spectral l^2 -distance between surfaces with simple eigen-systems.

Let (M, g) and (M, g') be two surfaces with simple eigen-systems $\Phi = \{(\lambda_i, \phi_i)\}_{i=0}^{\infty}$ and $\Phi' = \{(\lambda'_i, \phi'_i)\}_{i=0}^{\infty}$ respectively. Then the only freedom in determining the optimal basis is the sign of the eigenfunctions. If we write $\Gamma = \{\gamma = (\gamma_0, \gamma_1, \cdots) \mid \gamma_i \in \{1, -1\}\}$ and $\Phi'_{\gamma} = \{(\lambda'_i, \gamma_i \phi'_i)\}_{i=0}^{\infty}$, the spectral l^2 -distance will have the following form:

$$d(M, M') = max \left\{ \inf_{\gamma \in \Gamma} \int_{M} d_{\Phi}^{\Phi'_{\gamma}}(x, M') \mathrm{d}vol_{M}(x) , \\ \inf_{\gamma \in \Gamma} \int_{M'} d_{\Phi}^{\Phi'_{\gamma}}(M, y) \mathrm{d}vol_{M'}(y) \right\}$$
(6.16)

For surfaces with simple eigen-systems, this shows that the computation of the spectral l^2 -distance is a combinatorial optimization problem.

For two surfaces $M = \{V = \{v_i\}_{i=1}^N, T = \{T_l\}_{l=1}^L\}$ and $M' = \{V' = \{v'_i\}_{i=1}^{N'}, T' = \{T'_l\}_{l=1}^{L'}\}$, it is numerically intractable to compute infinite number of eigenfunctions to obtain the exact embedding defined in (6.4). However, since $\{\phi_i(x)/\sqrt{\lambda_i}\}_{i=1}^{\infty}$ is l^2 convergent, we can approximate the spectral l^2 -distance to the theoretical definition by using the first n eigenfunctions in the following way.

Let us write

$$I_{M}^{\Phi(n)}(x) = \left\{\frac{\phi_{i}(x)}{\sqrt{\lambda_{i}}}\right\}_{i=1}^{n}, I_{M'}^{\Phi_{\gamma}'(n)}(x) = \left\{\frac{\gamma_{i}\phi_{i}'(x)}{\sqrt{\lambda_{i}'}}\right\}_{i=1}^{n}$$

$$d_{\Phi(n)}^{\Phi'(n)}(x, M') = \min_{y \in M'} ||I_{M}^{\Phi(n)}(x) - I_{M'}^{\Phi_{\gamma}'(n)}(y)||_{2} ,$$

$$d_{\Phi(n)}^{\Phi'\gamma(n)}(M, y) = \min_{x \in M} ||I_{M}^{\Phi(n)}(x) - I_{M'}^{\Phi_{\gamma}'(n)}(y)||_{2} ,$$

$$d_{n}(M, M') = max\{\inf_{\gamma \in \Gamma} \int_{M} d_{\Phi(n)}^{\Phi_{\gamma}'(n)}(x, M') dvol_{M}(x) ,$$

$$\inf_{\gamma \in \Gamma} \int_{M'} d_{\Phi(n)}^{\Phi_{\gamma}'(n)}(M, y) dvol_{M'}(y) \}.$$
(6.17)

The integral on M can be approximated by:

$$\int_{M} d_{\Phi(n)}^{\Phi'_{\gamma}(n)}(x, M') \mathrm{d}s \approx \frac{1}{\operatorname{area}(M)} \sum_{i=1}^{N} d_{\Phi(n)}^{\Phi'_{\gamma}(n)}(v_i, M') \cdot A_i$$
(6.18)

where $A_i = \frac{1}{3} \sum_{v_i \in T_l} area(T_l)$. One can further show that:

$$d_n(M, M') \longrightarrow d(M, M')$$
, when $n \to \infty$ (6.19)

because $\{\phi_i(x)/\sqrt{\lambda_i}\}_{i=1}^{\infty}$ and $\{\gamma_i\phi'_i(x)/\sqrt{\lambda'_i}\}_{i=1}^{\infty}$ are l^2 convergent, and $\phi_i(x)/\sqrt{\lambda_i}$ and $\phi'_i(x)/\sqrt{\lambda'_i}$ approach zero quickly. This means the spectral l^2 -distance actually is dominated by the first n eigenfunctions. In this case, the optimal embedding basis of $d_n(\cdot, \cdot)$ will be the first n basis functions of the optimal embedding basis of the spectral l^2 -distance. Therefore, we just need to compute $d_n(\cdot, \cdot)$ to obtain the first n eigenfunctions of the optimal embedding basis.

To demonstrate the approximated distance and its ability of picking the optimal embedding basis for shape analysis, we show in Fig. 6.2 the embedding of two cortical surfaces with the first three eigenfunctions. Out of the eight possible embeddings, the one achieving the distance $d_3(M, M')$ is highlighted inside the red circle. While the two surfaces have significantly different poses in the Euclidean space, their embeddings are very close to each other, which reflects the fact that these two surfaces share very similar geometry. This example clearly illustrates the power of the optimal embedding in characterizing the intrinsic geometry of surfaces.



Figure 6.2: The first column: two input surfaces for computing their spectral l^2 distance. The last four columns: 8 possible embedding due to sign ambiguity of the first three eigenfunctions.

Because the spectral l^2 -distance depends completely on the intrinsic geometry of surfaces, we expect intuitively its value will reflect the similarity between surfaces. To show this property, we computed the pairwise spectral l^2 -distance of six surfaces: two vervet cortical surfaces, two caudate and two hippocampal surfaces using the first 20 LB eigenfunctions. To visualize the embedding, we used the multi-dimension scaling (MDS) to project these surfaces into a 2D plane as shown in Fig. 6.3. It clearly shows that similar surfaces cluster together under the spectral l^2 -distance.



Figure 6.3: MDS embedding results with the spectral l^2 -distance.

6.4 Applications: Sulci Identification on Vervet Cortical Surfaces

With the spectral l^2 -distance and the resulting optimal embedding, we can compare surfaces in a common space. This is a critical component in many different applications such as identifying meaningful and stable parts across a large group of surfaces with similar intrinsic geometry, constructing correspondences between surfaces, and classifying surfaces. As a demonstration, we apply our spectral l^2 distance to the problem of automated sulci identification in 3D medical image analysis. The identification of major sulci is one of the critical steps in cortical surface analysis [90]. However, the manual labeling of sulcal regions becomes impractical with the increasing availability of large data set. Therefore, it is important to find a robust way to identify the major sulci automatically. In this section, we develop a robust approach to identify sulcal regions based on the spectral l^2 -distance.

6.4.1 Cortical parcellation

To identify sulcal regions of a given cortical surface, the first step is to parcellate the cortical surface into sulcal and gyral regions. Using the mean curvature as an image defined on the surface, we extend the convexified version of Chan-Vese (CV) [20, 18] model to 3D triangulated surfaces for the extraction of sulcal regions (see Chapter 2). Let $I: M \to \mathbb{R}$ be an image on a surface M. The parcellation



Figure 6.4: First column: two different views of CV segmentation. The surfaces are color coded with its mean curvature and the red contours mark the boundary of the sulcal and gyral regions. Second column: sulcal regions (top) and the corresponding sulcal lines (bottom).

of the cortical surfaces then can be obtained by solving the following convexified version of CV segmentation model on M;

$$\arg\min_{\substack{0 \le u \le 1 \\ c1, c2}} \left(\int_M |\nabla_M u| + \mu \int_M u((c_1 - I)^2 - (c_2 - I)^2) \right)$$
(6.20)

Once we have the sulcal regions, we can compute the sulcal lines with the Reeb graph of their eigenfunctions [88]. In Fig. 6.4, we show the segmentation result obtained by applying the CV segmentation model to a vervet cortex. The resulting sulcal region and sulcal lines are also shown.

6.4.2 Sulci identification

Because all cortical surfaces share similar geometry, they will cluster together in the embedding space determined by the spectral l^2 -distance. As we illustrated in section 3, these surfaces overlap quite well in the embedding space. For major sulci on cortical surfaces, their relative positions on the cortex are stable and so are their locations in the embedding space. This suggests that the same sulcal line from different cortical surfaces will form clusters in the embedding space. To achieve the sulcal identification task, then the natural question is how many basis functions we should choose to find the optimal embedding and achieve robust performance. The intuition is that it suffices to choose a number n such that corresponding sulcal regions are well clustered in the embedding space and the distance among clusters is much larger than the distance between the surfaces. As shown in Fig. 6.5, we find in our experience that n = 3 or 4 works well to identify sulcal regions of vervet cortical surfaces.

Assuming we have a large group of cortical surfaces, we use template matching to build an automated approach to label sulci on vervet cortical surfaces. Let Mdenote a template cortical surface that has a set of manually labeled sulci to be identified. For an arbitrary vervet cortical surface M', we find the major sulci on M' by comparing it with M in the embedding space determined by the spectral l^2 distance. The detailed algorithm for the automated sulcal identification process is as follows:



Figure 6.5: The first column: two vervet cortical surfaces with CV segmentation results; the second column top: sulcal regions in the Euclidean space; the second column bottom: sulcal regions embedded into eigen-spaces by the first three functions of optimal embedding basis.

- Use the CV segmentation model to extract sulcal regions for both M and M'. Let the sulcal regions of M, M' be denoted by {l₁, ..., l_k}, {l'₁, ..., l'_{k'}}, respectively. For the template cortical surface M, the labeling of its sulci is known a priori.
- 2. Compute the spectral l^2 -distance between M and M' to obtain the optimal embedding basis $\{\phi_i/\sqrt{\lambda_i}\}, \{\phi'_i/\sqrt{\lambda'_i}\}$ for M, M' respectively.
- 3. To identify the sulcal regions $\{l'_1, \dots, l'_{k'}\}$ of M', we compute the sulcal region correspondence $\sigma : \{l'_1, \dots, l'_{k'}\} \to \{l_1, \dots, l_k\}$. For each sulcal region l'_j of the target surface M', we compute its spectral pre- l^2 -distance in the embedding space to each sulcal region of M with respect to $\{\phi_i/\sqrt{\lambda_i}\}, \{\phi'_i/\sqrt{\lambda'_i}\}$. We label l'_j to a sulcal region $\sigma(l'_j)$ of M who attains the shortest distance to l'_j .

4. To find the sulcal lines, the Reeb graph of the LB eigenfunction is computed for each sulcal region.

6.5 Experimental Results

In this section, we present experimental results to demonstrate our sulcal identification method using the spectral l^2 -distance. In the first experiment, we demonstrate the pose and scale invariance of our method by applying it to identify sulci on cortical surfaces that have very different Euclidean coordinate representations. In the second experiment, we apply our method to a large data set of 698 vervet cortical surfaces to validate the robustness of our method and demonstrate its potential in brain mapping studies.

6.5.1 Invariance of our method

In the first experiment, we illustrate the pose and scale invariance of our method by comparing a template brain and five other brains with different poses and scales. The results are shown in Fig 6.6. To visualize the embedding images of two surfaces, we choose the first three eigenfunctions and compute $d_3(\cdot, \cdot)$ to obtain the optimal embedding basis. The five surfaces shown in the second row have various pose and scales differences in the Euclidean space as compared to the template. As shown in the third row, the positions of similar sulci in the Euclidean spaces are misaligned. However, similar sulci will automatically cluster together in the embedding space under the optimal basis as shown in the fourth row. This demonstrates that the spectral l^2 -distance and the embedding based on its associated optimal basis reflect intrinsic geometry and they are rotation, translation and scale invariant.



Figure 6.6: The first row: the template cortical surface with sulcal regions marked in red; the second row: target cortical surfaces with sulcal regions marked by different colors; the third row: sulcal regions of the template and target cortical surfaces in the Euclidean space; the fourth row: sulcal regions of the template and target cortical surfaces in the embedding space.



Figure 6.7: Green: central sulcus; Red: arcuate sulcus; Blue: principal sulcus.

6.5.2 Application to brain mapping studies

In this experiment, we apply our algorithm to a large data set to demonstrate its robustness and application in brain mapping studies. The input data are cortical surfaces from 349 vervet brains in a study of genetics and brain morphometry. Because left and right cortical surfaces are separated for each brain, we have overall 698 surfaces. To label the major sulci on all the brains, we pick a template surface whose sulci are manually labeled and then apply the template matching approach to identify corresponding landmark curves on the whole data set. For all the 698 cortical surfaces, the approximated spectral l^2 -distance d_4 was used to compute the optimal embedding basis.

As an illustration, we show in Fig. 6.7 the identification results of eight surfaces. The template is shown in the center with manually labeled sulcal lines. Guided by the template matching process, our method automatically found the

sulcus	mean length (mm)	p-value
central sulcus (L)	26.5253	
central sulcus (R)	24.9897	5.2386e-17
arcuate sulcus (L)	22.5437	
arcuate sulcus (R)	21.2125	3.1261e-10
principal sulcus (L)	16.1234	
principal sulcus (R)	14.8789	3.4633e-008

Table 6.1: Statistics of sulcal lines on the 698 surfaces.

sulcal lines on the eight surfaces as shown in Fig. 6.7. We can clearly see that our method is robust to different poses and local variations.

Using the extracted sulcal lines, we can derive geometric measurement and perform statistical analysis for group studies. To demonstrate the potential of our results for brain mapping studies, we computed the length of the central sulcus, arcuate sulcus and principal sulcus for all cortical surfaces and compared the asymmetry of these three sulci between the left and right hemisphere. For all 349 brains, the length differences of the three sulci on the left and right cortical surface are plotted in Fig. 6.8. For the length difference of each sulcus, we applied a t-test to test the statistical significance. In Table 6.1, we list the mean length of all sulci and the p-values obtained from testing the left/right asymmetry. We can see from the results that the central sulcus, arcuate sulcus and principal sulcus on the left cortical surface of vervet brains are longer than the corresponding sulci on the right cortical surface.



Figure 6.8: Length differences of central $sulcus(\circ)$, arcuate sulcus(+) and principal $sulcus(\cdot)$ for the 349 vervet brains.

6.6 Conclusions and Future Work

In summary we proposed a general frame work to define a mathematically rigorous distance between surfaces by using the eigen-system of the LB operator. This distance captures the intrinsic geometry of surfaces and is invariant to pose and scale differences. As an application, we developed a robust approach to identify major sulci of vervet cortical surfaces. In our future research, we will investigate more applications of the spectral l^2 -distance, such as surface mapping and surface classification, and extend our sulci identification method to human brains.

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