

Reconstruction of Binary Functions and Shapes from Incomplete Frequency Information

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Abstract

The characterization of a binary function by partial frequency information is considered. We show that it is possible to reconstruct the binary signal from incomplete frequency measurements via solving a simple linear optimization problem. We further prove that if the binary function is spatially structured, e.g. a general black-write image or an indicator function of a shape, then it can be recovered from very few low frequency measurements in general. These results would lead to efficient methods of sensing, characterizing and recovering a binary signal or a shape as well as other applications like deconvolution of binary function from low-pass filter. Numerical results are demonstrated to validate the theoretical arguments.

1 Introduction

This paper concerns the reconstruction of binary signal. Binary signal is an important class of signals appearing in a variety of applications like shape processing, bar code and handwriting recognition, obstacle detection, image segmentation and so on, see e.g. [22, 26, 15, 1] and many others. Even some problems about general (not binary) signal can be solved via the binary lifting of the signal, i.e. treating the epigraph of the signal as a binary signal in a higher dimensional region. This idea has been successfully applied to many applications like Markov random fields optimization, see e.g. [19, 27].

One of the major difficulties in reconstruction of a binary function is that the binary constraint is non-convex, therefore it is often approached by means of double-well potential functional or other nonlinear schemes. In this paper we demonstrate that in general the binary function can be reconstructed precisely via a simple convex optimization problem when only partial frequency information is obtained (e.g. when the signal is blurred by a low-pass filter).

1.1 Main results

Let u_0 be the binary function, i.e. $u_0(x) \in \{0, 1\}$, $\forall x$. Let \mathcal{F} be the Fourier transform and S the selecting operator corresponding to the incomplete measurements b , our goal is to recover u_0 from $b = S\mathcal{F}u_0$ which is an underdetermined system. The main contribution of this work is showing that under certain conditions, u_0 can be exactly reconstructed by solving the convex relaxed optimization problem

$$\text{find } u \text{ s.t. } S\mathcal{F}u = b, \quad 0 \leq u \leq 1.$$

which is a little bit surprising because the solution of the problem seems not even necessarily unique at first glance. However, we proved that in many cases it is unique and equals to u_0 .

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- When the binary signal is spatially structured, i.e. the 1s and 0s are clustered, e.g. in a binary image or as an indicator function of a shape, with very few low frequency measurements taken, the solution of this convex optimization problem is deterministically unique and equals to the original binary signal. For a detailed statement, see theorem 2.5.
- When the binary signal is spatially structured but the frequency measurements are not taken only for low frequencies but in an arbitrary way, we give a sufficient condition to determine if the measurements are enough to recover the signal via the convex problem. For a detailed statement, see theorem 2.12.
- If the binary signal has no spatial structure, i.e. the 1s and 0s are randomly appearing, we show that this relaxation works with overwhelming probability when the measurements are taken more than a half of the size of the signal, and the probability tends to 1 when the size of the signal gets larger. For a detailed statement, see theorem 2.15.
- We also propose a very efficient algorithm especially designed for this convex problem (see algorithm 1). Numerical experiments are given in section 4.

1.2 Related works

It is not new to people that under certain circumstances, the binary constraint can be automatically satisfied by imposing the convex relaxation, i.e. the box constraint $0 \leq u \leq 1$. For example, when solving the image segmentation, multi-label and many other problems based on total variation model (see e.g. [7, 3, 27]), people noticed that although the original problem seems to be non-convex, the global minimizer can be obtained by solving the relaxed convex problem and the solution will be automatically (almost) binary. However, this approach works only because of the special structure of the variational model and the theoretical analysis strongly depends on the coarea formula of the total variation.

In contexts of regression and approximation, it is known for a long time that an L_∞ regression (sometimes referred to as Chebyshev or minimax regression), where the penalty function is given by the L_∞ norm, or a deadzone-linear penalty regression, where the penalty function is given by the deadzone function $(|\cdot| - a)^+$, will lead to a distribution of residual that many elements have the value right at the boundary of the feasible domain (see e.g. [2], Chapter 6). It is related with this research because when the feasible domain is interval $[0, 1]$, a function with many values right at the boundary is nothing but a binary signal. In fact, in section 2.1 we will show that our convex relaxation of the problem can be equivalently reformulated as an L_∞ or deadzone penalty minimization problem.

We also noticed that in a very recent research [11, 14], the authors showed that a random binary signal can be recovered with certain probability by means of the relaxed box constraint. In their work the major mathematical tool is delicate geometric face-counting of random polytopes. We also get a basically similar result in section 2.6, but from different approach.

The idea of recovering a signal from partial frequency measurements can be easily connected with compressed sensing [5, 6, 12], which takes advantage of the prior assumption on the sparsity of the signal and reconstructs the signal via L_1 minimization. However, this work is substantially different from the compressed sensing. Although the binary function is a special case of piecewise constant function whose derivative is sparse, it is actually stronger than simply being piecewise constant and therefore stronger results can be expected. Indeed, none of the results given in this work can be deduced from the standard compressed sensing theory directly. Some of them are even in a very different nature. For example, in compressed sensing the frequency measurements should be taken randomly to guarantee the restricted isometry property, while in the reconstruction of the structured binary function, the low frequency measurements actually play a more important role than high frequency measurements as we will discuss below. On the other hand, many results given in this paper are deterministic, while most results in compressed sensing are intrinsically stochastic.

The present research is somehow related to a seminal work on the reconstruction of signals from partial frequency information [13], where the spatial structures and patterns in both time

and frequency domain are also used to guarantee the uniqueness of the signal reconstruction as in our work. In [34], the authors defined the degree of freedom contained in a sparse or piecewise polynomial signal as the rate of innovation of the signal and showed that the signal can thus be recovered by so many samplings. This is similar in some sense with what we prove in theorem 2.5. However, this theory requires a certain pattern of sampling and the reconstruction involves a factorization of polynomial. In [4] the authors showed that if an underdetermined system admits a very sparse positive solution and the matrix has a row-span intersecting the positive orthant, then the solution is actually unique. We will further discuss the resemblance between this work and ours in section 2.3.

1.3 Notations and conventions

In this paper all signals are processed as with periodic boundary condition. The h -dimensional discrete signal is defined on $\{1, \dots, N\}^h$ where N is assumed to be even without loss of generality. Here for simplicity we assume the domain is equi-long along each dimension. The h -dimensional continuous signal is defined on $\mathbb{T}^h = [0, 1]^h$ where 0 and 1 are identified due to the periodic boundary condition.

We use \mathcal{F} to denote the Fourier transform. Depending on the context, it can be transform from discrete signal on grid to discrete coefficients or from continuous periodic function to discrete coefficients. When we talk about discrete Fourier transform, we use the following convention:

$$a_k = \sum_{x \in [1, N]^h} u(x) e^{-2\pi i \langle k, \frac{x}{N} \rangle}, \quad u(x) = \frac{1}{N^h} \sum_{k \in [-\frac{N}{2}, \frac{N}{2} - 1]^h} a_k e^{2\pi i \langle k, \frac{x}{N} \rangle}$$

where $\{a_k\}$ are the Fourier coefficients defined on symmetric support ($N/2$ is treated as same as $-N/2$). Smaller $|k|$ corresponds to lower frequency. Since u is always real, $\{a_k\}$ satisfies $a_{-k} = \overline{a_k}$. Notice that $\mathcal{F}^\top = N^h \mathcal{F}^{-1}$.

We use S to denote the selecting operator. S is a diagonal matrix where the selected positions have value 1 and others are 0.

1.4 Contents

The paper is organized as follows. Section 2 discusses the theoretical results. In section 3 the algorithm to solve the convex problem is proposed. Numerical experiments are shown in section 4 and conclusion is given in section 5. To make the main text more concise, we put all proofs into Appendix except those theorems and corollaries immediately concluded from the discussion in the context.

2 Theory

2.1 General reconstruction theory

Suppose u_0 is a discrete binary signal, i.e. $u_0(x) \in \{0, 1\}, \forall x$. Consider a linear system $Au_0 = b$ where $A = S\mathcal{F}$ is a degenerated matrix, \mathcal{F} is the Fourier transform and S is the selecting operator. The meaning of this system is clear: some partial frequency information of the binary signal is given, and we want to reconstruct u_0 from the incomplete measurements. This leads to the following problem (P_0):

$$P_0 : \quad \text{find } u \quad \text{s.t.} \quad Au = b, \quad u(x) \in \{0, 1\}. \quad (1)$$

The problem (P_0) is non-convex due to the binary condition, and the following convex problem is the tight relaxation of (P_0):

$$P_1 : \quad \text{find } u \quad \text{s.t.} \quad Au = b, \quad 0 \leq u \leq 1. \quad (2)$$

We want to show that (P_1) can be used to recover u_0 under certain conditions. The following theorem elaborates the condition when this relaxation works.

Theorem 2.1. *Assume u_0 is a binary solution of $Au_0 = b$. There exists no nonzero $v \in \{Av = 0\}$ such that*

$$\begin{cases} v(x) \leq 0, & \text{when } u_0(x) = 1 \\ v(x) \geq 0, & \text{when } u_0(x) = 0 \end{cases} \quad (3)$$

if and only if u_0 is the unique solution of (P_1) , i.e. solving (P_1) recovers u_0 .

If the size of the signal is N , then the criteria (3) on v determines an orthant in \mathbb{R}^N depending on u_0 . We denote this orthant as \mathbb{O}_{u_0} . The theorem tells us that as long as the kernel space of A intersects \mathbb{O}_{u_0} at nowhere but the origin, solving (P_1) is enough to recover u_0 .

This condition is ‘negative’, i.e. it requires the nonexistence of such a vector v . The following statement, sometimes referred to as the Gordan-Stiemke theorem of the alternative, will lead to a ‘positive’ criteria.

Lemma 2.2 (Alternative Theorem, see e.g. [2]). *One and only one of the two following problems is feasible: (1). Find $0 \neq v \geq 0$ s.t. $Av = 0$; (2). Find $v = A^\top \eta$ s.t. $v > 0$.*

Geometrically, this theorem says that if P is a subspace in \mathbb{R}^N , \mathbb{O} is the first orthant, then either $P \cap \mathbb{O} = \{0\}$ or $P^\perp \cap \text{int}(\mathbb{O}) = \emptyset$, but not both. The statement considers the first orthant only, but obviously it is true for any other given orthant. Apply this lemma on theorem 2.1, we immediately get the ‘positive’ version of the criteria:

Theorem 2.3. *Assume u_0 is a binary solution of $Au_0 = b$. There exists $v = A^\top \eta$ such that*

$$\begin{cases} v(x) < 0, & \text{when } u_0(x) = 1 \\ v(x) > 0, & \text{when } u_0(x) = 0 \end{cases} \quad (4)$$

if and only if u_0 is the unique solution of (P_1) , i.e. solving (P_1) recovers u_0 .

Unfortunately, there is no explicit formula to determine if an arbitrary subspace passes through an orthant. Indeed, it is very equivalent with any general linear programming feasibility problem and thus has no closed-form discriminant. However, for some special cases we can still give deterministic or stochastic results, as we will explain in the following subsections.

We want to remark that there are many alternative linear programming problems that can recover the signal as well. In fact, assume $J(u)$ is a convex function on u satisfying

$$J(u) < J(v), \forall u \in [0, 1]^N, v \notin [0, 1]^N. \quad (5)$$

It is easy to see that if u_0 is a binary solution of $Au_0 = b$, then u_0 is a unique solution of (P_1) implies that u_0 is a unique solution of the following convex problem:

$$\min_u J(u) \quad \text{s.t.} \quad Au = b. \quad (6)$$

Therefore solving (6) can also recover u_0 under the condition in theorem 2.1 or 2.3. There are many simple functions satisfying (5). One of the simplest examples is $J(u) = \|2u - 1\|_\infty$. Another example is the deadzone penalty $J(u) = \tilde{J}(\|2u - 1\| - 1)^+$ where \tilde{J} is any convex function with $\tilde{J}(0) = 0$ and $\tilde{J}(v) > 0$ for $v \neq 0$, e.g. $\tilde{J}(v) = \|v\|_p$ for $p \geq 1$.

2.2 Reconstruction of 1D binary signal from low frequency measurements

So far the discussion is based only on the fact that the signal is binary. In most practical applications, the signal is often not only binary, but also structured, i.e. the 1s and 0s are spatially clustered, not randomly distributed. This property could help us reconstruct the signal.

Let us consider 1D case first. Assume $u_0(x)$ is a periodic discrete binary signal defined on $\{1, \dots, N\}$. Since we are considering the structured signal, $u_0(x)$ consists of many intervals with constant value 1 or 0. If the first and last intervals have the same value, we treat them as one merged interval under the periodic boundary condition. Therefore, the total number of intervals is always even, thus $u_0(x)$ can be represented as

$$u_0 = \sum_{j=1}^{2d} \xi_j \mathbf{1}_{I_j}, \quad \xi_j \in \{0, 1\}. \quad (7)$$

$\{I_j\}$ is a partition of $\{1, \dots, N\}$ where each I_j is a consecutive interval.

From theorem 2.3, u_0 can be recovered from (P_1) if and only if there exists $v = A^\top \eta$ such that

$$\begin{cases} v(x) < 0 \text{ in } I_j & \text{if } \xi_j = 1 \\ v(x) > 0 \text{ in } I_j & \text{if } \xi_j = 0 \end{cases} \quad (8)$$

We want to show that this condition is always satisfied for certain types of A . Recall that when partial frequency information is given, $A = S\mathcal{F}$ where S is a sampling operator that corresponds to the known frequencies. If $v = A^\top \eta = \mathcal{F}^\top(S\eta)$, then v is a band-limit signal whose spectrum can be represented as $S\eta$, i.e. located inside the known frequencies. Therefore, the above condition means that the relaxation method is valid as long as we can use only those known frequencies to construct a band-limit signal that satisfies (8). Since (8) describes the zero-crossing position of v , it imposed restraint on the spectrum of v , and therefore on S .

The relationship between the zero-crossings of a signal and its spectrum information is not a new problem in signal processing, readers are referred to [23, 28, 20, 33] for some classic theories. The following result is natural from the perspective of trigonometric interpolation:

Lemma 2.4. *Let $\mathbb{T} = [0, 1]$ where 0 and 1 are identified, i.e. $\mathbb{T} \cong S^1 = \{z : |z| = 1\}$. Given $2n$ points on \mathbb{T} who define $2n$ intervals on \mathbb{T} , there exists a real trigonometric polynomial, whose spectrum is limited in $[-n, n]$, vanishing only at those points and changes signs alternatively on those intervals.*

This conclusion, combining with theorem 2.3, leads to the following deterministic result which states that the number of low frequency measurements we need is to reconstruct the binary function is basically the number of the jumps contained in the signal, no matter how large the size of the signal is. Comparing to the full size of the signal, this is a very big saving in general.

Theorem 2.5. *If $u_0(x)$ is a 1-D binary signal that can be represented as in (7) with $2d$ consecutive intervals of ones and zeros, then by knowing the Fourier coefficients $\{a_k\}$ for $|k| \leq d$, we can recover u_0 through the convex problem (P_1) . (Notice that $u_0(x) \in \mathbb{R}, \forall x$ implies $a_k = \overline{a_{-k}}, \forall k$, so essentially we only need to know $\{a_k\}$ for $0 \leq k \leq d$.) This result is optimal, i.e. precise reconstruction via solving (P_1) is impossible if knowing even less.*

Although theorem 2.5 only holds when the lowest frequency information is given, it is still very useful, because in many practical problems the low frequency measurements are far easier to obtain than the high frequency measurements. The deconvolution problem with a low-pass filter kernel, for example, can be treated as reconstruction from the lowest frequency information.

Heuristically, theorem 2.5 can be understood as follows: if the low frequency measurements are given, then the permitted perturbation can be with higher frequencies only and thus strongly oscillating around zero. Therefore, by controlling the lower and upper bound of the signal as in (P_1) , the oscillating perturbation would be eliminated, and thus the solution is uniquely determined.

2.3 Discussions and generalizations

First, it is easy to see that theorem 2.5 can be directly extended to cosine transform as well, based on the fact that cosine transform of a signal is basically nothing but the Fourier transform of the even extension of the signal.

Corollary 2.6. *If $u_0(x)$ is a 1-D binary signal that can be represented as in (7) with $2d$ consecutive intervals of ones and zeros, then by knowing the discrete cosine transform coefficients $\{a_k\}$ for $0 \leq k \leq 2d$, we can recover u_0 through the convex relaxation (P_1) .*

We also want to mention that if the signal is only bounded from one side, i.e. instead of knowing that the signal is binary, we know the signal is nonnegative, then similar argument would lead to a theorem concerning the reconstruction of sparse nonnegative signal. Indeed, using theorem 2.1 and lemma 2.2 we can obtain the following theorem (the proof is similar and hence omitted):

Theorem 2.7. *If $u_0 \geq 0$ is supported on $K = \{x : u_0(x) = 0\}$, then u_0 is the unique solution of $Au = b$, $u \geq 0$ if and only if there exists $v = A^\top \eta$ such that $v|_K = 0$, $v|_{K^c} > 0$.*

Let $A = SF$ and S also select the low frequency measurements, then theorem 2.7 implies (thanks to lemma 2.4 as well):

Theorem 2.8. *If $u_0(x)$ is a 1-D nonnegative sparse signal supported on $K = \{x : u_0(x) = 0\}$ with $|K| = d$, then by knowing the Fourier coefficients $\{a_k\}$ for $|k| \leq d$, we can recover u_0 through the convex problem $Au = b$, $u \geq 0$.*

This result is closely related with the theorems proved in [4] which said that a nonnegative solution of a linear system is unique if the solution is sparse enough and the matrix has a row-span intersecting the positive orthant. It worth mentioning that the quantity of the needed low frequency measurements in this case approximately equals two times the quantity of the ‘spikes’ of u_0 (other than the number of jumps of u_0 in theorem 2.5). This coincides the observation in [34] that the degree of freedom of a d -sparse signal is $2d$ (for each spike there is one degree for position and one for amplitude), and thus $2d + 1$ measurements are in principal enough.

Our result may further be generalized to basis other than the trigonometric functions. Indeed, the duality of lemma 2.4 stating that a signal without lower frequency components must have many sign changes, some times referred to as the Sturm-Hurwitz theorem [33], plays a critical role here. This properties can be extended to other basis that has similar oscillating pattern, such as some wavelet basis or the eigenfunctions of the regular Sturm-Liouville problems [16]. However, the generalization along this line would be beyond the scope of this paper.

2.4 Reconstruction of 2D binary signal from low frequency measurements

Multidimensional case is more complicated than 1D case due to several reasons. There is no fundamental algebraic theorem for multivariable polynomials. The Sturm-Hurwitz theorem that describes the zero-crossings of function with spectrum gap does not exist in higher dimension as well. In 1D, the complexity of a binary can be simply characterized by the number of jumps as in theorem 2.5. In higher dimension, the binary function may have only one connected part but with complicated shape and the complexity is not easy to calibrate.

Since theorem 2.3 is still valid in multidimensional case, a multidimensional binary signal can be reconstructed by low frequency measurements as long as the jump set of the binary signal corresponds the zero levelset of a low frequency function. Unfortunately, to the author’s knowledge, no criteria has been known to determine if a given shape can be realized as the zero levelset of function with only low frequency components. In [10, 9, 29, 30, 31, 18, 35], some results concerning the relationship between the levelset of a function and its Fourier transform are shown. In [24] it has been proved that under continuous Fourier transform, a function with given levelset curve can be approximated to any degree of accuracy by a band-limited function with given spectrum support. However, this result is barely useful in practice because it requires virtually infinitely high resolution in the frequency domain.

Heuristically, if a function has only low frequency components, we can imagine that its levelset would not be too complicated. Here we give a way to measure this complexity. The basic idea is that since in 1D case the complexity of a binary signal is determined by the number of jumps inside the signal, in higher dimensional we can define an ‘average number of

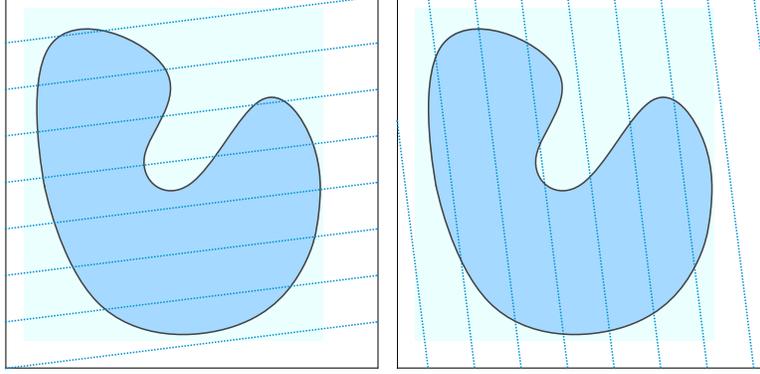


Figure 1: Two gratings on \mathbb{T}^2 with a binary function.

zero-crossings' as illustrated in Fig. 1. The following discussion will be in 2D, but it can be naturally extended to higher dimensional cases.

Define $\mathbb{T}^2 = [0, 1] \times [0, 1]$ with opposite boundaries identified, then $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. For $\theta \in (-\pi/4, \pi/4]$, define

$$L_{s,\theta}(t) = (t, s + t \tan \theta) \pmod{1}, \quad s \in [0, 1], t \in [0, 1]$$

to be a *grating* along angle θ starting from the left edge of the square (see Fig. 1 left). For $\theta \in (\pi/4, 3\pi/4]$, similarly define

$$L_{s,\theta}(t) = (s + t \cot \theta, t) \pmod{1}, \quad s \in [0, 1], t \in [0, 1]$$

to be the grating along angle θ starting from the bottom edge of the square (see Fig. 1 right). Assume u is a binary function whose jump set consists of analytic curves. For given θ and s the line segment $L_{s,\theta}$ will intersect the jump set of u finite times. We denote this number as $\#L_{s,\theta}$ and define the average of $\#L_{s,\theta}$ over s as

$$K_\theta = \cos \theta \int_0^1 \#L_{s,\theta} ds$$

for $\theta \in (-\pi/4, \pi/4]$ and

$$K_\theta = \sin \theta \int_0^1 \#L_{s,\theta} ds$$

for $\theta \in (\pi/4, 3\pi/4]$. The existence of the multiplier $\cos \theta$ and $\sin \theta$ is due to the fact that for gratings with different angle θ , s is not an equi-long variable, while the distance between $L_{s,\theta}$ and the origin is rotational invariant and more intrinsic which equals $s \cos \theta$ for $\theta \in (-\pi/4, \pi/4]$ and $s \sin \theta$ for $\theta \in (\pi/4, 3\pi/4]$. K_θ is called *average directional number of zero-crossings* in this paper which essentially describes the average quantity of sign changes along the direction θ . It is easy to see the connection of this quantity and the number of jumps for a 1D binary signal. Indeed, by Cauchy-Crofton formula, $\int K_\theta d\theta$ is nothing but 2 times the perimeter of the shape, i.e. the total variation of u_0 , while in 1D case the number of jumps also equals to the total variation of the binary signal. We will show that K_θ somehow characterizes the complexity of the shape as elaborated in the next theorem.

Theorem 2.9. *Assume $u(x, y)$ is a 2D binary function with analytic jump curve, the average directional number of zero-crossings of u along angle θ is denoted as K_θ . If there exists a band-limited real function $v(x, y) = \sum_{(j,k) \in \Omega} a_{jk} e^{2\pi i(jx+ky)}$ defined on \mathbb{T}^2 where $\Omega = \{(j, k) : \sqrt{j^2 + k^2} \leq d\}$ such that the jump set of $u(x, y)$ corresponds to the zero levelset of $v(x, y)$, then $K_\theta \leq 2d, \forall \theta$.*

Corollary 2.10. *Assume $u_0(x, y)$ is a discrete 2D binary function defined on $\{1, \dots, N\}^2$ and $u(x, y)$ is a binary function defined on continuous domain \mathbb{T}^2 with analytic jump curve such that $u(x/N, y/N) = u_0(x, y)$ for $(x, y) \in \{1, \dots, N\}^2$, the average directional number of zero-crossings of u along angle θ is denoted as K_θ . If the reconstruction of u_0 by linear programming problem (P_1) from low frequency measurements in $\Omega = \{(j, k) : \sqrt{j^2 + k^2} \leq d\}$ is available, then $d \geq \frac{1}{2} \max_\theta K_\theta$.*

The meaning of theorem 2.9 and corollary 2.10 is clear: the average directional number of zero-crossings of the levelset of a band-limited function is bounded by the diameter of the support of the spectrum. If we denote the jump set of $u(x, y)$ as Γ and the perimeter $|\Gamma|$, then by Cauchy-Crofton formula, the condition $d \geq \frac{1}{2} \max_\theta K_\theta$ further implies $d \geq \frac{1}{\pi} |\Gamma|$, which can be seen as a natural generalization of the 1D case (see lemma 2.4 and theorem 2.5). However, unlike 1D case, this theorem just gives the necessary condition, not a sufficient one. While we cannot prove the other direction of the theorem at this stage, numerical results (see section 4) show that most regular shapes can be reconstructed by a fairly small fraction of the frequencies. We conjecture that the condition can be improved to $d \geq \frac{1}{2} |\Gamma|$ which is a better analogue to the 1D case (where we have exactly $d = \frac{1}{2} |\Gamma|$). We also conjecture that a sufficient condition could be strictly given as well.

2.5 Reconstruction of binary signal from arbitrary frequency measurements

If S is an arbitrary frequency selector, i.e. not necessarily select the lowest frequencies, it is not easy to give a sufficient and necessary condition to determine if the reconstruction is possible since there is no way to quantify the zero-crossings simply from the irregular support of the spectrum. However, in [21] the authors show that given the support of the spectrum of a trigonometric polynomial, the size of the largest non-zero circular region of the polynomial is bounded. Similar idea leads to the following lemma (for simplicity we assume the domain is equi-long along each dimension):

Lemma 2.11. *Let $\Omega \subseteq [-N/2, N/2 - 1]^h \setminus \{0\}$ be the symmetric support of the spectrum of a real h -dimensional signal v , i.e. $v(x) = \sum_{k \in \Omega} a_k e^{2\pi i \langle k, \frac{x}{N} \rangle}$ for $x \in \{1, \dots, N\}^h$. Then for any disjoint union partition of $\Omega = \bigsqcup_{i=1}^h \Omega_i$ (where each pair $\pm k$ have to be inside same Ω_i), v cannot be nonnegative (or nonpositive) on a cuboid with size $L = (L_1, \dots, L_h)$ where $L_i > \frac{1}{2} \sum_{l \in \{k_i : k \in \Omega_i\}} \lceil \frac{N}{2|l|} \rceil, \forall i$.*

Lemma 2.11 holds for any dimensional case. It immediately implies the following theorem, which states that a binary signal with a constant cuboid large enough can be recovered by partial frequency information.

Theorem 2.12. *Assume $u_0(x)$ is an h -D discrete binary signal defined on $x \in \{1, \dots, N\}^h$. Let Ω be the symmetric set of the known frequencies. Suppose u_0 is constant on a cuboid with size $L = (L_1, \dots, L_h)$, if there is a disjoint union partition of $\Omega^c = \bigsqcup_{i=1}^h \Omega_i$ such that $L_i > \frac{1}{2} \sum_{l \in \{k_i : k \in \Omega_i\}} \lceil \frac{N}{2|l|} \rceil, \forall i$, then u_0 can be recovered from (P_1) .*

We want to remark though that this theorem gives only a sufficient condition since lemma 2.11 is not tight.

It is worth mentioning that the conclusion of theorem 2.12 tells us the ‘importance’ of each frequency is roughly determined by its reciprocal. That is to say, knowing lower frequency measurements is more important for reconstruction of the binary signal than knowing the high frequency measurements. This coincides with the intuition we learn from theorem 2.5 and differs from the case of sparse reconstruction as in the compressed sensing problems, where the measurements should be spread out in the frequency domain as much as possible.

2.6 Reconstruction of random binary signal

If the binary function is random, i.e. the orthant \mathbb{O}_{u_0} is randomly chosen, there is no deterministic way to guarantee if a certain subspace passes through it, but the probability can be estimated. From now on we denote the kernel of A by K_A , the image of A^\top by I_A , the rank of A by r , then $\dim(K_A) = N - r$, $\dim(I_A) = r$. We say an r -dimensional linear subspace is in general position if the projections of any r axes of \mathbb{R}^N onto the subspace are linearly independent, and we say A is in general position if $Ax = 0$ is in general position. The following result is known by mathematicians no later than 1950's (see [8] for a brief review), saying that any r -dimensional subspace in \mathbb{R}^N in general position will pass through a fix number of orthants of \mathbb{R}^N :

Lemma 2.13 (see e.g. [8]). *Any r -dimensional subspace in \mathbb{R}^N in general position passes through $2 \sum_{i=0}^{r-1} \binom{N-1}{i}$ orthants of \mathbb{R}^N .*

We denote $P_{r,N} = \sum_{i=0}^r \binom{N}{i} / 2^N$, which is nothing but the cumulative distribution of the function of the standard binomial distribution with $p = \frac{1}{2}$. For a random binary signal u_0 , we say u_0 have no 'preference' on orthant if for any two orthants \mathbb{O}_1 and \mathbb{O}_2 ,

$$\text{Prob}(\mathbb{O}_{u_0} = \pm \mathbb{O}_1) = \text{Prob}(\mathbb{O}_{u_0} = \pm \mathbb{O}_2), \quad (9)$$

then since there are 2^N orthants in total, we have

$$P \left(I_A \cap \text{int}(\mathbb{O}_{u_0}) \neq \emptyset \right) = \frac{2 \sum_{i=0}^{r-1} \binom{N-1}{i}}{2^N} = P_{r-1, N-1} \quad (10)$$

According to theorem 2.3, this is equivalent to say:

Theorem 2.14. *If u_0 is a random binary signal with size N without preference on orthants, then given a matrix A in general position with rank r , the probability that u_0 can be recovered from linear problem (P_1) is $P_{r-1, N-1}$.*

It is well known that $P_{r,N}$ can be approximated by $\Phi \left(\frac{2r-N}{\sqrt{N}} \right)$ where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$ is the cumulative distribution function of the normal distribution. By Hoeffding's inequality, the tail of $P_{r,N}$ is bounded by

$$\begin{cases} P_{r,N} \leq \frac{1}{2} \exp \left(-\frac{(2r-N)^2}{2N} \right) & \text{when } r < N/2 \\ P_{r,N} \geq 1 - \frac{1}{2} \exp \left(-\frac{(2r-N)^2}{2N} \right) & \text{when } r > N/2 \end{cases} \quad (11)$$

Therefore, if $r/N \rightarrow \rho$ as $N \rightarrow \infty$, then

$$\begin{cases} P_{r,N} \leq \frac{1}{2} \exp \left(-\left(\rho - \frac{1}{2}\right)^2 N \right) \rightarrow 0 & \text{when } \rho < 1/2 \\ P_{r,N} \geq 1 - \frac{1}{2} \exp \left(-\left(\rho - \frac{1}{2}\right)^2 N \right) \rightarrow 1 & \text{when } \rho > 1/2 \end{cases} \quad (12)$$

which is illustrated in Fig. 2.

The discussion above can be summarized as the following theorem, which basically says that if the measurements are taken more than a half of the size of the signal, the probability that the convex relaxation works will tend to 1 when N goes to infinity.

Theorem 2.15. *If u_0 is a random binary signal with size N without preference on orthants, A is a matrix in general position with rank r , when N is large, the probability that u_0 can be recovered from linear problem (P_1) can be approximated by $\Phi \left(\frac{2r-N-1}{N-1} \right)$ where Φ is the cumulative distribution function of the normal distribution. If $\frac{r-1}{N-1} \rightarrow \rho > 1/2$ as $N \rightarrow \infty$, then u_0 can be recovered from (P_1) with overwhelming probability at least $1 - \frac{1}{2} e^{-c(N-1)}$ where $c = \left(\rho - \frac{1}{2}\right)^2$.*

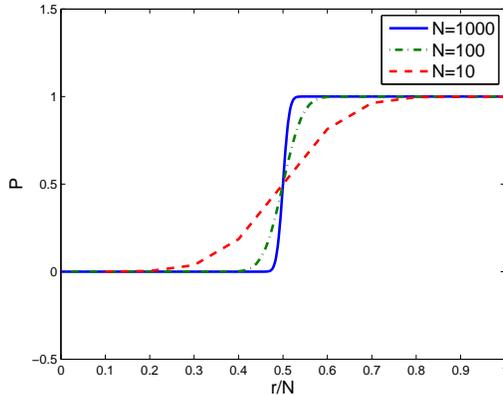


Figure 2: $P_{r,N}$ as a function of r/N .

3 Solving the Optimization Problem

As stated in section 2.1, there are many convex models that can recover the binary signal. Since the measurement might be noised in practice, we choose to reconstruct the function via the following optimization problem:

$$\min_u \|Au - b\|^2 \quad \text{s.t.} \quad 0 \leq u \leq 1. \quad (13)$$

First we discuss the robustness of this model. Let u_0 is the true binary function and $b = Au_0$ is the clean measurement. Assume b is contaminated by noise ϵ . For corrupted measurement $\tilde{b} = b + \epsilon$, we want to investigate if this model will still lead to the correct answer. Let

$$B(u(x)) = \begin{cases} 1, & u(x) \geq 1/2 \\ 0, & u(x) < 1/2 \end{cases}$$

be the thresholding operator that maps any function to its closest binary function. The following theorem shows that the model is robust to small perturbation. In section 4 we will show that the more measurements are given, the more robust the reconstruction would be, which is not surprising.

Theorem 3.1. *If u_0 is the unique solution of (P_1) , $\tilde{b} = b + \epsilon$ is the corrupted measurement, \tilde{u} is the minimizer of the optimization problem $\min_u \|Au - \tilde{b}\|_2^2$ s.t. $0 \leq u \leq 1$, then when $\|\epsilon\|$ is small enough, $B(\tilde{u}) = u_0$.*

Since (13) is a standard bounded least square problem, it can be solved by many existing optimization algorithms automatically. However, we will propose an algorithm that are specifically developed for this problem. It will only utilize the discrete Fourier transform without explicitly storing and multiplying the matrix A , which is very large in practical problems and will make most out-of-the-box optimization packages very inefficient.

Our algorithm will be based on the split Bregman method introduced in [17] and modified in [32] for solving the non-negative least square problem. We replace (13) with an equivalent problem

$$\min_u \|Au - b\|^2 \quad \text{s.t.} \quad u = P(d).$$

where $P(d)$ is defined component-wisely by

$$P(d) = \begin{cases} 1 & d \geq 1 \\ d & 0 < d < 1 \\ 0 & d \leq 0 \end{cases}.$$

This constrained problem can be solved iteratively by

$$\begin{cases} (d^{k+1}, u^{k+1}) = \min_{d,u} \frac{\lambda}{2} \|Au - b^k\|_2^2 + \|u - P(d) - v^k\|_2^2 \\ v^{k+1} = v^k + P(d^{k+1}) - u^{k+1} \\ b^{k+1} = b^k + b - Au^{k+1} \end{cases}$$

There last two lines are called Bregman steps and can be understood as the gradient ascent steps in the augmented Lagrangian method [25, 17, 36]. The first line can be solved exactly respectively on d and u , giving rise to the following iteration:

$$\begin{cases} d^{k+1} = P(u^k - v^k) \\ u^{k+1} = (\lambda A^\top A + I)^{-1} (\lambda A^\top b^k + P(d^{k+1}) + v^k) \\ v^{k+1} = v^k + P(d^{k+1}) - u^{k+1} \\ b^{k+1} = b^k + b - Au^{k+1} \end{cases} \quad (14)$$

Here the first, third and last lines contain only trivial computations. For the second line, we can notice that when $A = S\mathcal{F}$, $(\lambda A^\top A + I)^{-1} = \mathcal{F}^{-1}(N\lambda S + I)^{-1}\mathcal{F}$ where N is the size of the signal. Since $N\lambda S + I$ is nothing but a diagonal matrix, the whole operator can be calculated efficiently and precisely. Therefore we get the algorithm 1. Numerical results in the next section would show that this algorithm works very well.

If the given information is not partial Fourier measurements but a signal after filtering, i.e. $b = Au = K\mathcal{F}u$ where K is a filter in frequency domain, then this algorithm still works after small modification. The only point that needs to be noticed is that now $(\lambda A^\top A + I)^{-1} = \mathcal{F}^{-1}(N\lambda K^\top K + I)^{-1}\mathcal{F}$.

In some problems, the norm $\|Au - b\|^2$ can also be preconditioned, e.g. to prevent the effect of noise on high frequencies. We can minimize $\|Au - b\|_M^2 = (Au - b)^\top M(Au - b)$ where M is a certain preconditioner in frequency domain. Again, the algorithm still works without many modifications except the inverse operator becoming $(\lambda A^\top MA + I)^{-1}$ now.

Algorithm 1 The Split Bregman Algorithm for Solving (13)

Initialize: Let $b_0 = b$. Start from initial guess $u = \mathcal{F}^{-1}b$.

while $\|Au - b_0\|^2$ not small enough **do**

$d \leftarrow P(u - v)$

$u \leftarrow (\lambda A^\top A + I)^{-1} (\lambda A^\top b + P(d) + v)$

$v \leftarrow v + P(d) - u$

$b \leftarrow b + b_0 - Au$

end while

$u \leftarrow B(u)$.

4 Numerical Results

First we numerically show that the binary signal can in general be characterized by very few frequency measurements, especially for structured binary functions. Fig. 3 shows several 1D and 2D signals.

- The first one is a 1D binary signal. It contains 15 constant intervals with value 1 and 15 intervals with 0, so by theorem 2.5 it can be fully determined by Fourier coefficients a_k for $|k| \leq 16$, no matter how long the signal actually is (the length of the signal shown here is 400).
- The second one is a binary image corresponds to a geometrical shape, the size is 200×200 . Experiment shows that it can be fully determined by Fourier coefficients a_k for $|k| \leq 5$.

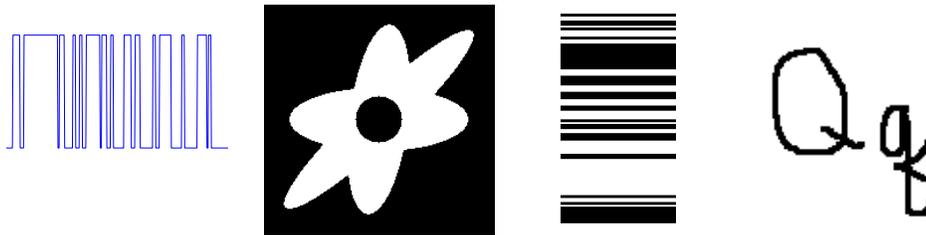


Figure 3: Several binary signals: a random 1D binary signal, a geometrical shape, a barcode and a handwriting image.

The needed Fourier coefficients for characterization of the shape account for 0.2% of the total Fourier coefficients.

- The third one is a barcode image, which can also be treated substantially as a 1D signal. It contains 15 black bars and 15 white bars, so the Fourier coefficients needed is a_k for $|k_1| \leq 16$, where k_1 is the component of k along the horizontal dimension, no matter how large the image actually is (the width of the barcode shown here is 400).
- The last one is an image with handwriting letters, the size is 100×100 . Experiment shows that it can be fully determined by Fourier coefficients a_k for $|k| \leq 10$. The needed Fourier coefficients for characterization of the image account for 3.17% of the total Fourier coefficients.

To show that these binary signal can be recovered from the low frequency measurements, we let them be filtered by a low-pass Gaussian kernel whose band corresponds to the partial frequencies. As long as the needed low frequencies information are precisely given, the reconstruction would be available. However, the numerical experiments also show that the more measurements are known, the fast the reconstruction would be, which means that the algorithm would find the correct binary signal easier. Fig. 4 demonstrates the filtered signal and the reconstruction. The curves in Fig. 5 show that the reconstruction time decreases when the measurements are given more. The x -axis means the extra radius of the support of the given frequency information, while the y -axis shows the logarithm of computational time.

To demonstrate that the reconstruction is robust, we now recover the signal from the low-pass filtered measurements with noise. It is well known that deblurring with high noise present is a difficult task. Our results show that even with very high noise and strong blurring, the results are still making sense. In Fig. 6-9 each group is showing a signal with different level of noise. The information of the blurring kernel and the noise level is given in the captions. The computational time costs are also recorded. All computation are done on a Matlab environment with a 2.8 GHz Intel CPU.

What if the known frequency measurements are even less? Fig. 10 shows the reconstructions if the measurements are severely incomplete. As one would expect, it leads to a ‘low resolution’ approximation of the original signal. This observation seems indicating a potential multi-resolution analysis method for shapes.

5 Conclusion

The reconstruction of binary function is difficult primarily because the nonconvex nature of the problem. In this work we proved that even with very few frequency measurements, the binary function can actually be reconstructed via solving a very simple convex problem. We also discussed the numerical implementation of this method.

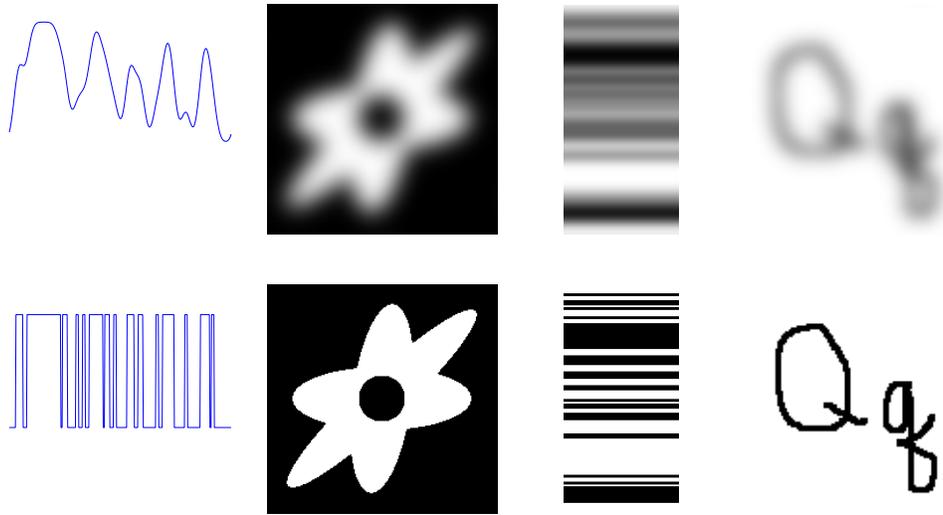


Figure 4: The signals after low-pass filtering and the reconstructions.

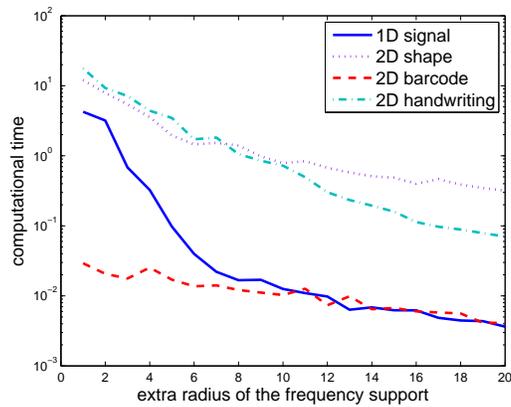


Figure 5: Time consumption (in seconds) versus the extra radius of the support of the given frequency information.

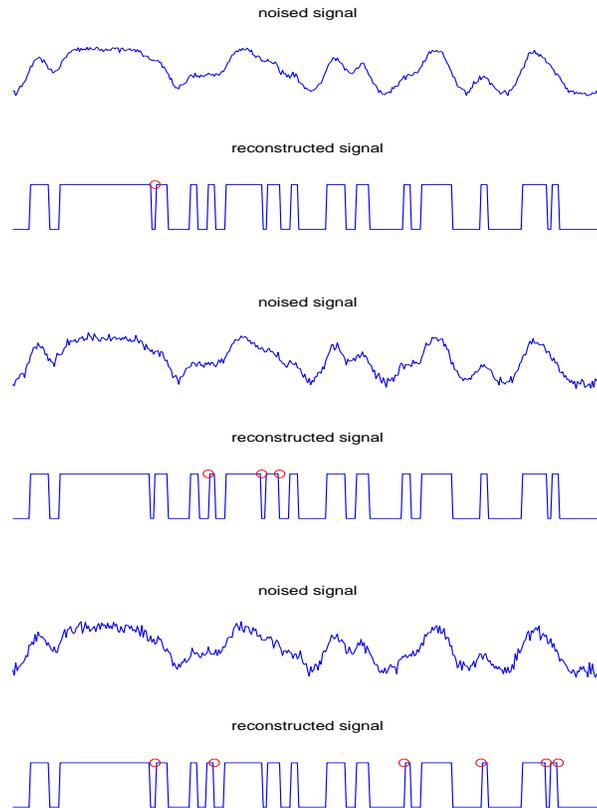


Figure 6: The first signal with Gaussian blurring filter $\sigma = 5$ generated by Matlab command `fspecial`. The variance of Gaussian noise are respectively 0.03, 0.05, 0.07 generated by Matlab command `randn`. The positions of the bars in the reconstruction are sometime not precisely equal to the original due to the present of the noise. The minor difference between the reconstruction and the true signal are highlighted by red circles. The average computational time for reconstructions are respectively 0.05s, 0.04s, 0.03s.

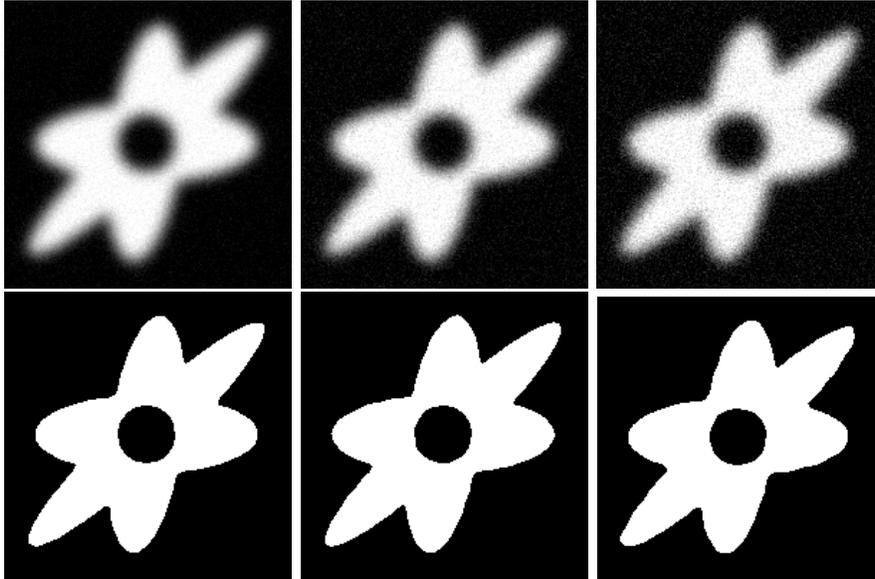


Figure 7: The first image with Gaussian blurring filter $\sigma = 5$ generated by Matlab command `fspecial`. The amplitude of Gaussian noise are respectively 0.03, 0.05, 0.07 generated by Matlab command `randn`. The average computational time for reconstructions are respectively 0.81s, 0.81s, 0.80s.

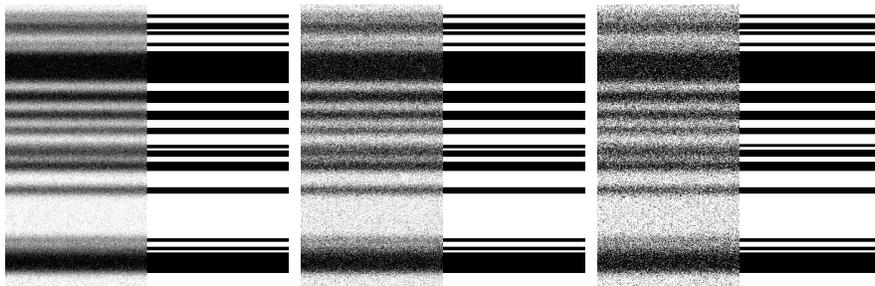


Figure 8: The second image with Gaussian blurring filter $\sigma = 5$ generated by Matlab command `fspecial`. The amplitude of Gaussian noise are respectively 0.1, 0.2, 0.3 generated by Matlab command `randn`. The noised image is on the left, the reconstruction is on the right. The average computational time for reconstructions are respectively 0.04s, 0.03s, 0.03s.

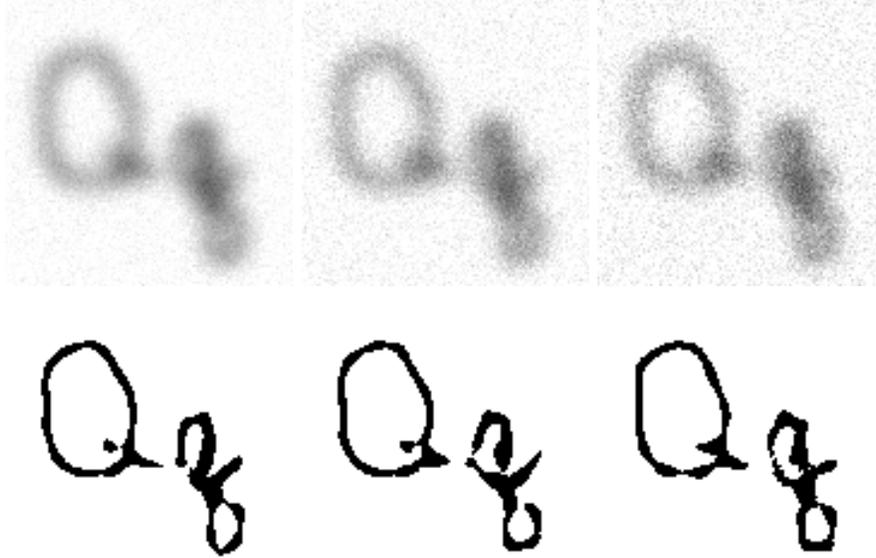


Figure 9: The last image with Gaussian blurring filter $\sigma = 5$ generated by Matlab command `fspecial`. The amplitude of Gaussian noise are respectively 0.03, 0.05, 0.07 generated by Matlab command `randn`. The average computational time for reconstructions are respectively 1.54s, 1.20s, 0.98s.

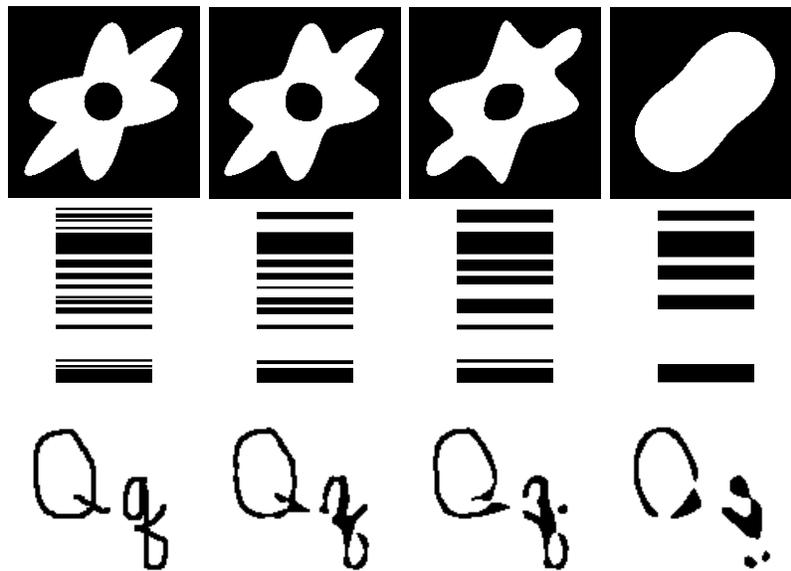


Figure 10: The reconstruction of the images with even less Fourier measurements. First row from left to right: original first image which needs Fourier coefficients a_k for $|k| \leq 5$; reconstruction with a_k given for $|k| \leq 4, 3, 2$ respectively. Second row from left to right: original second image which needs a_k for $|k| \leq 16$; reconstruction with a_k given for $|k| \leq 14, 11, 8$ respectively. Third row from left to right: original third image which needs a_k for $|k| \leq 10$; reconstruction with a_k given for $|k| \leq 8, 6, 4$ respectively.

There are several directions in which this research could be further pursued. Some of them have been discussed in section 2.3 and 2.4. Other potential question include: How can we investigate more properties of a given binary function by using fewer measurements? Is that possible to even characterize the motion and evolution of a shape by this method? These questions promise an interesting path on this field.

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A Proofs

Theorem. 2.1.

Proof. If (P_1) has another solution other than u_0 , denoted as u' , then let $v = u' - u_0$, we have $Av = Au_0 - Au' = 0$. Moreover, since $u'(x) \in [0, 1], \forall x$, then when $u_0(x) = 0$, $v(x) = u'(x) - u_0(x) \geq 0$; when $u_0(x) = 1$, $v(x) = u'(x) - u_0(x) \leq 0$, which contradicts the given condition on v . \square

Lemma. 2.4.

Proof. This conclusion is natural in the context of trigonometric interpolation. We denote the $2n$ points by $\{c_k = e^{2\pi i \alpha_k}\}_{k=1}^{2n} \subset S^1 = \{z : |z| = 1\}$. Let

$$u(z) = \frac{C}{z^n} \prod_{k=1}^{2n} (z - c_k)$$

where $C = \prod_{k=1}^{2n} c_k^{-1/2}$. Then $u(z)$ can be written as $u(z) = \sum_{k=-n}^n a_k z^k$, i.e. u is a trigonometric polynomial with spectrum limited in $[-n, n]$. Since $\{c_i\} \subset S^1$, $C \in S^1$, for $z \in S^1$ we have

$$\overline{u(z)} = \frac{\bar{C}}{\bar{z}^n} \prod_{k=1}^{2n} (\bar{z} - \bar{c}_k) = \frac{z^n}{C} \prod_{k=1}^{2n} \left(\frac{z - c_k}{z c_k} \right) = \frac{C}{z^n} \prod_{k=1}^{2n} (z - c_k) = u(z)$$

therefore $u(z)$ is real on S^1 . If we treat $u(z) = u(e^{2\pi i t})$ as a function defined on $[0, 1]$, it is easy to check that u vanishes on and only on $\{\alpha_k\}$ and $\frac{d}{dt}u(e^{2\pi i t})$ does not vanish on $\{\alpha_k\}$, and the conclusion follows. \square

Theorem. 2.5.

Proof. By theorem 2.3 we only need to construct a discrete signal $v = A^\top \eta = \mathcal{F}^{-1}(S\eta)$ that changes sign only at the boundaries of the intervals, where S is the sampling operator selecting $\{a_k : |k| \leq d\}$. Let the starting points of each interval are $\{s_i\}$, then by lemma 2.4 we can find a trigonometric polynomial $\sum_{k=-d}^d a_k e^{2\pi i k t}$ that changes sign only at $\{\frac{1}{N}(s_i - \frac{1}{2})\}$. Let $v(x) = \sum_{k=-d}^d a_k e^{2\pi i k \frac{x}{N}}$, $x \in \{1, \dots, N\}$, then v (or $-v$) satisfies the requirement. To show that this result is optimal, we only need to notice that a trigonometric polynomial with order less than d cannot have $2d$ zeros due to the fundamental algebraic theorem. \square

Theorem. 2.9.

Proof. Since we consider on a torus \mathbb{T}^2 , in this proof every coordinates are automatically mod by 1 without explicitly written to make the notations clear.

Without loss of generality, we only prove the case when $\theta \in (-\pi/4, \pi/4]$. The other case can be automatically proved by switching the $x - y$ coordinates. Since the zero levelset of $v(x, y)$ are analytic curves, $K_\theta(s)$ is a piece-wise constant function with respect to both s and θ with finitely many jumps, and it is easy to see that we can only prove the theorem for θ such that $\tan \theta \in \mathbb{Q}$ and $\theta \neq 0, 1$ (those θ are dense in $(-\pi/4, \pi/4]$).

For any $\theta \neq 0$ s.t. $\tan \theta \in \mathbb{Q}$, assume $\tan \theta = p/q$, $p < q \in \mathbb{Z}$ are coprime. Let

$$F_{s,\theta}(t) = (t, s + t \tan \theta) \pmod{1}, \quad t \in \mathbb{R}$$

be a linear flow on \mathbb{T}^2 , then $F_{s,\theta}(t)$ is a periodic function with period q since $F_{s,\theta}(t+q) = F_{s,\theta}(t)$.

The main idea of the proof is based on the following observation: a whole period of $F_{s,\theta}$ is actually composed of q segments that belong to the grating $L_{\cdot,\theta}$. Therefore, counting the average intersection along the line segments inside the grating L can be replaced by counting the intersection along F . The latter is easier because it can be reduced to counting the 1D zero crossings of a trigonometric polynomial inside a period.

We will first show that $F_{s,\theta}(t)$ can be split to q segments, each of which corresponds to a line segment inside the grating L . Indeed, when $t \in [m, m+1]$ for $m \leq q-1$, $m \in \mathbb{Z}$, it is easy to check $F_{s,\theta}(t) = L_{s+mp/q,\theta}(t-m)$, so when t goes from 0 to q , $F_{s,\theta}(t)$ can be seen as the connected version of q line segments $\{L_{s+mp/q,\theta} : m = 0, \dots, q-1\}$. Since p and q are coprime, elementary number theory tells us that

$$\left\{ \text{mod} \left(\frac{mp}{q}, 1 \right) : m = 0, \dots, q-1 \right\} = \left\{ \text{mod} \left(\frac{m}{q}, 1 \right) : m = 0, \dots, q-1 \right\}.$$

Similar to the notation of $\#L_{s,\theta}$, we use $\#F_{s,\theta}$ to denote the intersection of $F_{s,\theta}$, $t \in [0, q]$ with the jump set of u_0 , then

$$\#F_{s,\theta} = \sum_{m=0}^{q-1} \#L_{s+mp/q,\theta} = \sum_{m=0}^{q-1} \#L_{s+m/q,\theta}$$

Therefore, we have

$$\begin{aligned} K_\theta &= \cos \theta \int_0^1 \#L_{s,\theta} ds = \cos \theta \sum_{m=0}^{q-1} \int_{\frac{m}{q}}^{\frac{m+1}{q}} \#L_{s,\theta} ds \\ &= \cos \theta \sum_{m=0}^{q-1} \int_0^{\frac{1}{q}} \#L_{s+m/q,\theta} ds = \cos \theta \int_0^{\frac{1}{q}} \left(\sum_{m=0}^{q-1} \#L_{s+m/q,\theta} \right) ds \\ &= \cos \theta \int_0^{\frac{1}{q}} \#F_{s,\theta} ds \end{aligned}$$

Now we start to evaluate $\#F_{s,\theta}$. Since $u_0(x, y)$ is corresponding to the zero levelset of $v(x, y)$, $\#F_{s,\theta}$ is equal to the number of zero-crossings of $v(x, y)$ along $F_{s,\theta}$. Let $\tilde{v}(t) = v(F_{s,\theta}(t))$, recall $v(x, y) = \sum_{(j,k) \in \Omega} a_{jk} e^{2\pi i(jx+ky)}$, we have

$$\tilde{v}(qt) = \sum_{(j,k) \in \Omega} a_{jk} e^{2\pi i(jqt+k(s+pt))} = \sum_{(j,k) \in \Omega} a_{jk} e^{2\pi iks} e^{2\pi i(jq+kp)t}$$

Since $F_{s,\theta}(t)$ is periodic with period q , so is $\tilde{v}(t)$, thus $\tilde{v}(qt)$ is periodic with period 1. Since $\sqrt{j^2 + k^2} \leq d$ in Ω , we have

$$|jq + kp| = \sqrt{(j^2 + k^2)(p^2 + q^2) - (jp - kq)^2} \leq d\sqrt{p^2 + q^2}$$

That is to say, $\tilde{v}(qt)$ can be expanded as a trigonometric polynomial with order no more than $d\sqrt{p^2 + q^2}$, therefore the zero-crossing of $\tilde{v}(qt)$ is no more than $2d\sqrt{p^2 + q^2}$, i.e. $\#F_{s,\theta} \leq 2d\sqrt{p^2 + q^2}$. Plug it into $K_\theta = \cos \theta \int_0^{1/q} \#F_{s,\theta} ds$, we have $K_\theta \leq 2d \cos \theta \sqrt{p^2 + q^2}/q = 2d$. \square

Lemma. 2.11.

Proof. This lemma is similar to a theorem in [21]. Since our conclusion cannot be directly deduced from it, we prove our lemma from scratch here.

When v has only one pure frequency $\pm k$, it is easy to check the conclusion manually. For general v we prove by induction on the size of Ω . Suppose $v(x) = \sum_{k \in \Omega} a_k e^{2\pi i \langle k, x/N \rangle}$ and $\Omega = \bigsqcup_{i=1}^h \Omega_i$ is a disjoint union partition of Ω . For any index j such that $\Omega_j \neq \emptyset$, pick $l^0 \in \{k_j : k \in \Omega_j\}$, let $s = (s_1, \dots, s_h)$ where $s_j = \text{sign}(l^0)$ and $s_i = 0$ for $i \neq j$, let $t = (t_1, \dots, t_h)$ where $t_j = \text{sign}(l^0) \cdot \lceil \frac{N}{2|l^0|} \rceil$ and $t_i = 0$ for $i \neq j$. For any $k^0 \in \Omega$ such that $k_j^0 = l^0$, since $\langle k^0, s \rangle = l^0 s_j = |l^0|$, $\langle k^0, t \rangle = l^0 t_j = |l^0| \cdot \lceil \frac{N}{2|l^0|} \rceil \in [\frac{N}{2}, \frac{N}{2} + |l^0|)$, then $0 < 2\pi \langle k^0, \frac{s}{N} \rangle \leq \pi \leq 2\pi \langle k^0, \frac{t}{N} \rangle < 2\pi \langle k^0, \frac{s}{N} \rangle + \pi$. It is easy to check by simple calculation that there exist $\xi_1 \geq 0$, $\xi_2 > 0$ s.t. $1 + \xi_1 e^{i2\pi \langle k^0, \frac{s}{N} \rangle} + \xi_2 e^{i2\pi \langle k^0, \frac{t}{N} \rangle} = 0$. Therefore, let $T_t v = v(\cdot + t)$ be the translate of v by a positive integer vector t under periodic boundary condition, consider

$$\begin{aligned} \tilde{v} &:= v + \xi_1 \cdot T_s v + \xi_2 \cdot T_t v \\ &= \sum_{k \in \Omega} a_k e^{2\pi i \langle k, \frac{s}{N} \rangle} + \xi_1 \cdot \sum_{k \in \Omega} a_k e^{2\pi i \langle k, \frac{s+s}{N} \rangle} + \xi_2 \cdot \sum_{k \in \Omega} a_k e^{2\pi i \langle k, \frac{s+t}{N} \rangle} \\ &= \sum_{k \in \Omega} a_k \left(1 + \xi_1 e^{2\pi i \langle k, \frac{s}{N} \rangle} + \xi_2 e^{2\pi i \langle k, \frac{t}{N} \rangle} \right) e^{2\pi i \langle k, \frac{s}{N} \rangle}. \end{aligned}$$

The quantity inside the bracket is zero for all $k^0 \in \Omega_j$ such that $k_j^0 = l^0$, so the support of the spectrum of \tilde{v} is strictly smaller than Ω without including those k^0 . (It actually cannot include $-k^0$ as well because of the symmetry of the spectrum of real signal.) On the other hand, assume that v is nonnegative (or nonpositive) on a cuboid with size $L = (L_1, \dots, L_h)$, it is easy to see that $\tilde{v} = v + \xi_1 \cdot T_s v + \xi_2 \cdot T_t v$ is nonnegative (or nonpositive) on a cuboid with size $\tilde{L} = (\tilde{L}_1, \dots, \tilde{L}_h)$ where $\tilde{L}_j = L_j - |t_j| = L_j - \lceil \frac{N}{2|l^0|} \rceil$ and $\tilde{L}_i = L_i$ for $i \neq j$. If $L_i > \frac{1}{2} \sum_{l \in \{k_i : k \in \Omega_i\}} \lceil \frac{N}{2|l|} \rceil, \forall i$, then $\tilde{L}_j > \frac{1}{2} \sum_{l \in \{k_j : k \in \Omega_j\}, l \neq \pm l^0} \lceil \frac{N}{2|l|} \rceil$ and $\tilde{L}_i > \frac{1}{2} \sum_{l \in \{k_i : k \in \Omega_i\}} \lceil \frac{N}{2|l|} \rceil$ for $i \neq j$, which contradicts the inductive assumption. \square

Theorem. 3.1.

Proof. We first consider the \tilde{u} that solves $\min_u \|Au - \tilde{b}\|_2^2$ s.t. $\tilde{u} - u_0 \in \mathbb{O}_{u_0}$, i.e. $\tilde{u}(x) \geq u_0(x)$ if $u_0(x) = 0$ and $\tilde{u}(x) \leq u_0(x)$ if $u_0(x) = 1$. We will show that when $\epsilon = \tilde{b} - b$ is small enough, $|\tilde{u}(x) - u_0(x)| < 1/2, \forall x$, therefore \tilde{u} will also solve $\min_u \|Au - \tilde{b}\|_2^2$ s.t. $0 \leq u \leq 1$. On the other hand, since $|\tilde{u}(x) - u_0(x)| < 1/2, \forall x$, we have $B(\tilde{u}) = u_0$, which is what we need.

Let $w = \tilde{u} - u_0$, since $Au_0 = b$ and $\tilde{b} = b + \epsilon$, saying that \tilde{u} solves

$$\min_u \|Au - \tilde{b}\|_2^2, \quad \tilde{u} - u_0 \in \mathbb{O}_{u_0}$$

is equivalent to saying that w solves

$$\min_w \|Aw - \epsilon\|_2^2, \quad w \in \mathbb{O}_{u_0}.$$

Therefore, all we need to show is that when $\|\epsilon\|$ is small enough, the minimizer w satisfies $|w(x)| < 1/2, \forall x$.

Let P_w be the projection operator

$$P_w u(x) = \begin{cases} u(x) & w(x) \neq 0 \\ 0 & w(x) = 0 \end{cases}$$

then I_{P_w} (the image of projection P_w) defines a coordinate plane and w belongs to it. Since $w \in \mathbb{O}_{u_0}$, it is easy to see that w has a small neighborhood on I_{P_w} that also belongs to \mathbb{O}_{u_0} , and w must be the minimizer of $\min \|Aw - \epsilon\|_2^2$ inside this neighborhood. Because the problem is

convex, we know that w is also the the minimizer of $\min_w \|Aw - \epsilon\|_2^2$ s.t. $u \in I_{P_w}$. Therefore, Aw is the minimizer of the problem $\min_v \|v - \epsilon\|_2^2$ s.t. $v \in I_{AP_w}$ (the image of projection AP_w), i.e. Aw is nothing but the projection of ϵ onto I_{AP_w} . We can let P go over all possible coordinate planes (this is a finite set) and find all projections of ϵ onto I_{AP} , denoted by ϵ_P , then for any minimizer w of $\min_w \|Aw - \epsilon\|_2^2$ s.t. $w \in \mathbb{O}_{u_0}$, Aw would belong to the set $\{\epsilon_P\}$.

Recall that u_0 is the unique solution of (P_1) , from theorem 2.3 we know that there exists $v = A^\top \eta \in \text{int}(\mathbb{O}_{u_0})$. Without loss of generality we assume $\min_x \{|v(x)|\} = 1$. Since $\{\epsilon_P\}$ is a finite set in which each element depends on ϵ linearly, obviously there exists $h > 0$ such that when $\|\epsilon\| < h$, $\langle \epsilon_P, \eta \rangle < 1/2, \forall P$, i.e. $\langle Aw, \eta \rangle < 1/2$ for all minimizers of $\min_w \|Aw - \epsilon\|_2^2$ s.t. $w \in \mathbb{O}_{u_0}$. Since $\langle Aw, \eta \rangle < 1/2 \Rightarrow \langle w, A^\top \eta \rangle = \langle w, v \rangle < 1/2$, combining with the facts $w \in \mathbb{O}_{u_0}$ and $v \in \text{int}(\mathbb{O}_{u_0})$, we know that $|w(x)| < 1/(2|v(x)|) \leq 1/2, \forall x$. \square

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