Gaussian beam decomposition for seismic migration

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SUMMARY

In reflection seismology, seismic waves are recorded at the surface of the earth and migrated to image the subsurface. In mathematical terms, this is a boundary-value problem for the wave equation, where the boundary data are the recorded seismic wavefields. Gaussian beams, which are localized solutions of the wave equation, can be used for seismic migration. However, before Gaussian beams can be used for a solution of the boundary-value problem, the recorded data have to be represented in a compatible form. We present a new method for decomposing seismic data as a sum of Gaussian beams, based on transforming boundary data to a propagating wavefield. A simple synthetic seismic migration example illustrates our method and shows how different classes of Gaussian-beam parameters (such as curvature and extent) control the accuracy and sparseness of data representation.

INTRODUCTION

Gaussian beams, which are localized solutions of the wave equation, have been used successfully in seismic migration applications (Hill, 1990, 2001; da Costa et al., 1989; Alkhalifah, 1995; Nowack et al., 2003; Gray, 2005; Popov et al., 2010; Protasov and Tcheverda, 2011). Gaussian beam migration retains the advantages of Kirchhoff migration (such as the ability to handle steep dips) while being able to handle the issue of multipathing, which presents significant difficulties for traditional Kirchhoff implementations (Gray et al., 2002). In addition to their role in imaging, Gaussian beams also provide a convenient data analysis tool: by decomposing seismic data into beams, it is possible to track the role and contribution of each beam in the data processing flow (Hill et al., 2002).

To use Gaussian beam methods in imaging, one must first represent the recorded seismic data in a compatible form. Tanushev et al. (2009) present a method for decomposing a wavefield into a superposition of Gaussian beams. The method is a greedy iterative approach that represents as much of the wave energy as possible using a single Gaussian beam through a local optimization of the Gaussian beam parameters. Then, the Gaussian beam wavefield is subtracted from the total wavefield and then procedure is iterated.

Using a similar optimization procedure directly on the boundary data is inherently difficult, since the Gaussian beam parameters depend on time and while for a fixed time the Gaussian beam has small essential support (the set on which the magnitude of the beam is bigger than a small threshold), its essential support may be quite spread out in the time-space domain.

To avoid this difficulty, we propose a method for transforming boundary data into a wavefield, performing a Gaussian beam decomposition of this wavefield, then propagating the Gaussian beam parameters back to the boundary to obtain a decomposition of the boundary data into a superposition of Gaussian beams. The boundary data transformation can be viewed as the solution operator for the wave equation which maps the given boundary data to the wavefield data at the final time T. The domain of dependence and cone of influence (i.e. the finite speed of wave propagation) for the wave equation guarantee that the same information will be contained in the boundary data and in the wavefield. For the solution operator, we use the a constant-coefficient wave equation. Since the Gaussian beam parameters are propagated back to the boundary, the choice of the wave equation is arbitrary as long as it is used consistently. For simplicity, we assume marine data (constant water velocity at the surface) and use the classical wave equation with the water velocity. Thus, the solution can be expressed via Fourier transform, and the Gaussian beam coefficients can be propagated back to the boundary using exact formulas or by integrating the ray-tracing system numerically.

In the example presented in this paper, we use a modified version of the greedy algorithm described by Tanushev et al. (2009) to decompose the wavefield that is obtained by transforming the boundary data. This method is designed for cases in which the wave energy concentrates in relatively small regions. However, we note that any decomposition method can be used, including methods based on expectation maximization (Ariel et al., 2011), methods based on the FBI transform (Qian and Ying, 2010), and others. We further note that our approach can be applied not only with Gaussian beams, but also to other recently proposed localized beam-type representations, such as curvelets (Chauris and Nguyen, 2008), wavepackets (Qian and Ying, 2010; Duchkov et al., 2010), frozen Gaussians (Lu and Yang, 2011), etc.

THEORY

In this section, we briefly review some mathematical properties of the wave equation and Gaussian beams. For a detailed construction of Gaussian beams with the similar notation, we refer the reader to Ralston (1982) or Tanushev (2008). We will use the shorthand notation $\vec{\mathbf{x}} = (\vec{x}, z)$.

We consider the wave equation,

$$\begin{aligned} u_{tt} - c^2(\vec{x},z)\triangle u = 0 \quad &\text{for} \quad (t,\vec{x},z) \in [0,T]\times\mathbb{R}^{n-1}\times\mathbb{R}_- \,, \\ u = f(t,\vec{x}) \quad &\text{on} \quad z = 0 \,, \\ u = u_t = 0 \quad &\text{at} \quad t = 0 \,, \end{aligned} \tag{1}$$

for $n = 2$ or 3. We will assume that the data $f(t,\vec{x})$ given at $z = 0$ is a compactly supported function in $(t,\vec{x})$. We will refer to this data as "boundary data", as opposed to a "wavefield" which will refer to the wavefield and its time derivative at a fixed t. Since boundary data do not contain any information about the z derivative of the wavefield, we will also make the

assumption that all of the wave energy that we are interested in propagating is in the positive z direction.

The initial value problem for the wave equation (1) is well posed in the energy norm,

$$\|u(t,\cdot)\|_E = \left(\int_{\mathbb{R}^n} \left[|u_t(t,\vec{\mathbf{x}})|^2 + |\nabla u(t,\vec{\mathbf{x}})|^2 \right] d\vec{\mathbf{x}} \right)^{1/2} .$$

Associated with this norm is the pointwise energy function,

$$E[u,u_t](t,\vec{\mathbf{x}}) = |u_t(t,\vec{\mathbf{x}})|^2 + |\nabla u(t,\vec{\mathbf{x}})|^2 .$$

Gaussian beams are approximate localized solutions of the initial value problem for the wave equation that can be written as $u_{GB}(t,\vec{\mathbf{x}}) \sim a(t) e^{ik\phi(t,\vec{\mathbf{x}})}$, with $k \gg 1$ and ϕ given by the Taylor polynomial

$$\phi(t,\vec{\mathbf{x}}) = \vec{\mathbf{p}}(t) \cdot (\vec{\mathbf{x}} - \vec{\mathbf{y}}(t)) + \frac{1}{2} (\vec{\mathbf{x}} - \vec{\mathbf{y}}(t)) \cdot M(t) (\vec{\mathbf{x}} - \vec{\mathbf{y}}(t)) .$$

Thus, the parameters that define the Gaussian beam wavefield are the large parameter k, the Gaussian beam center $\vec{\mathbf{y}}(t)$, the oscillation direction $\vec{\mathbf{p}}(t)$, the complex valued Hessian matrix $M(t)$, and the amplitude $a(t)$. All of the time dependent parameters satisfy ODEs in time and can be easily propagated in time given initial values. The Hessian matrix $M(t)$ will always have a positive definite imaginary part provided that it does so initially, see Ralston (1982). This positive definite property gives Gaussian beams their name and the localization of the wavefield. Gaussian beams satisfy the wave equation asymptotically as $k \to \infty$.

Since the wave equation is linear, we can take N Gaussian beams with different sets of initial parameters and sum them to obtain a more general approximate solution to the wave equation. This approximate solution will have initial data of the form

$$\sum_{j=1}^{N} \left(a_j(t) e^{ik_j \phi_j(t,\vec{\mathbf{x}};\vec{\mathbf{y}}_j)} \Big|_{t=0} , \partial_t \left[a_j(t) e^{ik_j \phi_j(t,\vec{\mathbf{x}};\vec{\mathbf{y}}_j)} \right] \Big|_{t=0} \right)$$

and boundary data of the form

$$\sum_{j=1}^{N} a_j(t) e^{ik_j \phi_j(t,\vec{\mathbf{x}};\vec{\mathbf{y}}_j)} \Big|_{z=0} .$$

To use Gaussian beams for migration, we must find sets of parameters so that the above sums approximate the given data.

BOUNDARY DATA DECOMPOSITION

The decomposition algorithm for boundary data includes three major components: transformation of the boundary data into a wavefield, decomposition of the wave field into a superposition of Gaussian beams, and propagation of Gaussian beam parameters back to the boundary. We describe the details of each of these steps separately. A schematic representation of the algorithm is shown in Figure 1 for the 1-D spatial dimension case.

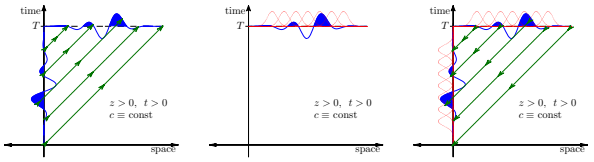

Figure 1: Schematic description of the boundary data decomposition algorithm in 1-D: (a) propagate wave to final time T, (b) decompose the wavefield at time T, (c) pull back Gaussian beam parameters to $z = 0$.

Transformation of Boundary Data into a Wavefield

As described in the introduction, the first step is to transform the boundary data, $f(t,\vec{x})$, into a wavefield $(u(t,\vec{\mathbf{x}}), u_t(t,\vec{\mathbf{x}}))$ at time $t = T$. Naturally, this transformation should keep all the recorded wave information while not introducing any new information. We assume that the boundary data , $f(t,\vec{x})$, is given for $t \in [0,T]$, has compact support in $\vec{x}$, and that $f(0,\vec{x}) = 0$ and $f(T,\vec{x}) = 0$. We consider the following propagation equation,

$$\begin{aligned} u_{zz} - u_{tt} + \triangle_{\vec{x}} u = 0 \quad &\text{for} \quad (t,\vec{x},z) \in [0,T] \times \mathbb{R}^{n-1} \times \mathbb{R}_+ , \\ u = f(t,\vec{x}) \quad &\text{on} \quad z = 0 , \\ u = u_t = 0 \quad &\text{at} \quad t = 0 , \end{aligned} \tag{2}$$

so that the propagation is considered in the positive z direction.

Recall that the Laplace transform is defined by

$$\mathscr{L}[g](s) = \int_0^\infty e^{-st} g(t) dt \quad \text{with} \quad s \in \mathbb{C} ,$$

and extend the given boundary data $f(t,\vec{x})$ by zero to all of $[0,\infty) \times \mathbb{R}^{n-1}$. We take the Fourier transform in $\vec{x} \to \xi$ and the Laplace transform in $t \to s$ of u and denote it by $\tilde{u} = \mathscr{L}[\mathscr{F}[u]]$. Then, equation (2) becomes

$$\begin{aligned} \tilde{u}_{zz} - (s^2 + |\xi|^2)\tilde{u} = 0 \quad &\text{for} \quad (s,\xi,z) \in \mathbb{C} \times \mathbb{R}^{n-1} \times \mathbb{R}_+ , \\ \tilde{u} = \tilde{f}(s,\xi) \quad &\text{on} \quad z = 0 , \end{aligned} \tag{3}$$

after recalling that $\mathscr{L}[g''](s) = s^2 \mathscr{L}[g](s) - sg(0) - g'(0)$ and applying the initial conditions for u. The general solution of equation (3) is given by (Gazdag, 1978)

$$\tilde{u}(s,\xi,z) = c_1 e^{z\sqrt{s^2+|\xi|^2}} + c_2 e^{-z\sqrt{s^2+|\xi|^2}} .$$

To satisfy the boundary conditions, we impose that as t increases all waves are propagating into $z > 0$. Before we impose this condition on the solution, let us recall the inversion formula for the Laplace transform,

$$\mathscr{L}^{-1}[G](t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} e^{st} G(s) ds ,$$

where γ is chosen so that $G(s)$ is well defined for all s on the integration contour. Since in our application, all of the functions are compactly supported in t, we can choose $\gamma = 0$ and then the inversion formula above is simply the inverse Fourier transform in $\tau \to t$, with $s = -i\tau$:

$$\mathscr{L}^{-1}[\tilde{u}](t,\xi,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\tau t} \tilde{u}(-i\tau,\xi,z) d\tau .$$

Thus, the propagation into the $z > 0$ domain can be described in the Fourier domain. The propagating Fourier modes of solutions of the wave equation, $\exp(i(\xi \cdot \vec{x} + \zeta z + \tau t))$, must satisfy the relation $\tau^2 = |\xi|^2 + \zeta^2$. Since τ, ξ and ζ are real, we have that $\tau^2 - |\xi|^2 \geq 0$ and $\zeta = \pm\sqrt{\tau^2 - |\xi|^2}$. This observation has two implications: first, $f(-i\tau, \xi) \equiv 0$ for $\tau^2 < |\xi|^2$ and second, the Fourier modes which propagate in the positive z-direction take the form $e^{i(\xi\cdot\vec{x} + z\sqrt{\tau^2-|\xi|^2} - \tau t)}$ for $\tau \geq 0$ and $e^{-i(\xi\cdot\vec{x} + z\sqrt{\tau^2-|\xi|^2} + \tau t)}$ for $\tau < 0$. Consequently, we obtain

$$\tilde{u}(-i\tau, \xi, z) = \begin{cases} \tilde{f}(-i\tau, \xi) e^{iz\sqrt{\tau^2-|\xi|^2}}, & \tau \geq 0\,, \\ \tilde{f}(-i\tau, \xi) e^{-iz\sqrt{\tau^2-|\xi|^2}}, & \tau < 0\,. \end{cases}$$

The above formulation defines a unique wavefield at $t = T$, given by

$$u(T, x, z) = \mathscr{L}^{-1}[\mathscr{F}^{-1}[\tilde{u}]]\Big|_{t=T}\,,$$
$$u_t(T, x, z) = \mathscr{L}^{-1}[-i\tau\mathscr{F}^{-1}[\tilde{u}]]\Big|_{t=T}\,.$$

In order to obtain a numerical solution, we compute the Fourier transform in the $\vec{x}$ variable using the fast Fourier transform (FFT). We compute the Laplace and inverse Laplace transforms by discretizing the integrals directly. We use an adaptive grid for the inverse Laplace transform to minimize the errors and computational time.

Wavefield Decomposition

Once the wavefield is computed at time T, we decompose it into Gaussian beams. For the decomposition, we use a variation of the algorithm proposed by Tanushev et al. (2009). This algorithm assumes that the large parameter k is unique and known. This assumption is not adequate for seismic data, since there is no single dominant frequency in the data. We determine each frequency band adaptively from the wavefield that is being decomposed in a greedy manner. Guided by the Fourier transform of a plane wave modulated by a Gaussian,

$$\mathscr{F}_{y\to\eta}\left[e^{ik\vec{\mathbf{p}}\cdot\vec{\mathbf{y}} - k|\vec{\mathbf{y}}|^2/2)}\right] = ce^{-(\eta + k\vec{\mathbf{p}})^2/2k}\,,$$

we use $\rho_k(\eta) = e^{-(|\eta|-k)^2/2k}$ to define the frequency band. To find k at each step of the algorithm, we scan through the possible values for k and pick the band that contains the most energy. The algorithm can be summarized as follows. Let $j = 1$ and $(u^0, u_t^0) = (u, u_t)$. Then,

1. Let k_j be given by

$$k_j = \underset{0<k<k_{\max}}{\arg\max}\, E\left[u^{j-1} * \left(\mathscr{F}^{-1}\rho_k\right), u_t^{j-1} * \left(\mathscr{F}^{-1}\rho_k\right)\right]\,.$$

2. Define $(w^j, w_t^j) = (u^{j-1}, u_t^{j-1}) * (\mathscr{F}^{-1}\rho_{k_j})$.

3. Apply the algorithm of Tanushev et al. (2009) to (w^j, w_t^j) with $k = k_j$ and obtain L_j Gaussian beams with wavefields, $(u_{GB}^{j,\ell}, \partial_t u_{GB}^{j,\ell})$, $\ell = 1, \ldots, L_j$. Briefly:

- **a.** Find the Gaussian beam center: $\vec{\mathbf{y}}_\ell = \arg\max_{\vec{x}}\{E[v^\ell, v_t^\ell]\}$
- **b.** Find the propagation direction: with $G(\vec{\mathbf{x}}) = \exp(-k|\vec{\mathbf{x}} - \vec{\mathbf{y}}_\ell|^2/2)$, let $\vec{\mathbf{p}}_\ell = \frac{1}{k}\arg\max_\xi\{|\mathscr{F}[v^j(\vec{\mathbf{x}})G(\vec{\mathbf{x}})]| + |\mathscr{F}[v_t^j(\vec{\mathbf{x}})G(\vec{\mathbf{x}})/k]|\}$
- **c.** Let the Hessian matrix be given by $M_\ell = iI$, with I the identity matrix.
- **d.** Choose $\phi_t^\ell = \pm|\vec{\mathbf{p}}_\ell|$ and a_ℓ by an inner product so that the Gaussian beam $(u_{GB}^{j,\ell}, \partial_t u_{GB}^{j,\ell})$ defined by these parameters best approximates (v^ℓ, v_t^ℓ).
- **e.** **i.** *Version 1.* No optimization.
  - **ii.** *Version 2.* Optimize the position $\vec{\mathbf{y}}_\ell$ and propagation direction $\vec{\mathbf{p}}_\ell$ locally.
  - **iii.** *Version 3.* Optimize $\vec{\mathbf{y}}_\ell$, $\vec{\mathbf{p}}_\ell$ and the real part of M_ℓ locally.
  - **iv.** *Version 4.* Optimize all of the Gaussian beam parameters locally.
- **e.** Let $(v^{\ell+1}, v_t^{\ell+1}) = (v^\ell - u_{GB}^{j,\ell}, v_t^\ell - \partial_t u_{GB}^{j,\ell})$, increment ℓ and repeat from step **3a.** until $E[v^{\ell+1}, v_t^{\ell+1}]$ is small enough or a maximum number of beams is reached.

6. Let $(u^j, u_t^j) = (u^{j-1} - \sum_\ell^{L_j} u_{GB}^{j,\ell}, u_t^{j-1} - \sum_\ell^{L_j} \partial_t u_{GB}^{j,\ell})$.

7. Increment j and repeat starting with step **1.**, until $E[u^j, u_t^j]$ is smaller than a predetermined tolerance or a maximum number of Gaussian beams has been reached.

"*Version 1*" of the algorithm is simply the initialization steps of the decomposition algorithm of Tanushev et al. (2009) and "*Version 4*" is the entire algorithm. The four versions of the algorithm are progressively more computationally intensive. However, with more optimization, the Gaussian beam representation of the wavefield is more sparse. We show a comparison the different versions in figure 2 for the synthetic seismic example from the next section.

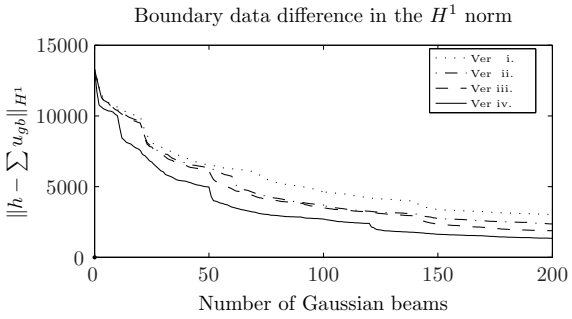

Figure 2: A comparison between the different versions of the decomposition algorithm. Note that the jagged structure of the H^1 norm difference is due to the fact that from each frequency band a fixed number of beams is extracted.

Back Propagation of Gaussian Beam Parameters

Once the Gaussian beam parameters are obtained in the decomposition step above, they need be propagated back to the $z = 0$ surface. Since the velocity is constant in the $z > 0$ region, the rays are straight lines and the time t at which they intersect the $z = 0$ plane can be found analytically. All Gaussian beam parameters are propagated back to this time t using the Gaussian beam ODEs.

SYNTHETIC SEISMIC EXAMPLE

We use a simple synthetic example to illustrate the proposed method. The seismic velocity is given by $c(z) = 1.5 + z$ km/s (linearly increasing with depth). The synthetic zero-offset data is shown in Figure 3. We apply version 4 of the decomposition algorithm described in this paper with 10 Gaussian beams per frequency band and 20 frequency bands for a total of 200 beams. After the Gaussian beam parameters have been recorded on the boundary data surface, the beams are migrated into the subsurface using standard Gaussian-beam techniques. Several stages of the results of the decomposition and migration are shown in Figure 4. We note that the migration was carried out by propagating each Gaussian beam to half of time at which the Gaussian beam appears on the boundary surface $z = 0$ (zero offset migration).

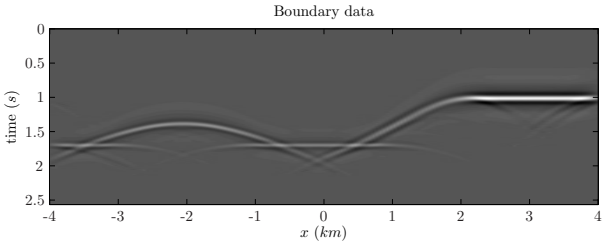

Figure 3: Zero-offset synthetic seismic data.

SUMMARY

We propose an algorithm for decomposing seismic data into a superposition of Gaussian beams with the ultimate goal that the Gaussian beams will be used for seismic imaging. The method has three main components: First, the recorded seismic data are transformed from boundary data into a wavefield at a fixed time. Second, this wavefield is then decomposed into Gaussian beams using a previously developed method. Finally, the Gaussian beam parameters are propagated to represent the seismic data at the surface. Although we present only a simple 2-D zero-offset example, the method is general and applicable to prestack 3-D data decompositions.

The reason for developing this algorithm rather than decomposing the recorded seismic data directly into Gaussian beams is that at a fixed time, the absolute value of the wavefield of a single beam is a Gaussian distribution. In contrast, the absolute value of the wavefield can have a much more complicated behavior on the boundary. We further note that this difficulty exists not only for Gaussian beams, but also for other localized beam-type wave representations and any other wave propagation methods that rely on localized-wave solutions. Thus, the algorithm presented here can be used with other beam type approximations, with the second and third step replaced by the appropriate decomposition and propagation.

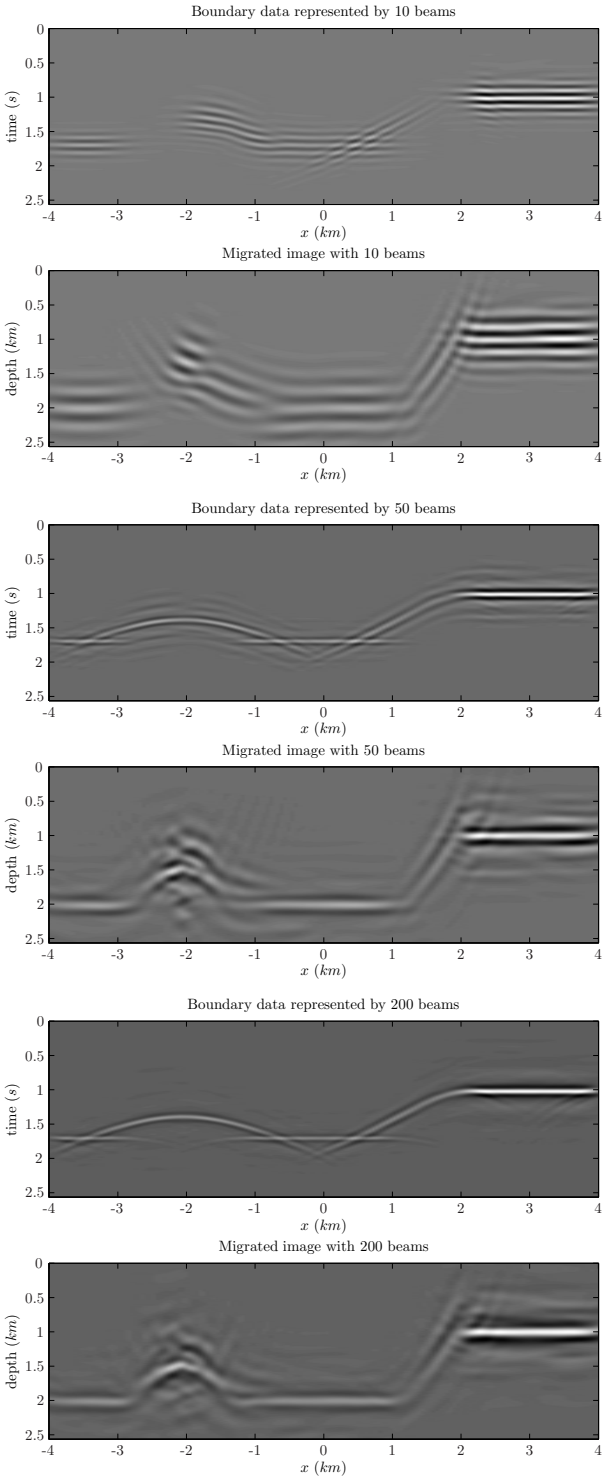

Figure 4: Boundary data representation with Gaussian beams and the associated migrated image.

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