Heat Source Identification Based on L_1 Constrained Minimization

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January 25, 2011

Abstract

The inverse problem of finding sparse initial data from the solutions to the heat equation is considered. The initial data is assumed to be a sum of an unknown but finite number of Dirac delta functions at unknown locations. Point-wise values of the heat solution at only a few locations are used in an L_1 constrained optimization to find such sparse initial data. A concept of domain of effective sensing is introduced to speed up the already fast Bregman iterative algorithm for L_1 optimization. Furthermore, an algorithm which successively adds new measurements at intelligent locations is introduced. By comparing the solutions of the inverse problem that are obtained from different number of measurements, the algorithm decides where to add new measurements in order to improve the reconstruction of the sparse initial data.

1 Introduction

Heat source identification problems have important applications in many branches of engineering and science. For example, an accurate estimation of pollutant source [3, 4] is a crucial environmental safeguard in cities with dense populations. Typically, a recovery of the unknown source is a reverse process in time. The major difficulty in establishing any numerical algorithm for approximating the solution is the severe ill-posedness of the problem. It appears that the mathematical analysis and numerical algorithms for inverse heat source problems are still very limited. For the kind of problem we consider in this paper, where we want to find the initial condition with known measurements in the future time, existing methods need many measurements [14] or have stability issues [2]. In this paper, we treat the source identification problem as an optimization problem. Our goal is to invert the heat equation to get the sparse initial condition. In other words, the problem is can formulated as an L_0 minimization problems with constraints. However, it is difficult to solve the L_0 problem since it is a nonconvex and NP-hard problem.

In compressed sensing [7], we can solve an L_0 problems by solving its L_1 relaxation when the associated matrix has the restricted isometry property (RIP) [6]. The heat operator does not satisfy RIP, but we can adopt the idea of substituting L_0 with L_1 for sparse optimization. We will show numerical results which indicate the effectiveness of this strategy. Our approach to this problem is to solve a L_1 minimization problem with constraints. We apply the Bregman iterative method [9, 13] to solve the constrained problem as a sequence of unconstrained subproblems. To solve these subproblems, we use the greedy coordinate descent method developed in [11] for solving the L_1 unconstraint problem, which was shown to be very efficient for sparse recovery.

For the heat source identification problem, the theory of compressive sensing does not apply. It is thus unclear if constrained L_1 minimization provides a good solution to our problem. In other hand, because of the inapplicability of the compressive sensing theory, there is room for finding specialized measurement locations for better solutions to our inverse problem. Hence, this paper is our attempt to understand the following questions:

- Is L_1-regularization adequate for inverse problems involving point sources?
- In which way can additional data improve the inversion?

In related work, the author [1] discussed optimal experimental design for ill-posed problems and suggest a numerical framework to efficiently achieve such a design in a statistical manner.

In section 2, we give a more detailed introduction of the heat and related source identification problem and related source problems. In section 3, we give a complete algorithm for solving the heat source identification problem and some algorithmic methods for improving efficiency. In section 4, we show two dimensional examples. In section 5, we obtain a useful stability estimation for a simple case. In section 6, we consider solving the heat source problem in a different setting. By using successive samples, we can get better results than random sampling. Finally, section 7 summarizes and discusses future directions. In the appendix, we explain about the proof of the stability in detail.

The contributions of this paper are:

- Using L_1 minimization for heat source identification
- Prove the stability under a simple case in terms of Wasserstein distance
- Invent a successive sampling strategy
- Propose ideas of exclusion region and support restriction to reduce the problem size

2 Source Problems

2.1 1D Heat Equation

We begin with a simple case. We consider the 1D heat equation with periodic boundaries. Consider the heat equation

$$\begin{cases} u_t = \Delta u & \text{on } \Omega \\ u(x,0) = u_0 \end{cases} \tag{1}$$

where Ω is the unit interval $(0,1)$. The initial condition u_0 is sparse, meaning its form is:

$$u_0(x) = \sum_{k=1}^{K} \alpha_k \delta(x - s_k),$$

where there are K spikes (delta impulses) located at the s_k. Physically, this initial condition describes K hot particles striking a cold plate Ω at the initial time and (1) describes the ensuing heat evolution.

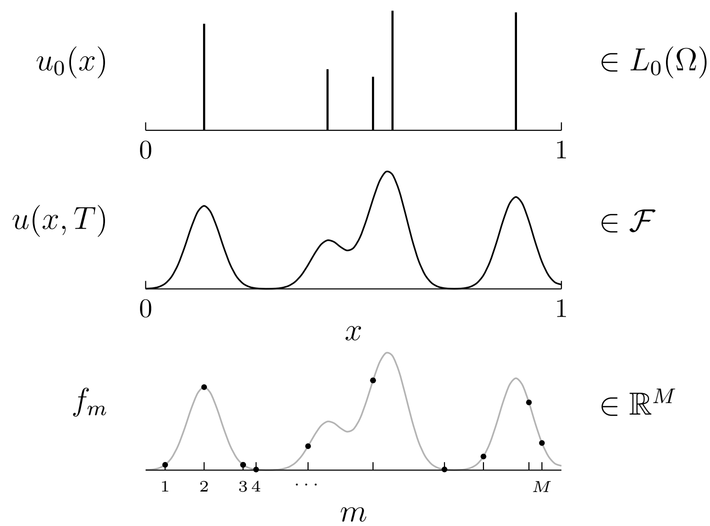


Figure 1: Depiction of the source problem: given samples $f_m = u(x_m, T)$, recover the sparse initial condition u_0.

The problem we consider is the following: if we observe (possibly noisy) measurements $f_m = u(x_m, T)$, $m = 1, \ldots, M$, then without knowing K, (α_k), or (s_k) in advance, can we recover u_0?

Let $\mathcal{G}$ denote the linear Green's operator such that $u = \mathcal{G}u_0$, that is, for the one-dimensional heat equation with periodic boundaries,

$$u(x,t) = (\mathcal{G}u_0)(x,t) = \int_0^1 G_t(x-y)u_0(y)\,dy \tag{2}$$

where $G_t(x) = \frac{1}{\sqrt{4\pi t}} \sum_{k\in\mathbb{Z}} \exp\big(-\frac{(x+k)^2}{4t}\big)$.

We pose the solution of the source problem as the following optimization problem:

$$\arg \min_{u_0 \in L_0(\Omega)} \|u_0\|_0 \quad \text{subject to} \quad f_m = (\mathcal{G}u_0)(x_m, T),\ m = 1, \dots, M. \tag{3}$$

Or, if the measurements f_m are noisy, we instead solve

$$\arg \min_{u_0 \in L_0(\Omega)} \|u_0\|_0 \quad \text{subject to} \quad |f_m - (\mathcal{G}u_0)(x_m, T)| \le \epsilon,\ m = 1, \dots, M. \tag{4}$$

As required by the application, we may have the additional constraint $u_0 \ge 0$. These sparsity optimizations closely resemble the recovery step in compressed sensing, and we will use techniques from compressed sensing for their solution.

2.2 Linear PDEs

Our approach applies more generally to problems where u_0 is sparse and linearly related to the the known measurements f_m.

Consider on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ source problems of the form

$$\begin{cases} \partial_t u = \sum_{i,j} \partial_{x_i}\big(a_{i,j}(x)\partial_{x_j} u\big) + \sum_i b_i(x)\partial_{x_i} u + c(x)u + g(x,t) \\ u(x,0) = u_0(x) \end{cases} \tag{5}$$

with periodic or Neumann boundary conditions where the initial condition $u_0 \in L_0(\Omega)$ is unknown and sparse. We suppose that a, b, c, and g satisfy appropriate conditions, so that $u(x,t)$ belongs to a suitable (linear) function space $\mathcal{F}$ on $\Omega \times [0,T]$. The solution is sampled by a linear operator $\mathcal{S} : \mathcal{F} \to \mathbb{R}^M$, for example, sampling point values

$$f_m = u(x_m, t_m), \qquad m = 1, \dots, M. \tag{6}$$

Other interesting choices are sampling the derivative values $\partial_{x_i} u(x_m, t_m)$ or some weighted local averages $(\varphi * u)(x_m, t_m)$.

Given $f = \mathcal{S}(u)$, the source problem is to recover u_0. While u_0 is unknown, we suppose that all other information (Ω, a, b, c, g, $\mathcal{S}$, f) is known.

The initial boundary value problem (5) is inhomogeneous due to the forcing term g. We decompose as $u = u_p + u_h$, let u_p denote the particular solution with $u_0 \equiv 0$ and u_h the homogeneous solution with $g \equiv 0$. Then there exists a linear operator $\mathcal{G} : L_0(\Omega) \to \mathcal{F}$ such that $u_h = \mathcal{G}(u_0)$.

We solve the source problem as the following optimization problem:

$$\arg\min_{u_0 \in L_0(\Omega)} \|u_0\|_0 \quad \text{subject to} \quad f = \mathcal{S}(u_p + \mathcal{G}u_0). \tag{7}$$

By linearity, the constraint is equivalently $\mathcal{S}\mathcal{G}u_0 = f - \mathcal{S}u_p$.

2.3 Sparsity in a Transformed Domain

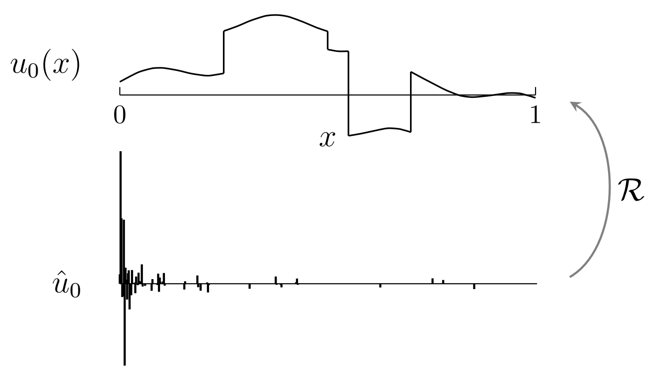


Figure 2: Our approach also applies when u_0 sparse under a transformed representation. Here we show a piecewise smooth function and its Cohen-Daubechies-Feauveau 9/7 wavelet transform.

Rather than considering u_0 itself as sparse, we can also consider u_0 as sparse under some transformed basis or frame. For example, a function like

$$u_0(x) = \sum_{k=1}^{K} c_k \cos(s_k x) + d_k \sin(s_k x)$$

has a sparse Fourier representation. If u_0 is piecewise smooth, it is has an approximately sparse wavelet representation, see Figure 2. Let $\mathcal{R}$ be a linear operator (e.g., inverse Fourier or inverse wavelet transform) such that $u_0 = \mathcal{R}\hat{u}_0$ for some sparse function $\hat{u}_0$. Then (7) becomes

$$\arg\min_{\hat{u}_0 \in L_0} \|\hat{u}_0\|_0 \quad \text{subject to} \quad f = \mathcal{S}(u_p + \mathcal{G}\mathcal{R}\hat{u}_0). \tag{8}$$

2.4 Discretization

Discrete approximations of these problems need to be made for numerical implementation. Partition Ω into N elements so that u_0 is approximated as a vector v in $\mathbb{R}^N$. Let G, S denote linear finite-dimensional approximations of operators $\mathcal{G}$, $\mathcal{S}$ such that $\mathsf{SG}v \approx \mathcal{S}\mathcal{G}u_0$. Then SG is an $M \times N$ matrix, and the recovery problem is

$$\arg\min_{v \in \mathbb{R}^N} \|v\|_0 \quad \text{subject to} \quad \mathsf{SG}v = f - \mathsf{S}v_p, \tag{9}$$

where v_p is a discrete approximation of u_p. The term $\mathsf{S}v_p$ does not change the problem significantly since it can be absorbed, $f \leftarrow f - \mathsf{S}v_p$; henceforth we omit $\mathsf{S}v_p$.

The matrix SG may be constructed by the following procedure. Set $v = (0, \ldots, 0, 1, 0, \ldots)$ with the nth component equal to 1 and zero otherwise. Approximate (5) as $\mathsf{G}v$, then sample the solution to obtain $f = \mathsf{SG}v \in \mathbb{R}^M$. This f is the nth column of matrix SG.

Similarly if u_0 is sparse in a transformed domain, we solve (9) but with SGR, where R is an $N \times N$ matrix approximating $\mathcal{R}$.

Let A denote the $M \times N$ constraint matrix SG or SGR. In analogy to compressed sensing, A is the sensing matrix.

3 Solving the CS Problem

While (9) is a natural way to pose the problem, it is not so easy to solve. There are two challenges in solving (9). First, the L_0-norm is nonconvex, meaning the existence and uniqueness of solutions are not guaranteed, and on a practical level, the nondifferentiability of the L_0-norm precludes the use of gradient-based minimization methods. Second, inverting the matrix A is an ill-conditioned process since heat diffusion may make two different initial conditions appear increasingly similar over time, hence the solution is extremely sensitive to the given measurements.

We try to overcome these challenges is by replacing the L_0 by L_1,

$$\arg\min_{v} \|v\|_1 \quad \text{subject to} \quad \mathsf{A}v = f - \mathsf{S}v_p. \tag{10}$$

The L_1-norm is convex, thus providing existence of solutions, and it is piecewise differentiable. As demonstrated in the compressive sensing literature, the L_1-norm tends to favor sparse solutions and makes for an effective approximation of L_0. Furthermore, it has been shown that under some general conditions [6], L_0 minimization and L_1 minimization yield the same solution—though unfortunately, this theory does not apply to (10).

In the following sections, we discuss the solution of (10) using the Bregman iteration algorithm.

3.1 Bregman Iteration

Bregman iterative techniques consider minimizing a problem of the form

$$\arg\min_u J(u) \quad \text{subject to} \quad H(u) = 0 \tag{11}$$

or

$$\arg\min_u J(u) \quad \text{subject to} \quad H(u) \leq \epsilon \tag{12}$$

where J and H are two convex functions on a Hilbert space $\mathcal{H}$ and $\min_u H(u) = 0$.

Define the *Bregman distance* as

$$D_J^p(u, \tilde{u}) = J(u) - J(\tilde{u}) - \langle p, u - \tilde{u} \rangle_{\mathcal{H}}, \quad p \in \partial J(\tilde{u}). \tag{13}$$

Note that this is not a distance in the usual sense; it is not symmetric. The constrained minimization (11) is solved by the *Bregman iteration algorithm*:

$$\begin{cases} \text{Initialize: } u^0 = \mathbf{0}, p^0 = \mathbf{0} \\ \textbf{for } k = 0, 1, \ldots \\ \quad u^{k+1} = \arg\min_u D_J^{p^k}(u, u^k) + \lambda H(u) \\ \quad p^{k+1} = p^k - \lambda \nabla H(u^{k+1}) \end{cases} \tag{14}$$

where λ is a positive parameter. For our application with $J(u) = \|u\|_1$ and constraint $H(u; f) = \frac{1}{2} \|\mathcal{A}u - f\|_2^2$, the Bregman iteration algorithm is

$$\begin{cases} \text{Initialize: } u^0 = \mathbf{0},\ p^0 = \mathbf{0} \\ \textbf{for } k = 0, 1, \ldots \\ \quad u^{k+1} = \arg\min_u \|u\|_1 - \left\langle p^k, u \right\rangle + \lambda \|\mathcal{A}u - f\|_2^2 \\ \quad p^{k+1} = p^k - \lambda \mathcal{A}^*(\mathcal{A}u - f) \end{cases} \tag{15}$$

Equivalently, by refactoring $\left\langle p^k, u \right\rangle + \lambda \|\mathcal{A}u - f\|_2^2$, the sequence (p^k) is concisely expressed as adding residuals to f:

$$\begin{cases} \text{Initialize: } u^0 = \mathbf{0},\ f^0 = f \\ \textbf{for } k = 0, 1, \ldots \\ \quad u^{k+1} = \arg\min_u \|u\|_1 + \lambda \|\mathcal{A}u - f^k\|_2^2 \\ \quad f^{k+1} = f^k + \lambda \mathcal{A}^*(\mathcal{A}u - f^k) \end{cases} \tag{16}$$

The Bregman iteration algorithm can be stopped for example when the $\|u^{k+1} - u^k\|$ is less than a tolerance. Similarly, to solve with an inequality constraint like $\|\mathcal{A}u - f\|_2^2 \leq \epsilon$, the algorithm should be stopped for the first k where $\left\|\mathcal{A}u^k - f\right\|_2^2 \leq \epsilon$.

Theorem 1. *1. Monotonic decrease in H:*

$$H(u^{k+1}) \leq H(u^{k+1}) + D_J^{p^k}(u^{k+1}, u^k) \leq H(u^k);$$

2. Converge to the exact minimizer of H: If $\tilde{u}$ minimizes $H(\cdot)$ and $J(\tilde{u}) < \infty$, then $H(u^k) \leq H(\tilde{u}) + J(\tilde{u})/k$;

3. Convergence with noisy data: Let $H(\cdot) = H(\cdot; f)$ and suppose $H(\tilde{u}; f) \leq \epsilon$ and $H(\tilde{u}; g) = 0$; then $D_J^{p^{k+1}}(\tilde{u}, u^{k+1}) < D_J^{p^k}(\tilde{u}, u^k)$ as long as $H(u^{k+1}; f) > \epsilon$.

3.2 Shrinkage

The Bregman iteration algorithm allows us to solve the constrained minimization problem (10) by solving a sequence of unconstrained problems,

$$\arg\min_u \|u\|_1 + \lambda \|\mathcal{A}u - f^k\|_2^2. \tag{17}$$

The subproblem in fact has an efficient closed-form solution.

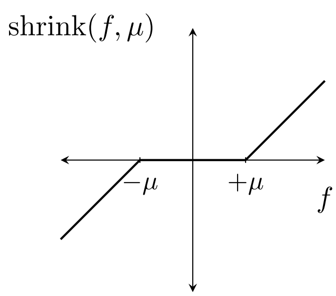
Consider the one-dimensional case where u is a scalar, then it is easy to solve the problem

$$u^\star = \arg\min_{u \in \mathbb{R}} |u| + \lambda (u - f)^2. \tag{18}$$

The solution to (18) is obtained by *shrinkage*, also known as soft thresholding:

$$u^\star = \text{shrink}(f\,, \tfrac{1}{2\lambda}) \equiv \text{sign}(f)\big(|f| - \tfrac{1}{2\lambda}\big)^+. \tag{19}$$

The shrink operator is illustrated in Figure 3.



$$\text{shrink}(f, \mu) \equiv \text{sign}(f)\big(|f| - \mu\big)^+ = \begin{cases} f + \mu & \text{if} \quad f \leq -\mu, \\ 0 & \text{if} \quad -\mu \leq f \leq +\mu, \\ f - \mu & \text{if} \quad +\mu \leq f. \end{cases}$$

Figure 3: Shrinkage

Similarly if u is constrained to be nonnegative, the scalar problem is

$$\arg\min_{u \geq 0} u + \lambda (u - f)^2,$$

and the minimizer is $u^\star = (f - \frac{1}{2\lambda})^+$.

In the multidimensional case where u is a vector, the Bregman subproblem (17) is much harder. We lose the explicit expression of the solution. Instead, we apply the coordinate descent method developed in [11] to solve $\arg\min_u \|u\|_1 + \lambda\|\mathcal{A}u - f^k\|_2^2$. Since we ultimately seek a sparse solution, the process of finding the solution should give preference to sparsity. Instead of proceeding through all the coordinates, we choose only to update coordinates most likely to be the spikes and decrease the energy the most. Therefore, we choose a greedy coordinate algorithm which was introduced in [11], and we also proved the convergence of the algorithm.

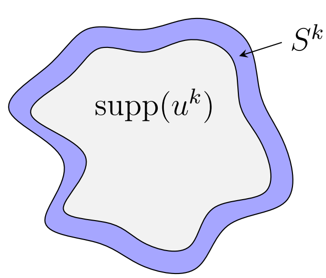
Algorithm (*Greedy Coordinate Descent*):

Precompute: $w_j = \|a_j\|_2^2$;
Normalization: $A(\cdot, i) = A(\cdot, i)/w_i$;
Initialization: $u = 0$, $\beta = A^* f$;
Iterate until converge:
$\tilde{u} = \text{shrink}(\beta, \frac{1}{2\lambda})$;
$j = \arg\max_i |u_i - \tilde{u}_i|$,
then $u_i^{k+1} = u_i^k$, $i \neq j$,
$u_j^{k+1} = \tilde{u}_j$;
$\beta^{k+1} = \beta^k - |u_j^k - \tilde{u}_j|(A^*A)e_j$,
$\beta_j^{k+1} = \beta_j^k$.

In the algorithm, the computation of $\tilde{u}$ and β is essential. $\tilde{u}$ can be obtained by the shrinkage formula with $O(n)$ complexity; for efficiency, β should be updated recursively by adding the difference between two iterations. Every step in the loop has complexity $O(n)$, so combined with its preference for sparsity, this algorithm is very efficient for our problem.

3.3 Support Restriction

We have two strategies to accelerate the solutions to these heat type equations. The first idea is to solve for u_0 only on its apparent support.



Empirically, we observe that early iterations of Bregman iterations tend to produce blurry approximations of the solution and later iterations sharpen the initial approximation into spikes. Therefore, in solving for u^{k+1}, it is reasonable to expect that $\mathrm{supp}(u^{k+1})$ is similar to $\mathrm{supp}(u^k)$. Let S^k be a set containing $\mathrm{supp}(u^k)$ and solve for u^{k+1} with its support restricted to S^k,

$$u^{k+1} = \arg\min\big\{\|u\|_1 + \lambda\,\|\mathcal{A}u - f\|_2^2 : \mathrm{supp}(u) \subset S^k\big\}. \tag{20}$$

For example, S^k may be a morphological dilation of $\mathrm{supp}(u^k)$. It is important that S^k is strictly larger than $\mathrm{supp}(u^k)$ to prevent the iteration from becoming trapped within an incorrect support. If we find a solution which is also the minimizer on its dilated support, then this solution is a local minimizer and a global minimizer due to the convexity.

S^k has to include $\mathrm{supp}(u^k)$ as a closed subset. In our numerical examples, we enlarge $\mathrm{supp}(u^k)$ by including all its connected neighbors in the discretized sense. That is, we increase $\mathrm{supp}(u^k)$ by one pixel in each direction. Then S^k is the smallest set including $\mathrm{supp}(u^k)$ as a closed subset in the discretized sense.

3.4 Domain Exclusion

The second idea is to eliminate a region from consideration when a measurement is very small. Suppose that the stengths of the delta functions in u_0 are bounded from below by $\alpha_{\min} > 0$. Then, since $\mathcal{A}$ is nonnegative, this implies

$$\begin{aligned} f_m \equiv (\mathcal{A}u_0)_m = \int \mathcal{A}(m,x)u_0(x)\,dx &= \sum_{k=1}^{K} \alpha_k \mathcal{A}(m,s_k) \\ &\geq \alpha_{\min} \sum_{k=1}^{K} \mathcal{A}(m,s_k). \end{aligned} \tag{21}$$

Thus for a spike to exist at location s, we must have $f_m \geq \alpha_{\min}\mathcal{A}(m,s)$ for all m. The contrapositive of this statement gives a way to identify regions of the domain that cannot have spikes:

$$\Omega_{\mathrm{z}} \equiv \bigcup_{m=1}^{M} \big\{s \in \Omega : f_m < \alpha_{\min}\mathcal{A}(m,s)\big\}. \tag{22}$$

Similarly with noisy measurements $|f_m^{\mathrm{exact}} - f_m^{\mathrm{noisy}}| \leq \epsilon$,

$$\Omega_{\mathrm{z}} \equiv \bigcup_{m=1}^{M} \big\{s \in \Omega : f_m^{\mathrm{noisy}} + \epsilon < \alpha_{\min}\mathcal{A}(m,s)\big\}. \tag{23}$$

Beware that the validity of this strategy requires that $\mathcal{A}$ is nonnegative. Otherwise, cancelations could occur such that the bound (21) does not hold.

For the periodic heat equation point-value sampling $f_m = u(x_m, t_m)$, the exclusion condition simplifies to $f_m < \alpha_{\min} G_{t_m}(x_m - s)$, see Figure 4.

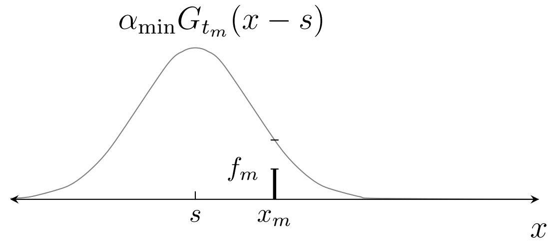


Figure 4: Exclusion for the periodic heat equation with point-value sampling: a small measurement $f_m < \alpha_{\min} G_{t_m}(x_m - s)$ implies that there cannot be a spike at $x = s$.

The Bregman algorithm (16) including support restriction and exclusion regions is performed as

$$\begin{cases} \text{Use (22) to determine } \Omega_{\mathrm{z}} \\ \text{Initialize: } u^0 = \mathbf{0},\ f^0 = f, S^0 = \Omega \backslash \Omega_{\mathrm{z}} \\ \textbf{for } k = 0, 1, \ldots \\ \quad u^{k+1} = \arg\min\big\{ \|u\|_1 + \lambda \|\mathcal{A}u - f^k\|_2^2 : \operatorname{supp}(u) \subset S^k \big\} \\ \quad f^{k+1} = f^k + \lambda \mathcal{A}^*(\mathcal{A}u - f^k) \\ \quad S^{k+1} = \big(\operatorname{supp}(u^{k+1}) \oplus B\big) \backslash \Omega_{\mathrm{z}} \end{cases} \tag{24}$$

4 Numerical Examples

4.1 Inverting the Heat Equation

Consider inverting the heat equation problem,

$$\begin{cases} u_t = u_{xx} + u_{yy}, & t > 0 \\ u_0 = \sum_k c_k \delta(x - x_k, y - y_k), & t = 0 \end{cases} \tag{25}$$

on the unit square $(0,1) \times (0,1)$ with periodic boundary conditions. We suppose that M point-value samples $f_m = u(x_m, y_m, T)$, $m = 1, \ldots, M$, have been observed at some final time $T > 0$. The source problem is to recover u_0 from these observations.

First, we discretize the region using a uniform $N \times N$ grid and a discrete δ function:

$$\delta(x - x_k, y - y_k) = \begin{cases} N^2, & (x,y) = (x_k, y_k) \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Let G be an $N \times N$ matrix such that $\mathsf{G}u_0$ is a finite difference approximation of $u(x, y, T)$,

$$(\mathsf{G}u_0)_k = u(x_k, y_k, T).$$

The constraint matrix A is formed by selecting the rows of G corresponding to the points of observation (x_m, y_m). In other words, the observations f are a downsampled version of the information on the complete grid, $f = \mathsf{S}(\mathsf{G}u_0)$.

Figure (5) shows an experiment using the linearized Bregman algorithm to recover the sparse u_0 by solving the problem:

$$\min_u \|u\|_1 \qquad \text{subject to } (\mathsf{G}u)(x_m, y_m) = f_m. \tag{27}$$

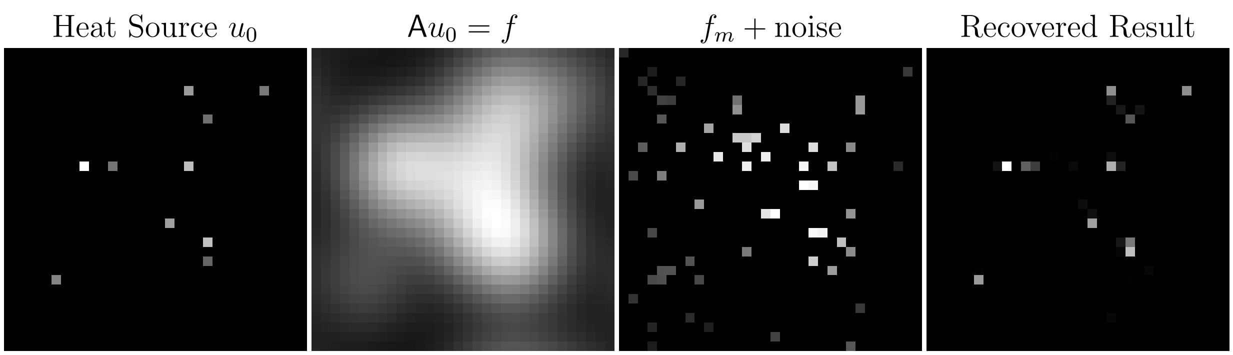


Figure 5: Recovery of $f(x)$ at $T = 0.01$ using 60 randomly selected measurements and 1% noise on a 32×32 grid.

If observations f_m are obtained at different times, the L_1 minimization model for recovery is

$$\min_u \|u\|_1 \qquad \text{s.t. } (\mathsf{G}_{t_m}u)(x_m, y_m) = f_m. \tag{28}$$

4.2 Inverting with Spatially-Varying Conductivity

Consider the equation with sparse initial condition,

$$\begin{cases} u_t = \operatorname{div}(a\nabla u) & x \in \Omega = (0,1) \times (0,1) \\ u = \sum_k c_k \delta(x - x_k) & t = 0 \end{cases}$$

with Neumann boundary condition. We sample u at time $T = 0.01$ and try to recover the initial condition using compressed sensing.

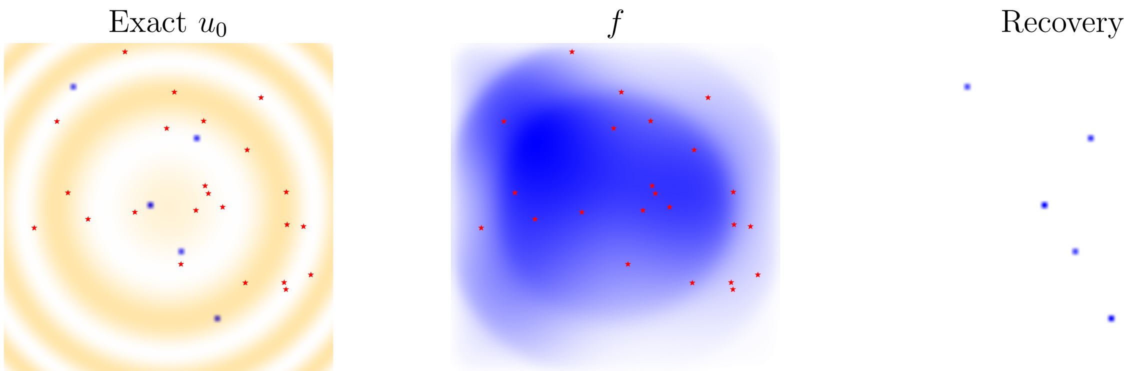


Figure 6: The left plot: the orange shows the distribution of $a(x)$; red stars indicate sampling locations and blue dots represents heat source locations. The middle plot: the heat distribution at time T is shown in blue. The right figure shows the recovery. Here $a(x)$ is a positive smooth function.

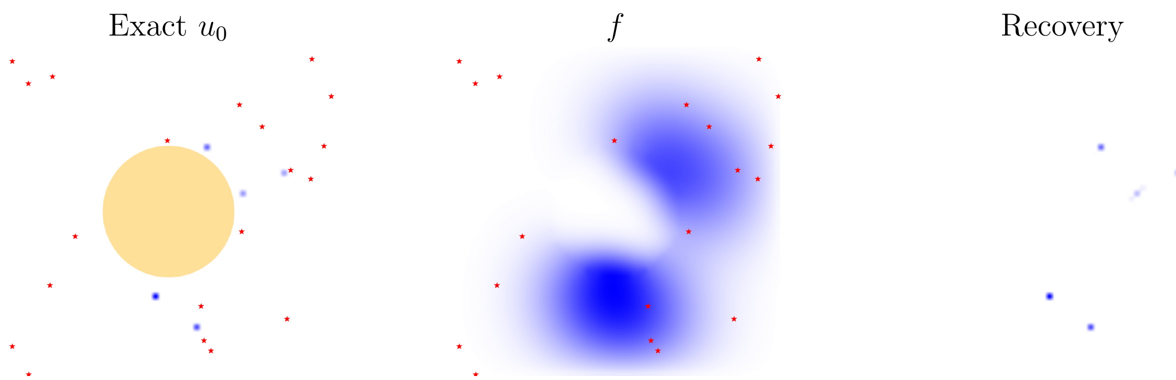


Figure 7: The orange circle indicates the distribution of $a(x)$, and $a(x) = 0.2$ inside of the circle and $a(x) = 1$ elsewhere.

5 Inequality

There is no rigorous proof to ensure that L_1 minimization for inverting the heat operator enhances the sparsity. However, the following inequality suggests it is true under some limited conditions. In one dimensional space, if the true solution has only one spike, the L_1 minimizer will be very close to the true solution under the Wasserstein distance. This implies a sense of stability of the spike locations of the solution. Intuitively, two signals that are close under the Wasserstein distance will also be close after the heat diffusion process. In our future work, we hope to verify this intuition for more general cases.

Theorem 2. *Suppose that u has one spike, $u(x) = \alpha\delta(x - x_1)$, where $\alpha > 0$. Let s_j, $j = 1, 2, \ldots, J$ denote the sampling locations and suppose there are two samples s_1 and s_2 such that $s_1 < x_1 < s_2$ and $s_2 - s_1 < \sqrt{2T}$. Let $f(s_j) = (Au)_{s_j}$ and $S = [s_2 - \sqrt{2T}, s_1 + \sqrt{2T}]$.*

For any v of the form

$$v(x) = \sum_j \beta_j \delta(x - y_j) \text{ and } \hat{f}(s_j) = (Av)_{s_j}. \tag{29}$$

satisfying $\beta_j > 0$, $\|v\|_1 \le \|u\|_1$, and $\|\hat{f} - f\|_\infty \le \epsilon$, there exist $C' > 0$ and $C'' > 0$ such that

$$1 \ge \frac{\sum_{j:y_j \in S} \beta_j}{\alpha} \ge 1 - C'\epsilon \tag{30}$$

$$\frac{\sum_{j:y_j \in S} \beta_j |y_j - x_1|^2}{\alpha} \le C''\epsilon. \tag{31}$$

From $\|v\|_1 \le \|u\|_1$, we know $\sum_{y_j \in S} \beta_j \le \alpha$, and the inequality (31), so

$$\left(\frac{\sum_{y_j \in S} \beta_j |y_j - x_1|}{\alpha}\right)^2 \le \left(\frac{\sum_{y_j \in S} \beta_j}{\alpha}\right)\left(\frac{\sum_{y_j \in S} \beta_j |y_j - x_1|^2}{\alpha}\right) \le \frac{\sum_{y_j \in S} \beta_j |y_j - x_1|^2}{\alpha} \le C''\epsilon \tag{32}$$

So

$$\frac{\sum_{j:y_j \in S} \beta_j |y_j - x_1|}{\alpha} \le C'''\sqrt{\epsilon}. \tag{33}$$

We can derive some simple conclusion from the above theorem: when the true sparse solution has only one spike, the recovery obtained by L_1 minimization should be close to the true solution. They have close L_1 norm and Wasserstein distance after normalization.

Theorem 3. *Suppose $u = \alpha\delta(x - x_1)$, $Au = f_0$, $\|f_0 - f\|_\infty \le \epsilon$, and*

$$v = \arg\min \|u\|_1 \text{ s.t. } \|Au - f\|_\infty \le \epsilon. \tag{34}$$

Let s_j, $j = 1, 2, \ldots, J$ denote the sampling locations and suppose there are two samples s_1 and s_2 such that $s_1 < x_1 < s_2$ and $s_2 - s_1 < \sqrt{2T}$. Then

$$\|u\|_1 - \|v\|_1 \le C_1\epsilon \tag{35}$$

and there are $S \subseteq [x_1 - \sqrt{2T}, x_1 + \sqrt{2T}]$ and $C_2 > 0$, such that

$$\left\| \tfrac{\|v_S\|_1}{\alpha} u - v_S \right\|_W \le C_2\sqrt{\epsilon}. \tag{36}$$

Proof. Since both u and v satisfy the constraint, but v is the minimizer, so $\|v\|_1 \le \|u\|_1$, and

$$\|Au - Av\|_\infty \le \|Au - f\|_\infty + \|f - Av\|_\infty \le 2\epsilon. \tag{37}$$

Using the theorem, there are $S \subseteq [x_1 - \sqrt{2T}, x_1 + \sqrt{2T}]$ and $C > 0$, such that

$$\frac{\sum_{j:y_j \in S} \beta_j |y_j - x_1|}{\alpha} \le C\sqrt{2\epsilon}. \tag{38}$$

Denote $v|_S = v_S$, then the Wasserstein distance between $\frac{\|v_S\|_1}{\alpha}u$ and v_S is

$$\left\| \tfrac{\|v_S\|_1}{\alpha}u - v_S \right\|_W = \frac{\sum_{y_j \in S} |y_j - x_1|}{\alpha} \leq C\sqrt{2\epsilon}. \tag{39}$$

□

6 Successive Sampling

In the previous sections, we developed a way to solve a heat source identification problem from a fixed set of observations. Therefore, we consider the random sampling scenario because it is the best for compressed sensing. However, taking random observations may not be the best strategy for heat source identification. Random sampling works well for compressed sensing because of the incoherence of the operator $\mathcal{G}$. The strong patterns in our operator $\mathcal{G}$ suggest that something more structured would be better than random sampling. For example, if we happened to know a region has very low heat distribution, then it is certain that it is impossible to have strong heat sources there. When we are choosing our sample locations, we may want to concentrate in the strong heat distribution area or explore unsampled areas. Therefore, if we have a chance to pick the next sample location, we should consider the existing information instead of picking a random location.

Here we consider solving the source problem in an adaptive or online kind of approach according to the following procedure. We want to come up with a better sampling strategy than random sampling. Since we want the adopt the information existing for picking the next sampling location, the whole process for solving our problem is the following:

1. Solve the heat source identification problem with k samples;
2. Use the solution u_k to select a $(k+1)$th sample;
3. Iterate.

Let us give a mathematical statement of this problem: Let $X_k = (x_1, x_2, \ldots, x_k)$, $T_k = (t_1, t_2, \ldots, t_k)$ and the measurements $f_j = f(x_j, t_j)$, $j = 1, 2, \cdots, k$. We denote $F_k = f = (f_1, f_2, \ldots, f_k)^T$, and $A_k : \mathbb{R}^N \to \mathbb{R}^k$, satisfies $A_k u = F_k$. We denote the solution from k measurements by u_k.

Suppose the spike amplitudes are bounded from below by $\alpha_{\min} > 0$, which is a plausible assumption since we can treat small spikes as noise but real heat sources. Define the *covering region of x* as the set

$$\{y \in \Omega : \mathcal{G}(\alpha_{\min}\delta_y)(x) \geq \text{threshold}\}.$$

This set describes the domain of dependence of $u(x,T)$. We define a way to measure to what extend a point x is covered by samples s_j,

$$V(x) = \mathcal{G}\big(\sum_j \delta_{s_j}\big).$$

It is equivalent to place single heat sources on all sample locations as an initial condition, then computing the total heat distribution. The bigger $V(x)$ is, the more information available at x.

To choose the next sample location, there are two competing objectives: to refine locally or to explore further. Our approach is to prioritize local refinement. We need refine locally since we want to improve the local resolution as we discover a plausible heat source cluster, usually to locate the accurate heat sources. Also we want to explore further to enlarge the effective coverage as a necessary stopping condition.

Improving local resolution: If u_k varies significantly from u_{k-1}, then we conclude both u_k and u_{k-1} are not close to the true solution. We need more information to identify the heat source inside the existing covering region. So we choose the next sampling location x_{k+1} by comparing the difference between the two solution u_k and u_{k-1}, and pick the one where they differ the most.

$$x_{k+1} = \arg \max_{x:x\notin B_r(x_j)} |G_\sigma * u_k - G_\sigma * u_{k-1}|,$$

where G_σ is a Gaussian with variance σ. We usually choose σ small and the role of G_σ is as a smoother. $B_r(x_j)$ is a ball with center in x_j and radius r since we want to exclude small regions around existing samples. In the other case, we are satisfied with the heat sources found inside the existing covering region. We then want to discover heat sources outside of the existing covering region. Therefore, we sample outside the covering region by selecting a point where V has minimal magnitude.

$$x_{k+1} = \arg \min_{x:x\notin B_r(x_j)} |V(x)|.$$

Compared with random sampling, these two criteria use more information in the space domain, approach the heat source more quickly and does not waste samples in the region which cannot contain heat sources.

6.1 Numerical Experiment

We consider 2D heat flow with periodic boundaries with the following initial condition u_0 and final time $f = u(\cdot, T)$.

By following our proposed adaptive sampling procedure, we have the following sequence of solutions:

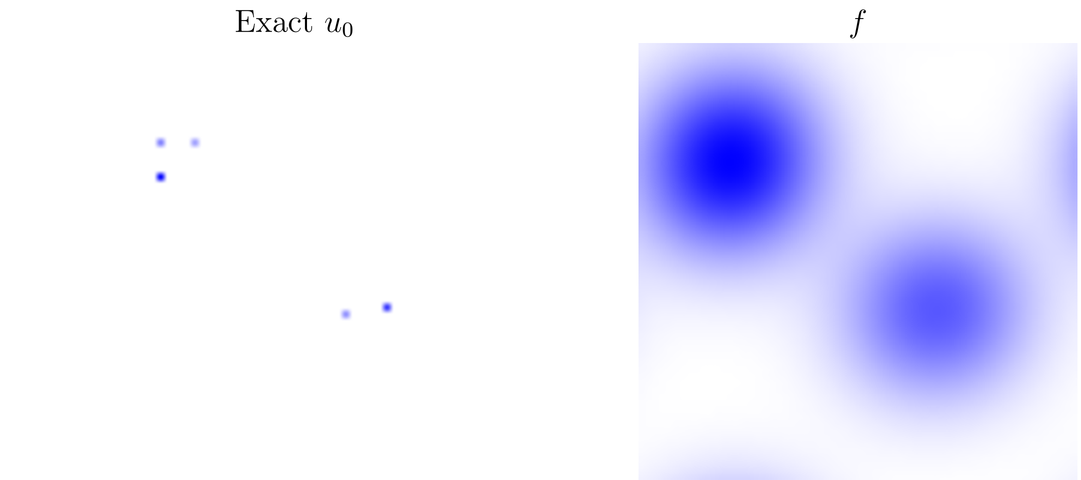


Figure 8: Left: heat sources at $t = 0$; right: heat distribution at T.

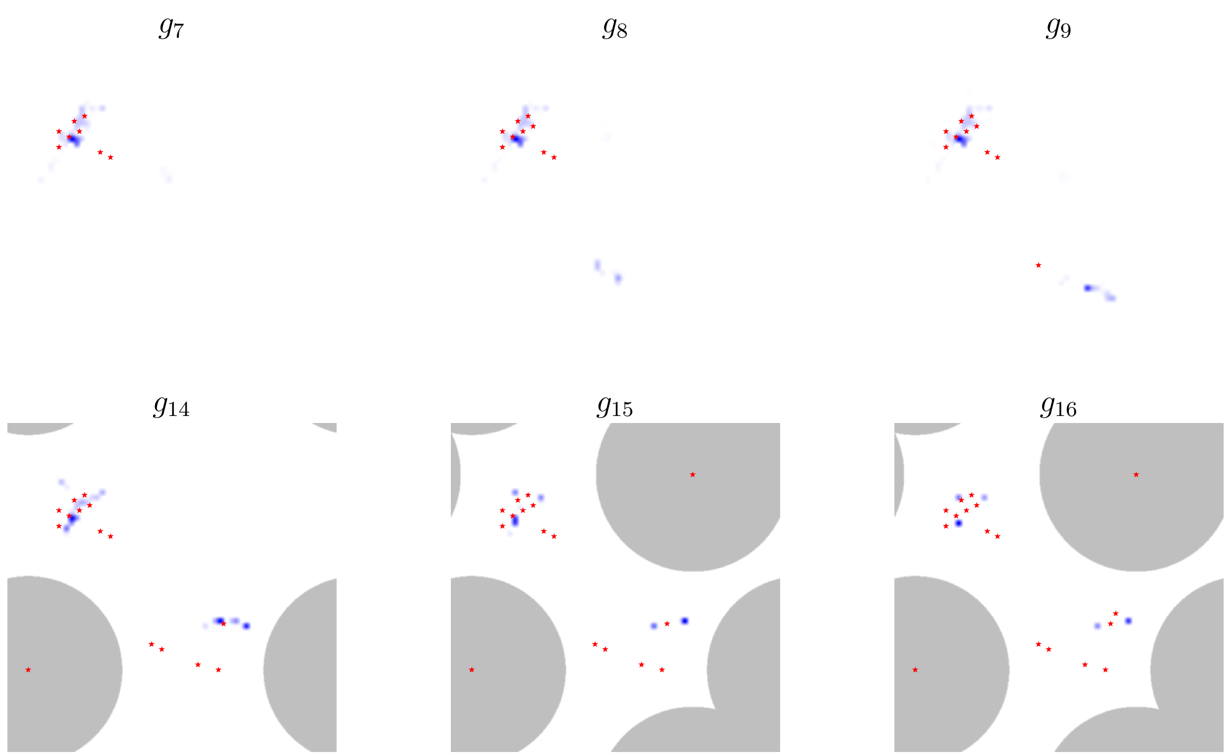


Figure 9: A simulation of the adaptive sampling and recovery results. We are showing the 7th, 8th, 9th, 14th, 15th, 16th steps. Blue shows the recovery; exclusion region shows in gray and sampling locations show in red stars.

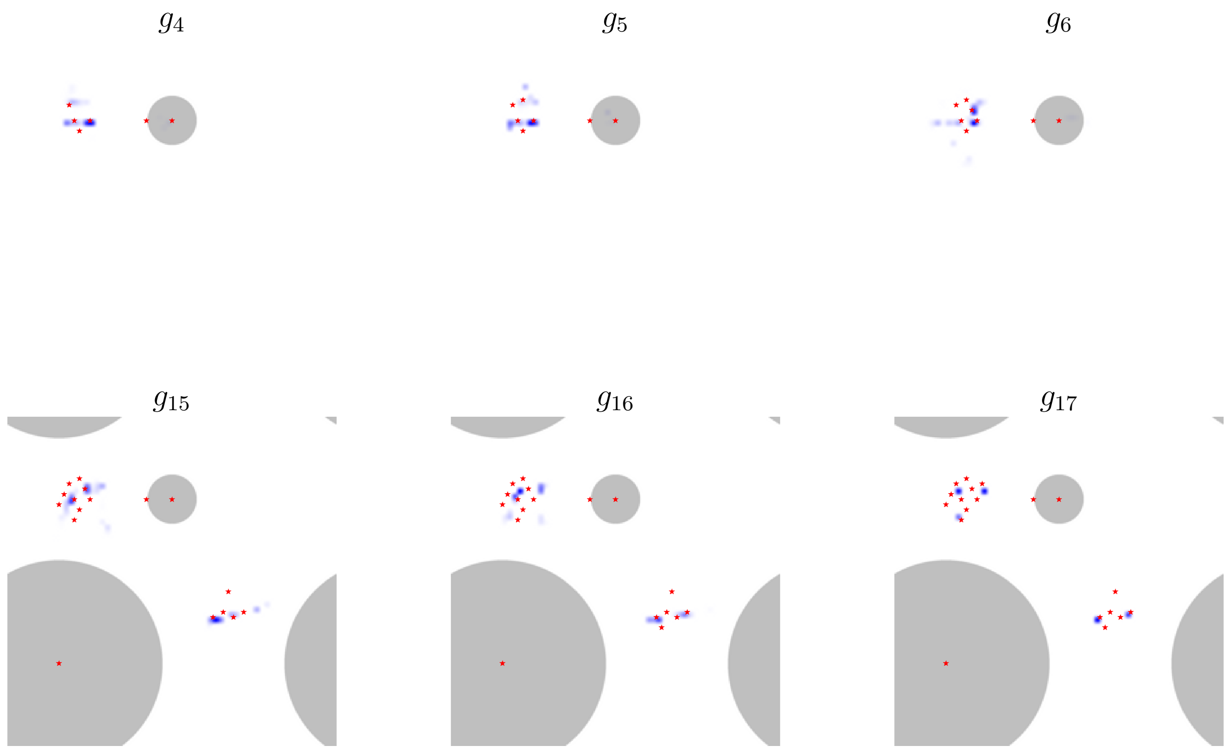


Figure 10: An another simulation starts at a different random chosen location under the same setup. We are showing the 4th, 5th, 6th, 15th, 16th, 17th steps.

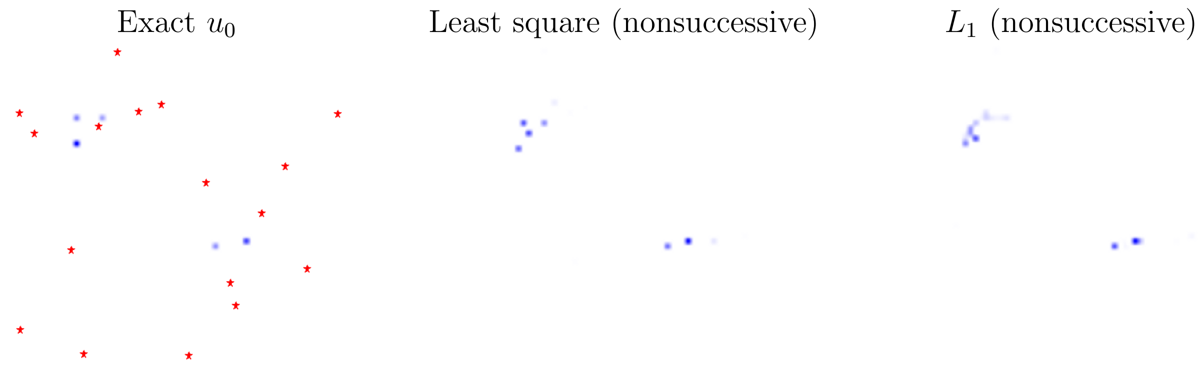


Figure 11: Random sampling is worse than successive sampling.

The red dots denote sampling locations, the solution value is shown as a blue gradient, and the exclusion regions are shown in gray.

A comparison: With the same number of random samples, the solutions of least squares and L_1 minimization are not as accurate as the successive greedy solution.

7 Conclusion

The heat source identification problem can be solved by L_1 minimization. For two dimensions, numerical experiments suggest that we can recover the sparse initial condition by using 4 times more measurements than the number of total initial spikes. If we can solve in a successive manner, then we can use even fewer measurements. As for the stability of our method, as the noise increases, we need more measurements to obtain accurate solutions. In the future, we want to work on the error estimation and theoretical analysis. We are also interested in more general equations and high dimensional problems.

A Inequality

Lemma 1. *Consider function* $W(x) = -g'(s_2 - x_1)g(x - s_1) - g'(x_1 - s_1)g(s_2 - x)$*, we claim that* $W(x)$ *has one maximal at* $x = x_1$ *and* $W(x_1) - W(x) \geq C||x_1 - x||_2^2$ *as* $x \in [s_2 - \sqrt{2T}, s_1 + \sqrt{2T}]$.

Proof. When $x \geq 0$, $g'(x) \leq 0$; when $x \leq 0$, $g'(x) \geq 0$. Since $s_1 < x_1 < s_2$, so $-g'(s_2 - x_1)$ and $-g'(s_2 - x_1)$ are positive.

$$W'(x_1) = -g'(s_2 - x_1)g'(x_1 - s_1) + g'(x_1 - s_1)g'(s_2 - x_1) = 0. \tag{40}$$

So x_1 is an extreme.

$$W''(x_1) = -g'(s_2 - x_1)g''(x_1 - s_1) - g'(x_1 - s_1)g''(s_2 - x_1) \tag{41}$$

When $|x| \le \sqrt{2T}$, $g''(x) \ge 0$ since $g(x)$ is convave. So when $x \in [s_2 - \sqrt{2T}, s_1 + \sqrt{2T}]$, $g''(x - s_1) \le 0$ and $g''(s_2 - x) \le 0$, so

$$W''(x) = -g'(s_2 - x_1)g''(x - s_1) - g'(x_1 - s_1)g''(s_2 - x) \le 0 \tag{42}$$

Suppose $x_1 \in [s_2 - \sqrt{2T}, s_1 + \sqrt{2T}]$, when $x \in [s_2 - \sqrt{2T}, s_1 + \sqrt{2T}]$,

$$W(x) = W(x_1) + W'(x_1)(x - x_1) + W''(\theta)(x - x_1)^2/2, \ \textit{where}\ \theta \in [x, x_1] \tag{43}$$

Denote $M = \inf_{\theta \in [s_2 - \sqrt{2T}, s_1 + \sqrt{2T}]} \{-W''(\theta)/2\}$, then $M > 0$, then

$$W(x_1) - W(x) \ge M(x - x_1)^2, \ \textit{where}\ \theta \in [s_2 - \sqrt{2T}, s_1 + \sqrt{2T}] \tag{44}$$

If $x \ge s_1 + \sqrt{2T}$, denote $p = x - (s_1 + \sqrt{2T}) > 0$, since $g(x)$ is convex when $x \ge \sqrt{2T}$, so

$$g(p + \sqrt{2T}) \le g(\sqrt{2T}) + pg'(\sqrt{2T}) \tag{45}$$

$$\begin{aligned} W(x) &= -g'(s_2 - x_1)g(p + \sqrt{2T}) - g'(x_1 - s_1)g(p + (s_1 + \sqrt{2T} - s_2)) \\ &\le -g'(s_2 - x_1)[g(\sqrt{2T}) + pg'(\sqrt{2T})] - g'(x_1 - s_1)g(s_1 + \sqrt{2T} - s_2) \\ &= W(s_1 + \sqrt{2T}) - p(g'(s_2 - x_1)g'(\sqrt{2T})). \end{aligned}$$

So

$$\begin{aligned} W(x_1) - W(x) &\ge W(x_1) - W(s_1 + \sqrt{2T}) + p(g'(s_2 - x_1)g'(\sqrt{2T})) \\ &\ge M(x_1 - (s_1 + \sqrt{2T}))^2 + p(g'(s_2 - x_1)g'(\sqrt{2T})) \\ &= M(x_1 - (s_1 + \sqrt{2T}))^2 + (x - (s_1 + \sqrt{2T}))(g'(s_2 - x_1)g'(\sqrt{2T})) \end{aligned}$$

Similarly, If $x \le s_2 - \sqrt{2T}$, denote $pp = (s_2 - \sqrt{2T}) - x > 0$, then

$$W(x_1) - W(x) \ge M(x_1 - s_2 + \sqrt{2T})^2 + ((s_2 - \sqrt{2T}) - x)(g'(x_1 - s_1)g'(\sqrt{2T})) \tag{46}$$

□

Theorem 4. *Suppose that u has one spike, $u(x) = \alpha\delta(x - x_1)$, where $\alpha > 0$. Let s_j, $j = 1, 2, \ldots, J$ denote the sampling locations and suppose there are two samples s_1 and s_2 such that $s_1 < x_1 < s_2$ and $s_2 - s_1 < \sqrt{2T}$. Let $f(s_j) = (Au)_{s_j}$ and $S = [s_2 - \sqrt{2T}, s_1 + \sqrt{2T}]$. For any v of the form*

$$v(x) = \sum_j \beta_j \delta(x - y_j) \ \textit{and}\ \hat{f}(s_j) = (Av)_{s_j}. \tag{47}$$

satisfying $\beta_j > 0$, $\|v\|_1 \leq \|u\|_1$, *and* $\|\hat{f} - f\|_\infty \leq \epsilon$, *there exist* $C' > 0$ *and* $C'' > 0$ *such that*

$$1 \geq \frac{\sum_{j:y_j \in S} \beta_j}{\alpha} \geq 1 - C'\epsilon \tag{48}$$

$$\frac{\sum_{j:y_j \in S} \beta_j |y_j - x_1|^2}{\alpha} \leq C''\epsilon. \tag{49}$$

Proof. $s_1 < x_1 < s_2$, denote $g(x) = \frac{e^{-\frac{x^2}{4T}}}{\sqrt{4\pi T}}$,

$$\alpha W(x_1) = \alpha(-g'(s_2 - x_1)g(x_1 - s_1) - g'(x_1 - s_1)g(s_2 - x_1)) = -g'(s_2 - x_1)f(s_1) - g'(x_1 - s_1)f(s_2) \tag{50}$$

$$\begin{aligned}\sum \beta_j W(y_j) &= \sum \beta_j(-g'(s_2 - x_1)g(x_1 - s_1) - g'(x_1 - s_1)g(s_2 - x_1)) \\ &= -g'(s_2 - x_1)\hat{f}(s_1) - g'(x_1 - s_1)\hat{f}(s_2)\end{aligned}$$

So

$$\alpha W(x_1) - \sum \beta_j W(y_j) \leq (-g'(s_2 - x_1) - g'(x_1 - s_1))\epsilon = \tilde{C}\epsilon. \tag{51}$$

Suppose $y_1, \ldots y_l \leq s_2 - \sqrt{2T}$; $y_{l+1}, \ldots, y_k \in [s_2 - \sqrt{2T}, s_1 + \sqrt{2T}]$ and $y_{k+1}, \ldots, y_m \geq s_1 + \sqrt{2T}$, then

$$\begin{aligned}&\alpha W(x_1) - \sum \beta_j W(y_j) \geq (\alpha - \sum \beta_j)W(x_1) + \sum \beta_j (W(x_1) - W(y_j)) \\ &\geq (\alpha - \sum \beta_j)W(x_1) \\ &+ \sum_{j=1}^{l} \beta_j \{M(x_1 - (s_1 + \sqrt{2T}))^2 + (x - (s_1 + \sqrt{2T}))(g'(s_2 - x_1)g'(\sqrt{2T}))\} \\ &+ \sum_{j=l+1}^{k} \beta_j M(x_1 - y_j)^2 \\ &+ \sum_{j=k+1}^{m} \beta_j \{M(x_1 - s_2 + \sqrt{2T})^2 + ((s_2 - \sqrt{2T}) - x)(g'(x_1 - s_1)g'(\sqrt{2T}))\}\end{aligned}$$

Denote $C_1 = \min(M(x_1 - (s_1 + \sqrt{2T})), (g'(s_2 - x_1)g'(\sqrt{2T})))$ and $C_2 = \min(M(x_1 - s_2 +$

$\sqrt{2T}), g'(x_1-s_1)g'(\sqrt{2T}))$, then $C_1>0$ and $C_2>0$, and $C=\min(C_1,C_2)$.

$$\begin{aligned}
\alpha W(x_1)-\sum\beta_jW(y_j) \geq (\alpha-\sum\beta_j)W(x_1)& \\
+\sum_{j=1}^{l}\beta_jC_1|x-x_1| &+ \sum_{j=l+1}^{k}\beta_jM(x_1-y_j)^2+\sum_{j=k+1}^{m}\beta_jC_2|x-x_1| \\
\geq (\alpha-\sum\beta_j)W(x_1)&+C\Big\{\sum_{j=1}^{l}\beta_j|x-x_1| \\
+\sum_{j=k+1}^{m}\beta_j|x-x_1|\Big\}&+M\sum_{j=l+1}^{k}\beta_j(x_1-y_j)^2.
\end{aligned}$$

Therefore

$$(\alpha-\sum\beta_j)W(x_1)+C\{\sum_{j=1}^{l}\beta_j|x-x_1|+\sum_{j=k+1}^{m}\beta_j|x-x_1|\}+M\sum_{j=l+1}^{k}\beta_j(x_1-y_j)^2\leq\alpha\tilde{C}\epsilon \quad (52)$$

We know that $\alpha-\sum\beta_j\geq 0$, and

$$\begin{aligned}
&\sum_{j=1}^{l}\beta_j|x-x_1|+\sum_{j=k+1}^{m}\beta_j|x-x_1| \\
\geq&\sum_{j=1}^{l}\beta_j|x_1+\sqrt{2T}-s_2|+\sum_{j=k+1}^{m}\beta_j|s_1+\sqrt{2T}-x_1| \\
\geq& C_3(\sum_{j=1}^{l}\beta_j+\sum_{j=k+1}^{m}\beta_j)
\end{aligned}$$

so

$$(\alpha-\sum\beta_j)W(x_1)+C_3\{\sum_{j=1}^{l}\beta_j+\sum_{j=k+1}^{m}\beta_j\}+M\sum_{j=l+1}^{k}\beta_j(x_1-y_j)^2\leq\alpha\tilde{C}\epsilon \quad (53)$$

so

$$\sum\beta_j\geq\alpha(1-\frac{\tilde{C}}{W(x_1)}\epsilon). \quad (54)$$

and

$$\frac{C_3\{\sum_{j=1}^{l}\beta_j+\sum_{j=k+1}^{m}\beta_j\}+M\sum_{j=l+1}^{k}\beta_j(x_1-y_j)^2}{\sum\beta_j}\leq\frac{1}{\frac{1}{\tilde{C}}-\frac{1}{W(x_1)}\epsilon}\epsilon \quad (55)$$

So

$$\frac{\sum_{j=l+1}^{k}\beta_j}{\sum\beta_j}\geq 1-\frac{1}{\frac{C_3}{\tilde{C}}-\frac{C_3}{W(x_1)}\epsilon}\epsilon \quad (56)$$

So

$$\frac{\sum_{j=l+1}^{k}\beta_j}{\alpha} \geq (1 - \frac{1}{\frac{C_3}{\tilde{C}} - \frac{C_3}{W(x_1)}\epsilon}\epsilon)(1 - \frac{\tilde{C}}{W(x_1)}\epsilon)$$
$$\geq 1 - [\frac{1}{\frac{C_3}{\tilde{C}} - \frac{C_3}{W(x_1)}\epsilon} + \frac{\tilde{C}}{W(x_1)}]\epsilon.$$

and

$$\frac{\sum_{j=l+1}^{k}\beta_j(x_1 - y_j)^2}{\alpha} \leq \frac{\tilde{C}}{M}\epsilon \tag{57}$$

□

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