Local and Global Well-Posedness for Aggregation Equations and Patlak-Keller-Segel Models with Degenerate Diffusion

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July 11, 2010

Abstract

Recently, there has been a wide interest in the study of aggregation equations and Patlak-Keller-Segel (PKS) models for chemotaxis with degenerate diffusion. The focus of this paper is the unification and generalization of the well-posedness theory of these models. We prove local well-posedness on bounded domains for dimensions $d \geq 2$ and in all of space for $d \geq 3$, the uniqueness being a result previously not known for PKS with degenerate diffusion. We generalize the notion of criticality for PKS and show that subcritical problems are globally well-posed. For a fairly general class of problems, we prove the existence of a critical mass which sharply divides the possibility of finite time blow up and global existence. Moreover, we compute the critical mass for fully general problems and show that solutions with smaller mass exists globally. For a class of supercritical problems we prove finite time blow up is possible for initial data of arbitrary mass.

1 Introduction

Nonlocal aggregation phenomena have been studied in a wide variety of biological applications such as migration patterns in ecological systems [14, 55, 47, 26, 15] and Patlak-Keller-Segel (PKS) models of chemotaxis [25, 49, 27, 33, 36]. Diffusion is generally included in these models to account for the dispersal of organisms. Classically, linear diffusion is used, however recently, there has been a widening interest in models with degenerate diffusion to include over-crowding effects [55, 15]. The parabolic-elliptic PKS is the most widely studied model for aggregation, where the nonlocal effects are modeled by convolution with the Newtonian or Bessel potential. On the other hand, in population dynamics, the nonlocal effects are generally modeled with smooth, fast-decaying kernels. However, all of these models are describing the same mathematical phenomenon: the competition between nonlocal aggregation and diffusion. For this reason, we are interested in unifying and extending the local and global well-posedeness theory of general aggregation models with degenerate diffusion of the form

$$u_t + \nabla \cdot (u\vec{v}) = \Delta A(u) \quad \text{in } [0,T) \times D, \tag{1a}$$

$$\vec{v} = \nabla \mathcal{K} * u. \tag{1b}$$

Mathematical works most relevant this paper are those with degenerate diffusion [8, 52, 53, 54, 11, 35, 34] and those from the classical PKS literature [31, 24, 13, 12]. See also [32].

Existence theory is complicated by the presence of degenerate diffusion and singular kernels such as the Newtonian potential. Bertozzi and Slepčev in [8] prove existence and uniqueness of models with general diffusion but restrict to non-singular kernels. Sugiyama [54] proved local existence for models with power-law diffusion and the Bessel potential for the kernel, but uniqueness of solutions was left open. We extend the work of [8] to prove the local existence of (1) with degenerate diffusion and singular kernels including the Bessel and Newtonian potentials. The existing work on uniqueness of these problems included a priori regularity assumptions [35] or the use of entropy solutions [15] (see also [18]). The Lagrangian method introduced by Loeper in [44] estimates the difference of weak solutions in the Wasserstein distance and is very useful for inviscid problems or problems with linear diffusion [43, 6, 21]. In the presence of nonlinear diffusion, it seems more natural to approach uniqueness in H^{-1}, where the diffusion is monotone (see [56]).

This is the approach taken in [4, 8], which we extend to handle singular kernels such as the Newtonian potential, proving uniqueness of weak solutions with no additional assumptions, provided the domain is bounded or $d \geq 3$. The new insight is using a more refined estimate to handle the lower regularity of $\nabla K * u$, similar to the traditional proof of uniqueness of $L^1 \cap L^\infty$-vorticity solutions to the 2D Euler equations [59, 45].

There is a natural notion of criticality associated with this problem, which roughly corresponds to the balance between the aggregation and diffusion. For problems with homogeneous kernels and power-law diffusion, $\mathcal{K} = c\,|x|^{2-d}$ and $A(u) = u^m$, a simple scaling heuristic suggests that these forces are in balance if $m = 2-2/d$ [11]. If $m > 2-2/d$ then the problem is subcritical and the diffusion is dominant. On the other hand, if $m < 2-2/d$ then the problem is supercritical and the aggregation is dominant. For the PKS with power-law diffusion, Sugiyama showed global existence for subcritical problems and that finite time blow up is possible for supercritical problems [54, 53, 52]. We extend this notion of criticality to general problems by observing that only the behavior of the solution at high concentrations will divide finite time blow up from global existence (see Definition 6). We show global well-posedness for subcritical problems and finite time blow up for certain supercritical problems.

If the problem is critical, it is well-known that in PKS there exists a critical mass, and solutions with larger mass can blow up in finite time [13, 31, 10, 24, 12, 16, 11, 52, 53]. For linear diffusion, the same critical mass has been identified for the Bessel and Newtonian potentials [13, 16]; however for nonlinear diffusion, the critical mass has only been identified for the Newtonian potential [11]. In this paper we extend the free energy methods of [11, 24, 12] to estimate the critical mass for a wide range of kernels and nonlinear diffusion, which include these known results. For a smaller class of problems, including standard PKS models, we show this estimate is sharp.

The problem (1) is formally a gradient flow with respect to the Euclidean Wasserstein distance for the *free energy*

$$\mathcal{F}(u(t)) = S(u(t)) - \mathcal{W}(u(t)), \tag{2}$$

where the *entropy* $S(u(t))$ and the *interaction energy* $\mathcal{W}(u(t))$ are given by

$$S(u(t)) = \int \Phi(u(x,t))dx,$$
$$\mathcal{W}(u(t)) = \frac{1}{2}\int\int u(x,t)\mathcal{K}(x-y)u(y,t)dxdy.$$

For the degenerate parabolic problems we consider, the *entropy density* $\Phi(z)$ is a strictly convex function satisfying

$$\Phi''(z) = \frac{A'(z)}{z}, \quad \Phi'(1) = 0, \quad \Phi(0) = 0. \tag{3}$$

See [20] for more information on these kinds of entropies. Although there is a rich theory for gradient flows of this general type when the kernel is regular and λ-convex [46, 3, 19], the kernels we consider here are more singular and the notion of displacement convexity introduced in [46] no longer holds. Although the rigorous results of the gradient flow theory may not be applicable, the free energy (2) is still the important dissipated quantity in the global existence and finite time blow up arguments. The free energy has been used by many authors for the same purpose, see for instance [52, 13, 11, 5, 12]. For the remainder of the paper we only consider initial data with finite free energy, although the local existence arguments may hold in more generality.

There is a vast literature of related works on models similar to (1). For literature on PKS we refer the reader to the review articles [30, 29]; see also [28, 23, 16] for parabolic-parabolic Keller-Segel systems. For the inviscid problem, see the recent works of [37, 5, 4, 6, 19]. For a study of these equations with fractional linear diffusion see [38, 39, 9]. When the diffusion is sufficiently nonlinear and the kernel is in L^1, (1) may be written as a regularized interface problem, a notion studied in [50]. Critical mass behavior is also a property of the marginal unstable thin film equation [58, 7] and the mass-critical nonlinear Schrödinger equation [57].

Outline of Paper. In Section 1 we state the relevant definitions and notation. Furthermore, we give a summary of the main results but reserve the proofs for subsequent sections. In Section 2 we prove the uniqueness result. Local existence is proved in Section 3. The first result is proved on bounded domains in $d \geq 2$ and the second is proved on all space for $d \geq 3$. In Section 4 we prove a continuation theorem. The global well-posedness results are proved in Section 5. In addition, we prove a uniform boundedness theorem. Finally, in Section 6 we prove the finite time blow up results.

1.1 Definitions and Assumptions

We consider either $D = \mathbb{R}^d$ with $d \geq 3$ or D smooth, bounded and convex with $d \geq 2$, in which case we impose no-flux conditions

$$(-\nabla A(u) + u\nabla \mathcal{K} * u) \cdot \nu = 0 \text{ on } \partial D \times [0, T), \tag{4}$$

where ν is the outward unit normal to D. We neglect the case $D = \mathbb{R}^2$ for technicalities introduced by the logarithmic potential.

We denote $D_T := (0, T) \times D$. We also denote $\|u\|_p := \|u\|_{L^p(D)}$ where L^p is the standard Lebesgue space. We denote the set $\{u > k\} := \{x \in D : u(x) > k\}$, if $S \subset \mathbb{R}^d$ then $|S|$ denotes the Lebesgue measure and $\mathbf{1}_S$ denotes the standard characteristic function. In addition, we use $\int f dx := \int_D f dx$, and only indicate the domain of integration where it differs from D. We also denote the weak L^p space by $L^{p,\infty}$ and the associated quasi-norm

$$\|f\|_{L^{p,\infty}} = \left(\sup_{\alpha>0} \alpha^p \lambda_f(\alpha)\right)^{1/p},$$

where $\lambda_f(\alpha) = |\{f > \alpha\}|$ is the distribution function of f. Given an initial condition $u(x, 0)$ we denote its mass by $\int u(x, 0)dx = M$. In formulas we use the notation $C(p, k, M, ..)$ to denote a generic constant, which may be different from line to line or even term to term in the same computation. In general, these constants will depend on more parameters than those listed, for instance those associated with the problem such as $\mathcal{K}$ and the dimension but these dependencies are suppressed. We use the notation $f \lesssim_{p,k,...} g$ to denote $f \leq C(p, k, ..)g$ where again, dependencies that are not relevant are suppressed.

We now make reasonable assumptions on the kernel which include important cases of interest, such as when $\mathcal{K}$ is the fundamental solution or Green's function of an elliptic PDE. To this end we state the following definition.

Definition 1 (Admissible Kernel)**.** We say a kernel $\mathcal{K}$ is *admissible* if $\mathcal{K} \in W^{1,1}_{loc}$ and the following holds:

(R) $\mathcal{K} \in C^3 \setminus \{0\}$.

(KN) $\mathcal{K}$ is radially symmetric, $\mathcal{K}(x) = k(|x|)$ and $k(|x|)$ is non-increasing.

(MN) $k''(r)$ and $k'(r)/r$ are monotone on $r \in (0, \delta)$ for some $\delta > 0$.

(BD) $D^2\mathcal{K}(x) = \mathcal{O}(|x|^{-d})$ as $|x| \to 0$, $|x| > 0$.

If the domain $D = \mathbb{R}^d$ we must impose an additional decay requirement:

(D) $D^2\mathcal{K}(x) = \mathcal{O}(|x|^{-d})$ as $x \to \infty$.

These conditions imply that if $\mathcal{K}$ is singular, the singularity is restricted to the origin. Note also, that the Newtonian and Bessel potentials are both admissible for all dimensions $d \geq 2$; hence, the PKS and related models are included in our analysis.

We now make precise what kind of nonlinear diffusion we are considering.

Definition 2 (Admissible Diffusion Functions)**.** We say that the function $A(u)$ is an admissible diffusion function if:

(D1) $A \in C^1([0, \infty))$ with $A'(z) > 0$ for $z \in (0, \infty)$.

(D2) $A'(z) > c$ for $z > z_c$ for some $c, z_c > 0$.

(D3) $\int_0^1 A'(z)z^{-1}dz < \infty$.

This definition includes power-law diffusion $A(u) = u^m$ for $m > 1$. Note that **(D3)** requires the diffusion to be degenerate at $u = 0$, however it is permitted to behave linearly at infinity. Furthermore, on bounded domains condition **(D3)** can be relaxed without any significant modification to the methods. Following [8], the notions of weak solution are defined separately for bounded and unbounded domains.

Definition 3 (Weak Solutions on Bounded Domains)**.** Let $A(u)$ and $\mathcal{K}$ be admissible, and $u_0(x) \in L^\infty(D)$ be non-negative. A non-negative function $u : [0,T] \times D \to [0,\infty)$ is a weak solution to (1) if $u \in L^\infty(D_T)$, $A(u) \in L^2(0,T,H^1(D))$, $u_t \in L^2(0,T,H^{-1}(D))$ and

$$\int_0^T \int u\phi_t \, dxdt = \int u_0(x)\phi(0,x)dx + \int_0^T \int (\nabla A(u) - u\nabla\mathcal{K} * u) \cdot \nabla\phi \, dxdt, \tag{5}$$

for all $\phi \in C^\infty(\overline{D_T})$ such that $\phi(T) = 0$.

It follows that $u\nabla\mathcal{K} * u \in L^2(D_T)$; therefore, definition 3 is equivalent to the following,

$$\langle u_t(t), \phi\rangle = \int (-\nabla A(u) + u\nabla\mathcal{K} * u) \cdot \nabla\phi \, dx, \tag{6}$$

for all test functions $\phi \in H^1$ for almost all $t \in [0,T]$. Above $\langle\cdot,\cdot\rangle$ denotes the standard dual pairing between H^1 and H^{-1}. Similarly for $\mathbb{R}^d$ we define the following notion of weak solution as in [8].

Definition 4 (Weak Solution in $\mathbb{R}^d$, $d \geq 3$)**.** Let A and $\mathcal{K}$ be admissible, and $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be non-negative. A function $u : [0,T]\times\mathbb{R}^d \to [0,\infty)$ is a weak solution of (1) if $u \in L^\infty((0,T)\times\mathbb{R}^d) \cap L^\infty(0,T,L^1(\mathbb{R}^d))$, $A(u) \in L^2(0,T,\dot{H}^1(\mathbb{R}^d))$, $u\nabla K * u \in L^2(D_T)$, $u_t \in L^2(0,T,\dot{H}^{-1}(\mathbb{R}^d))$, and for all test functions $\phi \in \dot{H}^1(\mathbb{R}^d)$ for a.e $t \in [0,T]$ (6) holds.

Since (1) conserves mass, the natural notion of criticality is with respect to the usual mass invariant scaling $u_\lambda(x) = \lambda^d u(\lambda x)$. A simple heuristic for understanding how this scaling plays a role in the global existence is seen by examining the case of power-law diffusion and homogeneous kernel, $A(u) = u^m$ and $\mathcal{K}(x) = |x|^{-d/p}$. Under this mass invariant scaling the free energy (2) becomes,

$$\mathcal{F}(u_\lambda) = \lambda^{dm-d}S(u) - \lambda^{d/p}\mathcal{W}(u).$$

As $\lambda \to \infty$, the entropy and the interaction energy are comparable if $m = (p+1)/p$. We should expect global existence if $m > (p+1)/p$, as the diffusion will dominate as u grows, and possibly finite time blow up if $m < (p+1)/p$ as the aggregation will instead be increasingly dominant. We consider inhomogeneous kernels and general diffusion, however for the problem of global existence, only the behavior as $u \to \infty$ will be important, in contrast to the problem of local existence. Noting that $|x|^{-d/p}$ is, in some sense, the representative singular kernel in $L^{p,\infty}$ leads to the following definition. This critical exponent also appears indirectly in [42].

Definition 5 (Critical Exponent)**.** Let $d \geq 3$ and $\mathcal{K}$ be admissible such that $\mathcal{K} \in L^{p,\infty}_{loc}$ for some $d/(d-2) \leq p < \infty$. Then the *critical exponent* associated to $\mathcal{K}$ is given by

$$1 < m^\star = \frac{p+1}{p} \leq 2 - 2/d.$$

If $D^2\mathcal{K}(x) = \mathcal{O}(|x|^{-2})$ as $x \to 0$, then we take $m^\star = 1$.

Remark 1. The case $m^\star = 1$ implies at worst a logarithmic singularity as $x \to 0$ and if $d = 2$ then all admissible kernels have $m^\star = 1$.

Now we define the notion of criticality. It is easier to define this notion in terms of the quantity $A'(z)$, as opposed to using $\Phi(z)$ directly.

Definition 6 (Criticality)**.** We say that the problem is *subcritical* if

$$\liminf_{z\to\infty} \frac{A'(z)}{z^{m^\star-1}} = \infty,$$

critical if

$$0 < \liminf_{z\to\infty} \frac{A'(z)}{z^{m^\star-1}} < \infty,$$

and *supercritical* if

$$\liminf_{z\to\infty} \frac{A'(z)}{z^{m^\star-1}} = 0.$$

Notice that in the case of power-law diffusion, $A(u) = u^m$, subcritical, critical and supercritical respectively correspond to $m > m^\star, m = m^\star$ and $m < m^\star$. Moreover, in the case of the Newtonian or Bessel potential, $m^\star = 2 - 2/d$ and the critical diffusion exponent of the PKS models discussed in [53, 52, 11] is recovered.

1.2 Summary of Results

The proof of local existence follows the work of Bertozzi and Slepčev [8], where (1) is approximated by a family of uniformly parabolic problems. The primary new difficulty, due to the singularity of the kernel, is obtaining uniform a priori L^∞ bounds, which is overcome here using the Alikakos iteration [2]. Solutions are first constructed on bounded domains.

Theorem 1 (Local Existence on Bounded Domains, $d \geq 2$)**.** *Let $A(u)$ and $\mathcal{K}(x)$ be admissible. Let $u_0(x) \in L^\infty(D)$ be a non-negative initial condition, then* (1) *has a weak solution u on $[0,T] \times D$, for some $T > 0$. Additionally, $u \in C([0,T]; L^p(D))$ for $p \in [1,\infty)$.*

In dimensions $d \geq 3$ we also construct local solutions on $\mathbb{R}^d$ by taking the limit of solutions on bounded domains.

Theorem 2 (Local Existence in $\mathbb{R}^d$, $d \geq 3$)**.** *Let $A(u)$ and $\mathcal{K}(x)$ be admissible. Let $u_0(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be a non-negative initial condition, then* (1) *has a weak solution u on $\mathbb{R}^d_T$, for some $T > 0$. Additionally, $u \in C([0,T]; L^p(\mathbb{R}^d))$ for all $1 \leq p < \infty$ and the mass is conserved.*

The free energy is a dissipated quantity for weak solutions and is the key tool for the global theory.

Proposition 1 (Energy Dissipation)**.** *Weak solutions to* (1) *satisfy the energy dissipation inequality for a.e. t,*

$$\mathcal{F}(u(t)) + \int_0^t \int \frac{1}{u} \left|A'(u)\nabla u - \nabla \mathcal{K} * u\right|^2 dxdt \leq \mathcal{F}(u_0(x)). \tag{7}$$

As in [8], uniqueness holds on bounded, convex domains in $d \geq 2$ or on $\mathbb{R}^d$ for $d \geq 3$. The proof also holds for more general diffusion (e.g. fast diffusion) or no diffusion at all.

Theorem 3 (Uniqueness)**.** *Let $D \subset \mathbb{R}^d$ for $d \geq 2$ be bounded and convex, then weak solutions to* (1) *are unique. The conclusion also holds on $\mathbb{R}^d$ for $d \geq 3$.*

We also prove the following continuation theorem, which generalizes similar theorems used in for instance [13, 11]. The proof follows the well known approach of first bounding intermediate L^p norms and using Alikakos iteration [2] to conclude the solution is bounded in L^∞ (for instance, see [34, 54, 11, 22, 31, 13]).

Theorem 4 (Continuation)**.** *The weak solution to* (1) *has a maximal time interval of existence $T_\star$ and either $T_\star = \infty$ or $T_\star < \infty$ and*

$$\lim_{t \nearrow T_\star} \|u\|_p = \infty,$$

for all $(2-m)/(2-m^\star) < p \leq \infty$. Here m is such that $1 \leq m \leq m^\star$ and $\liminf_{z\to\infty} A'(z) z^{1-m} > 0$.

For the case $m^\star = 2-2/d$, Blanchet et al. [11] identified the critical mass for the problem with the Newtonian potential, $\mathcal{K} = c_d |x|^{d-2}$, and $A(u) = u^m$. The authors show that if $M < M_c$ then the solution exists globally and if $M > M_c$ then the solution may blow up in finite time. There M_c is identified as

$$M_c = \left(\frac{2}{(m^\star - 1)C_0 c_d \omega_d^{(d-2)/d}} \right)^{1/(2-m^\star)},$$

where ω_d is the volume of the unit sphere in $\mathbb{R}^d$ and C_0 is the best constant in the Hardy-Littlewood-Sobolev inequality given below in Lemma 4. It is natural to ask the same question for more general cases. In this work we generalize these results to include inhomogeneous kernels and general nonlinear diffusion. First, we state the generalization of the finite time blow up results.

Theorem 5 (Finite Time Blow Up for Critical Problems: $m^\star > 1$)**.** *Let $\mathcal{K}$ and $A(u)$ be admissible and satisfy*

(B1) $\mathcal{K}(x) = c|x|^{2-d/\gamma} + o(|x|^{2-d/\gamma})$ *as* $x \to 0$ *for some* $c > 0$ *and* $1 \leq \gamma < d/2$.

(B2) $x \cdot \nabla \mathcal{K}(x) \leq (2 - d/\gamma)\mathcal{K}(x) + C_1 + C_2 |x|^2$ *for all* $x \in \mathbb{R}^d$, *for some* $C_1, C_2 \geq 0$.

(B3) $A'(z) = m\overline{A}z^{m-1} + o(z^{m-1})$ *as* $z \to \infty$ *for some* $m > 1, \overline{A} > 0$.

(B4) $A(z) \leq (m-1)\Phi(z)$ *for all* $z > R$, *for some* $R > 0$.

Suppose the problem is critical, that is $m = m^\star = 1 + 1/\gamma - 2/d$. *Then the critical mass* M_c *satisfies*

$$M_c = \left(\frac{2\overline{A}}{(m^\star - 1)C_0 c \omega_d^{(d/\gamma-2)/d}} \right)^{1/(2-m^\star)},$$

and for all $M > M_c$ *there exists a solution to* (1) *which blows up in finite time with* $\|u_0\|_1 = M$.

Theorem 6 (Finite Time Blow Up for Supercritical Problems)**.** *Let $\mathcal{K}$ satisfy **(B1)** and **(B2)** in Theorem 5 for some γ, $1 \leq \gamma < d/2$ and $A(u)$ satisfy **(B3)** and **(B4)** in Theorem 5 with $1 < m < m^\star = 1 + 1/\gamma - 2/d$. Then for all $M > 0$ there exists a solution which blows up in finite time with $\|u_0\|_1 = M$.*

The Newtonian and Bessel potentials both satisfy these conditions (Lemma 2.2, [52]), and so the results apply to PKS with degenerate diffusion. However, condition **(B2)** is somewhat restrictive, as it does not allow the homogeneity to be violated severely in the limit $x \to 0$. Power-law diffusion satisfies conditions **(B3)** and **(B4)**; however, **(B4)** is also restrictive, for example, $A(u) = u^m - u$ for u large does not satisfy the condition.

The accompanying global existence theorem is significantly more inclusive than the blow up theorems, both in the kinds of kernels and nonlinear diffusion considered. As in Theorem 5, the estimate of the critical mass only depends on the leading order term of an asymptotic expansion of the kernel at the origin and the growth of the entropy at infinity. The approach used here and in [11, 13] relies on using the energy dissipation inequality (7) and the continuation theorem (Theorem 4). The third key component is an inequality which relates the interaction energy $\mathcal{W}(u)$ to the entropy $S(u)$. For $m^\star > 1$ this is the Hardy-Littlewood-Sobolev inequality given in Lemma 4. In this case, the estimate of the critical mass is given by (8).

Theorem 7 (Global Well-Posedness for $m^\star > 1$)**.** *Suppose $m^\star > 1$. Then we have the following:*

(i) If the problem is subcritical, then the solution exists globally (i.e. $T_\star = \infty$).

(ii) If the problem is critical then there exists a critical mass $M_c > 0$ such that if $\|u_0\|_1 = M < M_c$, then the solution exists globally. The critical mass is estimated below in (8).

Proposition 2 (Critical Mass For $m^\star > 1$)**.** *If $\mathcal{K} = c|x|^{2-d/\gamma} + o(|x|^{2-d/\gamma})$ as $x \to 0$ for some $c \geq 0$ and γ, $1 \leq \gamma < d/2$, then M_c satisfies,*

$$\lim_{z \to \infty} \frac{\Phi(z)}{z^{m^\star}} - \frac{C_0}{2} c \omega_d^{(d/\gamma-2)/d} M_c^{2-m^\star} = 0. \tag{8}$$

If $c = 0$ or $\lim_{z\to\infty} \Phi(z) z^{-m^\star} = \infty$ then we define $M_c = \infty$. Note that $m^\star = 1 + 1/\gamma - 2/d$.

Remark 2. Note that $\| \, |x|^{2-d/\gamma} \, \|_{L^{p,\infty}} = \omega_d^{(d/\gamma-2)/d}$.

Remark 3. By Lemma 19, if $\mathcal{K} \in L^{d/(d/\gamma-2),\infty}_{loc}$ for some $1 \leq \gamma < d/2$ then $\exists\, \delta, C > 0$ such that $\forall x$, $|x| < \delta$, $\mathcal{K}(x) \leq C\,|x|^{2-d/\gamma}$. Therefore, if the kernel does not admit an asymptotic expansion as in Proposition 2, the critical mass M_c can be estimated by,

$$\lim_{z\to\infty} \frac{\Phi(z)}{z^{m^\star}} - \frac{C_0}{2} C \omega_d^{(d/\gamma-2)/d} M_c^{2-m^\star} = 0.$$

Remark 4. Note, $\lim_{z\to\infty} \Phi(z) z^{-m^\star}$ is always well-defined but is not necessarily finite unless

$$\limsup_{z\to\infty} A'(z) z^{1-m^\star} < \infty.$$

If the problem is critical then necessarily $\lim_{z\to\infty} \Phi(z) z^{-m^\star} > 0$ so there always exists a positive mass which satisfies (8). Moreover, if the problem is subcritical then necessarily $\lim_{z\to\infty} \Phi(z) z^{-m^\star} = \infty$.

The case $m^\star = 1$ is analogous to the classical PKS problem in 2D, where linear diffusion is critical. For the 2D PKS, the critical mass is given by $M_c = 8\pi$ for both the Newtonian and Bessel potentials [13, 16]. In this work we treat the $m^\star = 1$ case for $d \geq 2$ on bounded domains, recovering the critical mass of the classical PKS, although **(D3)** technically requires the diffusion to be nonlinear and degenerate. The case $d \geq 3$ and $m^\star = 1$ is approached in [32], but the optimal critical mass is not identified. Our estimate is given below in (9). As above, the critical mass only depends on the asymptotic expansion of the kernel at the origin and the growth of the entropy at infinity. We first state the analogue of Theorem 5.

Theorem 8 (Finite Time Blow Up for Critical Problems $m^\star = 1$)**.** *Let D be a smooth, bounded and convex domain and $d \geq 2$. Suppose $\mathcal{K}$ satisfies*

(C1) $\mathcal{K}(x) = -c \ln |x| + o(\ln |x|)$ *as* $x \to 0$ *for some* $c > 0$ *.*

(C2) $x \cdot \nabla \mathcal{K}(x) \leq -c + C\,|x|$ *for all* $x \in \mathbb{R}^d$*, for some* $C \geq 0$ *.*

(C3) $A(z) \leq \overline{A} z$ *for some* $\overline{A} > 0$*.*

Then the critical mass M_c satisfies

$$M_c = \frac{2d\overline{A}}{c},$$

and for all $M > M_c$ there exists a solution which blows up in finite time with $\|u_0\|_1 = M$.

The corresponding global existence theorem includes more general kernels and nonlinear diffusion. The proof is similar to Theorem 7, except that the logarithmic Sobolev inequality (Lemma 5) is used in place of the Hardy-Littlewood-Sobolev inequality.

Theorem 9 (Global Well-Posedness for $m^\star = 1$ on Bounded Domains)**.** *Suppose $m^\star = 1$ and $d \geq 2$, let D be bounded, smooth and convex. Then we have the following:*

(i) If the problem is subcritical, then the solution exists globally.

(ii) If the problem is critical then there exists a critical mass, $M_c > 0$, such that if $\|u_0\|_1 = M < M_c$, then the solution exists globally. The critical mass is estimated below in (9)*.*

Proposition 3 (Critical Mass for $m^\star = 1$ on Bounded Domains)**.** *If $\mathcal{K}(x) = -c \ln |x| + o(\ln |x|)$ as $x \to 0$ for some $c \geq 0$, then M_c satisfies,*

$$\lim_{z\to\infty} \frac{\Phi(z)}{z \ln z} - \frac{c}{2d} M_c = 0. \tag{9}$$

If $c = 0$ or $\lim_{z\to\infty} \Phi(z)(z \ln z)^{-1} = \infty$ then we define $M_c = \infty$.

Remark 5. By **(BD)** and **(MN)**, $\exists\, \delta, C > 0$ such that $\forall x$, $|x| < \delta$, $\mathcal{K}(x) \leq -C \ln x$. Therefore, if the kernel does not have the asymptotic expansion required in Proposition 3 then the critical mass M_c may be estimated as,

$$\lim_{z\to\infty} \frac{\Phi(z)}{z \ln z} - \frac{C}{2d} M_c = 0.$$

Remark 6. These theorems include many known global existence and finite time blow up results in the literature including [53, 52, 53, 8, 11, 34]. Our main contributions to the existing theory is the unification of these results and the estimate of the critical mass for inhomogeneous kernels and general nonlinear diffusion. In the case of the Newtonian potential Blanchet et al. showed in [11] that solutions at the critical mass also exist globally. See [24, 10, 12] for the corresponding result for classical 2D PKS.

The entropy could potentially control higher norms than $L^{m^\star}$ and so the energy dissipation inequality may be used assert that the solution cannot grow as $t \to \infty$.

Theorem 10 (Uniform Boundedness)**.** *Let u be a global solution to* (1) *and let $m > 1$ such that $m \geq m^\star$ and $\lim_{z\to\infty} \Phi(z)z^{-m} > 0$. Suppose that any of the following conditions are satisfied:*

(i) $m > d$,

(ii) $D^2\mathcal{K} \in L^p$ *for some p satisfying*

$$\frac{dm}{dm + m - d} < p, \tag{10}$$

(iii) $\nabla\mathcal{K} \in L^s$, *for some s satisfying* $1 \leq s/(s-1) \leq m$.

then $u \in L^\infty((0,\infty) \times D)$.

1.3 Properties of Admissible Kernels

Definition 1 implies a number of useful characteristics which we state here and reserve the proofs for the Appendix 8.4. First, we have that every admissible kernel is at least as integrable as the Newtonian potential.

Lemma 1. *Let $\mathcal{K}$ be admissible. Then $\nabla\mathcal{K} \in L^{d/(d-1),\infty}$. If $d \geq 3$, then $\mathcal{K} \in L^{d/(d-2),\infty}$.*

In general, the second derivatives of admissible kernels are not locally integrable, but we may still properly define $D^2\mathcal{K} * u$ as a linear operator which involves a Cauchy principal value integral. By the Calderón-Zygmund inequality (see e.g. [Theorem 2.2 [51]]) we can conclude that this distribution is bounded on L^p for $1 < p < \infty$. The inequality also provides an estimate of the operator norms, which is of crucial importance to the proof of uniqueness.

Lemma 2. *Let $\mathcal{K}$ be admissible and $\vec{v} = \nabla\mathcal{K} * u$. Then $\forall p$, $1 < p < \infty$, $\exists C(p)$ such that $\|\nabla\vec{v}\|_p \leq C(p)\|u\|_p$ and $C(p) \lesssim p$ for $2 \leq p < \infty$.*

One can further connect the integrability of the kernel with the integrability of the derivatives at the origin, which provides a natural extension of Lemma 2 through the Young's inequality for $L^{p,\infty}$.

Lemma 3. *Let $d \geq 3$ and $\mathcal{K}$ be admissible. Suppose γ is such that $1 < \gamma < d/2$. Then $\mathcal{K} \in L^{d/(d/\gamma-2),\infty}_{loc}$ if and only if $D^2\mathcal{K} \in L^{\gamma,\infty}_{loc}$. The same holds for $\nabla\mathcal{K} \in L^{d/(d/\gamma-1),\infty}_{loc}$. In particular, $m^\star = 1 + 1/\gamma - 2/d$ for some $1 < \gamma < d/2$ if and only if $D^2\mathcal{K} \in L^{\gamma,\infty}_{loc}$. Moreover, $m^\star = 1$ if and only if $D^2\mathcal{K} \in L^{d/2,\infty}_{loc}$.*

The following lemma clarifies the connection between the critical exponent and the interaction energy.

Lemma 4. *Consider the Hardy-Littlewood-Sobolev type inequality, for all $f \in L^p$, $g \in L^q$ and $\mathcal{K} \in L^{t,\infty}$ for $1 < p, q, t < \infty$ satisfying $1/p + 1/q + 1/t = 2$,*

$$\left| \int\int f(x)g(y)\mathcal{K}(x-y)dxdy \right| \lesssim \|f\|_p\|g\|_q\|\mathcal{K}\|_{L^{t,\infty}}. \tag{11}$$

See [40]. In particular, let $\mathcal{K}$ be admissible and $m^\star > 1$. Then for all $u \in L^1 \cap L^{m^\star}$ and $\mathcal{K} \in L^{p,\infty}_{loc}$,

$$\int u\mathcal{K} * u dx \leq C_0\|\mathcal{K}\mathbf{1}_{B_1(0)}\|_{L^{p,\infty}}\|u\|_1^{2-m^\star}\|u\|_{m^\star}^{m^\star} + \|\mathcal{K}_{\mathbb{R}^d\setminus B_1(0)}\|_\infty\|u\|_1^2. \tag{12}$$

Here C_0 is taken to be the best constant for which this inequality holds for all u and $\mathcal{K}$ (see for instance [40, 42]).

If $m^\star = 1$ then we will need the logarithmic Sobolev inequality, as in for instance [24, 12].

Lemma 5 (Logarithmic Sobolev inequality [17])**.** *Let $d \geq 2$ and $0 \leq f \in L^1$ be such that $\left|\int f \ln f dx\right| < \infty$. Then,*

$$-\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)f(y)\ln|x-y|\,dxdy \leq \frac{\|f\|_1}{d}\int_{\mathbb{R}^d} f\ln f dx + C(\|f\|_1). \tag{13}$$

2 Uniqueness

We now prove the uniqueness of weak solutions stated in Theorem 3.

Proof. (**Theorem 3**) The proof follows [4, 8] and estimates the difference of weak solutions in H^{-1}, motivated by the fact that the nonlinear diffusion is monotone in this norm [56]. To this end, if the domain is bounded, we define $\phi(t)$ as the zero mean strong solution of

$$\Delta\phi(t) = u(t) - v(t) \text{ in } D \tag{14}$$

$$\nabla\phi(t)\cdot\nu = 0, \text{ on } \partial D, \tag{15}$$

where ν is the outward unit normal of D. If the domain is $\mathbb{R}^d$ for $d \geq 3$, we let $\phi(t) = -\mathcal{N} * (u - v)$ where $\mathcal{N}$ is the Newtonian potential. In either case, by the integrability and boundedness of weak solutions $u(t)$ and $v(t)$ we can conclude $\phi(t) \in L^\infty(D_T)\cap C([0,T];\dot{H}^1)$, $\nabla\phi(t) \in L^\infty(D_T)\cap L^2(D_T)$ and ϕ_t solves,

$$\Delta\phi_t = \partial_t u - \partial_t v.$$

Then since $\|u(t) - v(t)\|_{H^{-1}} = \|\nabla\phi(t)\|_2$, we will show that $\|\nabla\phi(t)\|_2 = 0$. During the course of the proof, we integrate by parts on a variety of quantities. If the domain is bounded, then the boundary terms will vanish due to the no-flux conditions (4),(15). Moreover the computations are justified as $\nabla\mathcal{K}*u, \nabla A(u), \nabla\mathcal{K}*v, \nabla A(v), \nabla\phi \in L^2(D_T)$.

By the regularity of $\phi(t)$ and the no-flux boundary conditions (15), (4) we have possibly up to a set of measure zero,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\nabla\phi(t)|^2\,dx &=< \nabla\phi(t), \partial_t\nabla\phi(t) > \\ &= - < \partial_t u(t) - \partial_t v(t), \phi(t) > .\end{aligned}$$

Therefore, using $\phi(t)$ in the definition of weak solution we have,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\nabla\phi(t)|^2\,dx &= \int(\nabla A(u(t)) - \nabla A(v(t)))\cdot\nabla\phi(t)dx \\ &\quad - \int(u-v)(\nabla\mathcal{K}*u)\cdot\nabla\phi dx \\ &\quad - \int v(\nabla\mathcal{K}*(u-v))\cdot\nabla\phi dx. \\ &:= I_1 + I_2 + I_3.\end{aligned}$$

We drop the time dependence for notational simplicity. Since A is increasing, we have the desired monotonicity of the diffusion,

$$I_1 = -\int(A(u) - A(v))(u-v)dx \leq 0.$$

We now concentrate on bounding the advection terms.

By computations found in [8], the following bound holds,

$$I_2 \lesssim \int\left|D^2\mathcal{K}*u\right||\nabla\phi|^2\,dx.$$

In those computations, the convexity of D and the monotone attraction of the kernel **(KN)** is used to imply

$$\int_{\partial D} \nabla \mathcal{K} * u \cdot \nu dS \leq 0,$$

which is necessary to account for boundary terms arising from integration by parts. By Hölder's inequality, Lemma 2 and $\nabla \phi \in L^\infty(D_T)$ for $p \geq 2$,

$$\begin{aligned}
\int \left|D^2 \mathcal{K} * u\right| |\nabla \phi|^2 \, dx &\leq \|D^2 \mathcal{K} * u\|_p \left(\int |\nabla \phi|^{2p/(p-1)} \, dx \right)^{(p-1)/p} \\
&\lesssim p \|u\|_p \|\nabla \phi\|_\infty^{2/p} \left(\int |\nabla \phi|^2 \, dx \right)^{(p-1)/p} \\
&\lesssim p \left(\int |\nabla \phi|^2 \, dx \right)^{(p-1)/p},
\end{aligned}$$

where the implicit constant depends only on the uniformly controlled L^p norms of u and v.

As for I_3, we compute as in [8]. By the computations in the proof of Lemma 2 we may justify integration by parts on the inside of the convolution, that is for all p, $2 \leq p < \infty$,

$$\| \sum_j \int \partial_i K(x-y) \partial_{jj} \phi dx \|_p \lesssim_p \|\nabla \phi\|_p.$$

Applying this inequality for $p = 2$ and Cauchy-Schwarz implies,

$$I_3 \lesssim \|v\|_\infty \|\nabla \phi\|_2^2.$$

Letting $\eta(t) = \int |\nabla \phi(t)|^2 \, dx$, we get the differential inequality,

$$\frac{d}{dt}\eta(t) \leq \hat{C} p \max(\eta(t)^{1-1/p}, \eta(t)),$$

where $\hat{C}$ again depends only on the uniformly controlled L^p norms of u, v. The differential equality does not have a unique solution, but all of the solutions are absolutely continuous integral solutions bounded above by the maximal solution $\overline{\eta}(t)$. By continuity, for $t < 1/\hat{C}$ the maximal solution is given by $\overline{\eta}(t) = (\hat{C}t)^p$, hence,

$$\eta(t) \leq \overline{\eta}(t) = (\hat{C}t)^p.$$

For $t < 1/(2\hat{C})$ we then have

$$\eta(t) \leq \overline{\eta}(t) \leq 2^{-p},$$

and we take $p \to \infty$ to deduce that for $a.e. t \in [0, 1/(2\hat{C}))$, $\eta(t) = 0$, therefore the solution is unique. This procedure may be iterated to prove uniqueness over the entire interval of existence since the time interval only depends on uniformly controlled norms. □

3 Local Existence

3.1 Local Existence in Bounded Domains

Let $\tilde{A}(z)$ be a smooth function on $\mathbb{R}^+$ such that $\tilde{A}'(z) > \eta$ for some $\eta > 0$. In addition, let $\vec{v}$ be a given smooth velocity field with bounded divergence. Classical theory gives a global smooth solution to the uniformly parabolic equation

$$u_t = \Delta \tilde{A}(u) - \nabla \cdot (u \vec{v}) \tag{16}$$

(see [41]). The solutions obey the global L^∞ bound

$$\|u\|_{L^\infty(D)} \leq \|u_0\|_{L^\infty(D)} \, e^{\left\|(\nabla\cdot\vec{v})_-\right\|_{L^\infty(D_T)} t}. \tag{17}$$

We take advantage of this theory to prove existence of weak solutions to (1) by regularizing the degenerate diffusion and the kernel. Consider the modified aggregation equation

$$u_t^\epsilon = \Delta A^\epsilon(u^\epsilon) - \nabla\cdot\left(u^\epsilon\left(\nabla\mathcal{J}_\epsilon\mathcal{K} * u^\epsilon\right)\right), \tag{18}$$

with corresponding no-flux boundary conditions (4). We define

$$A^\epsilon(z) = \int_0^z a_\epsilon'(z)dz, \tag{19}$$

where $a_\epsilon'(z)$ is a smooth function, such that $A'(z)+\epsilon \leq a_\epsilon'(z) \leq A'(z)+2\epsilon$, and the standard mollifier is denoted $\mathcal{J}_\epsilon v$. We first prove existence of solutions to the regularized equation (18), this is stated formally in the following proposition.

Proposition 4 (Local Existence for the Regularized Aggregation Diffusion Equation)**.** *Let $\epsilon > 0$ be fixed and $u_0(x) \in C^\infty(\overline{D})$, then* (18) *has a classical solution u on D_T for all $T > 0$.*

We obtain the proof of Proposition 4 directly from Theorem 12 in [8]. The proof requires a bound on $\|\nabla A^\epsilon\|_{L^2(D_T)}$, for some $T > 0$. We state this lemma for completeness but reference the reader to [8] for a proof.

Lemma 6 (Uniform Bound on Gradient of $A(u)$)**.** *Let $\epsilon > 0$ be fixed and $u^\epsilon \in L^\infty(D_T)$ be a solution to* (18)*. There exists a constant $C = C(T, \|\nabla\mathcal{J}_\epsilon\mathcal{K} * u^\epsilon\|_{L^\infty(D)}, \|u^\epsilon\|_\infty)$ such that:*

$$\|\nabla A^\epsilon(u^\epsilon)\|_{L^2(D_T)} \leq C. \tag{20}$$

Remark 7. The estimate given by (20) is independent of ϵ.

Proposition 4 gives a family of solutions $\{u^\epsilon\}_{\epsilon>0}$. To prove local existence to the original problem (1) we first need some a priori estimates which are independent of ϵ. Mainly, we obtain an independent-in-ϵ bound on the L^∞ norm of the solution and the velocity field. This is the main difference in the local existence theory from [8]. Due to the singularity of the kernels significantly more is required to obtain these a priori bounds. We first state a lemma, due to Kowalczyk [34]. The Alikakos iteration is used in the proof [2], and is easily adapted to dimensions $d \geq 2$ and $\mathbb{R}^d$ (see Appendix 8.3).

Lemma 7 (Iteration Lemma)**.** *Let $0 < T \leq \infty$ and assume that there exists a $c > 0$ and $u_c > 0$ such that $A'(u) > c$ for all $u > u_c$. Then if $\|\nabla\mathcal{K} * u\|_\infty \leq C_1$ on $[0,T]$ then $\|u\|_\infty \leq C_2(C_1)\max\{1, M, \|u_0\|_\infty\}$ on the same time interval.*

Lemma 8 (L^∞ Bound of Solution)**.** *Let $\{u^\epsilon\}_{\epsilon>0}$ be the classical solutions to* (18) *on D_T, with smooth, non-negative, and bounded initial data $\mathcal{J}_\epsilon u_0$. Then there exists $C = C(\|u_0\|_1, \|u_0\|_\infty)$ and $T = T(\|u_0\|_1, \|u_0\|_p)$ for any $p > d$ such that for all $\epsilon > 0$,*

$$\|u^\epsilon(t)\|_{L^\infty(D)} \leq C \tag{21}$$

for all $t \in [0,T]$.

Proof. For simplicity we drop the ϵ. The first step is to obtain an interval for which the L^p norm of u is bounded. Following the work of [31] we define the function $u_k^\epsilon = (u^\epsilon - k)_+$, for $k > 0$. Due to conservation of mass the following inequality provides a bound for the L^p norm of u given a bound on the L^p norm of u_k,

$$\|u\|_p^p \leq C(p)(k^{p-1}\|u\|_1 + \|u_k\|_p^p). \tag{22}$$

We look at the time evolution of $\|u_k\|_p$ and make use of the parabolic regularization (19).

Step 1:

$$\begin{aligned}
\frac{d}{dt}\|u_k\|_p^p &= p\int u_k^{p-1}\nabla\cdot(\nabla A^\epsilon(u) - u\nabla\mathcal{J}_\epsilon\mathcal{K}*u)\,dx \\
&= -p(p-1)\int A^{\epsilon'}\nabla u_k\cdot\nabla u dx - p(p-1)\int u u_k^{p-2}\nabla\mathcal{J}_\epsilon\mathcal{K}*u\,dx. \\
&\leq -\frac{4(p-1)}{p}\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx + p(p-1)\int u_k^{p-1}\nabla u_k\cdot\nabla\mathcal{J}_\epsilon\mathcal{K}*u\,dx \\
&\quad + kp(p-1)\int u_k^{p-2}\nabla u_k\cdot\nabla\mathcal{J}_\epsilon\mathcal{K}*u\,dx,
\end{aligned}$$

where we used the fact that for $l>0$

$$u(u_k)^l = (u_k)^{l+1} + ku_k^l. \tag{23}$$

Hence, integrating by parts once more gives

$$\begin{aligned}
\frac{d}{dt}\|u_k\|_p^p &\leq \frac{4(p-1)}{p}\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx - (p-1)\int u_k^p\Delta\mathcal{J}_\epsilon\mathcal{K}*udx - kp\int u_k^{p-1}\Delta\mathcal{J}_\epsilon\mathcal{K}*udx \\
&\leq -C(p)\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx + C(p)\|u_k\|_{p+1}^p\|\Delta\mathcal{J}_\epsilon\mathcal{K}*u\|_{p+1} + C(p)k\|u_k\|_p^{p-1}\|\Delta\mathcal{J}_\epsilon\mathcal{K}*u\|_p \\
&\leq -C(p)\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx + C(p)\left(\|u_k\|_{p+1}^{p+1} + \|u\|_{p+1}^{p+1}\right) + C(p)k\left(\|u_k\|_p^p + \|u\|_p^p\right).
\end{aligned}$$

In the last inequality we use Lemma 2. Now, using (22) we obtain that

$$\frac{d}{dt}\|u_k\|_p^p dx \leq -C(p)\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx + C(p)\|u_k\|_{p+1}^{p+1} + C(p,k)\|u_k\|_p^p + C(p,k,M).$$

An application of the Gagliardo-Nirenberg-Sobolev inequality gives that for any p such that $d < 2(p+1)$ (see Lemma 17 in the Appendix):

$$\|u\|_{p+1}^{p+1} \lesssim \|u\|_p^{\alpha_2}\left\|u^{p/2}\right\|_{W^{1,2}}^{\alpha_1},$$

where $\alpha_1 = d/p,\ \alpha_2 = 2(p+1)-d$. From the inequality $a^r b^{(1-r)} \leq ra + (1-r)b$ (using that $a = \delta\left\|u^{p/2}\right\|_{W^{1,2}}^2$ and $r = \alpha_1/2$) we obtain

$$\|u\|_{p+1}^{p+1} \lesssim \frac{1}{\delta^{\beta_1}}\|u\|_p^{\beta_2} + r\delta^2\left\|\nabla u^{p/2}\right\|_2^2 + r\delta^2\|u\|_p^p.$$

Above $\beta_1,\ \beta_2 > 1$. For k large enough we have that $A'(u) > c > 0$ over $\{u > k\}$; hence, if we choose δ small enough we obtain the final differential inequality:

$$\frac{d}{dt}\|u\|_p^p \lesssim C(p)\|u_k\|_p^{\beta_2} + C(p,k,r\delta)\|u_k\|_p^p + C(p,k,\|u_0\|_1). \tag{24}$$

The inequality (24) in turns gives a $T_p = T(p) > 0$ such that $\|u_k\|_p$ is bounded on $[0,T_p]$. Inequality (22) gives that $\|u\|_p$ remains bounded on the same time interval. Next we prove that the velocity field is bounded in $L^\infty(D)$ on some time interval $[0,T]$. This then allows us invoke Lemma 7 and obtain the desired bound.

Step 2:

Since $\nabla\mathcal{K} \in L^1_{loc}$ we have for all $p \geq 1$, uniformly in $x_0 \in D$,

$$\|\vec{v}\|_{L^p(B_1(x_0)\cap D)} = \|\nabla\mathcal{K}*u\|_{L^p(B_1(x_0)\cap D)} \lesssim \|u\|_p.$$

By Lemma 2 we also have, for all $p, 1 < p < \infty$,

$$\|\nabla \vec{v}\|_p = \|D^2\mathcal{K} * u\|_p \lesssim \|u\|_p.$$

By Morrey's inequality we have $\vec{v} \in L^\infty(D_T)$ by choosing some $p > d$, and Lemma 7 concludes the proof. Note that the bound depends on the geometry of the domain through the constant on the Gagliardo-Nirenberg-Sobolev inequality (Lemma 17). However, this constant is related to the regularity of the domain, and not directly to the diameter of the domain. □

In addition to the a priori estimates the proof of Theorem 1 requires precompactness of $\{u^\epsilon\}_{\epsilon>0}$ in $L^1(D_T)$.

Lemma 9 (Precompactness in $L^1(\Omega_T)$)**.** *The sequence of solutions obtained via Proposition 4, $\{u_\epsilon\}_{\epsilon>0}$, which exist on $[0,T]$, is precompact in $L^1(D_T)$.*

The proof of Lemma 9 follows exactly the work in [8]. The key is to prove that the sequence satisfies the Riesz-Frechet-Kolmogorov Criterion. This relies on the fact that $\|A(u^\epsilon)\|_{L^2(0,T;H^1(D))} \leq C$ uniformly.

Proof. (**Theorem 1**) For a given $\epsilon > 0$, if we regularize the initial condition $u_0^\epsilon(x) = \mathcal{J}_\epsilon u_0(x)$, Proposition 4 gives a solution u^ϵ to (18). Furthermore, the proof of Proposition 4 and Lemma 8 gave uniform-in-ϵ bounds on $\|A^\epsilon(u)\|_{L^2(0,T,H^1(D))}$, $\|u^\epsilon\|_{L^\infty(D_T)}$, and $\|u_t^\epsilon\|_{L^2(0,T,H^{-1}(D))}$. By Lemma 8, all solutions exist on $[0,T]$, with T independent of ϵ. Also, recalling that $A^\epsilon(z) \geq A(z)$ and $a'_\epsilon(z) \geq A'(z)$ gives that

$$\|A(u^\epsilon)\|_{L^2(0,T,H^1(D))} \leq C,$$

where C is independent of ϵ. Since $L^2(0,T,H^1(D))$ is weakly compact there exists a ρ such that some subsequence of $\{u^\epsilon\}_{\epsilon>0}$ converges weakly, i.e $A(u^{\epsilon_j}) \rightharpoonup \rho$ in $L^2(0,T,H^1(D))$. Precompactness in L^1 implies strong convergence of u^{ϵ_j} to some $u \in L^1(D_T)$; therefore, $A(u) = \rho$. In fact, the $L^\infty(D_T)$ bound on u^{ϵ_j} gives strong convergence in $L^p(D_T)$, for $1 \leq p < \infty$, via interpolation. Also, Young's inequality gives

$$\begin{aligned}
\left\|u^{\epsilon_j}\nabla\mathcal{J}_{\epsilon_j}\mathcal{K} * u^{\epsilon_j} - u\nabla\mathcal{K} * u\right\|_{L^1(D_T)} &\leq \|u\|_{L^\infty(D_T)}\left\|\nabla\mathcal{J}_{\epsilon_j}\mathcal{K} * u^{\epsilon_j} - \nabla\mathcal{K} * u\right\|_{L^1(D_T)} \\
&\quad + \left\|\nabla\mathcal{J}_{\epsilon_j}\mathcal{K} * u^{\epsilon_j}\right\|_{L^\infty(D_T)}\|u^{\epsilon_j} - u\|_{L^1(D_T)} \\
&\lesssim \left(\|u\|_{L^\infty(D_T)}\|\nabla\mathcal{K}\|_{L^1_{loc}} + \|\nabla\mathcal{K} * u^{\epsilon_j}\|_{L^\infty(D_T)}\right)\|u^{\epsilon_j} - u\|_{L^1(D_T)}. \quad (25)
\end{aligned}$$

Therefore, by interpolation u satisfies (5). Furthermore, we obtain that $u \in C([0,T];H^{-1}(D))$. To prove that $u(t)$ is continuous with respect to the weak L^2 topology one uses standard density arguments. Since D is a bounded, u is therefore also continuous in the weak L^1 topology. To prove continuity in the strong L^2 topology we define $F(z) = \int_0^z A(s)ds$ and show that it is continuous in the strong L^1 topology. Indeed, Lemma 14 in the Appendix, see [8] for a proof, gives

$$\lim_{h\to 0}\left|\int (F(u(t)) - F(u(t+h)))\,dx\right| = \lim_{h\to 0}\int_t^{t+h} < u_\tau, A(\tau) > \,d\tau. \quad (26)$$

Recall that $\|A(u)\|_{L^\infty(D_T)} \leq A(\|u\|_{L^\infty(D_T)})$ and so $A(u) \in L^2(0,T,H^{-1}(D))$. Therefore, the left hand side of (26) goes to 0 as $h \to 0$. Now, we can invoke Lemma 15 in Appendix, [8], to obtain that $u \in C([0,T];L^2(D))$. Using interpolation the L^∞ bound of u gives that $u \in C([0,T];L^p(D))$, for $1 \leq p < \infty$. □

3.2 Local Existence in $\mathbb{R}^d$

Now we consider solutions to (1) in $\mathbb{R}^d$ for $d \geq 3$. We obtain such solution by taking the limit of the solutions in balls centered on the origin with increasing radius n, denoted by B_n.

Proof. (**Theorem 2**) Let B_n be defined as above and consider the truncation of the initial condition on B_n, i.e. $u_0^n = \mathbf{1}_{B_n}u_0$. By Theorem 1, we have a family of solutions $\{u_n\}_{n>0}$ on B_n for all $t \in [0,T]$. Define a new sequence, $\{\tilde{u}_n\}_{n>0}$, where $\tilde{u}_n$ is the zero extension of u_n. The previous work for bounded domains gives the uniform bounds

$$\|\tilde{u}_n\|_{L^\infty(\mathbb{R}^d_T)} \leq C_1, \tag{27}$$
$$\|\nabla A(\tilde{u}_n)\|_{L^2(\mathbb{R}^d_T)} \leq C_2. \tag{28}$$

The bounds may be taken independent of n since the constant in the Gagliardo-Nirenberg-Sobolev inequality, Lemma 17, does not depend directly on the diameter of the domain and may be taken uniform in $n \to \infty$.

Therefore, there exist $u, w \in L^2(\mathbb{R}^d_T)$ for which $\tilde{u}_n \rightharpoonup u$ and $\nabla A(\tilde{u}_n) \rightharpoonup w$ in $L^2(\mathbb{R}^d_T)$. Furthermore, (27) implies $\|u\|_{L^\infty(\mathbb{R}^d_T)} \leq C_1$. Precompactness of $\{\tilde{u}_n^\epsilon\}_{\epsilon>0}$ in $L^1(B_n)$ for fixed $n > 0$ and Theorem 2.33 in [1] gives that $\{\tilde{u}_n\}_{n>0}$ is precompact in $L^1_{loc}(\mathbb{R}^d_T)$. Therefore, up to a subsequence, not renamed, $\tilde{u}_n \to u$ in $L^1_{loc}(\mathbb{R}^d_T)$; thus, $w = \nabla A(u)$. Also, the L^∞ bound gives that $\tilde{u}_n \to u$ in $L^p_{loc}(\mathbb{R}^d)$ for $1 \leq p < \infty$.

In addition, we have the estimate

$$\|\tilde{u}_n \nabla \mathcal{K} * \tilde{u}_n\|_{L^2(\mathbb{R}^d_T)} \leq \|\nabla \mathcal{K} * \tilde{u}_n\|_{L^\infty(\mathbb{R}^d_T)} \|\tilde{u}_n\|_{L^2(\mathbb{R}^d_T)}. \tag{29}$$

Therefore, we can extract a subsequence that converges weakly to some $w_1 \in L^2(\mathbb{R}^d_T)$. Since $u\mathbf{1}_{B_n} \in L^\infty(0,T,L^1(\mathbb{R}^d))$ and $u\mathbf{1}_{B_n} \nearrow u$ by monotone convergence $u \in L^\infty(0,T,L^1(\mathbb{R}^d_T))$. Once again, from the estimates performed in the bounded domains $\tilde{u}_n \nabla \mathcal{K} * \tilde{u}_n \to u \nabla \mathcal{K} * u$ in $L^1_{loc}(\mathbb{R}^d_T)$. Therefore, we can identify $w_1 = u\nabla\mathcal{K} * u$.

We now show that $u \in C([0,T]; L^1_{loc}(\mathbb{R}^d))$, which we know to be true, implies that $u \in C([0,T]; L^1(D))$. Let $t_n \to t \in [0,T]$ then for all $R > 0$ we have,

$$\int |u(t_n) - u(t)|\, dx = \int_{B_R} |u(t_n) - u(t)|\, dx + \int_{\mathbb{R}^d \setminus B_R} |u(t_n) - u(t)|\, dx. \tag{30}$$

The first term on the right hand side of (30) can be bounded by $\epsilon/2$, provided n is chosen large enough, since $u \in C([0,T]; L^1_{loc}(\mathbb{R}^d))$. To bound the second term we first show that $A(u) \in L^1(\mathbb{R}^d_T)$. By **(D3)** we can deduce $\lim_{z\to 0} A(z)z^{-1} = 0$. Then, for $k > 0$ there exists some $0 < C_k < \infty$ such that if $z < k$ then $A(z) \leq Cz$. Hence,

$$\begin{aligned}\int A(u)dx &= \int_{\{u<k\}} A(u)dx + \int_{\{u\geq k\}} A(u)dx \\ &\leq CM + A(\|u\|_\infty)\lambda_A(k) < \infty.\end{aligned}$$

Therefore, $\|A(u)\|_{L^1(\mathbb{R}^d_T)} \leq C(M, \|u\|_\infty)T$. Now, let $w(x)$ be a smooth radially-symmetric cut-off function with $w(x) = 0$ for $|x| < 1/2$ and $w(x) = 1$ for $|x| \geq 1$. Then consider the quantity, $M_R(t) = \int uw(x/R)dx$. Then formally,

$$\frac{d}{dt} M_R(t) = \frac{1}{R} \int uv \cdot (\nabla w)(x/R) dx + \frac{1}{R^2} \int A(u)(\Delta w)(x/R) dx.$$

Estimating terms in L^∞ gives,

$$\frac{d}{dt} M_R(t) \lesssim \frac{\|v\|_\infty \|u\|_1}{R} + \frac{1}{R^2} \int A(u) dx.$$

Formally, then

$$M_R(t) \lesssim M_R(0) + M\|v\|_{L^1((0,t);L^\infty)} R^{-1} + \|A(u)\|_{L^1((0,t)\times\mathbb{R}^d)} R^{-2}. \tag{31}$$

Since $A \in L^1((0,t)\times\mathbb{R}^d)$ and $M_R(0) \to 0$ as $R \to \infty$, by choosing R sufficiently large, the last term of (30) can be bounded by $\epsilon/2$. Hence, implies that $u \in C([0,T]; L^1(\mathbb{R}^d))$. Furthermore, via interpolation we obtain that $u \in C([0,T]; L^p(\mathbb{R}^d))$ for $1 \leq p < \infty$.

Conservation of mass can be proved similarly using a cut-off function $w(x) = 1$ for $|x| \leq 1/2$ and $w(x) = 0$ for $|x| \geq 1$, see the proof of Theorem 15 in [8] for a similar proof. □

We are left to prove the energy dissipation inequality (7). As expected, the approach is to regularize the energy and take the limit in the regularizing parameters.

Proof. (**Proposition 1**) Define

$$h(u) = \int_1^u \frac{A'(s)}{s} ds,$$

then $\Phi(u) = \int_0^u h(s)ds$. The regularized entropy is defined similarly with $a'_\epsilon(u)$, as defined in (19), taking the place of $A'(u)$. Given a smooth solution u^ϵ to (18) one can verify,

$$\mathcal{F}_\epsilon(u^\epsilon(t)) + \int_0^t \int \frac{1}{u^\epsilon} \left| a'_\epsilon(u^\epsilon)\nabla u^\epsilon - u^\epsilon \nabla \mathcal{J}_\epsilon \mathcal{K} * u^\epsilon \right|^2 dx d\tau = \mathcal{F}_\epsilon(u^\epsilon(0)). \tag{32}$$

Here $\mathcal{F}_\epsilon(u(t))$ denotes the free energy with the regularized entropy and kernel. Once again we take the limit ϵ approaches zero to obtain (7). We first show that the entropy converges.

Step 1: The parabolic regularization gives

$$\begin{aligned} h(z) + \epsilon \ln z \le h_\epsilon(z) \le h(z) + 2\epsilon \ln z \quad & \text{for } 1 \le z, \\ h(z) + 2\epsilon \ln z \le h_\epsilon(z) \le h'(z) + \epsilon \ln z \quad & \text{for } z \le 1. \end{aligned}$$

Therefore, writing $\Phi(u) = \int_0^1 h(s)ds + \int_1^u h(s)ds$ one observes that

$$\Phi(u) - 2\epsilon \le \Phi_\epsilon(u) \le \Phi(u) + 2\epsilon(u \ln u)_+. \tag{33}$$

This will allow us to show convergence of the entropy. In fact,

$$\begin{aligned} \left| \int \Phi_\epsilon(u^\epsilon) - \Phi(u) dx \right| &\le \int \left| \Phi_\epsilon(u^\epsilon) - \Phi(u^\epsilon) \right| dx + \int \left| \Phi(u^\epsilon) - \Phi(u) \right| dx \\ &\overset{(33)}{\le} 2\epsilon \int (1 + u^\epsilon \ln u^\epsilon)_+ dx + \|\Phi\|_{C^1([0,\|u^\epsilon\|_\infty])} \int |u^\epsilon - u| \, dx. \\ &\le 2\epsilon \left(|D| + \| \ln u^\epsilon \|_\infty \|u_0^\epsilon\|_1 \right) + C\|u^\epsilon - u\|_1. \end{aligned}$$

Conservation of mass, boundedness of smooth solutions, and precompactness in L^1_{loc} imply there exists a subsequence, such that as $\epsilon_j \to 0$,

$$\int \Phi_{\epsilon_j}(u_j^\epsilon) dx \to \int \Phi(u) dx.$$

Step 2: To show convergence of the interaction energy we need that for $a.e$ $t \in (0, T)$

$$\int u^\epsilon(t) \mathcal{J}_\epsilon \mathcal{K} * u^\epsilon(t) dx \to \int u(t) \mathcal{K} * u(t) dx. \tag{34}$$

Since $\mathcal{K} \in L^1_{loc}(D)$ we know that $\|\mathcal{K} * u\|_{L^\infty}$ is bounded; hence, replacing $\nabla\mathcal{K}$ with $\mathcal{K}$ in (25) gives the desired result. Finally, we are left to deal with the entropy production functional.

Step 3: From Lemma 10 in [20],

$$\int \frac{1}{u} \left| A'(u)\nabla u - u\nabla\mathcal{K} * u \right|^2 dx \le \liminf_{\epsilon \to 0} \int \frac{1}{u^\epsilon} \left| a'_\epsilon(u^\epsilon)\nabla u^\epsilon - u^\epsilon \nabla \mathcal{J}_\epsilon \mathcal{K} * u^\epsilon \right|^2 dx. \tag{35}$$

We also note that this was proved in [8]. The proof of (35) relies on a result due to Otto in [48], refer to Lemma 16 in the Appendix. In our case, $u^\epsilon \in L^1(D_T)$ and $J_\epsilon = \nabla A^\epsilon(u^\epsilon) - u^\epsilon \nabla \mathcal{K} * u^\epsilon \in L^1_{loc}(D_T)$. Furthermore, up to a sequence not renamed, $u^\epsilon \rightharpoonup u \in L^2$ and $J_\epsilon \rightharpoonup J$ in L^2, therefore, we can apply Lemma 16.

For the energy dissipation estimate in $\mathbb{R}^d$ we again consider the family of solutions $\{u_r\}$ to (1) on B_r

(for simplicity let u_r denote the zero-extension of the solutions). Since $u_n(0)\mathbf{1}_{B_n} \nearrow u(0)$ by monotone convergence we obtain that $\mathcal{F}(u_n(0)) \to \mathcal{F}(u(0))$. Noting that $\mathcal{K} \in L^{d/(d-2)}$ allows us to make a modification to (29) and obtain that $u_n\mathcal{K} * u_n \rightharpoonup u\mathcal{K} * u$ in $L^2(\mathbb{R}^d_T)$. Furthermore, (34) implies that $u_n\mathcal{K} * u_n \to u\mathcal{K} * u$ in L^1_{loc}. We are left to verify the uniform integrability over all space. First note that Morrey's inequality implies

$$\begin{aligned}\|\mathcal{K} * \tilde{u}_n\|_\infty &\lesssim \|\nabla\mathcal{K} * u\|_\infty + \|\mathcal{K} * u_n\|_p \\ &\leq \|\nabla\mathcal{K} * u\|_\infty + \|\mathcal{K}\|_{L^{d/(d-2),\infty}} \|u_n\|_{dp/(d+2p)}.\end{aligned}$$

Hence, taking p sufficiently large we obtain that $\mathcal{K} * u_n$ is bounded in $L^\infty(D_T)$. Therefore,

$$\int_{\mathbb{R}^d\setminus B_k} u_n\mathcal{K} * u_n dx \leq \|\mathcal{K} * u_n\|_\infty \int_{\mathbb{R}^d\setminus B_k} u_n dx.$$

This fact along with (31) gives that for any $\epsilon > 0$ there exists a k_ϵ sufficiently large such that for all $k > k_\epsilon$

$$\int_{\mathbb{R}^d\setminus B_k} \tilde{u}_n\mathcal{K} * \tilde{u}_n dx \leq \epsilon.$$

This gives convergence of the interaction energy. The result follows from the weak lower semi-continuity of the entropy production functional and $\int \Phi(u)dx$ in L^2. □

4 Continuation Theorem

Continuation of weak solutions, Theorem 4, is a straightforward consequence of the local existence theory and the following lemma, which follows substantially the recent work in [11, 34, 13]. This lemma provides a more precise version of Lemma 8 and has a similar proof.

Lemma 10. *Let $\{u^\epsilon\}_{\epsilon>0}$ be the classical solutions to* (18) *on D_T, with non-negative initial data $\mathcal{J}_\epsilon u_0$. Suppose there exists T_0, $0 < T_0 < \infty$, such that*

$$\sup_{\epsilon>0} \lim_{k\to\infty} \sup_{t\in(0,T_0)} \|(u^\epsilon - k)_+\|_{\frac{2-m}{2-m^\star}} = 0, \tag{36}$$

where m is such that $1 \leq m \leq m^\star$ and $\liminf_{z\to\infty} A'(z)z^{1-m} > 0$. Then there exists $C = C(M, \|u_0\|_\infty, T_0)$ such that for all $\epsilon > 0$,

$$\sup_{t\in(0,T_0)} \|u^\epsilon(t)\|_\infty \leq C.$$

Proof. (**Lemma 10**) Let $\overline{q} = (2-m)/(2-m^\star) \geq 1$. We first bound intermediate L^p norms over the same interval, $(0, T_0)$. Then we use Morrey's inequality and Lemma 7 to finish the proof.

Step 1:

We have two cases to consider, $m^\star = 2 - 2/d$ and $m^\star < 2 - 2/d$, which occurs if $D^2K \in L^{\gamma,\infty}_{loc}$ for $\gamma > 1$ (Lemma 3). In the former we show that for any $p \in (\overline{q}, \infty)$ we have $u^\epsilon(t)$ uniformly bounded in $L^\infty(0, T_0; L^p)$. In the latter case we only show that for $\overline{q} < p \leq \gamma/(\gamma-1)$ we have $u^\epsilon(t)$ uniformly bounded in $L^\infty(0, T_0; L^p)$. In either case, this is sufficient to apply Lemma 7 and conclude the proof.

Let $k > 0$ be some constant to be determined later and let $u_k = (u-k)_+$. We have dropped the ϵ and time dependence for notational convenience. By conservation of mass and (22), it suffices to control $\|u_k\|_p$ for any $k > 0$. Thus, using the parabolic regularization, (19), and (22) we obtain

$$\frac{d}{dt}\|u_k\|_p^p \leq -p(p-1)\int u_k^{p-2}A'(u)\left|\nabla u\right|^2 dx + p(p-1)\int (u_k^{p-1} + ku_k^{p-2})\nabla u \cdot \mathcal{J}_\epsilon\nabla\mathcal{K} * udx.$$

Then,

$$\frac{d}{dt}\|u_k\|_p^p \leq -4(p-1)\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx - \int((p-1)u_k^p + kpu_k^{p-1})\mathcal{J}_\epsilon\Delta\mathcal{K} * udx.$$

Since the constants are not relevant, we treat the cases together only noting minor differences when they appear. If $m = 2 - 2/d$ we may use Hölder's inequality and then Lemma 2 to obtain a bound on the first term from the advection:

$$\left|\int u_k^p \mathcal{J}_\epsilon \Delta\mathcal{K} * udx\right| \lesssim_{p,\mathcal{K}} \|u_k\|_{p+1}^p \|u\|_{p+1}.$$

On the other hand, if $\gamma > 1$ we have from the generalized Hardy-Littlewood-Sobolev inequality (11) (Lemma (4)),

$$\left|\int u_k^p \mathcal{J}_\epsilon \Delta\mathcal{K} * udx\right| \lesssim_{p,\mathcal{K}} \|u_k\|_{\alpha p}^p \|u\|_t + C(M)\|u_k\|_p^p,$$

with the scaling condition $1/\alpha + 1/t + 1/\gamma = 2$. Choosing $t = \alpha p$ implies that

$$\frac{1}{\alpha} = \frac{2 - 1/\gamma}{1 + 1/p}. \tag{37}$$

Notice that from our choice of p then $1 \leq 1/p + 1/\gamma$; thus, $1/\alpha \leq 1$. Note that in the case when $m = 2 - 2/d$ then $t = \alpha p = p + 1$. In either case we use the Gagliardo-Nirenberg-Sobolev inequality (Lemma 17),

$$\|u_k\|_{\alpha p} \lesssim \|u_k\|_{\overline{q}}^{\alpha_2} \|u_k^{(p+m-1)/2}\|_{W^{1,2}}^{\alpha_1} \tag{38}$$

with

$$\alpha_1 = \frac{2d}{p}\left(\frac{(p - \overline{q}/\alpha)}{\overline{q}(2-d) + dp + d(m-1)}\right),$$

and

$$\alpha_2 = 1 - \alpha_1(p + m - 1)/2 > 0.$$

By the definition of $\overline{q}$ and (37) we have that,

$$\alpha_1(p+1)/2 = 1, \tag{39}$$

which implies,

$$\|u_k\|_{\alpha p}^{p+1} \lesssim \|u_k\|_{\overline{q}}^{\alpha_2(p+1)}\left(\int u_k^{m-1}\left|\nabla u_k^{p/2}\right|^2 dx + \int u_k^{p+m-1}dx\right). \tag{40}$$

If $d = 2$ then necessarily $m = m^\star = 1$ and this inequality will be sufficient. However, for $d \geq 3$, more work must be done. Define,

$$I = \int u_k^{m-1}\left|\nabla u_k^{p/2}\right|^2 dx.$$

Then, for $\beta_1 \leq \alpha_1$ and $(p + m - 1)\beta_1/2 < 1$,

$$\beta_1 = \frac{2d(1 - \overline{q}/(p + m - 1))}{\overline{q}(2-d) + dp + d(m-1)},$$

and $\beta_2 = 1 - \beta_1(p + m - 1)/2 > 0$, we have the following by Lemma 17,

$$\begin{aligned}\int u_k^{p+m-1}dx &\lesssim \|u_k\|_{\overline{q}}^{(p+m-1)\beta_2}\left(I + \int u_k^{p+m-1}dx\right)^{(p+m-1)\beta_1/2} \\ &\lesssim \|u_k\|_{\overline{q}}^{(p+m-1)\beta_2}\left(I^{(p+m-1)\beta_1/2} + \left(\int u_k^{p+m-1}dx\right)^{(p+m-1)\beta_1/2}\right).\end{aligned}$$

Therefore, by weighted Young's inequality for products,

$$\int u_k^{p+m-1}dx \lesssim \|u_k\|_{\overline{q}}^{(p+m-1)\beta_2}(1 + I) + \|u_k\|_{\overline{q}}^{\gamma}, \tag{41}$$

for some $\gamma > 0$, the exact value of which is not relevant. Putting (40),(41) together and using γ_i to denote exponents with $\gamma_i > 0$, we have the following estimate for k sufficiently large,

$$\begin{aligned}\left|\int u_k^p \mathcal{J}_\epsilon \Delta\mathcal{K} * u dx\right| &\lesssim_{p,\mathcal{K}} \|u_k\|_{\alpha p}^p \|u\|_{\alpha p} + C(M)\|u_k\|_p^p \\ &\lesssim \|u_k\|_{\alpha p}^{p+1} + \|u\|_{\alpha p}^{p+1} + C(M)\|u_k\|_p^p \\ &\overset{(22)}{\lesssim} \|u_k\|_{\alpha p}^{p+1} + C(M)\|u_k\|_p^p + C(k,M) \\ &\overset{(40),(39),(41)}{\lesssim} \mathcal{O}(\|u_k\|_{\overline{q}}^{\gamma_1})I + \mathcal{O}(\|u_k\|_{\overline{q}}^{\gamma_2}) + C(M)\|u_k\|_p^p + C(k,M). \end{aligned} \tag{42}$$

Here the $\mathcal{O}(\|u_k\|^{\gamma_i})$ terms are with respect to $\|u_k\|^{\gamma_i} \to 0$ and denote polynomials which consist of several terms of this form, all with positive exponents. The lower order terms in the advection can be controlled using Hölder's inequality and Lemma 2,

$$\begin{aligned}\left|\int u_k^{p-1} \mathcal{J}_\epsilon \Delta\mathcal{K} * u dx\right| &\lesssim_p \|u_k\|_p^{p-1}\|u\|_p \\ &\leq \|u_k\|_p^p + \|u\|_p^p \\ &\overset{(22)}{\lesssim} \|u_k\|_p^p + C(k,M). \end{aligned} \tag{43}$$

Combining the advection estimates (42) and (43),

$$\begin{aligned}\frac{d}{dt}\|u_k\|_p^p \leq &-C(p)\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx \\ &+ \mathcal{O}(\|u_k\|_{\overline{q}}^{\gamma_1})\int u_k^{m-1}\left|\nabla u_k^{p/2}\right|^2 dx \\ &+ C(k,M,p)\|u_k\|_p^p + C(k,M,p,\|u_k\|_{\overline{q}}). \end{aligned} \tag{44}$$

By definition of m, $\exists\ \delta > 0$ such that for k sufficiently large then $u > k$ implies $A'(u) > \delta u^{m-1}$. Then we have,

$$\begin{aligned}\frac{d}{dt}\|u_k\|_p^p \leq &-C(p)\left(\delta - \mathcal{O}(\|u_k\|_{\overline{q}}^{\gamma_1})\right)\int u^{m-1}\left|\nabla u_k^{p/2}\right|^2 dx \\ &+ C(k,M,p)\|u_k\|_p^p + C(k,M,p,\|u_k\|_{\overline{q}}). \end{aligned} \tag{45}$$

By the assumption (36), we choose k sufficiently large to ensure the first two terms are negative; thus, $\|u_k^\epsilon\|_p$ is bounded on $(0, T_0)$.

Step 2:

The control of these L^p norms will enable us to invoke Lemma 7 and conclude $u^\epsilon(t)$ is bounded uniformly in $L^\infty(D_{T_0})$. Since $\nabla\mathcal{K} \in L^1_{loc}$ we have, uniformly in $x_0 \in D$,

$$\|\vec{v}\|_{L^{\overline{q}}(B_1(x_0)\cap D)} = \|\nabla\mathcal{K} * u\|_{L^{\overline{q}}(B_1(x_0)\cap D)} \lesssim \|u\|_{\overline{q}}.$$

By (36) and conservation of mass $\|u\|_{\overline{q}}$ is uniformly bounded. If $\gamma > 1$ then,

$$\|\nabla\vec{v}\|_{d+1} = \|D^2\mathcal{K} * u\|_{d+1} \leq \|D^2\mathcal{K}\mathbf{1}_{B_1(0)}\|_{L^{\gamma,\infty}}\|u\|_p + C(M),$$

for $p = \gamma(d+1)/(d(\gamma-1) + 2\gamma - 1)$. Note that

$$1 < p = \frac{\gamma(d+1)}{d(\gamma-1) + 2\gamma - 1} \leq \frac{\gamma}{\gamma - 1}.$$

On the other hand, if $m^\star = 2 - 2/d$ then the above proof shows that $u^\epsilon(t)$ is bounded uniformly in $L^\infty((0,T_0); L^p)$ for all $p < \infty$. Therefore, by Lemma 2 we have $\|\nabla\vec{v}\|_p \lesssim \|u\|_p \lesssim 1$, for all $1 < p < \infty$. In either case, this is sufficient to apply Morrey's inequality and conclude that $\|\vec{v}\|_\infty$ is uniformly bounded on $(0, T_0)$. By Lemma 7 we then have that u^ϵ is uniformly bounded in $L^\infty(D_{T_0})$ and we have proved the lemma. As in Lemma 8, the uniform bounds depend on the domain but not it's diameter. □

Remark 8. The proof of this lemma directly implies global well-posedness in the subcritical case since (36) is only necessary in the critical and supercritical cases. Moreover, in the critical case, one may prove directly that there exists some M_0 such that if $M < M_0$ the solution is global. However, M_0 will generally depend on the constant of the Gagliardo-Nirenberg-Sobolev inequality, as in [53, 54, 31]. As discussed in the recent works of [11, 13], the use of a continuation theorem will allow for a more accurate estimate of the critical mass through the use of the free energy.

Proof. (**Theorem 4**) Suppose, for contradiction, that the weak solution cannot be continued past $T_\star < \infty$, but that for some p, $\overline{q} < p < \infty$,

$$\lim \inf_{t \nearrow T_\star} \|u\|_p < \infty.$$

This implies that for some $\eta > 0$, there exists a sequence $\{t_n\}$ with $t_n \to T_\star$ such that $\|u(t_n)\|_p \leq \eta$. By the proof of Lemma 8, for any $\eta > 0$ there exists a $\tau = \tau(\eta, M) > 0$ such that if $\|u_0\|_p < \eta$ then $\|u^\epsilon\|_p \leq C$ for all $\epsilon > 0$. Once again, $u^\epsilon(t)$ is the solution to the regularized problem (18). Let $u_k = (u^\epsilon - k)_+$. Since $p > \overline{q}$, for all $\delta > 0$ there exists k sufficiently large such that the following holds independent of ϵ,

$$\begin{aligned}\int u_k^{\overline{q}} dx &\leq \int \frac{u^p u^{\overline{q}}}{u^p} \mathbf{1}_{\{u>k\}} dx \\ &\leq \delta \left(\int u^p \mathbf{1}_{\{u>k\}} dx \right) \leq \delta \|u\|_p.\end{aligned}$$

Then by Lemma 10 we have $\sup_{t\in[0,\tau]} \sup_{\epsilon>0} \|u^\epsilon(t)\|_\infty < \infty$. Therefore, we may choose some t_n such that τ satisfies $t_n + \tau > T_\star$ and, by Theorems 1 and 2, we construct a solution $\tilde{u}(x,t)$ on the time interval $[t_n, t_n + \tau)$. By uniqueness, $\tilde{u}(x,t) = u(x,t)$ a.e. for $t \in [t_n, T_\star)$; hence, it is a genuine extension of the original solution $u(x,t)$. However, it exists on a longer time interval which is a contradiction. By interpolation we also conclude the result for $p = \infty$. □

5 Global Existence

We now prove Theorem 7. We first note that the entropy is bounded below uniformly in time.

Lemma 11. *Let $u(x,t)$ be a weak solution to* (1)*. Then,*

$$\int \Phi(u(t)) dx \geq -CM.$$

Proof. Let $h(z) = \int_1^z A'(s) s^{-1} ds$. By Definition 2, **(D3)**, for $z \leq 1$,

$$h(z) \geq -C > -\infty.$$

Therefore,

$$\begin{aligned}\int \Phi(u) dx = \int \int_0^u h(z) dz dx &\geq \int \mathbf{1}_{\{u\leq 1\}} \int_0^u h(z) dz + \mathbf{1}_{\{u\geq 1\}} \int_0^1 h(z) dz dx. \\ &\geq - \int \mathbf{1}_{\{u\leq 1\}} Cu - \mathbf{1}_{\{u\geq 1\}} C dx \\ &\geq -2C\|u\|_1.\end{aligned}$$

where the last line followed from Chebyshev's inequality. □

5.1 Theorem 7: $m^\star > 1$

Proof. (**Theorem 7**) As noted in Remark 8, the subcritical case follows from the proof of Theorem 4. Therefore, we only prove the second the assertion under the hypotheses of Proposition 2. Let $p = d/(d/\gamma - 2)$ for $1 \leq \gamma < d/2$ and note $\mathcal{K} \in L^{p,\infty}_{loc}$. By the energy dissipation inequality (7) we have for all time $0 \leq t < T_\star$,

$$S(u(t)) - \mathcal{W}(u(t)) \leq \mathcal{F}(u_0) := F_0. \tag{46}$$

We drop the time dependence of $u(t)$ for notational simplicity. By the assumption on $\mathcal{K}$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|\mathcal{K}(x)| \leq (c+\epsilon)\,|x|^{2-d/\gamma}$ for $|x| < \delta$. By Lemma 4 we have,

$$\int \Phi(u)dx - \frac{1}{2}C_0 M^{2-m^\star}(c+\epsilon)\| \, |x|^{2-d/\gamma} \, \|_{L^{p,\infty}} \|u\|_{m^\star}^{m^\star} \leq F_0 + C(\delta, M).$$

Noting that $\| \, |x|^{2-d/\gamma} \, \|_{L^{p,\infty}} = \omega_d^{1/p}$ and defining $C^\star = C_0 \omega_d^{1/p}/2$,

$$\int \Phi(u)dx - C^\star M^{2-m^\star}(c+\epsilon)\,\|u\|_{m^\star}^{m^\star} \leq F_0 + C(\delta, M).$$

By (8) and $M < M_c$, there exists $\epsilon > 0$ small enough and $\alpha, k > 0$ such that

$$\Phi(z) z^{-m^\star} - C^\star M^{2-m^\star}(c+\epsilon) \geq \alpha > 0, \text{ for all } z > k. \tag{47}$$

By Lemma 11 we have,

$$\int_{\{u>k\}} u^{m^\star}\left(\Phi(u)u^{m^\star} - C^\star M^{2-m^\star}(c+\epsilon)\right)dx - \int_{\{u<k\}} C^\star M^{2-m^\star}(c+\epsilon)\,u^{m^\star}dx \leq F_0 + C(\delta, M),$$

and by (47),

$$\alpha \int_{\{u>k\}} u^{m^\star}dx - C^\star M^{2-m^\star}(c+\epsilon)\int_{\{u<k\}} u^{m^\star}dx \leq F_0 + C(M, \delta).$$

By mass conservation we have that $\|u\|_{m^\star}$ is a priori bounded independent of time and Theorem 4 implies global existence. □

5.2 Theorem 9: $m^\star = 1$

The proof of Theorem 9 follows similarly, but requires the logarithmic Sobolev inequality (Lemma 5) as opposed to Lemma 4.

Proof. (**Theorem 9**) The subcritical case follows as in the proof of Theorem 4. Unlike in the proof of Theorem 7, Theorem 4 will not be directly applicable since we make an argument based on the traditional entropy $\int (u \ln u)_+ dx$, not on L^p norms, as no L^p norm is controllable through the free energy. However, we may use instead Lemma 10 directly by proving

$$\sup_{\epsilon>0} \sup_{t\in(0,\infty)} \int (u^\epsilon \ln u^\epsilon)_+ dx < \infty.$$

Here $u^\epsilon(t)$ denotes the classical solution to the regularized problem 18. The local existence theory (Theorem 1) together with Lemma 10 then provides a global solution. We drop the regularization parameter ϵ and argue formally, noting that the rigorous computations lead to a bound which does not depend on ϵ.

By the energy dissipation inequality (7) we again have (46). By the assumptions of Proposition 3, for all $\epsilon > 0$ there exists $\delta > 0$ such that,

$$\int \Phi(u)dx + (c+\epsilon)\frac{1}{2}\int\int_{|x-y|<\delta} u(x)u(y)\ln|x-y|\,dxdy \leq C(F_0, \delta, M).$$

By D bounded, the logarithmic Sobolev inequality (13) implies,

$$\int \Phi(u)dx - (c+\epsilon)\frac{M}{2d}\int u \ln u dx \leq C(F_0, \delta, M, \mathrm{diam} D).$$

Choosing $k > 0$ large and recalling Lemma 11 implies

$$\int_{\{u>k\}} u \ln u \left(\frac{\Phi(u)}{u \ln u} - (c+\epsilon)\frac{M}{2d}\right)dx - (c+\epsilon)\int_{\{u<k\}} u \ln u dx \leq C(F_0, \delta, M, \mathrm{diam} D).$$

As in the proof of Theorem 7, by conservation of mass, (9) and $M < M_c$, we may choose $\epsilon > 0$ small enough and k large enough such that

$$\int_{\{u>k\}} u \ln u dx \leq C(F_0, M, \mathrm{diam} D).$$

□

5.3 Uniform Boundedness

The proof of Theorem 7 also shows that the entropy $S(u)$ is bounded above independent of time. In subcritical cases, this implies more uniform control of the solution than what the boundedness of $\|u\|_{m^\star}$ provides. Together with Lemma 7 this may be exploited to prove Theorem 10.

Proof. (**Theorem 10**)

By conservation of mass, if $m > 1$ such that $0 < \lim_{z\to\infty} \Phi(z)z^{-m}$ then we have that $\|u\|_m \lesssim S(u) + \|u\|_1$. This can be seen by noting that for all z sufficiently large, $\Phi(z)z^{-m} > c$ for some $c > 0$.

Assertion (i): By Lemma 2,

$$\|\nabla \vec{v}\|_m = \|D^2\mathcal{K} * u\|_m \lesssim \|u\|_m,$$

which is bounded uniformly in time by the proof of Theorem 7. Since $\nabla \mathcal{K} \in L^1_{loc}$, we then have uniformly in $x_0 \in D$,

$$\|\vec{v}\|_{L^m(D\cap B_1(x_0))} = \|\nabla \mathcal{K} * u\|_{L^m(D\cap B_1(x_0))} \lesssim \|u\|_m.$$

This is sufficient to apply Morrey's inequality and conclude that $\vec{v} \in L^\infty((0,\infty)\times D)$, and the conclusion follows from Lemma 7.

Assertion (ii): The second assertion follows similarly. By Young's inequality and (10), if $1 < p < d$,

$$\|\nabla \vec{v}\|_r = \|D^2\mathcal{K} * u\|_r \leq \|D^2\mathcal{K}\|_p \|u\|_q,$$

for some q, r satisfying $r > d$, $1 < q \leq m$ and $1/p + 1/q = 1 + 1/r$. On $\mathbb{R}^d$ if $D^2\mathcal{K} \in L^p$ for some $p \geq d$, we may decompose $D^2\mathcal{K} = D^2\mathcal{K}\mathbf{1}_{B_1(0)} + D^2\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_1(0)}$ and since $D^2\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_1(0)} \in L^p$ for all $p > d/(d-1)$ we may use the above argument on each part of the kernel separately. It follows by interpolation that $\|\nabla \vec{v}\|_r$ is bounded independent of time. As above we also have, uniformly in $x_0 \in D$,

$$\|\vec{v}\|_{L^m(D\cap B_1(x_0))} \lesssim \|u\|_m.$$

Again by Morrey's inequality, we have that $\vec{v} \in L^\infty((0,\infty)\times D)$, and the conclusion follows from Lemma 7.

Assertion (iii): The third assertion follows immediately from Young's inequality and Lemma 7. □

6 Finite Time Blow Up

In this section we prove Theorem 6 and Theorem 5. We prove Theorem 6 as it is somewhat easier, though the technique is the same as that used to prove Theorem 5.

6.1 Supercritical Case: Theorem 6

For Theorem 6 we state the following lemma, which provides insight into the nature of the supercritical cases. The proof and motivation follows [11].

Lemma 12. *Define $\mathcal{Y}_\mathcal{M} = \{u \in L^1 \cap L^{m^\star} : u \geq 0, \|u\|_1 = M\}$. Suppose $\mathcal{K}$ satisfies **(B1)** for some $\gamma, 1 \leq \gamma < d/2$ and $A(u)$ satisfies **(B3)** for some $m > 1, \overline{A} > 0$. Suppose further that the problem is supercritical, that is, $m < m^\star = 1 + 1/\gamma - 2/d$. Then $\inf_{\mathcal{Y}_\mathcal{M}} F = -\infty$. Moreover, there exists an infimizing sequence with vanishing second moments which converges to the Dirac delta mass in the sense of measures.*

Proof. Let $0 < \theta < 1$, $\alpha = d/\gamma - 2$ and $p = d/(d/\gamma - 2)$. Then by Lemma 4 there exists $h^\star$ such that,

$$\theta C_0 \| \, |x|^{-\alpha} \, \|_{L^{p,\infty}} \leq \frac{\left| \int\int h^\star(x)h^\star(y) \, |x-y|^{-\alpha} \, dxdy \right|}{\|h^\star\|_1^{2-m^\star} \|h^\star\|_{m^\star}^{m^\star}} \leq C_0 \| \, |x|^{-\alpha} \, \|_{L^{p,\infty}}. \tag{48}$$

We may assume without loss of generality that $h^\star \geq 0$, since replacing $h^\star$ by $|h^\star|$ will only increase the value of the convolution. By density, we may take $h^\star \in C_c^\infty$ and therefore with a finite second moment. We denote $C^\star = C_0 \|c \, |x|^{-\alpha} \, \|_{L^{p,\infty}}$.

Let $\mu = \|h^\star\|_1^{1/d} M^{-1/d}$, $\lambda > 0$ and $h_\lambda(x) = \lambda^d h^\star(\lambda \mu x)$. First note, by **(B3)**, $\forall \epsilon > 0$, $\exists \, R > 0$ such that,

$$\begin{aligned} \int \Phi(h_\lambda) dx &= \int \int_0^{h_\lambda} \int_1^s \frac{A'(z)}{z} dz ds dx \\ &\leq \int \int_0^{h_\lambda} \int_R^{\max(s,R)} (m\overline{A} + \epsilon) z^{m-2} dz + \int_1^R \frac{A'(z)}{z} dz ds dx \\ &\leq \frac{\overline{A} + \epsilon}{m-1} \|h_\lambda\|_m^m + C(R) \|h_\lambda\|_1. \end{aligned} \tag{49}$$

By **(B1)** and $h^\star \in C_c^\infty$, $\forall \, \epsilon > 0$, $\exists \lambda > 0$ such that,

$$-\mathcal{W}(t) \leq -(c - \epsilon) \frac{\mu^{-2d+\alpha} \lambda^\alpha}{2} \int \int h^\star(x) h^\star(y) \, |x-y|^{-\alpha} \, dxdy. \tag{50}$$

Combining (50),(49) with (48) and Lemma 4, we have for λ, R sufficiently large,

$$\begin{aligned} \mathcal{F}(h_\lambda) \leq & \frac{\lambda^{dm-d} M}{(m-1)\|h^\star\|_1} (\overline{A} + \epsilon) \|h^\star\|_m^m - \lambda^\alpha (\theta - \epsilon) \frac{C^\star}{2} \left(\frac{\|h^\star\|_1}{M} \right)^{-2+\alpha/d} \|h^\star\|_1^{2-m^\star} \|h^\star\|_{m^\star}^{m^\star} \\ & + C(R) \mu^{-d} \|h^\star\|_1. \end{aligned}$$

By supercriticality, we have $\alpha = d/\gamma - 2 = dm^\star - d > dm - d$, and so for $\epsilon < \theta$, we take $\lambda \to \infty$ to conclude that for all values of the mass $M > 0$ we have $\inf_{\mathcal{Y}_\mathcal{M}} F = -\infty$. Moreover, since $h^\star \in C_c^\infty$, the second moment of h_λ goes to zero and h_λ converges to the Dirac delta mass in the sense of measures. □

Proof. (**Theorem 6**) We may justify the formal computations for weak solutions using the regularized problems and taking the limit but we do not include such details. Let

$$I(t) = \int |x|^2 \, u(x,t) dx.$$

Then,

$$\begin{aligned} \frac{d}{dt} I(t) &= 2d \int A(u) dx + 2 \int \int u(x) u(y) x \cdot \nabla \mathcal{K}(x-y) dxdy \\ &= 2d \int A(u) dx + \int \int (x-y) \cdot \nabla \mathcal{K}(x-y) u(x) u(y) dxdy, \end{aligned}$$

where the second integral was obtained by symmetrizing in x and y and the time dependence was dropped for notational simplicity. Using **(B2)** on $\mathcal{K}$,

$$\frac{d}{dt} I(t) \leq 2d \int A(u) dx + 2(2 - d/\gamma) \mathcal{W}(u) + C_1 M^2 + C_2 \int \int u(x) u(y) \, |x-y|^2 \, dxdy.$$

By **(D3)**, **(B4)** and Lemma 11,

$$\begin{aligned} \int A(u) dx &= \int_{\{u<R\}} A(u) dx + \int_{\{u>R\}} A(u) dx \\ &\leq C(M) + (m-1) \int_{\{u>R\}} \Phi(u) dx \\ &\leq C(M) + (m-1) \int \Phi(u) dx. \end{aligned}$$

Using that $2d(m-1) < 2d(m^\star - 1) = 2(d/\gamma - 2)$ we have,

$$\frac{d}{dt}I(t) \le 2d(m-1)\mathcal{F}(u) + C(M, C_1) + C_2 \int\int u(x)u(y)\,|x-y|^2\,dxdy.$$

We also have,

$$\int\int u(x)u(y)\,|x-y|^2\,dxdy \le 2\int\int u(x)u(y)\left(\left|x^2\right| + |y|^2\right)dxdy \le 4MI(t). \tag{51}$$

We use the energy dissipation inequality (7) to bound the first term and (51) to bound the third term,

$$\frac{d}{dt}I(t) \le 2d(m-1)\mathcal{F}(u_0) + C(M, C_1) + 4C_2MI(t).$$

Integrating we get,

$$I(t) \le e^{4C_2Mt}\left(I(0) + \frac{1}{2C_2M}(\mathcal{F}(u_0) + C(M, C_1))\right) - \frac{1}{C_2M}(\mathcal{F}(u_0) + C(M, C_1)).$$

Therefore, the solution blows up in finite time if

$$I(0) + \frac{1}{2C_2M}(\mathcal{F}(u_0) + 2C(C_1, M)) < 0.$$

By Lemma 12, we may always find initial data with any given mass $M > 0$ such that this is true, since there exists infimizing sequences with vanishing second moments. The final assertion follows from Theorem 4. □

6.2 Critical Case: Theorems 5 and 8

The proof of Theorem 5 follows the proof of Theorem 6.

Lemma 13. *Define $\mathcal{Y}_\mathcal{M} = \left\{u \in L^1 \cap L^\infty : u \ge 0, \|u\|_1 = M\right\}$. Suppose $\mathcal{K}$ satisfies **(B1)** for some $\gamma, 1 \le \gamma < d/2$ and $A(u)$ satisfies **(B3)** for $m > 1$ and $\overline{A} > 0$. Suppose further that the problem is critical, that is, $m = m^\star = 1 + 1/\gamma - 2/d$ and let M_c satisfy (8). If M satisfies $M > M_c$, then $\inf_{\mathcal{Y}_\mathcal{M}} F = -\infty$. Moreover, there exists an infimizing sequence with vanishing second moments which converges to the Dirac delta mass in the sense of measures.*

Proof. We may proceed as in the proof of Lemma 12, but instead choose $\theta \in \left((M_c/M)^{2-m^\star}, 1\right)$. Let $\alpha = d/\gamma - 2$ and $p = d/(d/\gamma - 2)$. By optimality of C_0, as before there exists $h^\star$ such that,

$$\theta C_0 \|\,|x|^{-\alpha}\,\|_{L^{p,\infty}} \le \frac{\left|\int\int h^\star(x)h^\star(y)\,|x-y|^{-\alpha}\,dxdy\right|}{\|h^\star\|_1^{2-m^\star}\|h^\star\|_{m^\star}^{m^\star}} \le C_0\|\,|x|^{-\alpha}\,\|_{L^{p,\infty}}. \tag{52}$$

As above, we assume $h^\star \ge 0$ and $h^\star \in C_c^\infty$. Denote $C^\star = C_0c\|\,|x|^{-\alpha}\,\|_{L^{p,\infty}}$.

Let $\mu = \|h^\star\|_1^{1/d}M^{-1/d}$, $\lambda > 0$ and $h_\lambda(x) = \lambda^d h^\star(\lambda\mu x)$. By **(B1)** and **(B3)**, $\forall\,\epsilon > 0$ there exists a λ and R sufficiently large such that by $h^\star \in C_c^\infty$,

$$\begin{aligned}\mathcal{F}(h_\lambda) \le & \frac{\lambda^{dm-d}M}{(m^\star-1)\|h^\star\|_1}(\overline{A}+\epsilon)\|h^\star\|_{m^\star}^{m^\star} + C(R)\mu^{-d}\|h^\star\|_1 \\ & - \frac{(\theta-\epsilon)C^\star}{2}\left(\frac{\|h^\star\|_1}{M}\right)^{-2+\alpha/d}\lambda^\alpha\|h^\star\|_1^{2-m^\star}\|h^\star\|_{m^\star}^{m^\star}\end{aligned}$$

However in this case $\alpha = dm - d$ and $m = m^\star$, therefore by (52) and Lemma 4,

$$\mathcal{F}(h_\lambda) \le \lambda^{dm^\star - d}\|h^\star\|_{m^\star}^{m^\star}\left[\frac{M(\overline{A}+\epsilon)}{(m^\star-1)\|h^\star\|_1} - \frac{(\theta-\epsilon)C^\star}{2}\left(\frac{\|h^\star\|_1}{M}\right)^{-2+\alpha/d}\|h^\star\|_1^{2-m^\star}\right].$$

Using $\alpha = d/\gamma - 2$ and $m^\star = 1 + 1/\gamma - 2/d$

$$\mathcal{F}(h_\lambda) \leq \lambda^{dm^\star - d} \frac{\|h^\star\|_{m^\star}^{m^\star}}{\|h^\star\|_1} \left[\frac{M(\overline{A} + \epsilon)}{(m^\star - 1)} - \frac{(\theta - \epsilon)}{2} C^\star M^{2-\alpha/d} \right].$$

Then since $\overline{A}/(m^\star - 1) = C^\star M_c^{2-m^\star}/2$ and $\alpha/d - 1 = 2 - m^\star$ we have,

$$\mathcal{F}(h_\lambda) \leq \lambda^{dm^\star - d} \frac{\|h^\star\|_m^m}{2\|h^\star\|_1} C^\star M^{2-\alpha/d} \left[\left(1 + \frac{\epsilon}{\overline{A}}\right) \left(\frac{M_c}{M}\right)^{2-m^\star} - (\theta - \epsilon) \right].$$

Since $\theta > (M_c/M)^{2-m^\star}$ we may take ϵ sufficiently small and $\lambda \to \infty$ to conclude that $\inf_{\mathcal{Y}_\mathcal{M}} F = -\infty$. As before, h_λ converges to the Dirac delta mass in the sense of measures. □

Proof. (Theorem 5) The theorem follows from a Virial identity as in Theorem 6. □

Proof. (Theorem 8) As in Theorem 6 we have by **(C2)** and **(C3)**,

$$\begin{aligned} \frac{d}{dt} I(t) &= 2d\overline{A} \int A(u)dx + \int \int u(x)u(y)(x-y) \cdot \nabla K(x-y) dxdy \\ &\leq 2dM \left(\overline{A} - \frac{cM}{2d} \right) + C_1 M^{3/2} I^{1/2}. \end{aligned}$$

Clearly, if $M > M_c$ then $I \to 0$ in finite time if $I(0)$ is sufficiently small. □

7 Conclusion

The prior treatments of (1) have restricted attention to either very singular kernels (PKS) or very smooth kernels (as in [8]). Moreover, most work has been restricted to power-law diffusion. We extend these approaches to develop a unified theory which applies to general nonlinear, degenerate diffusion and attractive kernels which are no more singular than the Newtonian potential. Existence arguments may apply to more singular kernels or unbounded initial data, however, to the authors' knowledge, Lemma 2 or something analogous must be available for any known uniqueness argument to hold. We generalize the existing notions of criticality for PKS and show that the critical mass phenomenon observed in PKS is a generic property of critical aggregation diffusion models. We extend the free energy methods of [24, 13, 11] to obtain the sharp critical mass for a class of models with general nonlinear diffusion and inhomogeneous kernels. In particular, we show that the critical mass depends only on the singularity of the kernel at the origin and the growth of the entropy at infinity. The results presented here hold on bounded, convex domains for $d \geq 2$ and on $\mathbb{R}^d$ for $d \geq 3$. The $\mathbb{R}^2$ case remains open, as does the behavior of solutions with critical mass (treated for homogeneous problems in [12, 11]). The long term qualitative properties of the solutions are currently known only in certain cases.

8 Appendix

8.1 Auxiliary Lemmas

Lemma 14. *Let F be a convex C^1 function and $f = F'$. Assume that $f(u) \in L^2(0, T, H^1(D))$, $u \in H^1(0, T, H^{-1}(D))$ and $F(u) \in L^\infty(0, T, L^1(D))$. Then for almost all $0 \leq s, \tau, \leq T$ the following holds:*

$$\int (F(u(x,\tau)) - F(u(x,s)))\ dx = \int_s^\tau \langle u_t, f(u(t)) \rangle\ dt.$$

Lemma 15. *Let $F(u,t) \in C^2([0,\infty),[0,\infty))$ be a convex function such that $F(0) = 0$ and $F'' > 0$ on $(0,\infty)$. Let f_n, for $n = 1, 2, ...,$ and f be a non-negative function on D bounded from above by $M > 0$. Furthermore, assume that $f_n \rightharpoonup f$ in $L^1(D)$ and $F(f_n) \to F(f)$ in $L^1(D)$, then $\|f_n - f\|_{L^2(D)} \to 0$ as $n \to 0$.*

Lemma 16 (Weak Lower-semicontinuity)**.** *Let ρ_ϵ be non-negative $L^1_{loc}(D_T)$ and f_ϵ a vector valued function in $L^1_{loc}(D_T)$ such that $\forall \phi \in C_c^\infty(\overline{D_T}) and \xi \in C_c^\infty(\overline{D_T}, \mathbb{R}^d)$*

$$\int_{D_T} \rho_\epsilon \phi dxdt \to \int_{D_T} \rho \phi dxdt$$
$$\int_{D_T} f_\epsilon \cdot \xi dxdt \to \int_{D_T} f \cdot \xi dxdt.$$

Then

$$\int_{D_T} \frac{1}{\rho} |f|^2 dxdt \leq \liminf_{\epsilon \to 0} \int_{D_T} \frac{1}{\rho_\epsilon} |f_\epsilon|^2 dxdt$$

8.2 Gagliardo-Nirenberg-Sobolev Inequality

Gagliardo-Nirenberg-Sobolev inequalities are the main tool for obtaining L^p estimates of PKS models and are used in many works, for instance [34, 11, 53, 31]. The following inequality follows by interpolation and the classical Gagliardo-Nirenberg-Sobolev inequality.

Lemma 17 (Inhomogeneous Gagliardo-Nirenberg-Sobolev)**.** *Let $d \geq 2$ and $D \subset \mathbb{R}^d$ satisfy the cone condition (see e.g. [1]). Let $f : D \to \mathbb{R}$ satisfy $f \in L^p \cap L^q$ and $\nabla f^k \in L^r$. Moreover let $1 \leq p \leq rk \leq dk$, $k < q < rkd/(d-r)$ and*

$$\frac{1}{r} - \frac{k}{q} - \frac{s}{d} < 0. \tag{53}$$

Then there exists a constant C_{GNS} which depends on s, p, q, r, d and the dimensions of the cone for which D satisfies the cone condition such that

$$\|f\|_{L^q} \leq C_{GNS} \|f\|_{L^p}^{\alpha_2} \|f^k\|_{W^{s,r}}^{\alpha_1}, \tag{54}$$

where $0 < \alpha_i$ satisfy

$$1 = \alpha_1 k + \alpha_2, \tag{55}$$

and

$$\frac{1}{q} - \frac{1}{p} = \alpha_1 \left(\frac{-s}{d} + \frac{1}{r} - \frac{k}{p}\right). \tag{56}$$

Proof. We may assume that f is Schwartz then argue by density. Let β satisfy $\max(q, rk) < \beta < rkd/(d-r)$. First note by the Gagliardo-Nirenberg-Sobolev inequality, [Theorem 5.8, [1]], we have

$$\begin{aligned} \|f^k\|_{\beta/k} &\lesssim_{\beta,k,r,s} \|f^k\|_r^{1-\theta} \|f^k\|_{W^{s,r}}^\theta \\ &\leq \|f^k\|_{p/k}^{(1-\theta)(1-\mu)} \|f^k\|_{\beta/k}^{(1-\theta)\mu} \|f^k\|_{W^{s,r}}^\theta, \end{aligned}$$

for $\mu \in (0,1)$ determined by interpolation and $\theta = s^{-1}(d/r - dk/\beta) \in (0,1)$. Moreover, the implicit constant does not depend directly on the size of the domain. Therefore,

$$\|f^k\|_{\beta/k} \lesssim \|f\|_p^{(1-\theta)(1-\mu)/(1-\mu(1-\theta))} \|f^k\|_{W^{s,r}}^{\theta/(1-\mu(1-\theta))}.$$

Now, where $\lambda \in (0,1)$ determined by interpolation,

$$\begin{aligned} \|f\|_q &\leq \|f\|_p^{(1-\lambda)} \|f^k\|_{\beta/k}^{\lambda/k} \\ &\lesssim \|f\|_p^{(1-\lambda)+(1-\theta)(1-\mu)/(1-\mu(1-\theta))} \|f^k\|_{W^{s,r}}^{\lambda\theta/(k-k\mu(1-\theta))}. \end{aligned}$$

□

8.3 Alikakos Iteration Lemma 7

In this section we briefly indicate how to generalize the proof of Lemma 7, due to Kowalzyck [34], to $d \geq 2$ and unbounded domains. Following the notation therein, we use the notation $n_m = (n-m)_+$ instead of u_k as in our proofs.

In [34], the inequality $n_m^{p-1} \leq n_m^{p+1} + 1$ is used to imply

$$\int n_m^{p-1} dx \leq \int n_m^{p+1} + |D|\,.$$

On unbounded domains we may use Chebyshev's inequality and $m \geq 2\|n\|_1$ to imply,

$$\int n_m^{p-1} dx \leq \int n_m^{p+1} + \frac{1}{2}.$$

An inhomogeneous Gagliardo-Nirenberg-Sobolev inequality is used. The inequality for $d \geq 3$ still applies and the constant does not directly depend on the domain size. Indeed, Lemma 17 applies with $k = 1$,

$$\|v\|_{L^2}^2 \leq \overline{C}\|v\|_{W^{1,2}}^{2d/(d+2)}\|v\|_{L^1}^{4/(d+2)} \leq \epsilon^{(d+2)/2d}\|v\|_{W^{1,2}}^2 + \frac{C(d,D)}{\epsilon^{(d+2)/4}}\|v\|_{L^1}^2,$$

which is sufficient to continue the proof contained therein.

8.4 Admissible Kernels

We now prove Lemmas 1,2 and 3. We begin with the following characterizations of $L^{p,\infty}$.

Lemma 18. *Let $F(x) = f(|x|) \in L^1_{loc} \cap C^0 \setminus \{0\}$ be monotone in a neighborhood of the origin. If $r^{-d/p} = o(f(r))$ as $r \to 0$, then $F \notin L^{p,\infty}_{loc}$.*

Proof. Since we have assumed f to be monotone in a neighborhood of the origin, without loss of generality we prove the assertions assuming $f \geq 0$ on that neighborhood, since corresponding work may be done if f is negative. For any $\alpha > 0$, by monotonicity, we have a unique $r(\alpha)$ such that $f(r) > \alpha, \forall r < r(\alpha)$. We thus have that $\lambda_f(\alpha) = \omega_d r(\alpha)^d$, where ω_d is the volume of the unit sphere in $\mathbb{R}^d$. By the growth condition on f and continuity we also have that for α sufficiently large,

$$\frac{1}{\epsilon} r(\alpha)^{-d/p} \leq f(r(\alpha)) = \alpha.$$

Now,

$$\alpha^p \lambda_f(\alpha) = \omega_d \alpha^p r(\alpha)^d.$$

Hence, by (8.4) we have $\forall\, \epsilon > 0$ there is a neighborhood of infinity such that,

$$\omega_d \alpha^p r(\alpha)^d \gtrsim \epsilon^{-p}.$$

We take $\epsilon \to 0$ to deduce that $F \notin L^{p,\infty}$.

□

Lemma 19. *Let $F(x) = f(|x|) \in L^1_{loc} \cap C^0 \setminus \{0\}$ be monotone in a neighborhood of the origin. Then $f \in L^{p,\infty}_{loc}$ if and only if $f = \mathcal{O}(r^{-d/p})$ as $r \to 0$.*

Proof. Since we have assumed f to be monotone in a neighborhood of the origin, without loss of generality we prove the assertions assuming $f \geq 0$ on that neighborhood.

First assume that $f \neq \mathcal{O}(r^{-d/p})$ as $r \to 0$, which implies that for all $\delta_0 > 0$ and every $C > 0$ there exists an $r_C < \delta_0$ such that

$$f(r_C) > C r_C^{-d/p}.$$

We now show that in a neighborhood of the origin, the function $f(r) - Cr^{-d/p}$ is strictly positive for $r < r_C$. Suppose not. Since both $f, r^{-d/p}$ are monotone, there exists r_0 such that $f(r) < Cr^{-d/p}$ for $r < r_0$. However, this contradicts $f \neq \mathcal{O}(r^{-d/\gamma})$ as $r \to 0$. Thus, we have that

$$f(r) > Cr^{-d/p}$$

in a neighborhood of the origin ($r < r_C$). Since for all $C > 0$ we can find a corresponding r_C, this is equivalent to $r^{-d/p} = o(f(r))$, and by Lemma 18 we have that $f \notin L^{p,\infty}$.

On the other hand, if $f = \mathcal{O}(r^{-d/p})$ as $r \to 0$ there exists $\delta > 0$ and $C > 0$ such that for all $r < \delta$,

$$f(r) \leq Cr^{-d/p}. \tag{57}$$

By monotonicity, for all $\alpha > 0$ there is a unique $r(\alpha) \in [0, \delta]$ such that

$$f(r) > \alpha, \text{ for } r < r(\alpha), \tag{58}$$

where we take $r(\alpha) = 0$ if $f(r) < \alpha$ over the entire neighborhood. By (57) and (58), we have, necessarily that $r(\alpha) \lesssim \alpha^{-p/d}$. Therefore,

$$\alpha^p \lambda_f(\alpha) = \alpha^p \omega_d r(\alpha)^d \lesssim 1,$$

which implies $f\mathbf{1}_{B_1(0)} \in L^{p,\infty}$. □

Remark 9. Similar statements may be made about the decay of $F(x)$ at infinity.

Proof. (**Lemma 1**) We first show that $\nabla\mathcal{K} \in L^{d/(d-1),\infty}$. Restrict to the neighborhood of the origin $(0, \delta)$ such that $k''(r)$ is monotone and bounded by Cr^{-d}

$$k'(r) - k'(\delta) = -\int_\delta^r k''(s)ds \leq C\int_\delta^r s^{-d}ds = C(d)r^{1-d} + C(d, \delta).$$

Therefore, $k'(r) = \mathcal{O}(r^{1-d})$ as $x \to 0$. The decay condition **(D)** similarly implies that $\nabla\mathcal{K} = \mathcal{O}(r^{1-d})$ as $r \to \infty$. By Lemma 19 this is sufficient to imply $\nabla\mathcal{K} \in L^{d/(d-1),\infty}$. The first assertion follows similarly. □

Proof. (**Lemma 2**) We compute second derivatives of the kernel $\mathcal{K}$ in the sense of distributions. Let $\phi \in C_c^\infty$, then by the dominated convergence theorem,

$$\begin{aligned}\int \partial_{x_i}\mathcal{K}\partial_{x_j}\phi dx &= \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} \partial_{x_i}\mathcal{K}\partial_{x_j}\phi dx \\ &= -\lim_{\epsilon\to 0}\int_{|x|=\epsilon} \partial_{x_j}\mathcal{K}(x)\frac{x_j}{|x|}\phi(x)dS - \mathrm{PV}\int \partial_{x_i x_j}\mathcal{K}\phi dx.\end{aligned}$$

By $\nabla\mathcal{K} \in L^{d/(d-1),\infty}$ and Lemma 19, we have $\nabla\mathcal{K} = \mathcal{O}(|x|^{1-d})$ as $x \to 0$. Therefore for ϵ sufficiently small, there exists $C > 0$ such that,

$$\begin{aligned}\left|\int_{|x|=\epsilon} \partial_{x_j}\mathcal{K}(x)\frac{x_j}{|x|}\phi(x)dS\right| &\leq C\int_{|x|=\epsilon} |x|^{1-d}|\phi(x)|\,dS \\ &= C\int_{|x|=1} |\epsilon x|^{1-d}|\phi(\epsilon x)|\,\epsilon^{d-1}dS = C\,|\phi(0)|\,.\end{aligned}$$

Similarly, we may define $D^2\mathcal{K} * \phi$ and we have,

$$\|D^2\mathcal{K} * \phi\|_p \leq C\|\phi\|_p + \|\mathrm{PV}\int \partial_{x_i x_j}\mathcal{K}(y)\phi(x-y)dy\|_p.$$

Therefore, the first term can be extended to a bounded operator on L^p for $1 \leq p \leq \infty$ by density. The admissibility conditions **(R)**,**(BD)**,**(KN)** and **(D)** are sufficient to apply the Calderón-Zygmund inequality

[Theorem 2.2 [51]], which implies that the principal value integral in the second term is a bounded linear operator on L^p for all $1 < p < \infty$. Moreover the proof provides an estimate of the operator norms,

$$\|\mathrm{PV}\int \partial_{x_i,x_j}\mathcal{K}(y)u(x-y)dy\|_p \lesssim \begin{cases} \frac{1}{p-1}\|u\|_p & 1 < p < 2 \\ p\|u\|_p & 2 \leq p < \infty. \end{cases}$$

□

Proof. (**Lemma 3**) The assertion that $D^2\mathcal{K} \in L^{\gamma,\infty}_{loc}$ implies $\mathcal{K} \in L^{d/(d/\gamma-2),\infty}_{loc}$ follows similarly as in Lemma 1.

Now we prove the second assertion. Let $\mathcal{K} \in L^{d/(d/\gamma-2),\infty}_{loc}$. We show that $D^2\mathcal{K} = \mathcal{O}(r^{-d/\gamma})$ as $r \to 0$. Assume for contradiction that $D^2\mathcal{K} \neq \mathcal{O}(r^{-d/\gamma})$ as $r \to 0$. This implies that $k'' \neq \mathcal{O}(r^{-d/\gamma})$ or that $k'(r)r^{-1} \neq \mathcal{O}(r^{-d/\gamma})$ as $r \to 0$. These two possibilities are essentially the same, so just assume that $k'' \neq \mathcal{O}(r^{-d/\gamma})$. By monotonicity arguments used in the proof of Lemma 19, this in turn implies $r^{-d/\gamma} = o(k'')$. However, this means that for all ϵ, there exists a $\delta(\epsilon) > 0$ such that for $r \in (0, \delta(\epsilon))$ we have,

$$\begin{aligned} k(r) - k(\delta(\epsilon)) &= -\int_{\delta(\epsilon)}^{r} k'(s)ds = \int_{\delta(\epsilon)}^{r}\int_{\delta(\epsilon)}^{s} k''(t)dtds + (r-\delta(\epsilon))k'(\delta(\epsilon)) \\ &\gtrsim \epsilon^{-1}r^{2-d/\gamma} + 1, \end{aligned}$$

which contradicts the fact that $k(r) = \mathcal{O}(r^{2-d/\gamma})$ as $r \to 0$ by Lemma 19.

The assertion regarding $\nabla\mathcal{K}$ is proved in the same fashion.

□

Acknowledgments

This work was supported NSF Grant DMS-0907931 for their financial support. The authors would like to thank Dong Li, Inwon Kim and Thomas Laurent for their helpful discussions.

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