UNIVERSITY OF CALIFORNIA
Los Angeles

Applied Partial Differential Equations in Crime Modeling and Biological Aggregation

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Nancy Rodríguez

2011

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2011

To my family.

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Acknowledgments

Words cannot explain the gratitude I feel towards my family, friends, mentors, committee members, fellow graduate students and the departmental staff. This work would not have been possible without their invaluable support.

It has been a pleasure working under the guidance of Andrea Bertozzi. I want to thank her for all her mathematical and professional guidance. Her advice and expertise have been invaluable. I would also like to thank my committee members: Jeff Brantingham, Lincoln Chayes, and Inwon Kim, for their many helpful discussions and letters of recommendations. The UCLA mathematics departmental staff members have also been extremely helpful in the completion of my PhD.

My family has supported me though all of my endeavors and I would not have been able to achieve anything without them. I want to thank my parents for having faith in me and supporting my decision to pursue a doctoral degree. My sister Marisa has been a great role model and an inspiration. I am so lucky to have her in my life and am very glad that we have managed to share many experiences throughout these busy times. My love and thanks also to my brother Diego, my little sister Sheccíd, to my aunt Socorrito, Jenny, Rubí, and my grandparents.

Meeting my boyfriend Paul was one of the best thing that happened during my graduate years. He has made my personal life more fulfilling and has been a tremendous help in my professional life. I want to thank him for the many fun moments we have shared, for the many proposals and letters that he has proof read, and for being his wonderful self.

Finally, I want to thank my fellow graduate students. My experience at UCLA has been extremely positive because I got to share it with you guys. In particular, I want to thank Jacob Bedrossian for being a great collaborator.

Last but not least, I want to thank my undergraduate professors. It is thanks to Ani Velo, Lynn Small, and Luckasz Pruski that I was able to attend graduate school.

Vita

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Publications

1. Bedrossian J., Rodríguez N., and Bertozzi, A. *Local and Global Well- Posedness for Aggregation Equations and Patlak-Keller-Segel Models with Degenerate Diffusion.* Nonlinearity, 24 (6), 1683–01714, 2011.

2. Rodríguez, N., and Bertozzi, A. *Local Existence and Uniqueness of Solutions to a PDE Model for Criminal Behavior*. M3AS, Special issue on Mathematics and Complexity in Human and Life Sciences, 20, 1425–1457, 2010.

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Abstract of the Dissertation

Applied Partial Differential Equations in Crime Modeling and Biological Aggregation

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Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2011

Professor Andrea L. Bertozzi, Chair

Recently, there has been sustained interest in the use of partial differential equation (PDE) models to obtain insight into biological and sociological phenomena. This work is separated into two parts were we study the well-posedness two such PDE systems.

In particular, in the first part we study a fully-parabolic system of PDEs for residential burglary 'hotspots' (spatio-temporal areas of high density of crime). Although crime is a ubiquitous feature of all societies, certain geographical locations have a higher propensity to crime than others. In fact, residential burglary data exhibit areas of high crime density surrounded by areas of low crime density. There have been many studies indicating that the "repeat and near-repeat victimization effect," which states that crime in an area induces more crime in that and neighboring areas, leads to the residential burglary hotspots seen in real data. Short *et al.* develop in [92] an agent-based statistical crime model whose dynamics rely on this repeated victimization effect. The formal continuum limit of this model is a nonlinear couple system of PDEs. This models exhibits the right qualitative behavior (with certain parameters), that is, the existence of

crime 'hotspots'. In this work we are concerned with the existence and uniqueness of solutions of this model. In two-dimensions we prove local existence and uniqueness of classical solutions with H^m-initial data on periodic domains. Furthermore, we prove a continuation argument that provides a sufficient condition for global existence of the solution. More specifically, we prove that $\|\nabla\rho\|_\infty$ is a controlling quantity; hence, the solution continues to exist while this quantity remains bounded. Finally, motivated by the relation between this PDE system and the Keller-Segel model for chemotaxis (the movement of cellular organisms in response to some chemical concentration in their environment), we conclude this section by studying a modified model; which provides insight into the global theory of the original model.

In the second part we study an aggregation equation with degenerate diffusion. Aggregation equations have been studied for a wide variety of biological applications in migration patterns in ecological systems and Patlak-Keller-Segel models for chemotaxis. We study the local and global well-posedness of weak-solutions of an equation that models the competition between aggregation (modeled via a convolution with a kernel) and over-crowding effects (modeled via general degenerate diffusion). We divide they system into three types: *subcritical, supercritical, and critical*. We prove global existence for *subcritical* problems, which correspond to the diffusion dominating aggregation. We prove finite time blow-up for a subclass of *supercritical* problems, which correspond to the aggregation dominating the diffusion. Finally, we show that there is a critical mass phenomena for the *critical* problems, which correspond to the aggregation and diffusion balancing out.

Part I

A PDE Model Residential Burglaries

CHAPTER 1

Introduction and Background

1.1 Introduction and Motivation

The study of crime hardly needs motivation, it is a phenomenon that affects all individuals. The city of Los Angeles, nicknamed the "Gang Capital of the Nation," is of particular interest. Violent and non-violent crimes from burglaries to drive-by-shootings have affected the citizens of this city since the beginning of the 20th century. One of the most frequently occurring crimes is residential burglaries, a crime which will affect most people at some point. The observation that residential burglaries are not spatially homogeneously distributed and that certain neighborhoods have more propensity to crime than others led Short *et al.* to study the dynamics of residential burglary *hotspots*[92]. A *hotspot* is a spatio-temporal aggregation of criminal occurrences and the understanding how they evolve can be extremely useful. For example, it can help the police force mobilize their resources optimally. This would ideally lead to the reduction or even eliminations of these crime hotspots. This spatial heterogeneity in crime can be seen in Figure 1.1. The figure on the left is a snapshot of The Times's database of Los Angeles crime reports. The Times creates and updates this crime map of over 200 neighborhoods in the Los Angeles area daily using data provided by the Los Angeles Police Department and the Los Angeles County Sheriff's Department. The figure on the right is a density map of residential burglary data from Long Beach, CA for three consecutive months (this figure taken from [92]). A theoretical understanding of dynamics of hotspots can help predict how these hotspots will change and thus aid law enforcement agencies fight crime.

Short *et al.* modeled the dynamics of hotspots using an agent-based statistical model based on the 'broken window' sociological effect [108].The idea of the 'broken window' effect is that crime in an area leads to more crime. It has been observed in the residential burglary data that houses which are burglarized have an increased probability of being burglarized again for some period of time after the initial burglary. This increased probability of burglary also affects neighboring houses and is referred to as the 'repeat near-repeat effect' [4, 60, 61, 62, 94]. Figure 1.2 shows burglary data from from Long Beach binned by two-week intervals (this figure was taken from [94]). The model is based on the assumption that criminal agents are walking randomly on a two-dimensional lattice and committing burglaries when encountering an opportunity. Furthermore, there is an attractiveness value assigned to every house, which refers to how easily the house can be burgled reduces negative consequences for the criminal agent. The criminal agents, in addition to walking randomly, have a biased movement toward areas of high attractiveness values and move with a speed inversely proportional to the value in their current position. Let $A(x, t)$ and $\rho(x, t)$ be the attractiveness value and the criminal density at position x and time t respectively, then the continuum limit of the agent-based model gives the following PDE model:

$$\frac{\partial A}{\partial t} = \eta \Delta A - A + A\rho + A^o, \tag{1.1a}$$

$$\frac{\partial \rho}{\partial t} = \Delta \rho - 2\nabla \cdot [\rho \nabla \chi(A)] + \overline{B} - A\rho; \tag{1.1b}$$

where $\chi(A) = \log(A)$. A formal derivation of this model can be found in [92]. From (1.1) we observe that criminal agents are being created at a constant rate $\overline{B}$ and are removed from the model when a burglary is committed. In essence, the number of burglaries being committed at time t and location x is given by the $A(x,t)\rho(x,t)$. Furthermore, the attractiveness value increases with each burglary. As we will discuss later the system (1.1) can be seen as a nonlinear version of the Keller-Segel model for chemotaxis with growth and decay. The Keller-Segel model is a reaction-diffusion system that models the movement of some mobile specie that is being influenced by an external chemo-attractant [26, 38, 43, 56, 106, 97]. In the Keller-Segel model literature the function $\chi(A)$ is referred to as the sensitivity function. Various forms of the sensitivity function have been analyzed including $\log A$ and A [71, 91]. For these cases global existence has been proved in one-dimension [36, 85]. Furthermore, in two-dimension global existence has been proved for small enough initial mass of the cell density [15, 28]. It is important to note that these models do not include growth or decay. Although the logarithmic sensitivity function has been analyzed most of the research done on the Keller-Segel model has been for $\chi(A) = A$. Recall that the model (1.1) is the continuum limit of an discrete agent-based model. In the discrete model the probability of an agent moving from node s to node n is given by the ratio of the attractiveness value at node n over the sum of attractiveness values of the neighboring nodes of node s. This gives the logarithmic sensitivity function we see in (1.1). Therefore, it makes sense for us to analyze the more complicated sensitivity function. In fact, we will see later that the logarithmic velocity field helps prevent finite time blow-up. From the numerical analysis performed in [92] this model seems to have appropriate qualitative properties, i.e. existence of hotspots. However, to show that this model is truly robust the unique existence of a solution, which does not blow-up in finite time, is essential.

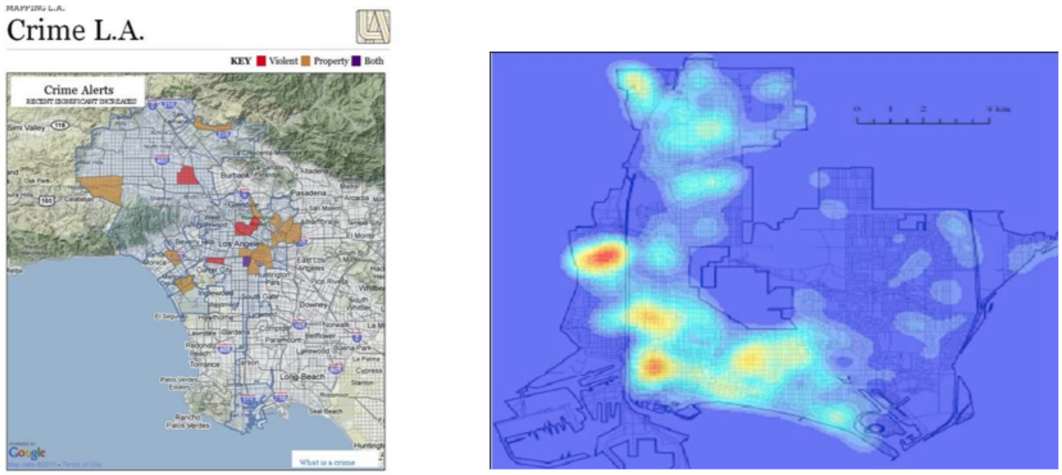

Figure 1.1: Left – Snap shot of The Time's database of Los Angeles crime report (red corresponds to violent crimes, orange corresponds to property crime, and purple corresponds to both). Right – Density map of burglary data from Long Beach, CA from June 2001–August 2001.

1.2 Background Work

Despite the recent introduction of the system (1.1) there has been a significant amount of research done on the model, particularly qualitative analysis, see for example [92, 93]. In the original work, Short and collaborators determined the parameter regimes that lead to hotspots via the use of linear stability analysis. Furthermore, in [93] the authors studied the suppression of hotspots via the use weakly-nonlinear analysis. The effect of different policing strategies on hotspots, see [63], and strategies to measure and model the 'repeat and near-repeat victimization' effect have also been studied, see [94]. In the following sections we give a brief summary of these works.

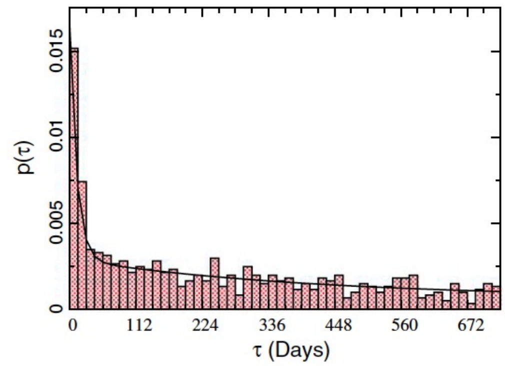

Figure 1.2: Probability distribution of the time interval τ between repeated offenses computed from residential burglary data from Long Beach, CA using a moving window method.

1.2.1 Linear Stability Analysis

The motivation for the development of (1.1) was the need to understand the spatio-temporal dynamics of residential burglary hotspots. Hence, one would expect the PDE system to reproduce similar behavior. In fact, one can study the existence and nonexistence of hotspots via linear stability. The system (1.1) has one flat steady-state given by

$$A_s = A^0 + \overline{B} \quad \text{and} \quad \rho_s = \frac{\overline{B}}{A^0 + \overline{B}}. \tag{1.2}$$

Linear stability analysis gives that the system has some unstable modes provided the parameters and steady-state solutions satisfy the following inequality

$$3\rho_s - \eta A_s - 1 > 2\sqrt{\eta A_s}. \tag{1.3}$$

When the parameters satisfy condition (1.3) then numerical simulations performed in [92] show that stationary hotspots appear (see Figure 1.3). When this condition is not satisfied then hotspots do not appear. Simulations of the discrete model show a third parameter regime: transient hotspots. This regime is

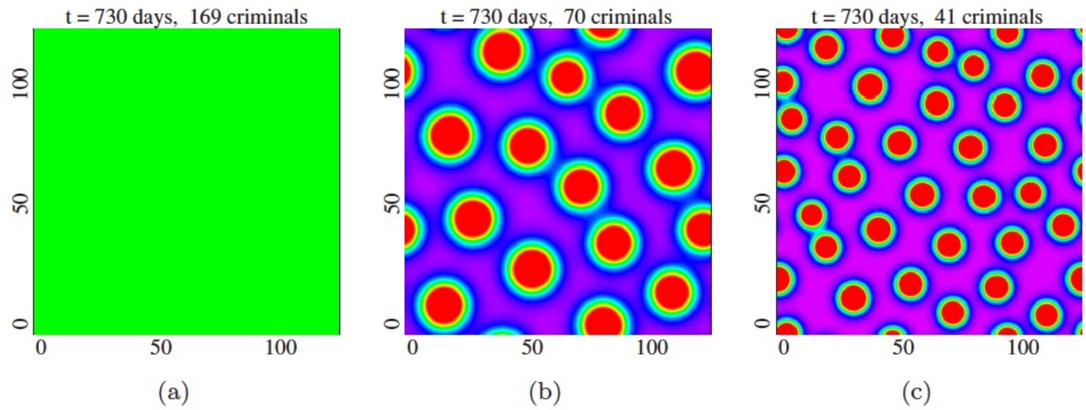

Figure 1.3: Numerical simulation of the continuous system. Figure (a) shows the hotspot-free parameter regime. Figures (b) and (c) show the hotspot regime with different hotspots sizes (Figured obtained from [92]).

not observed in the numerical simulations for continuous model. The transitivity is predicted to be a consequence of the stochasticity of the discrete model, which is lost in the continuous system. Examining (1.3) further, one can see that hotspots can only be formed if areas of high attractiveness value are separated enough, the diffusion coefficient η playing a key role in this.

1.2.2 The Effects of Law Enforcement Agents

One weakness of the (1.1) is that it lack the effect that law enforcement agents have on deterring and stopping crime. In [63] the authors extended the agent based model from [92] to include the effect of *law enforcement agents*. This enabled the authors to study the effectiveness of different police deployment strategies. While law enforcement agents play multiple roles in the reduction of crime only the deterring effect was considered. Hence, for example the incarceration of criminals was not considered. Studies have shown that presence of police force can be sufficient to prevent crime (see for example [37]). Two different methods of modifying the behavior of criminal agents in response to the presence of the police are discussed in this work. The first via the attractiveness value. That is, assuming that the presence of law enforcement agents decrease the attractiveness value of an area automatically leads to a reduction of criminal agents in that area. The second method is to consider the direct interaction between the law enforcement agents and the criminal agents. This interaction leads to the removal of the criminal agent, as they are motivated to return home.

The authors also considered and modeled various policing strategies. Three strategies are considered: *random policing, cops on the dots, and peripheral interdiction.* In the *random* policing strategy law enforcement are deployed on random routes, mathematically this lead to diffusion of the law enforcement agents. If $\kappa(x,t)$ corresponds to the police density then the corresponding equation is the heat equation

$$\kappa_t = \frac{1}{4}\Delta\kappa.$$

In the *cops on the dots* strategy the law enforcement agents behave similar to the criminal agents. They walk randomly with a bias towards areas of high attractiveness values. As expected, the corresponding equation is

$$\kappa_t = \frac{1}{4}\Delta\kappa - \frac{1}{2}\nabla\cdot(\kappa\nabla A).$$

Finally, the *peripheral interdiction* takes into account the limited resources. They use the fact that the area of a hotspot grows quadratically with the growth of the radius while the perimeter grow linearly. Hence, the strategy is to deploy the law enforcement agents to the perimeter of the hotspots. One disadvantage of this strategy is the difficulty of expressing this with a partial differential equation.

These different strategies where explored with computer simulations and the authors found that the *cops on the dots* and *peripheral interdiction* strategies are more effective in reducing crime than the *random* strategy. Furthermore, the authors observed that in certain cases seemingly eradicated hotspots re-emerged in the same or nearby locations. We discuss this in more detail in the following section.

1.2.3 Bifurcation Theory

While the linear stability analysis answers some questions on the qualitative behavior of (1.1), it does not provide a full picture and many questions remain unanswered. In particular, the question of the effectiveness of hotspot policing is especially important. This is a point of contention between criminologist that can be observed in the literature. Indeed, some studies claim that hotspot suppression can successfully eliminate hotspots (see for example [88, 66]), while others claim that this strategy only displaces crime (see for example [24]). In [93] the authors explore the question of whether or not the policing strategy of hotspot suppression is an effective strategy via the use of weakly-nonlinear analysis (see [40, 104]).

Another way to understand the linearly stability of the steady state (1.2) depends on the parameter A^o. If A^o is less than some critical A^* and linearly stable for $A^o > A^*$, where

$$A^o < A^o_* = \frac{2}{3}\overline{A} - \frac{1}{3}\eta\overline{A}^2 - \frac{2}{3}\overline{A}\sqrt{\eta\overline{A}}.$$

In fact, a deeper mathematical analysis shows that there are three different parameter regimes: linearly unstable, weakly nonlinearly unstable, and linearly stable. In the linearly stable regime hotspots are never formed. On the other hand, the two other regimes do result in hotspots. The linearly unstable regime parameters lead to *supercritical* hotspots. The weakly nonlinearly unstable regimes are those whose parameters lead to a steady state which is linearly stable. However, these parameters are close enough to the the critical A^* that the solutions develop hotspots in the slow time scale. Hence, parameters in this regime lead to *subcritical* hotspots. Interestingly, these two types of hotspots react differently to hotspot suppression. The authors of [93] showed the existence of two qualitatively different hotspots, supercritical and subcritical, which react very differently to crime suppression. The suppression was included in the reaction term of criminal density equation of (1.1). while suppression can destroy *sub*critical hotspots, it tends to only displace *super*critical hotspots. More specifically, for the supercritical case, the effect of crime suppression is to push the burglaries outwards, forming an annulus-shaped area of high crime density with interior and exterior areas of low crime density. These ring solutions then break up into multiple hotspots; hence, with respect to the model described by (1.1), crime suppression in supercritical hotspots is qualitatively different than suppression for subcritical hotspots.

1.3 Outline

In Chapter 2 we first prove local existence of classical solutions and a continuation argument, which gives a necessary and sufficient conditions for global existence in $\mathbb{R}^2$. In Chapter 3 we first discuss the Keller-Segel model for chemotaxis and its relation to residential burglaries system (1.1). We then explore the importance of the logarithmic velocity field in the prevention of finite time blow-up. Please refer to each individual chapter for a more detailed outline. This work is in collaboration with Andrea Bertozzi. In particular, Chapter 2 and §3.3 in Chapter 3 are part of published work, see [90].

CHAPTER 2

Existence and Uniqueness of Classical Solutions

In this section we analyze the well-posedness of classical solution the 1.1 in $\mathbb{R}^2$. Assuming that the criminals entering the city and the criminals leaving are approximately the same we consider no-flux boundary condition in a bounded domain $\Omega \subset \mathbb{R}^2$:

$$\frac{\partial A}{\partial \nu}|_{\partial\Omega} = 0 \quad and \quad (-\nabla\rho + 2\frac{\rho}{A}\nabla A) \cdot \vec{\nu}|_{\partial\Omega} = 0; \tag{2.1}$$

where ν is the outer normal vector. The initial conditions are given by:

$$A(0, x) = A_0(x), \ \rho(0, x) = \rho_0(x). \tag{2.2}$$

Outline: This chapter is divided into two sections. In §2.1 we prove local existence and uniqueness to (1.1) with no-flux boundary conditions. In §2.2 we prove a continuation theorem.

2.1 Local Existence and Uniqueness in $\mathbb{R}^2$

2.1.1 Definitions and Notation

We begin this section by establishing the notation that will be used throughout the paper. The proof of the main result follows the techniques used in [82] for the Navier-Stokes Equation in 3-D (see also [102] Taylor for symmetric hyperbolic systems). In the Keller-Segel literature there are two principal methods used to prove global existence of solutions to various versions of the model[57]. The first one involves finding L^∞ estimates for the advection term. The second method involves finding a Lyapunov function. Both of these methods use fixed point theory to obtain local solutions. Since we do not know of the existence of a Lyapunov function for (1.1) our method is more closely related to the first method mentioned. We use an abstract version of Picard's Theorem for ODEs to obtain a local solution to (1.1). We will see that global existence depend on some L^∞ estimates.

2.1.1.1 Notation

We have a initial-boundary value problem with no-flux boundary conditions. For simplicity assume that our domain is a square. This problem can be mapped into the periodic problem with symmetry on a domain four times the size of the original domain. This is true provided A^o and the initial data satisfy reflection symmetry, in which case the model preserves symmetry. Hence, from now on we work with periodic boundary conditions and $\Omega = \mathbb{T}^2$ unless otherwise specified. It is useful to define the following notation:

$$\int v dx = \int_\Omega v dx,$$
$$\|v\|_0^2 = \int_\Omega v^2 dx.$$

The notation $\|u\|_{L^p} = |u|_p$ will be used interchangeably throughout this work to denote that L^p-norm. Furthermore, for a multi-index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+ \cup \{0\}$, we define the $H^m(\Omega)$-norm as follows:

$$\|v\|_m := \left(\sum_{|\alpha| \leq m} \|D^\alpha v\|_0^2 \right)^{\frac{1}{2}}. \tag{2.3}$$

Finally, we define the spaces with their corresponding norms to be used:

- For X a Banach Space with norm $\|\cdot\|_X$, $C([0,T];X)$ is the space of continuous functions mapping $[0,T]$ into X. This space has the following norm:

$$\|v\|_{C([0,T];X)} := \sup_{0\le t\le T} \|v\|_X .$$

- $L_\infty(0,T;X)$ is the space of functions such that $v(t) \in X$ for *a.e.* $t \in (0,T)$ has finite norm:

$$\|v\|_{L_\infty(0,T;X)} := ess \sup_{t\in(0,T)} \|v(t)\|_X .$$

- $L^2(0,T;X)$ is the space of functions such that $v(t) \in X$ for *a.e.* $t \in (0,T)$ with finite norm:

$$\|v\|_{L^2(0,T;X)} := \left(\int_0^T \|v(t)\|_X^2 \, dt\right)^{\frac{1}{2}} .$$

Definition 1. *The space $C^{weak}([0,T];H^s(\Omega))$ denotes continuity on the interval $[0,T]$ with values in the weak topology of H^s. In other words, for any fixed $\Phi \in H^s$, $(\Phi, u(t))_s$ is a continuous scalar function on $[0,T]$. The inner-product of H^s is given by:*

$$(u,v)_s = \sum_{\alpha\le s} \int D^\alpha u \cdot D^\alpha v dx. \tag{2.4}$$

The Hilbert spaces we will be working on for most of the time is:

$$V^m = \{(u,v) \in H^m(\Omega) \times H^m(\Omega)\} . \tag{2.5}$$

Since we are working extensively with different bounds and the constants are not always important, we introduce the notation $A \lesssim B$ to mean that there exists a positive constant c such that $A \le cB$. This notation will be used when the constants are irrelevant and become tedious.

2.1.1.2 Main Result and Outline of its Proof

Our main contribution is to prove local existence and uniqueness of solutions to the system (1.1). More precisely, we prove the following theorem.

Theorem 1 (Local Existence of Solutions to the PDE Residential Burglaries Model)**.** *Given initial conditions* $(A_0(x), \rho_0(x)) \in V^m$ *for* $m > 3$ *such that* $A_0(x) > A^o$ *there exists a positive time,* $T > 0$*, such that* $A, \rho \in C([0,T]; C^2(\Omega)) \cap C^1([0,T]; C(\Omega))$ *form a unique solution to* (1.1) *on the time interval* $[0,T]$.

We first modify the system (1.1) by regularizing it, for the purpose of bounding differential operators in . This is useful because finding a family of solutions to the regularized system is straightforward. Given $v \in L^p(\mathbb{T}^2)$ for $1 \le p \le \infty$ we define the mollification of v by

$$J_\epsilon v(x) = \sum_{k \in \mathbb{Z}^2} \hat{v}(k) e^{-\epsilon^2 |k|^2 + 2\pi i k \cdot x}, \tag{2.6}$$

where $\hat{v}(k) = \int_\Omega v(x) e^{-2\pi i k \cdot x} dx$. The mollified function, $J_\epsilon v_\epsilon$, has many useful properties, some of which are summarized in the following lemma. For more details we refer the reader to [11]. Furthermore, a proof can be found in [51]. We note that this is analogous to mollification by convolution with smooth functions in $\mathbb{R}^2$. The interested reader is referred to [48].

Lemma 1 (Properties of Mollifiers)**.** *Let* J_ϵ *be a mollifier defined in* (2.6)*. Then* $J_\epsilon v \in C^\infty$ *and has the following properties:*

1. $\forall\ v\ \in C^1(\Omega)$ $J_\epsilon v \to v$ *uniformly and*

$$|J_\epsilon v|_\infty \le |v|_\infty .$$

2. *Mollifiers commute with distribution derivatives,*

$$D^\alpha J_\epsilon v = J_\epsilon D^\alpha v \quad \forall\ |\alpha| \le m,\ v \in H^m.$$

3. $\forall\, u, v \in L^2(\Omega)$,

$$\int_\Omega (J_\epsilon u) v dx = \int_\Omega (J_\epsilon v) u dx.$$

4. $\forall\, v \in H^s(\Omega)$, $J_\epsilon v$ *converges to* v *in* H^s *and the rate of convergence in the* H^{s-1} *norm is linear in* ϵ*:*

$$\lim_{\epsilon \searrow 0} \|J_\epsilon v - v\|_s = 0,$$

$$\|J_\epsilon v - v\|_{s-1} \leq C\epsilon \|v\|_s .$$

5. $\forall\, v \in H^m(\Omega),\ \gamma,\ k \in Z^+ \cup 0,$ *and* $0 \leq \epsilon \leq 1$*:*

$$\|J_\epsilon v\|_{m+\gamma} \leq \frac{c_{m\gamma}}{\epsilon^\gamma} \|v\|_m ,$$

$$\left|J_\epsilon D^k v\right|_\infty \leq \frac{c_k}{\epsilon^{N/2+\gamma-k}} \|v\|_k .$$

Once the original system has been regularized it is easy to show that the assumptions of the Picard Theorem on a Banach Space are satisfied by the regularized model for any fixed $\epsilon > 0$. We now state this theorem along with its natural continuation theorem. A proof of the following two theorems can be found in [53].

Theorem 2 (Picard Theorem on a Banach Space)**.** *Let* $O \subseteq \boldsymbol{B}$ *be an open subset of a Banach Space* $\boldsymbol{B}$*, and let* $F : O \to \boldsymbol{B}$ *be a mapping satisfying:*

1. *F(x) maps O to* $\boldsymbol{B}$

2. *F is locally Lipschitz continuous i.e. for any* $x \in O$ *there exists* $L > 0$ *and an open neighborhood* $U_x \subset O$ *of* x *such that for all* $x, \hat{x} \in U_x$ *we have*

$$\|F(x) - F(\hat{x})\|_{\boldsymbol{B}} \leq L \|x - \hat{x}\|_{\boldsymbol{B}}$$

Then for any $x_o \in O$, there exist a time T such that the ODE

$$\frac{dx}{dt} = F(x), \quad x|_0 = x(0) \in O$$

has a unique local solution $x \in C^1((-T,T);O)$.

Theorem 3 (Continuation on a Banach Space)**.** *Let $O \subseteq B$ be an open subset of a Banach Space B, and let $F : O \to B$ be a locally Lipschitz-continuous map. Then the unique solution $X \in C^1([0,T);O)$ to the autonomous ODE*

$$\frac{dx}{dt} = F(x), \quad x|_0 = x(0) \in O,$$

either exists globally in time, or $T < \infty$ and $X(t)$ leaves the open set O as $t \to T$.

We will see that the above theorem can be applied provided an appropriate functional framework is chosen. We use some calculus inequalities in the Sobolev Spaces to show that this theorem can be used to obtain a family of solutions which depend on the regularizing parameter ϵ. Refer to [82] for a proof of the following lemma in the case when $\Omega = \mathbb{R}^N$. The proof for the case when Ω is the torus follows exactly.

Lemma 2 (Calculus Inequalities in the Sobolev Spaces)**.**

1. *$\forall\, m \in \mathbb{Z}^+ \cup 0$, there exists $c \geq 0$ such that for all $u, v \in L^\infty(\Omega) \cap H^m(\Omega)$:*

$$\|uv\|_m \leq c \left\{ |u|_\infty \|D^m v\|_0 + \|D^m u\|_0 |v|_\infty \right\},$$

$$\sum_{0 \leq |\alpha| \leq m} \|D^\alpha(uv) - uD^\alpha v\|_0 \leq c \left\{ |\nabla u|_\infty \left\|D^{m-1} v\right\|_0 + \|D^m u\|_0 |v|_\infty \right\}.$$

2. *$\forall\, s > \frac{N}{2}$, $H^s(\Omega)$ is a Banach algebra. That is, there exists $c > 0$ such that for all $u, v \in H^s(\Omega)$:*

$$\|uv\|_s \leq c \|u\|_s \|v\|_s.$$

The next step is to pass to the limit as $\epsilon \to 0$. Energy estimates, which are independent of the regularizing parameter, are essential for this purpose.

2.1.2 Local Existence and Uniqueness for a Regularized Crime Model

We consider the following regularization of (1.1):

$$\frac{\partial A^\epsilon}{\partial t} = \eta J_\epsilon^2 \Delta A^\epsilon - A^\epsilon + \rho^\epsilon A^\epsilon + A^o, \tag{2.7a}$$

$$\frac{\partial \rho^\epsilon}{\partial t} = J_\epsilon(J_\epsilon \Delta \rho^\epsilon) - 2J_\epsilon[\nabla \cdot (\frac{\rho^\epsilon}{A^\epsilon} J_\epsilon \nabla A^\epsilon)] - \rho^\epsilon A^\epsilon + \overline{B}. \tag{2.7b}$$

This choice of regularization will become clear when we perform the energy estimate calculations. The goal of this section is to prove the local existence and uniqueness of solutions to the system (2.7) for fixed ϵ. Consider the function space for the solution to (2.7) to be the Banach Space V^2, $m = 2$ in (2.5), with norm $\|(A, \rho)\|_{V^2} := \|A\|_2 + \|\rho\|_2$.

Theorem 4 (Local Existence of solutions to the Regularized Residential Burglary Model)**.** *For any $\epsilon > 0$ and initial conditions $(A_0(x), \rho_0(x)) \in V^2$ such that $A_0(x) > A^o$ there exists a solution, $(A^\epsilon, \rho^\epsilon) \in C^1([0, T_\epsilon); V^2)$, for some $T_\epsilon > 0$, to the regularized system* (2.7)*. Furthermore, the following energy estimate is satisfied,*

$$\frac{d}{dt} \|(A^\epsilon, \rho^\epsilon)\|_{V^2} \leq c_3 \|(A^\epsilon, \rho^\epsilon)\|_{V^2}^3 + c_2 \|(A^\epsilon, \rho^\epsilon)\|_{V^2}^2 + c_1 \|(A^\epsilon, \rho^\epsilon)\|_{V^2}; \tag{2.8}$$

where c_1, c_2, and c_3 are constants that depend only on $\frac{1}{A^o}$, ϵ and η.

Proof. Define the map $F^\epsilon = [F_1^\epsilon, F_2^\epsilon] : O \subseteq V^2 \to X$. To use *Theorem 2* we need a suitable set O such that F^ϵ maps O to V^2, (*i.e.* $X = V^2$). Defined the function by:

$$F_1^\epsilon(A^\epsilon, \rho^\epsilon) = \eta J_\epsilon^2 \Delta A^\epsilon - A^\epsilon + \rho^\epsilon A^\epsilon + A^o, \tag{2.9a}$$

$$F_2^\epsilon(A^\epsilon, \rho^\epsilon) = J_\epsilon^2 \Delta \rho^\epsilon - 2J_\epsilon[\nabla \cdot (\frac{\rho^\epsilon}{A^\epsilon} J_\epsilon \nabla A^\epsilon)] - \rho^\epsilon A^\epsilon + \overline{B}. \tag{2.9b}$$

Hence, if $v^\epsilon = (A^\epsilon, \rho^\epsilon) \in V^2$, the original model reduces to an ODE in V^2.

$$\frac{dv^\epsilon}{dt} = F^\epsilon(v), \tag{2.10a}$$

$$v^\epsilon(0) = (A_0(x), \rho_0(x)). \tag{2.10b}$$

With this framework we can prove that the conditions of Theorem 2 are satisfied. Let $v_i^\epsilon = (A_i^\epsilon, \rho_i^\epsilon) \in V^2$ $(i = 1, 2)$, we drop ϵ for notational convenience. By definition of the V^2-norm and F we have:

$$\|F(v_1) - F(v_2)\|_{V^2} = \|F_1(v_1) - F_1(v_2)\|_2 + \|F_2(v_1) - F_2(v_2)\|_2 .$$

After substituting (2.9) above and using *(5)* of Lemma 1 and *(1)* of Lemma 2 we obtain a suitable bound for F_1. Initially we have:

$$\|F_1(v_1) - F_1(v_2)\|_2 \leq \eta \left\|J_\epsilon^2 \Delta(A_1 - A_2)\right\|_2 + \|A_1 - A_2\|_2 + \|\rho_1 A_1 - A_2\rho_2\|_2 .$$

The last term in the above inequality will appear repeatedly and can be bounded using (2) of Lemma 2 by:

$$\|\rho_1 A_1 - A_2\rho_2\|_2 \lesssim \|\rho_2\|_2 \|A_1 - A_2\|_2 + \|A_1\|_2 \|\rho_1 - \rho_2\|_2 . \tag{2.11}$$

Using (2.11) we easily obtain the final estimate for F_1:

$$\|F_1(v_1) - F_1(v_2)\|_2 \lesssim (\frac{\eta}{\epsilon^2} + 1 + \|\rho_2\|_2) \|A_1 - A_2\|_2 + \|A_1\|_2 \|\rho_1 - \rho_2\|_2 . \tag{2.12}$$

For F_2 we only state the final bound, refer to Appendix A.1 for more detailed computations. If we define the open set

$$O = \left\{ (u, v) \in V^2 : \left|\frac{1}{u}\right|_\infty < K_1, \|u\|_2 < L_1, \|v\|_2 < L_2 \right\},$$

we obtain similar estimates for F_2. In particular, if $v_1,\ v_2 \in O$ then

$$\|F_2(v_1) - F_2(v_2)\|_2 \lesssim \widetilde{C_1} \|A_1 - A_2\|_2 + \widetilde{C_2} \|\rho_1 - \rho_2\|_2 ; \tag{2.13}$$

where,

$$\widetilde{C_1} = \frac{K_1}{\epsilon^3}\left(\|\rho_1\|_2 + K_1\|A_1\|_1 |\rho_1|_\infty + K_1\|A_2\|_2\|\rho_2\|_2 + K_1^2\|A_2\|_2^2\|\rho_2\|_2\right)$$
$$+ \frac{K_1}{\epsilon}^3 \|A_1\|_1\|A_2\|_2\|\rho_2\|_2 + \frac{K_1}{\epsilon^2}|\rho_1|_\infty + \|\rho_2\|_2\,,$$
$$\widetilde{C_2} = \frac{1}{\epsilon^2} + \|A_1\|_2 + \frac{C_1^2}{\epsilon^3}\|A_2\|_2\|A_1\|_2\left(1 + K_1\|A_2\|_1 + K_1\|A_1\|_1\right).$$

The important thing to note is that $\widetilde{C_1}$ and $\widetilde{C_2}$ depend only on $\|A_i\|_2$, $\|\rho_i\|_2$, ϵ, and K_1 for $i = 1, 2$. Combining (2.12) and (2.13) gives:

$$\|F(v_1)-F(v_2)\|_{V^2} \leq C(\eta, L_1, L_2, K_1, \epsilon)\|A_1-A_2\|_2 + C(L_1, L_2, K_1, \epsilon)\|\rho_1-\rho_2\|_2\,. \tag{2.14}$$

Setting $A_2 = 0$ and $\rho_2 = 0$ we see that F does map O to V^2. Furthermore, $F : O \to V^2$ is locally Lipschitz therefore the conditions of Theorem 2are satisfied for fixed ϵ. Consequently, we obtain a family of unique local solutions to (2.7), $\{(A^\epsilon, \rho^\epsilon)\}_{\epsilon>0}$, such that $(A^\epsilon, \rho^\epsilon) \in C^1([0, T_\epsilon); V^2 \cap O)$. A careful look at the computations performed (see B.1.1) enables us to see that the constants in the above inequality are at most cubic in $\|(A, \rho)\|_{V^2}$. Once again, setting $A_2 = 0$ and $\rho_2 = 0$ in (2.14) from (2.10) we obtain the desired inequality (2.8). Note that the constants $c_1,\ c_2$ and c_3 depend solely on $C_1,\ \epsilon,$ and η. We by taking $K_1 = \frac{1}{A^o}$ we obtain the dependence on $\frac{1}{A^o}$. □

2.1.3 Local Existence and Uniqueness to Crime Model

In the previous section we successfully showed the unique existence of a solution to (2.7) on $[0, T_\epsilon)$ for fixed ϵ. The next step is to show that a subsequence of these solutions converge to a solution of the original system (1.1). To do this we need estimates that are independent of ϵ. The following section is devoted for this purpose.

2.1.3.1 Energy Estimates

From Theorem (4) we see that the time interval on which the solutions to (2.7) exist depend on ϵ. To be able to pass to the limit it is essential that we find a uniform time interval of existence. To obtain such an interval we look at energy estimates which are essential to show that the solution to (2.7) is in $C([0,T);V^m)$. We will see that provided m is chosen large enough then we obtain that the solution is classical. For simplicity from now on we denote $C_1 = \frac{1}{A^o}$.

Proposition 1 (Higher-Order Energy Estimates)**.** *Let $(A^\epsilon, \rho^\epsilon)$ be a solution to the regularized system (2.7) with initial conditions $(A^\epsilon(0), \rho^\epsilon(0)) \in V^m$, where V^m is defined by (2.5) for $m \geq 3$, such that $A_0(x) > A^o$. If M is chosen large enough then $E^\epsilon_m(t) = \frac{M}{2}\|A^\epsilon\|^2_m + \|\rho^\epsilon\|^2_m$ satisfies the following differential inequalities:*

- *For $m = 3$:* $\quad \frac{d}{dt}E^\epsilon_3(t) \lesssim C(M, C_1)(E^\epsilon_3)^{10} + C(A^o, \overline{B}, M).$

- *For $m > 3$:*

$$\frac{d}{dt}E^\epsilon_m(t) \lesssim C\left(M, C_1, |A|_\infty, |\rho|_\infty, |\nabla\rho^\epsilon|_\infty, |\nabla A^\epsilon|_\infty\right)E^\epsilon_m(t) + C(A^o, \overline{B}, M).$$

The proof of this proposition requires a sequence of lemmas. For these lemmas we let A^ϵ and ρ^ϵ be as in Proposition 1.

Lemma 3. *If M is an arbitrary constant then the following holds:*

$$\frac{M}{2}\frac{d}{dt}\|A^\epsilon\|^2_m \lesssim -M\eta\|J^\epsilon\nabla A^\epsilon\|^2_m + \frac{M}{2}\|A^o\|^2_0 + M\left(|\nabla A^\epsilon|_\infty + |\rho^\epsilon|_\infty + |A|_\infty\right)\|A^\epsilon\|^2_m + M\left(|\nabla A^\epsilon|_\infty + |A|_\infty\right)\|\rho^\epsilon\|^2_m. \quad (2.15)$$

Proof. Following standard procedure we first look at the time evolution equation of $\|A\|_m^2$. We drop ϵ for notational simplicity. Recalling the multi-index notation from Section 2.1 and using the chain rule we obtain:

$$\frac{1}{2}\frac{d}{dt}\|A\|_m^2 = \sum_{|\alpha|\leq m}\int (D^\alpha A)(D^\alpha A_t)dx.$$

For fixed α substitute in (2.7a) and obtain:

$$\begin{aligned}\int (D^\alpha A)(D^\alpha A_t)dx &= \int (D^\alpha A)D^\alpha(\eta J_\epsilon^2 \Delta A - A + A\rho + A^o)dx \\ &= -\eta\|J^\epsilon D^\alpha \nabla A\|_0^2 - \|D^\alpha A\|_0^2 + \int (D^\alpha A)(D^\alpha A^o)dx \\ &\quad + \int (D^\alpha A)(D^\alpha(A\rho))dx.\end{aligned}$$

Note that the third term of the last equality will only contribute when $\alpha = \vec{0}$. For now consider the case $\alpha \neq \vec{0}$. The Cauchy-Schwarz inequality gives:

$$\int (D^\alpha A)(D^\alpha A_t)dx \leq -\eta\|J^\epsilon D^\alpha \nabla A\|_0^2 - \|D^\alpha A\|_0^2 + \|D^\alpha A\|_0 \|D^\alpha(A\rho)\|_0 . \quad (2.16)$$

To simplify the computations we first look at the following claim. The derivation can be found in Appendix A.2 and uses part *(1)* of Lemma 2.

Claim 1:

$$\sum_{|\alpha|\leq m}\|D^\alpha u\|_0\|D^\alpha(uv)\|_0 \lesssim (|\nabla u|_\infty + |u|_\infty + |v|_\infty)\|u\|_m^2 + (|\nabla u|_\infty + |u|_\infty)\|v\|_m^2 .$$

Adding (2.16) over $|\alpha| \leq m$:

$$\frac{1}{2}\frac{d}{dt}\|A\|_m^2 \leq -\eta\|J^\epsilon \nabla A\|_m^2 - \|A\|_m^2 + \|A^o\|_0\|A\|_0 + \sum_{|\alpha|\leq m}\|D^\alpha A\|_0\|D^\alpha(A\rho)\|_0 .$$

Applying Cauchy-Schwarz Inequality to $M\|A^o\|_0\|A\|_0$ and *Claim 1* to the summation term gives the final result. □

Since the computations for ρ are more complicated we first look at the advection term.

Lemma 4. *For* $I_\alpha = \int \left\{ D^\alpha (J_\epsilon \nabla \rho^\epsilon) \cdot D^\alpha \left(\frac{\rho^\epsilon}{A^\epsilon} J^\epsilon \nabla A^\epsilon \right) \right\} dx$ *the following estimate holds for any* $0 < \delta < 1$*:*

$$2\sum_{|\alpha|\le m} I_\alpha \lesssim \delta \left\| J_\epsilon \nabla \rho^\epsilon \right\|_m^2 + \frac{(C_1C_2)^2}{\delta} \left\| J_\epsilon \nabla A^\epsilon \right\|_m^2 + \frac{1}{\delta} \left(C_1 \left| \nabla A^\epsilon \right|_\infty \right)^2 \left\| \rho^\epsilon \right\|_m^2$$
$$+ \frac{1}{\delta} \left(C_1 \left| \nabla \rho^\epsilon \right|_\infty + C_1^2 \left| \rho^\epsilon \right|_\infty \left| \nabla A^\epsilon \right|_\infty + \left| \rho^\epsilon \right|_\infty \sum_{k=0}^{m-1} \overline{C}_k C_1^{k+2} \left| \nabla A^\epsilon \right|_\infty^{k+1} \right)^2 \left\| A^\epsilon \right\|_m^2 .$$

The proof can be found in the appendix. We note that the power 10 in the energy inequality for the case when $m = 3$ in Proposition 1 is comes from that fact that we are taking multiple derivatives of $1/A$.

Lemma 5.

$$\frac{1}{2}\frac{d}{dt} \left\| \rho^\epsilon \right\|_m^2 \lesssim (1-\delta) \left\| J^\epsilon \nabla \rho^\epsilon \right\|_m^2 + \frac{1}{2} \left\| \overline{B} \right\|_0^2 + \frac{1}{2} \left\| \rho^\epsilon \right\|_0^2 + \beta_1 \left\| A^\epsilon \right\|_m^2 + \beta_2 \left\| \rho^\epsilon \right\|_m^2 \tag{2.17}$$
$$+ \frac{(C_1C_2)^2}{\delta} \left\| J_\epsilon \nabla A^\epsilon \right\|_m^2 .$$

where,

- $\beta_1 = \left|\nabla \rho\right|_\infty + \left|\rho\right|_\infty + \frac{C_1}{\delta} \left(\left|\nabla \rho^\epsilon\right|_\infty + C_1 \left|\rho^\epsilon\right|_\infty \left|\nabla A^\epsilon\right|_\infty + \left|\rho^\epsilon\right|_\infty \sum_{k=0}^{m-1} \overline{C}_k C_1^{k+1} \left|\nabla A^\epsilon\right|_\infty^{k+1} \right)^2,$
- $\beta_2 = \left|\nabla \rho^\epsilon\right|_\infty + \left|A^\epsilon\right|_\infty + \left|\rho\right|_\infty + \frac{1}{\delta} C_1^2 \left|\nabla A\right|_\infty^2.$

Proof. For fixed α substitute in (2.7b):

$$\int (D^\alpha \rho)(D^\alpha \rho_t)\, dx = \int (D^\alpha \rho) D^\alpha \left(J_\epsilon^2 \Delta \rho - 2 J_\epsilon \nabla \cdot \left(\frac{\rho}{A} J^\epsilon \nabla A \right) - A\rho + \overline{B} \right) dx$$
$$\le - \left\| J^\epsilon D^\alpha \nabla \rho \right\|_0^2 + \left\| D^\alpha \rho \right\|_0 \left\| D^\alpha \overline{B} \right\|_0 + \left\| D^\alpha \rho \right\|_0 \left\| D^\alpha (A\rho) \right\|_0$$
$$+ 2 \underbrace{\int D^\alpha (J_\epsilon \nabla \rho) \cdot D^\alpha \left(\frac{\rho}{A} J^\epsilon \nabla A \right) dx.}_{I_\alpha}$$

Simply using Lemma 4 and *Claim 1* we obtain the final estimate for ρ given by (2.17). ∎

Combining Lemma 3 and Lemma 5 gives the proof of Proposition 1.

Proof. (Proposition 1) Recalling that we have the estimate $|\rho|_\infty \le c\,\|\rho\|_2$ then $|\rho_0(x)|_\infty \le cL_2 =: C_2$. Combine (2.15) and (2.17) by first fixing $\delta < 1$ and then choosing $M > \frac{1}{\eta\delta}(C_1C_2)^2$. In fact, if $\delta_1 = (1-\delta) > 0$ and $\delta_2 = M\eta\delta - (C_1C_2)^2 > 0$ then:

$$\frac{d}{dt}E_m(t)+\delta_1\,\|J_\epsilon\nabla\rho^\epsilon\|_m^2+\delta_2\,\|J_\epsilon\nabla A^\epsilon\|_m^2 \le D_1\,\|A^\epsilon\|_m^2+D_2\,\|\rho^\epsilon\|_m^2+C(A^o,\overline{B},M). \tag{2.18}$$

where,

- $C(A^o,\overline{B},M) = \frac{M}{2}\,\|A^o\|_0^2 + \frac{1}{2}\,\left\|\overline{B}\right\|_0^2,$
- $D_1 = \beta_1 + M\left(|\nabla A^\epsilon|_\infty + |\rho^\epsilon|_\infty + |A|_\infty\right),$
- $D_2 = \beta_2 + M\left(|\nabla A^\epsilon|_\infty + |A|_\infty\right).$

Observe that the coefficients of $\|\rho^\epsilon\|_m^2$ and $\|A^\epsilon\|_m^2$ depend only on $|\nabla A^\epsilon|_\infty$, $|\nabla\rho^\epsilon|_\infty$, $|A^\epsilon|_\infty$, $|\rho^\epsilon|_\infty$, and C_1. From Sobolev embedding estimates we have $|\nabla u|_\infty \le c\,\|u\|_3$; hence, it is natural to first consider the case $m = 3$. This case is useful to get an initial estimate of T from (2.18). Indeed, we obtain the desired result for this case:

$$\frac{d}{dt}E_3(t) \lesssim C\left(M,C_1\right)\left(E_3\right)^{10} + C(A^o,\overline{B},M). \tag{2.19}$$

The power ten on E_3 in (2.19) comes from Lemma 4. Fortunately, this estimate is independent of the regularizing parameter ϵ. Hence, there exists a positive time, T, such that the H^3-norms of A and ρ are bounded on $[0,T]$. Considering the case where $m > 3$ gives the second desired inequality:

$$\frac{d}{dt}E_m(t) + \delta_1\,\|J_\epsilon\nabla\rho^\epsilon\|_m^2 + \delta_2\,\|J_\epsilon\nabla A^\epsilon\|_m^2 \lesssim CE_m(t) + C(A^o,\overline{B},M), \tag{2.20}$$

with $C = C\left(M, C_1, |A|_\infty, |\rho|_\infty, |\nabla\rho^\epsilon|_\infty, |\nabla A^\epsilon|_\infty\right)$. □

Remark 1. *Note that for the above argument we needed* $|\rho|_\infty < C_2$. *Due to the Sobolev Embedding Theorem the* L_∞*-norm is controlled by the* H^2*-norm. From Theorem 4 each* $\epsilon > 0$ *we know that* $\|\rho^\epsilon\|_2 < L_2$ *for* $t \in [0, T_\epsilon)$. *However, we know that* $[0, T] \subset [0, T_\epsilon)$.

The bound on the higher-norms of the regularized solutions prove to be extremely useful in multiple ways. To begin with, all higher-order norms are bounded on $[0, T]$. Moreover, we know that there exists some $\tau > 0$ such that $A^\epsilon(x, t) \geq A^o$ for all $(x, t) \in \Omega \times [0, \tau]$ if $A^\epsilon(x, 0) > A^o$. Indeed, if we define $A^\epsilon_* = \min_{x\in\Omega} A^\epsilon(x, t)$ then we have a point-wise bound on its time derivative thanks Proposition 1. In fact, we know that:

$$\begin{aligned} \left|\frac{dA^\epsilon_*}{dt}\right| &\leq \eta \left|\Delta A^\epsilon\right|_\infty + \left|A^\epsilon + A^\epsilon\rho^\epsilon + A^o\right|_\infty \\ &\leq \eta \left\|\Delta A^\epsilon\right\|_2 + \left\|A^\epsilon + A^\epsilon\rho^\epsilon + A^o\right\|_2, \end{aligned}$$

where we need $A^\epsilon \in H^m$ for $m > 4$ to use *1* of Lemma 1 and then Sobolev embedding estimates. Since $\|A^\epsilon\|_4$ is bounded independent of ϵ then $A^\epsilon_* > A^o$ on $[0, \tau]$ for some $\tau \in [0, T]$. For simplicity let $T = \min\{T, \tau\}$, from now on we interval $[0, T]$ to be the interval on which the higher-order norms are bounded and $A^\epsilon \geq A^o$. Now that we have a non-trivial interval on which all the higher-order norms are bounded we show that the family of solutions to the regularized system (2.7), $\{(A^\epsilon, \rho^\epsilon)\}_{\epsilon>0}$, form a Cauchy sequence in the L^2-norm. This enables us to obtain the necessary limiting functions A, ρ, which are a solutions to (1.1).

Lemma 6. *The family of solutions* $\{(A^\epsilon, \rho^\epsilon)\}_{\epsilon>0}$ *to* (2.7) *form a Cauchy sequence in* $C([0, T]; L^2(\Omega) \times L^2(\Omega))$. *In particular, there exists a constant* C *and a time* $T > 0$ *such that for all* ϵ *and* ϵ'

$$\sup_{0\leq t\leq T} \left\{ \left\|A^\epsilon - A^{\epsilon'}\right\|_0 + \left\|\rho^\epsilon - \rho^{\epsilon'}\right\|_0 \right\} \leq C \max(\epsilon, \epsilon').$$

Proof. Let $(A^\epsilon, \rho^\epsilon)$ and $(A^{\epsilon'}, \rho^{\epsilon'})$ solve their respective regularized systems (2.7) and satisfy the conditions of the lemma. Take the inner-product of $A^\epsilon - A^{\epsilon'}$ and $A^\epsilon_t - A^{\epsilon'}_t$.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|A^\epsilon - A^{\epsilon'}\right\|_0^2 &= \int \left(A^\epsilon - A^{\epsilon'}\right)\left(A^\epsilon_t - A^{\epsilon'}_t\right) dx \\ &= \int \left(A^\epsilon - A^{\epsilon'}\right)\left(\eta J_\epsilon^2 \Delta A^\epsilon - \eta J_\epsilon^2 \Delta A^{\epsilon'}\right) dx - \left\|A^\epsilon - A^{\epsilon'}\right\|_0^2 \\ &\quad + \int \left(A^\epsilon - A^{\epsilon'}\right)\left(A^\epsilon \rho^\epsilon - A^{\epsilon'}\rho^{\epsilon'}\right) dx = I_1 + I_2 + I_3.\end{aligned}$$

Since I_2 has a negative sign it is not problematic. The other two terms can be easily dealt with using *(4)* of Lemma 1.

$$\begin{aligned}I_1 &= -\eta\left\|J_{\epsilon'}\nabla(A^\epsilon - A^{\epsilon'})\right\|_0^2 + \eta\int (J_\epsilon^2 - J_{\epsilon'}^2)\Delta A^\epsilon (A^\epsilon - A^{\epsilon'}) dx \\ &\le -\eta\left\|J_{\epsilon'}\nabla(A^\epsilon - A^{\epsilon'})\right\|_0^2 + \eta \max(\epsilon, \epsilon')\left\|A\right\|_3 \left\|A^\epsilon - A^{\epsilon'}\right\|_0.\end{aligned}$$

For the last term,

$$\begin{aligned}I_3 &= \int \rho^\epsilon (A^\epsilon - A^{\epsilon'})^2 dx + \int A^{\epsilon'}\left(A^\epsilon - A^{\epsilon'}\right)\left(\rho^\epsilon - \rho^{\epsilon'}\right) dx \\ &\le \left(\left|\rho^\epsilon\right|_\infty + \frac{1}{2}\left|A^\epsilon\right|_\infty\right)\left\|A^\epsilon - A^{\epsilon'}\right\|_0^2 + \frac{1}{2}\left|A^\epsilon\right|_\infty \left\|\rho^\epsilon - \rho^{\epsilon'}\right\|_0^2.\end{aligned}$$

Combine these inequalities and return to the initial estimate to obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|A^\epsilon - A^{\epsilon'}\right\|_0^2 \le \left(\left|\rho^\epsilon\right|_\infty + \frac{1}{2}\left|A^\epsilon\right|_\infty - 1\right)\left\|A^\epsilon - A^{\epsilon'}\right\|_0^2 + \eta\max(\epsilon, \epsilon')\left\|A\right\|_3\left\|A^\epsilon - A^{\epsilon'}\right\|_0 \\ + \frac{1}{2}\left|A^\epsilon\right|_\infty\left\|\rho^\epsilon - \rho^{\epsilon'}\right\|_0^2. \qquad (2.21)\end{aligned}$$

Perform a similar computation for ρ:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\rho^\epsilon - \rho^{\epsilon'}\right\|_0^2 &= \int\left(\rho^\epsilon - \rho^{\epsilon'}\right)\left(\rho^\epsilon_t - \rho^{\epsilon'}_t\right) dx \\ &= \int\left(\rho^\epsilon - \rho^{\epsilon'}\right)\left(J_\epsilon^2 \Delta\rho^\epsilon - J_\epsilon^2 \Delta\rho^{\epsilon'}\right) dx + \int\left(\rho^\epsilon - \rho^{\epsilon'}\right)\left(A^\epsilon\rho^\epsilon - A^{\epsilon'}\rho^{\epsilon'}\right) dx \\ &\quad + \int\left(\rho^\epsilon - \rho^{\epsilon'}\right)\left(J_\epsilon\left(\nabla\cdot\frac{\rho^\epsilon}{A^\epsilon}J_\epsilon\nabla A^\epsilon\right) - J_{\epsilon'}\left(\nabla\cdot\frac{\rho^{\epsilon'}}{A^{\epsilon'}}J_{\epsilon'}\nabla A^{\epsilon'}\right)\right) dx \\ &= F_1 + F_2 + F_3.\end{aligned}$$

The terms F_1 and F_2 are dealt with exactly as was done for the attractiveness value. F_3 is not as straight forward but it can be simplified using Cauchy-Schwarz inequality:

$$F_3 \leq \left(\left\| J_\epsilon \left(\nabla \cdot \frac{\rho^\epsilon}{A^\epsilon} J_\epsilon A^\epsilon \right) \right\|_0 + \left\| J_{\epsilon'} \left(\nabla \cdot \frac{\rho^{\epsilon'}}{A^{\epsilon'}} J_{\epsilon'} \nabla A^{\epsilon'} \right) \right\|_0 \right) \left\| \rho^\epsilon - \rho^{\epsilon'} \right\|_0 .$$

We can extract an ϵ at the expense of a higher-order norm and the loss of a mollifier. For example we have:

$$\begin{aligned} \left\| J_\epsilon \left(\nabla \cdot \frac{\rho^\epsilon}{A^\epsilon} J_\epsilon \nabla A^\epsilon \right) \right\|_0 &\leq \epsilon \left\| \frac{\rho^\epsilon}{A^\epsilon} J_\epsilon \nabla A^\epsilon \right\|_2 \\ &\lesssim \epsilon \left\{ \left| \frac{\rho^\epsilon}{A^\epsilon} \right|_\infty \left\| D^2 \nabla A^\epsilon \right\|_0 + |\nabla A^\epsilon|_\infty \left\| D^2 \left(\frac{\rho^\epsilon}{A^\epsilon} \right) \right\|_0 \right\} . \end{aligned}$$

From the proof of Lemma 4, refer to the inequality (B.3), the above inequality has a bound that depends only on $\|\rho^\epsilon\|_2$, $\|A^\epsilon\|_3$, and C_1. Define $v^2 = \left\| A^\epsilon - A^{\epsilon'} \right\|_0^2 + \left\| \rho^\epsilon - \rho^{\epsilon'} \right\|_0^2$. Since $\|A^\epsilon\|_3$ and $\|\rho^\epsilon\|_2$ are bounded on $[0, T]$ then we have the following differential inequality:

$$\frac{d}{dt} v \lesssim C(\max(\epsilon, \epsilon') + v).$$

Notice that the constant depends on C_1, $\|\rho^\epsilon\|_2$ and $\|A^\epsilon\|_3$. The above differential inequality gives $v(t) \leq e^{Ct} \left(v(0) + \max(\epsilon, \epsilon') \right) - \max(\epsilon, \epsilon')$. Since $(A^\epsilon, \rho^\epsilon)$ and $(A^{\epsilon'}, \rho^{\epsilon'})$ satisfy the same initial conditions we have that $v(0) = 0$, which implies:

$$\sup_{0 \leq t < T} v(t) \leq C \max(\epsilon, \epsilon').$$

□

2.1.3.2 Existence and Uniqueness of Solutions to the Original Residential Burglary Model

We have all the tools to prove Theorem (1); however, we first state and prove the result for uniqueness of solutions. More precisely, if we assume that we have existence of a smooth enough solution to (1.1) then this solution must be unique.

Lemma 7 (Uniqueness of Smooth Solutions)**.** *Let* $(A_1, \rho_1), (A_2, \rho_2)$ *be local-in-time solutions, with a common interval of existence* $[0, T]$*, to the system* (1.1)*. Furthermore, suppose these solutions are smooth enough and with the same initial data in* V^m*, for* $m \geq 3$*, which satisfy the conditions stated in Theorem 1 then* $A_1 = A_2$ *and* $\rho_1 = \rho_2$ *on* $[0, T]$*.*

Proof. We consider the difference of both variables $u = A_1 - A_2$ and $v = \rho_1 - \rho_2$. From (1.1) we can see that u and v satisfy the following system:

$$u_t = \eta \Delta u - u + \rho_1 u + A_2 v, \tag{2.22a}$$

$$v_t = \Delta v - 2\nabla \cdot \left(\frac{\rho_1}{A_1} \nabla A_1 - \frac{\rho_2}{A_2} \nabla A_2 \right) - \rho_1 u - A_2 v. \tag{2.22b}$$

The time evolution of the L^2-norm of u multiplied by a constant M (the same M used in Lemma 1) satisfies the following inequality:

$$\frac{d}{dt} \frac{M}{2} \|u\|_0^2 \leq -M\eta \|\nabla u\|_0^2 + M \left(|\rho_1|_\infty + \frac{1}{2} |A_2|_\infty - 1 \right) \|u\|_0^2 + \frac{M}{2} |A_2|_\infty \|v\|_0^2 . \tag{2.23}$$

The above inequality can be seen simply by taking the L^2-inner product of u_t and u. Substituting (2.22a) for u_t into this inner product and integrating by parts gives (2.23). The same is done for v. The following inequality holds:

$$\frac{d}{dt} \frac{1}{2} \|v\|_0^2 \lesssim C_1^2 C_2^2 \|\nabla u\|_0^2 + C(|\rho_1|_\infty, |\nabla A_2|_\infty) \|u\|_0^2 + C(|\rho_1|_\infty, |\nabla A_2|_\infty, |A_1|_\infty) \|v\|_0^2 . \tag{2.24}$$

For detailed computations of the upper bound given by (2.24) refer to Appendix A.3. Define $F(t) = \frac{M}{2} \|u(t)\|_0^2 + \frac{1}{2} \|v(t)\|_0^2$, again choosing $M > \frac{1}{\eta}(C_1 C_2)^2$ then from (2.23) and (2.24) we see that $F(t)$ satisfies the following ode:

$$\frac{dF(t)}{dt} \leq C_M F(t). \tag{2.25}$$

In (2.25) the constant $C_M = C_M(M, |\rho_1|_\infty, |A_1|_\infty, |A_2|_\infty, |\nabla A_1|_\infty, C_1)$. We are set to apply a Grönwall's lemma [82]. Applying this lemma to (2.25) gives that $\sup_{0\le t\le T}\{F(t)\} \le F(0)e^{C_M T}$. All terms that compose C_M are bounded on the interval $[0,T]$. Since the two solutions satisfy the same initial conditions then $F(0) = 0$, which implies uniqueness of the solution.

□

We now progress to the proof of the main result.

Proof. (Theorem 1) From Theorem 4 we have that given the initial conditions in the hypothesis of Theorem 1, there exists a family of solutions $\{(A^\epsilon, \rho^\epsilon)\}_{\epsilon>0}$ to the regularized problem (2.7). These solutions exist on the time interval $[0, T_\epsilon)$. The interval of existence depends on the regularizing parameter; however, from Lemma 1 we know that the V^2-norm of the solutions are bounded independent of ϵ. This gives a uniform interval of existence $[0,T]$. Furthermore, from Lemma 6 we conclude that there exist $A,\ \rho \in C([0,T]; L^2(\Omega))$ such that:

$$\sup_{0\le t\le T}\{\|A^\epsilon - A\|_0 + \|\rho^\epsilon - \rho\|_0\} \le C\epsilon.$$

Therefore, the solutions converge strongly in the low-norm. We state an interpolation lemma needed to show strong convergence in intermediate norms. This lemma offers a connection between Lemma 1 and Lemma 6 which leads to the desired result.

Lemma 8 (Interpolation in Sobolev Spaces)**.** *Given $s \ge 0$, there exists a constant C_s so that for all $v \in H^s(\Omega)$, and $0 < s' < s$ the following inequality holds:*

$$\|v\|_{s'} \le \|v\|_0^{1-s'/s} \|v\|_s^{s'/s}.$$

To use Lemma 8 having strong convergence in the L^2-norm and some bounds on the higher norms is essential. For $m > 3$ we apply the above lemma to $\overline{A} = A^\epsilon - A$

and $\overline{\rho} = \rho^\epsilon - \rho$.

$$\sup_{0\le t\le T}\left\{\|\overline{A}\|_{m'} + \|\overline{\rho}\|_{m'}\right\} \lesssim \left(\|\overline{A}\|_0^{1-m'/m}\|\overline{A}\|_m^{m'/m} + \|\overline{\rho}\|_0^{1-m'/m}\|\overline{\rho}\|_m^{m'/m}\right)$$
$$\lesssim \left(\|\overline{A}\|_m^{m'/m}\epsilon^{1-m'/m} + \|\overline{\rho}\|_m^{m'/m}\epsilon^{1-m'/m}\right).$$

The estimate (2.20) implies that A^ϵ, ρ^ϵ are uniformly bounded in H^m, for $m \geq 2$. Therefore, the above inequality implies strong convergence in $C([0,T], V^{m'})$. Taking m' to be larger than three implies strong convergence in $C([0,T], C^2(\Omega))$ due to the Sobolev Embedding Theorem [46]. Now, we simply need to verify that the limits A and ρ actually satisfy (1.1). Since $(A^\epsilon, \rho^\epsilon) \to (A, \rho)$ from (2.7) we see that A_t^ϵ converges to $\eta\Delta A - A + A\rho + A^o$ in $C([0,T], C(\Omega))$. Correspondingly, ρ_t^ϵ converge to $\nabla\cdot\left[\nabla\rho - 2\frac{\rho}{A}\nabla A\right] + \overline{B} - A\rho$. Finally, since $A_t^\epsilon \to A_t$ and $\rho_t^\epsilon \to \rho_t$ then A and ρ are classical solutions of (1.1). Since the solutions satisfy the smoothness requirements of Lemma 7 they are unique. □

2.2 Continuation of the Solutions to the Residential Burglary Model

In the previous section we proved that if the initial data $(A(0,x), \rho(0,x)) \in V^m$ then there exists some positive time T, such that there exists a classical solution $(A(x,t), \rho(x,t))$ to (1.1) on $[0,T]$. We are interested in whether this solution can be continued for all time or if there exists a blow-up in finite time. A natural subsequent step is to prove a continuation argument which gives necessary and sufficient conditions for global existence. Recall that we used the Picard Theorem on a Banach Space to prove local existence, for fixed ϵ, to the regularized system (2.7) in Lemma 4. This theorem has a natural continuation argument. The family of solutions can be extended in time provided $|1/A^\epsilon|_\infty$, $\|A^\epsilon\|_m$, and $\|\rho^\epsilon\|_m$ remain bounded [82]. This argument does not directly apply to the solution of the original system and to prove a similar result we need the following theorem.

Theorem 5 (Continuity in the High Norms)**.** *Given initial conditions* $(A_0, \rho_0) \in V^m$*, for* $m > 3$*, which satisfy the conditions stated in Theorem 1. Let* $\{(A^\epsilon, \rho^\epsilon)\}_{\epsilon>0}$ *be the family of solutions to* (2.7) *and* (A, ρ) *be the solution described in Theorem 1. The following hold:*

1. $\{(A^\epsilon, \rho^\epsilon)\}_{\epsilon>0}$ *and* (A, ρ) *are uniformly bounded in* $C^{weak}([0, T]; V^m)$*.*
2. $(A, \rho) \in C([0, T]; V^m) \cap C^1([0, T]; V^{m-2})$*.*

Proof. From Lemma 1 we conclude that:

$$\sup_{0 \leq t \leq T} \|(A^\epsilon, \rho^\epsilon)\|_{V^m} \leq K. \tag{2.26}$$

Furthermore, automatically from (1.1):

$$\sup_{0 \leq t \leq T} \left\| \frac{\partial}{\partial t}(A^\epsilon, \rho^\epsilon) \right\|_{V^{m-2}} \leq \widetilde{K}. \tag{2.27}$$

We need to show that the limiting solution is continuous in the weak topology of $V^m(\Omega)$. From **Definition 1** in Section 2 it suffices to show that $(A, \phi_1)_m$ and $(\rho, \phi_2)_m$, where these inner-products are defined by (2.4), are continuous scalar functions $\forall\ \phi_1, \phi_2 \in H^m$. Actually, since H^{-m} is the dual of H^m we simply need to prove that for all $\psi \in H^{-m}$ the following is true: $(\psi, A^\epsilon)_{L^2} \rightharpoonup (\psi, A)_{L^2}$. The same needs to hold for ρ. Previously we proved that $A^\epsilon \to A$ in the intermediate norms, *i.e.* in $C([0, T); H^{m'})$, where $m' < m$. This implies that $A^\epsilon \rightharpoonup A$. Consider the L^2-inner product of $\psi \in H^{-m}$ and $A^\epsilon - A$:

$$(\psi, A^\epsilon - A)_{L^2} = (\psi - \phi_j, A^\epsilon)_{L^2} + (\phi_j, A^\epsilon - A)_{L^2} + (\phi_j - \psi, A)_{L^2}, \tag{2.28}$$

where $\{\phi_j\}_{j \in \mathbb{N}}$ is a sequence in $H^{-m'}$ which converges strongly in H^{-m} to ψ. Such a sequence exists because $H^{-m'}$ is dense in H^{-m}. These terms are bounded above on $[0,T]$:

$$(\psi-\phi_j, A^\epsilon)_{L^2} \leq \|\psi-\phi_j\|_{-m}\|A^\epsilon\|_m \leq K\delta/3,$$
$$(\phi_j, A^\epsilon - A)_{L^2} \leq \|\phi_j\|_{-m'}\|A^\epsilon - A\|_{m'} \leq K_2\delta/3,$$
$$(\phi_j-\psi, A)_{L^2} \leq \|\psi-\phi_j\|_{-m}\|A\|_m \leq K\delta/3.$$

These inequalities substituted into (2.28) gives that $(\psi, A^\epsilon - A)_{L^2} \to 0$. The same argument can be made for ρ and this wraps up the proof of part *1*.

We are left to prove that $(A,\rho) \in C([0,T];V^m(\Omega)) \cap C^1([0,T];V^{m-2}(\Omega))$. Thanks to part *1* it suffices to show that $\|A(t)\|_m$ and $\|\rho(t)\|_m$ are continuous functions in time. We take advantage of (2.20) by integrating it on the interval $[0,T]$:

$$E_m(T) + \delta_1\int_0^T \|J_\epsilon \nabla\rho^\epsilon\|_m^2\,dt + \delta_2\int_0^T \|J_\epsilon\nabla A^\epsilon\|_m^2\,dt \lesssim E_m(0) + \int_0^T \{CE_m(t) + D_0\}\,dt.$$

Applying Grönwall's Lemma we obtain that $E_m(T) \leq (E_m(0) - D_0/C)e^{CT} + D_0/C$. Taking the limit as $T \to 0^+$ we see that $E_m(t)$ is continuous at $t = 0^+$. Furthermore, being that $E_m(t)$ is bounded on $[0,T]$ and $\delta_1, \delta_2 > 0$ the inequality above implies that $(A,\rho) \in L^2([0,T];V^{m+1}(\Omega))$. Thus, for *a.e* $t_0 \in [0,T]$ then $(A(t_0),\rho(t_0)) \in V^{m+1}$. Indeed, the initial conditions have gained regularity. Take an arbitrarily small t_0 and let $(A(t_0),\rho(t_0))$ to be a new set of initial conditions. Running through the same existence and uniqueness arguments we obtain a solution (A,ρ) which exist on an interval $[t_0,T_1]$, $(A,\rho) \in C([t_0,T_1];V^{m'})$, where now $m' < m+1$. In view of the fact that for $m > 3$, E_m and E_{m+1} satisfy the same differential inequality then $T_1 \geq T$. Uniqueness and the arbitrary choice of t_0 implies that $(A,\rho) \in C([0,T];V^m)$. Furthermore, by virtue of the equation then $(A,\rho) \in C^1([0,T];V^{m-2})$. □

Remark 2. *From* (2.20) *we know we have control of the* V^{m+1} *norm as long as* *we have control* $|A(t_0)|_\infty$, $|\rho(t_0)|_\infty$, $|\nabla A(t_0)|_\infty$, $|\nabla\rho(t_0)|_\infty$ *and* M. *Furthermore, control of* $|1/A(t_0)|_\infty$ *implies control of* M.

Fortunately, we find that the terms mentioned in Remark 2 are interdependent and we can obtain a dominating term. However, before we discuss this we state and prove a regularity argument.

Theorem 6 (Regularity)**.** *The solutions* A, ρ *of the system* (1.1) *obtained from Theorem 1 are in the space* $C^\infty((0,T)\times\Omega)$.

Proof. Since $(A,\rho) \in C([0,T];V^m) \cap C^1([0,T];V^{m-2})$ from Sobolev embedding estimates $(A,\rho) \in C([0,T];C^{m-s}) \cap C^1([0,T];C^{m-2-s})$ for $s > 1$. This will give us smoothness in space. To obtain smoothness in time we simply look at the time-derivates of the system of equations (1.1) and use a bootstrap argument. □

Next we show that if the appropriate initial and boundary data are chosen for A then only control of $|\nabla\rho(t_0)|_\infty$ is needed to continue the solution. We prove this in the following sequence of lemmas. The first one states that $|\nabla\rho|_\infty$ and $|\nabla A|_\infty$ controls $|\rho|_\infty$ and $|A|_\infty$ respectively. This holds because there is a bound for the mass of ρ and A on any finite time interval.

Lemma 9. *Let* A *and* ρ *be solutions from Theorem 1 with initial conditions* $A_0(x)$ *and* $\rho_0(x)$, *for* $1 \le p \le \infty$ *the following estimate holds for* A *and* ρ *on* $[0,T]$ *for any* $T > 0$:

$$\|u(\cdot,t)\|_{L^p} \le c\,\|\nabla u\|_{L^p} + (\overline{B} + A^o)T, \tag{2.29}$$

for all $t \in [0,T]$.

Proof. Adding both equations in the system (1.1) we obtain that $\int \rho(x,t)dx \le \left(\overline{B} + A^o\right)t$. The same estimate holds for A. Since Ω is the unit torus the average value of a function u is given by $\overline{u} = \int u dx$. Now, by Poincaré inequality $\|u\|_{L^p} \leq c\|\nabla u\|_{L^p} + \|\overline{u}\|_{L^p}$. This gives the final result.

□

Furthermore, since there is a max principle for the attractiveness value equation we prove that if $A(x,0) > A^o \neq 0$ for all x then $A(x,t) \geq A^o$ during the interval of existence. We state this result formally in following lemma.

Lemma 10 (Lower-Bound of Attractiveness Value)**.** *Let* $\Omega = \mathbb{T}^2$ *and* $A,\ \rho \in C([0,T];C^2(\Omega)) \cap C^1([0,T];C(\Omega))$ *be a solutions to* (1.1) *with initial conditions:*

$$A(x,0) = A_0(x) > A^o,$$
$$\rho(x,0) = \rho_0(x).$$

Then $A(x,t) \geq A^o$ *in* Ω *for all* $t \in [0,T]$.

Proof. We see directly from (1.1) that $A, \rho \geq 0$. Let $w = A^o - A$ then w satisfies: $w_t = \eta\Delta w - w + w\rho - A^o\rho$. Since both ρ and A^o are nonnegative then we have:

$$w_t - \eta\Delta w - \lambda w \leq 0, \tag{2.30}$$

where $\lambda = \sup_{0\leq t\leq T} |\rho(\cdot,t) - 1|_\infty$. Then $w = e^{\lambda t}v$ satisfies (2.30) if v satisfies $v_t - \eta\Delta v \leq 0$. From the initial data we know that $w(x,0) < 0$ for all $x \in \Omega$ and the same is true for v. By continuity in time v must remain nonnegative for some nontrivial time interval say $0 < t < t_0$. Assume that at t_0 we have that $v(x_0,t_0) = 0$ for some x_0. This means that $v_t(x_0,t_0) \geq 0$ and since we have a maximum then $-\Delta v(x_0,t_0) \geq 0$ which is a contradiction unless $v(x,t_0) = 0$ for all $x \in \Omega$. Therefore, $v(x,t) \leq 0$ and since w and v have the same sign then $w(x,t) \leq 0$. This proves the result.

□

Lemma 10 tells us that if $|\rho|_\infty$ is bounded then $A > A^o$, provided we have appropriate initial and boundary data. We also need for the solutions to the regularized model to remain bounded from below. However, we know that this is true on $[0,T]$ as was discussed earlier. In addition, we prove that if $|\nabla\rho|_\infty$ remains bounded then $|\nabla A|_\infty$ also remains bounded. This will be demonstrated in the following two lemmas.

Lemma 11. *Let (A,ρ) satisfy (1.1) in the classical sense and assume that $\|\nabla\rho\|_\infty$ is bounded on $[0,T]$, for $T>0$ then*

$$\|\nabla A(\cdot,t)\|_{L^2}^2 \le \left(\|\nabla A(\cdot,0)\|_{L^2}^2 - \tilde{C}\right) e^{C(\eta,|\nabla\rho|_\infty,A^o,\overline{B})T} + \tilde{C}, \tag{2.31}$$

where $\tilde{C} = \tilde{C}(\eta,|\nabla\rho|_\infty, A^o, \overline{B})$. This holds $\forall t \in [0,T]$.

Proof.

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |\nabla A|^2\,dx &= \int \nabla A\cdot\nabla A_t dx \\
&\overset{(1.1)}{=} \int \nabla A\cdot\nabla\left(\eta\Delta A - A + A\rho + A^o\right)dx \\
&= -\eta\int |\Delta A|^2\,dx - \int |\nabla A|^2\,dx + \int \nabla A\cdot\nabla(A\rho)dx \\
&= -\eta\int |\Delta A|^2\,dx - \int |\nabla A|^2\,dx + \int |\nabla A|^2\rho dx + \int A\nabla A\cdot\nabla\rho dx \\
&\overset{\text{Cauchy-Schwarz}}{\le} -\eta\|\Delta A\|_{L^2}^2 + (|\rho|_\infty - 1)\|\nabla A\|_{L^2}^2 + |\nabla\rho|_\infty\left(\|A\|_{L^2}^2 + \|\nabla A\|_{L^2}^2\right) \\
&\overset{(2.29)}{\le} C(\eta,|\rho|_\infty,|\nabla\rho|_\infty)\|\nabla A\|_{L^2}^2 + C(|\nabla\rho|_\infty, A^o,\overline{B},T).
\end{aligned}$$

Integrating this and using (2.29) for $p=\infty$ gives the desired result (2.31). □

Lemma 12. *Let (A,ρ) satisfy (1.1) in the classical sense and assume that $|\nabla\rho|_\infty$ is bounded on $[0,T]$, for $T>0$ then*

$$|\nabla A(\cdot,t)|_\infty \le c_4\max\{|\nabla A(\cdot,0)|_\infty, \left(\|\nabla A(\cdot,0)\|_{L^2}^2 - \tilde{C}\right)e^{CT} + \tilde{C}\} \tag{2.32}$$

$\forall t\in[0,T]$. The constants C and $\tilde{C}$ are defined as in Lemma 11.

The proof of Lemma 12 uses the Moser-Alikakos iteration [2].

Proof. Let $s \geq 2$:

$$\begin{aligned}
\frac{1}{s}\frac{d}{dt}\int |\nabla A|^s \, dx &= \int |\nabla A|^{s-1} \nabla A_t dx \\
\overset{(1.1)}{=} & \int |\nabla A|^{s-1} \nabla \left(\eta \Delta A - A + A\rho + A^o\right) dx \\
&\leq -\eta(s-1)\int |\nabla A|^{s-2} |\Delta A|^2 \, dx - \int |\nabla A|^s \, dx + \int |\nabla A|^s \rho dx \\
&\quad + \int A \, |\nabla A|^{s-1} |\nabla \rho| \, dx \\
\overset{\text{Hölder's Ineq.}}{\leq} & -\frac{4\eta(s-1)}{s^2}\left\| \nabla \left(|\nabla A|^{s/2} \right) \right\|_{L^2}^2 + |\nabla \rho|_\infty \|\nabla A\|_{L^s}^{s-1} \|A\|_{L^2} \\
&\quad + \left(|\rho|_\infty - 1\right) \|\nabla A\|_{L^s}^s \\
\overset{(2.29)}{=} & -\frac{4\eta(s-1)}{s^2}\left\| \nabla \left(|\nabla A|^{s/2} \right) \right\|_{L^2}^2 + c_1 \|\nabla A\|_{L^s}^s + c_2,
\end{aligned}$$

where $c_1 = c_1(|\nabla \rho|_\infty, |\rho|_\infty, A^o, \overline{B})$ and $c_2 = c_2(A^o, \overline{B})$. Multiplying both sides by s, $s \geq 2$ gives:

$$\frac{d}{dt}\int |\nabla A|^s \, dx \leq -2\eta \left\| \nabla \left(|\nabla A|^{s/2} \right) \right\|_{L^2}^2 + sc_1 \|\nabla A\|_{L^s}^s + sc_2.$$

We need to make use of an extended Sobolev inequality:[49]

$$- \|\nabla u\|_{L^2}^2 \leq -\frac{(1-\epsilon)}{\epsilon} \|u\|_{L^2}^2 + \frac{\overline{c}}{\epsilon^2} \|u\|_{L^1}^2 . \tag{2.33}$$

A derivation of (2.33) can be found in Appendix A.4. Taking $u = |\nabla A|^{s/2}$ gives:

$$\frac{d}{dt}\int |\nabla A|^s \, dx \leq -\frac{2\eta(1-\epsilon)}{\epsilon} \|\nabla A\|_s^s + \frac{c_0}{\epsilon^2} \|\nabla A\|_{L^{s/2}}^s + c_1 s \|\nabla A\|_{L^s}^s + sc_2,$$

Choose $\epsilon = \frac{\eta}{sc_1 + \eta}$ noting that $s > \eta \in [0,1]$ (refer to [92]) then:

$$\frac{d}{dt}\int |\nabla A|_{L^s}^s \, dx \leq -c_1 s \|\nabla A\|_{L^s}^s + c_3 s^2 \|\nabla A\|_{L^{s/2}}^s + sc_2.$$

By multiplying both sides by $e^{c_1 st}$ the above inequality is equivalent to

$$\frac{d}{dt}\left\{ e^{c_1 st} \|\nabla A\|^s \right\} \leq e^{c_1 st}\left(c_3 s^2 \|\nabla A\|_{L^{s/2}}^2 + c_2 s \right).$$

Integrating this over $[0,t]$ gives:

$$e^{c_1 st}\|\nabla A(\cdot,t)\|_{L^s}^s \leq \|\nabla A(\cdot,0)\|_{L^s}^s + \sup_{0\leq\tau\leq t}\|\nabla A(\cdot,\tau)\|_{L^{s/2}}^s \int_0^t c_3 s^2 e^{c_1 s\tau}d\tau + \int_0^t c_2 e^{c_1 s\tau} s d\tau$$
$$\leq \|\nabla A(\cdot,0)\|_{L^s}^s + \sup_{0\leq\tau\leq t}\|\nabla A(\cdot,\tau)\|_{L^{s/2}}^s c_4 s(e^{c_1 st}-1) + c_5(e^{c_1 st}-1).$$

Therefore,

$$\|\nabla A(\cdot,t)\|_{L^s}^s \leq (|\nabla A(\cdot,0)|_\infty + c_6)^s + c_4 s \sup_{0\leq\tau\leq t}\|\nabla A(\cdot,t)\|_{L^{s/2}}^s, \tag{2.34}$$

where $c_6 = \max\{1, c_5\}$.

Define $M(s) = \max\{|\nabla A(\cdot,0)|_\infty + c_6, \sup_{0\leq t\leq T}\|\nabla A(\cdot,t)\|_{L^s}\}$. From (2.34) we conclude that

$$M(s) \leq (c_7 s)^{1/s} M(s/2). \tag{2.35}$$

Let $s = 2^k$ for $k \in \mathbb{N}$ the recursive relation (2.35) gives:

$$M(2^k) \leq (c_7)^{\sum_{j=1}^k 2^{-j}} (2)^{\sum_{j=1}^k j2^{-j}} M(1).$$

Since both sums $\sum_{j=1}^k 2^{-j}$ and $\sum_{j=1}^k j2^{-j}$ converge as $k\to\infty$ taking the limit as $s\to\infty$ we get:

$$|\nabla A(\cdot,T)|_\infty \leq \lim_{s\to\infty} M(s)$$
$$\leq (c_7)^{\sum_{j=1}^\infty 2^{-j}} (2)^{\sum_{j=1}^\infty j2^{-j}} M(1),$$

Applying Lemma 11 gives the final result.

□

From Theorem 5 and Lemma 10-12 proved above we obtain necessary and sufficient conditions for the continuation of the solution to (1.1).

Corollary 1. *Given initial conditions* $(A(x,0), \rho(x,0)) \in V^m$*,* $m \geq 4$ *such that* $A(x,0) > A^o$ *and 'no-flux' boundary conditions, there exist a maximal time of existence* $0 < T_{max} \leq \infty$ *and a unique solution* $(A(x,t), \rho(x,t)) \in C([0,T_{max}); V^m) \cap C^1([0,T_{max}); V^{m-2})$ *the the system* (1.1)*. Furthermore, if* T_{max} *is finite then* $\lim_{t \to T_{max}} |\nabla \rho|_\infty = \infty$.

CHAPTER 3

Modified Residential Burglaries Models

Though we succeeded in proving local existence and uniqueness of solutions to (1.1) the question of whether the solutions can be extended for all time has not been addressed. To be confident that we have a robust model, suitable for the target application, we need insight on global existence and/or possible finite time blow-up. Working with a strongly coupled system of nonlinear PDEs makes it difficult to apply the usual techniques to prove well-posedness. In this section, we take advantage from the relation of the original model (1.1) to a well-studied model of chemotaxis, known as the Keller-Segel model, to obtain insight into the global theory. Indeed, we see that from the work done in this model that we expect that the logarithmic velocity field will be necessary if we want all solutions, regardless of initial mass, to be global in time.

Outline: This chapter is divided into three sections. In §3.1 we discuss the Keller-Segel model for chemotaxis and its connection to the system (1.1). Motivated by this connection we considered alternate crime pattern formation models in §3.2. In particular, we study the behavior of solutions to a system with logarithmic velocity field vs. linear velocity field. Finally, in §3.3 we prove blow up of a modified parabolic-elliptic model.

3.1 Keller-Segel Model for Chemotaxis

There is an evident relation between the model for residential burglaries and the Keller-Segel model for chemotaxis, developed in [65] by Keller and Segel in 1971. Chemotaxis is the influence of a chemical substance in the environment on the movement of a mobile species. This process is key in cellular communications. Keller and Segel developed a general model for the chemotaxis phase of aggregation of slime mold, *i.e* Dictyostelium Discoidium in [65]. There has been a great deal of analysis on various versions of the Keller-Segel model since it was developed and research is still in progress [34, 41, 45, 47, 55, 68, 42]. Thus far the most studied version is:

$$\frac{\partial u}{\partial t} = \kappa \Delta u - \chi \nabla \cdot (u \nabla v), \tag{3.1a}$$

$$\epsilon \frac{\partial v}{\partial t} = k_c \Delta v - \alpha v + \beta u. \tag{3.1b}$$

with Neumann boundary conditions. In (3.1) u is the myxamoebae density of slime mold and v the chemo-attractant concentration. Comparing this model to (1.1) we can see that the chemo-attractant density is comparable to the attractiveness value. It is worth noting that chemotaxis is sometimes modeled by an elliptic-parabolic system; however, in the residential burglaries model the timescale of the change in attractiveness value is similar to the change in criminal density. From (3.1) we see that the myxamoebae move up gradients of chemo-attract concentration like criminals move up gradients of attractiveness value. Global existence and finite time blow-up of the (3.1) is highly dependent on the dimension. In one-dimension finite time blow-up cannot occur [36]. In two-dimensions it has been shown, by Corrias and Calvez [28], that the solution exist globally in time if the initial mass is below the critical quantity 8π. If the initial mass is above 8π then aggregation occurs in the case when $\epsilon = 0$

[55, 54]. As far as we know of the blow-up results for the fully parabolic system has not been proved. For higher dimensions, d, there exists a similar critical quantity that is governed by the $L^{d/2}$-norm of the initial myxamoebae density. Although the most studied version of the Keller-Segel model is (3.1) various variations of the model have also been analyzed. A comprehensive summary of much of this work can be found in [57] and [58]. It is not surprising that (3.1) is the most studied version of the Keller-Segel model since is possesses properties that facilitates mathematical analysis. There are three properties worth noting. First, the system (3.1) conserves mass of the cell density. Furthermore, one can express the chemo-attractant concentration as the convolution of the Bessel Kernel, $\mathcal{B}_\eta(z) = \frac{1}{4\eta\pi}\int_0^\infty \frac{1}{t}e^{-\frac{|z|^2}{4\eta t}-t}dt$, and the cell density. In two-dimensions this is especially useful for proving blow-up results given large enough initial mass of the cell density. Most importantly, his model, after non-dimensionalization, has a Lyapunov functional [28]:

$$\mathcal{F}(t) = \int u \log u \; dx - \int uv \; dx + \frac{1}{2}\int |\nabla v|^2 \, dx + \int \alpha v^2 dx.$$

This functional is key in proving global existence. This connection motivates us to study alternative residential burglary models.

3.2 Crime Models with General Velocity Field

In this section we assume that the attractiveness value increases proportionally with the criminal density, as opposed increasing with the number of crimes. Indeed, criminals walking around an area may automatically increase the vulnerability of a neighborhood. This is still modeling the 'broken-window' effect but not the 'repeat and near-repeat victimization effect'. This assumption leads to a linear equation for the attractiveness value. In particular, the model we now consider is

$$A_t = \eta\Delta A - A + \beta\rho + A_T(x) \tag{3.2a}$$

$$\rho_t = \nabla\cdot(\nabla\rho - \rho\nabla\chi(A)) - A\rho + \overline{B}(x), \tag{3.2b}$$

with $\beta \geq 0$ and $\eta \in [0,1]$ defined as in Chapter 1. Similar models have been studied in [16, 14, 17]. Note that the nonlinearities in the criminal density remain the same. From the modeling perspective criminals should be removed at the rate which crimes are committed. Furthermore, as we will see in the subsequent work the logarithmic velocity field plays an important role in the prevention of aggregation of criminals. We will pay special attention $\chi(A) = \log A$, which gives the original velocity field seen in (1.1), and linear $\chi(A) = A$. In this section we explore the difference between these two velocity fields. Recall, that the original system (1.1) is the formal limit of the agent based system modeling certain criminology theories deemed to be reasonable. The goal of studying the well-posedness of the the system is to obtain insight into the robustness of the model. Indeed, in Chapter 2 we sought classical solutions. However, physical solutions need not be classical. For example, finite time blow-up in the gradient of criminal density is perfectly reasonable. Physically, what is required is control of the criminal density it self. Hence, we expect that $|\rho|_\infty$ remains bounded for the solutions to be physical. We look for weaker solutions, which are physical solutions, but not necessarily classical. Furthermore, we will see the importance of the logarithmic velocity field for global existence of solutions.

3.2.1 Global Existence for Logarithmic Velocity Field

A key ingredient missing in our analysis of the original residential burglary system (1.1) is an energy functional that remains bounded for all time for its solutions. This is problematic because that we do not obtain enough control over the the criminal density to extend our local solution to a global one. One advantage (3.2) has over (1.1) is that the following functional

$$\mathcal{F}_{log}(t) = \int \rho(\log\rho - \frac{1}{2}\log A)dx \tag{3.3}$$

is a bounded above for all time for A and ρ solutions to (3.2). The functional $\mathcal{F}_{log}(t)$ is not necessarily dissipated; however, an upper bound proves to be sufficient for the purpose of our analysis. We take advantage of the specific form of the velocity field (3.2), *i.e.* $\nabla \log A$. For the purpose of this work we only provide a discussion of the importance of the control of $\mathcal{F}_{log}(t)$ for global existence, and all of the computations are formal. However, we note that this can all be made rigorous. We consider free boundary conditions for the attractiveness value and no-flux boundary conditions for the criminal density, *i.e.*

$$A = \mathcal{G}_\eta * (\beta\rho + A^o(x)) \quad \text{and} \quad (-\nabla\rho + \rho\nabla\chi(A)) \cdot \vec{\nu}|_{\partial\Omega} = 0, \tag{3.4}$$

where $\mathcal{G}_\eta$ is the fundamental solution to the $\partial_t - \eta\Delta + 1$ in $\mathbb{R}^n \times \mathbb{R}^+$. Now, from the integral representation of $A(x,t)$ one can prove that there exists a function $A_{min}(t)$ which is a lower bound for $A(x,t)$ for all $x \in \Omega$. Recall from Chapter 2 that if the solutions have enough regularity then $A(x,t)$ remains above A_o (provided the initial condition is larger than A_o). However, we can not assume such regularity a priori. We first state the relevant bounds on the mass of the criminal density and the attractiveness value.

Lemma 13 (Bounds on Mass of ρ and A)**.** *Let* $M_v = \int |v|\, dx$ *and let* A, ρ *be solutions to* (3.2) *with initial data* (A_0, ρ_0) *then* A, ρ *satisfy the following bounds*

$$M_\rho(t) \leq M_{\overline{B}}\, t + \|\rho_0\|_1\,, \tag{3.5a}$$

$$M_A(t) \leq C(\beta, M_{\overline{B}})t^2 + C(\|A^o\|_1\,, \|\rho_0\|_1)t + \|A_0\|_1\,. \tag{3.5b}$$

The proof of Lemma 3.5 is simple and we omit the details.

Proposition 2 (Bounds on the Energy Functional)**.** *Let A, ρ be solutions* (3.2) *with $\chi(A) = \log A$ and initial data, $(A_o(x),\ \rho_o(x)) \geq (A_o, 0)$, such that $\mathcal{F}_{log}(0)$ is positive and finite then $\mathcal{F}_{log}(t)$ is bounded from above $t > 0$, i.e.*

$$\mathcal{F}_{log}(t) \leq e^{-A_{min}t}\mathcal{P}(t) < \infty \tag{3.6}$$

Proof. We look at the evolution equation of $\mathcal{F}_{log}(t)$. We do not specify the domain of integration for notational simplicity.

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_{log}(t) =& \int\left(\log\rho - \frac{1}{2}\log A\right)\left(\nabla\cdot(\nabla\rho - \rho A^{-1}\nabla A) - A\rho + \overline{B}\right) \\
&+ \int \rho_t - \frac{1}{2}\int \rho A^{-1}A_t \\
=& -\int\frac{1}{\rho}\left|\nabla\rho\right|^2 + \frac{3}{2}\int A^{-1}\nabla\rho\cdot\nabla A - \frac{1}{2}\int\rho\left|\nabla A\right|^2 A^{-2} + \int\overline{B} - \int A\rho \\
&+ \underbrace{\int(\overline{B} - A\rho)(\log\rho - \frac{1}{2}\log A)}_{\mathcal{R}_1} \underbrace{- \frac{1}{2}\int\rho A^{-1}(\eta\Delta A - A + \beta\rho + A^o)}_{\mathcal{R}_2}
\end{aligned}$$

We can simplify the last term of the above inequality by integrating by parts

$$\begin{aligned}
-\mathcal{R}_2 =& \frac{\eta}{2}\int A^{-1}\nabla\rho\cdot\nabla A - \frac{\eta}{2}\int\rho A^{-2}\left|\nabla A\right|^2 \\
&+ \frac{1}{2}\int\rho - \frac{1}{2}\beta\int\rho^2 A^{-1} - \frac{1}{2}\int A^o(x)\rho A^{-1} \\
\leq& \frac{\eta}{2}\int A^{-1}\nabla\rho\cdot\nabla A - \frac{\eta}{2}\int\rho A^{-2}\left|\nabla A\right|^2 + \frac{1}{2}M_\rho(t).
\end{aligned}$$

Substituting this back into the evolution equation gives

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_{log}(t) + \int A\rho \leq& -\int\frac{1}{\rho}\left|\nabla\rho\right|^2 + \frac{1}{2}(3+\eta)\int A^{-1}\nabla\rho\cdot\nabla A \\
&- \frac{1}{2}(1+\eta)\int\rho\left|\nabla A\right|^2 A^{-2} + M_{\overline{B}} + \frac{1}{2}M_\rho(t) + \mathcal{R}_1 \\
\overset{C.S.}{\leq}& M_{\overline{B}} + \frac{1}{2}M_\rho(t) + \mathcal{R}_1.
\end{aligned}$$

Note that this holds for all $\eta = 1$ and the only term left to control is

$$\begin{aligned}\mathcal{R}_1 &= \int \overline{B}(\log\rho - \frac{1}{2}\log A) - \int A\rho(\log\rho - \frac{1}{2}\log A)\\ &\leq \left\|\overline{B}\right\|_\infty M_\rho(t) + \frac{1}{2}\left|\log A_{min}(t)\right| M_{\overline{B}} - A_{min}(t)\mathcal{F}_{log}(t).\end{aligned}$$

Since the initial energy is positive then the second term in the above equality will contribute to the decay of the energy. Combining everything gives

$$\frac{d}{dt}\mathcal{F}_{log}(t) + \int A\rho \leq \alpha(t) - A_{min}(t)\mathcal{F}_{log}(t),$$

where $\alpha(t) = M_{\overline{B}}(1 + \frac{1}{2}\left|\log A_{min}(t)\right|) + M_\rho(t)(\frac{1}{2} + \left|\overline{B}\right|_\infty)$. We can integrate this in time to obtain

$$\mathcal{F}_{log}(t) + \int_0^t \int A\rho \leq e^{-\int_0^t A_{min}(s)\,ds}\left(\mathcal{F}_{log}(0) - \frac{1}{A_{min}(t)}\int_0^t \alpha(s)ds\right) + \frac{1}{A_{min}(t)}\int_0^t \alpha(s)ds.$$

This proves (3.6). □

The importance of this Proposition 2 can be seen in the following corollary

Lemma 14 (Entropy Bound)**.** *Let* A *and* ρ *be weak solutions to* (3.2) *then* $\int \rho\log\rho dx \leq \mathcal{P}(t) < \infty$*, for all* $t > 0$*.*

Proof. We prove a lower bound we fix $t^* \geq 0$ and for notational simplicity we will suppress the time dependence let $M_\rho = M_\rho(t^*)$ and $d\mu = \frac{\rho}{M_\rho}$ note that then $\int d\mu = \frac{1}{M_\rho}\int \rho(x,t^*)dx = 1$. We will apply Jensen's inequality with the measure μ.

$$\begin{aligned}\int \rho(\log\rho - \log A)dx &= -M_\rho \int \frac{\rho}{M_\rho}\left(\log\frac{A}{\rho}\right)dx\\ &\geq -M_\rho \log\left(\int \frac{A}{M_\rho}dx\right)\\ &= M_\rho\log M_\rho - M_\rho\log M_A.\end{aligned}$$

This implies that

$$\int \rho \log \rho \geq \int \rho \log A + M_\rho \log M_\rho - M_\rho \log M_A.$$

Now, one use this to bound the entropy $\int \rho \log \rho \, dx$

$$\begin{aligned}\frac{1}{2}\int \rho \log \rho \leq \int \rho \log \rho - \frac{1}{2}\rho \log \rho \leq \int \rho \left(\log \rho - \frac{1}{2}\log A \right) dx + \mathcal{P}_2(t) \\ \leq \mathcal{F}_{log}(t) + \mathcal{P}_2(t).\end{aligned}$$

where $\mathcal{P}_2(t) := M_\rho(t) \left(\log M_A(t) - M_\rho(t) \right)$. □

Lemma 14 gives us a global bound on the entropy, which turns out to be key in extending the solution to global ones.

Remark 3. *We can easily see the importance of the logarithmic velocity field via the Young type inequality*

$$\int |uv| \;\, dx \leq \int u \log u \; dx + \int e^{v-1} dx$$

Indeed, this provides a better bound that provided by the Lemma. For the logarithmic case we have $\int |\rho \log A| \leq \int \rho \log \rho dx + e^{-1} \int A$*. Then*

$$\frac{1}{2}\int \rho \log \rho \leq \int \rho \log \rho - \frac{1}{2}\rho \log \rho \leq \mathcal{F}_{log}(t) + \frac{e^{-1}}{2} M_A(t).$$

3.2.2 Linear Velocity Field

In this section we consider a linear velocity field, $\chi(A) = A$. It will be come apparent in this section why the log velocity field is important for the global existence. The corresponding energy functional for (3.2) with this velocity field is

$$\begin{aligned}\mathcal{F}_L(t) &= \int \rho(\log \rho - A) dx + \frac{1}{2}\int A^2 dx + \frac{\eta}{2}\int |\nabla A|^2 \, dx - \int A A^o dx \\ &:= \mathcal{S}_L(t) + \mathcal{W}_L(t), \end{aligned} \tag{3.7}$$

where $\mathcal{S}_L(t) = \int \rho(\log \rho - A)dx$ and the remaining part corresponds to $\mathcal{W}_L(t)$. As in the previous section, this energy remains bounded along trajectories of the dynamical system associated with our system with linear velocity field.

Proposition 3 (Bounds on Energy for the Linear Velocity Field)**.** *Let A, ρ be solutions* (3.2) *with $\chi(A) = A$ and initial data, $(A_o(x),\ \rho_o(x)) \geq (A_o, 0)$, such that $\mathcal{F}_L(0)$ is positive and finite then $\mathcal{F}_L(t)$ is bounded from above $t > 0$.*

Proof.

$$\begin{aligned}\frac{d}{dt}\mathcal{S}_L(t) = &\int (\log \rho - A)\left(\nabla \cdot (\nabla \rho - \rho \nabla A) - A\rho + \overline{B}(x)\right) dx \\ &+ \int \rho_t - \int A_t \rho \, dx \\ = &- \int \rho \left|\nabla(\log \rho - A)\right|^2 dx + \int (\log \rho - A)(\overline{B} - A\rho)dx \\ &+ \int \overline{B}dx - \int A\rho dx - \int A_t\rho \, dx.\end{aligned}$$

Now, multiplying (3.2a) by A_t and integrating gives

$$\int |A_t|^2 \, dx = -\frac{\eta}{2}\frac{d}{dt}\int |\nabla A|^2 \, dx - \frac{1}{2}\frac{d}{dt}\int A^2 dx + \beta \int \rho A_t + \int A_o A_t dx.$$

From this we see that the time evolution of $\mathcal{F}_L(t)$ is bounded as follows

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_L(t) \leq &- \int \rho \left|\nabla(\log \rho - 2A)\right|^2 dx - \int |A_t|^2 \, dx - \int A\rho dx \\ &+ (\beta - 2)\int \rho A_t dx + M_{\overline{B}} + \int (\log \rho - 2(A))(\overline{B} - A\rho)dx.\end{aligned}$$

The last term in the above inequality can be bounded as was done in §3.2.1. Hence, if $2 \geq \beta$ the energy $\mathcal{F}_L(t)$ remains bounded for all $t > 0$. □

As in the case with logarithmic velocity field we have an upper bound on $\mathcal{F}_L(t)$. However, recall that control of $\int \rho \log |\rho| \, dx$ is ultimately what provided the global existence. To obtain this we need a lower bound on $\mathcal{F}_L(t)$ as well. To explore this issue we will make use of a generalized Moser-Trudinger inequality (see [35]-Prop. 2.3).

Lemma 15 (Generalized Moser-Trudinger Inequality)**.** *Let* $v \in H^1(\Omega)$*,* Ω *with piecewise* C^2 *boundary and minimal interior angle of corners* θ *then*

$$\int_\Omega e^v \leq C \exp\left(\frac{1}{|\Omega|}\left|\int_\Omega v dx\right| + \frac{1}{8\theta}\int_\Omega |\nabla v|^2 \, dx\right).$$

Our objective now is to try to obtain a lower bound for $\mathcal{S}_L$, following the proof for the lower bound of $\mathcal{F}_{log}(t)$ we get

$$\begin{aligned}
\mathcal{S}_L(t^*) &= -M_\rho \int \frac{\rho}{M_\rho} \log\left(\frac{e^A}{\rho}\right) dx \\
&\geq -M_\rho \log\left(\frac{1}{M_\rho}\int e^A dx\right) \\
&\geq M_\rho \log M_\rho - M_\rho \log \int e^A dx.
\end{aligned}$$

Immediately we see that this lower bound is not as nice that the one obtained from the logarithmic velocity field. We use Lemma 15 to bound the last term in the above inequality

$$\begin{aligned}
\log\left(\int e^A dx\right) &\leq \log\left\{C \exp\left(\frac{1}{|\Omega|}\left|\int_\Omega A\right| + \frac{1}{8\theta}\int |\nabla A|_2^2\right)\right\} \\
&\leq \log C + \frac{1}{|\Omega|} M_A + \frac{1}{8\theta}\int |\nabla A|_2^2 .
\end{aligned}$$

We conclude that

$$\begin{aligned}
0 &\leq \mathcal{S}_L(t) + M_\rho(t) C + \frac{1}{|\Omega|} M_A^2(t) + \frac{M_\rho(t)}{8\theta} |\nabla A|_2^2 - M_\rho(t) \log M_\rho(t) \\
&= \mathcal{F}_L + \left(\frac{M_\rho(t^*)}{8\theta} - \frac{\eta}{2}\right) \int |\nabla A|^2 \, dx - \frac{1}{2}\int |A|^2 \, dx + \mathcal{B}(t)
\end{aligned}$$

Therefore, if

$$2M_\rho(t) < 8\theta\eta, \tag{3.8}$$

then this implies that $\mathcal{F}_L$ controls $|\nabla A|_2$, which then provides controls of $\int \rho \log |\rho| \, dx$. However, since our bound on the mass of ρ is only linear we cannot verity that (3.8) holds for all $t > 0$.

3.3 Blow-up for Model with Linear Velocity Field and Bounded Mass

In the previous section we saw that we could not prove global existence to (3.2) with a linear velocity field for arbitrary mass. This problem was caused because the linear upper-bound on the mass was was not good enough. Motivated by this problem we consider another modification to (3.2). This will ease the mathematical analysis while maintaining fundamental assumptions made in [92]. The model we propose is:

$$\epsilon\frac{\partial A}{\partial t} = \eta\Delta A - A + \beta\rho + A^o(x), \tag{3.9a}$$

$$\frac{\partial \rho}{\partial t} = \Delta\rho - 2\nabla\cdot(\rho\nabla A) + \overline{B}(x) - f(A)\rho. \tag{3.9b}$$

From now on we work in all of $\mathbb{R}^2$. Notice that now A^o and $\overline{B}$ are functions of the space variable and must have sufficient decay as $|x| \to \infty$. Model (3.9) makes three simplifications to (1.1). First, the advection speed is now given simply by $|\nabla A|$. The second modification is that the attractiveness value increases with the number of criminals with constant of proportionality β, *i.e* we replace $A\rho$ with $\beta\rho$ in (1.1a). We have no reason to believe that this modification will decrease the accuracy of the model. Finally, the criminal density decays with a rate of $f(A)$ and we assume that $f(A)$ has a lower and upper bound.

3.3.1 Useful Properties of the Modified Residential Burglaries Model

The model (3.9) does not possess these exact properties; however, it does possess ones which are useful enough. For $v \in L^1(\Omega)$ let $M_v(t) = \int_\Omega v(x,t)dx$. As an example of a useful property if $A,\ \rho$ are solutions to (3.9) then $M_\rho(t)$ is bounded above and below. Define $f_{min} = \min_{A\in\mathbb{R}^+} f(A)$ then:

$$M_\rho(t) \leq e^{-f_{min}t}\left(M_\rho(0) - \frac{M_{\overline{B}}}{f_{min}}\right) + \frac{M_{\overline{B}}}{f_{min}}. \tag{3.10}$$

Replacing f_{min} with f_{max} gives a similar lower bound for $M_\rho(t)$. Another key property is the explicit expression of the attractiveness value in terms of the criminal density in the quasi-static case, *i.e* $\epsilon = 0$:

$$A(x) = \mathcal{B}_\eta * (\beta\rho + A^o) \tag{3.11}$$

We conjecture that solutions to (3.9) satisfy an energy functional whose upper bound can be controlled with time. Being that this is beyond the scope of this paper we only mention that proving such an energy functional is important for proving global existence via the Lyapunov functional method discussed in the introduction.

3.3.2 Blow-up of a Modified Residential Burglaries Model

In this section we explore the possibility of blow-up in finite time of the solution to the modified residential burglaries model (3.9) in the case where $\epsilon = 0$. It turns out, that similar to the Keller-Segel model, if the lower-bound on the mass of the criminal density is large enough there is mass concentration on a set of measure zero. Let $M_\rho^{min} = \min\{M_\rho(0), M_{\overline{B}}/f_{max}\}$ and $M_\rho^{max} = \max\{M_\rho(0), M_{\overline{B}}/f_{min}\}$ and for a function v we denote the finite second moment by $I_v = \int |x|^2 v dx$. We state this blow-up result in the following theorem.

Theorem 7 (Blow-up of a Modified Residential Burglaries Model)**.** *Let* $(A(x,0), \rho(x,0)) \in L^1(\mathbb{R}^2)$ *be initial data such that* $\left(\beta M_\rho^{min} - 4\pi\right) M_\rho^{min} > \pi I_{\overline{B}}$. *Furthermore, let* ρ *be the non-negative smooth solution to* (3.9b) *and that* A *has reached a steady state and is defined by* (3.11)*, then* A, ρ *are a non-negative* *smooth solutions to* (3.9) *(when* $\epsilon = 0$*). Then, if the initial second moment is small enough. That is if*

$$\int |x|^2 \rho dx \leq \frac{1}{K^2}\left[\left(\frac{\beta}{\pi}M_\rho^{min} - 4\right)M_\rho^{min} - I_{\overline{B}}\right]^2, \tag{3.12}$$

where $I_{\overline{B}}(x) = \int \overline{B}dx$, $K = \left[\frac{2\beta}{\pi}C\left(M_\rho^{max}\right)^{3/2} + a_1\left(M_\rho^{max}\right)^{1/2}\right]$,
$a_1 = 4\left\|\nabla\mathcal{B}_\eta(x)\right\|_1 |A^o(x)|_\infty$ *and* C *a constant, then there exists a finite time singularity.*

Proof. Consider the time evolution of the second moment of ρ, $I(t) = \int_{\mathbb{R}^2}|x|^2\rho dx$:

$$\begin{aligned}\frac{dI}{dt} &= \int_{\mathbb{R}^2}|x|^2\left(\Delta\rho - 2\nabla\cdot(\rho\nabla A) - f(A)\rho + \overline{B}\right)dx\\ &\leq 4\int_{\mathbb{R}^2}\rho\, dx + 4\beta\int_{\mathbb{R}^2}\rho\left(\mathbf{x}\cdot\nabla\mathcal{B}_\eta * \rho\right)dx + 4\int_{\mathbb{R}^2}\rho\left(\mathbf{x}\cdot\nabla\mathcal{B}_\eta * A^o(x)\right)dx + I_{\overline{B}}.\end{aligned} \tag{3.13}$$

The third term of on the right in the above inequality can be bounded above using Cauchy-Schwarz Inequality and Young's inequality for convolutions [110]:

$$\begin{aligned}4\int_{\mathbb{R}^2}\rho|\mathbf{x}||\nabla\mathcal{B}_\eta * A^o(x)|dx &\leq 4|\nabla\mathcal{B}_\eta * A^o(x)|_\infty\int\rho|\mathbf{x}|dx.\\ &\leq \underbrace{4\left\|\nabla\mathcal{B}_\eta\right\|_{L^1}|A^o(x)|_\infty}_{a_1}M_\rho^{1/2}(t)\sqrt{I(t)}\end{aligned} \tag{3.14}$$

We use the explicit expression of the gradient of the Bessel Kernel, $\nabla\mathcal{B}_\eta(z) = -\frac{1}{2\pi}\frac{z}{|z|^2}\int_0^\infty e^{-s-\frac{|z|^2}{4\eta s}}ds$, to bound the second term. Let $g_\eta(z) = \int_0^\infty e^{-s-\frac{|z|^2}{4\eta s}}ds$ and $d\mathcal{A} = dxdy$ then, omitting the time dependence, we obtain

$$\begin{aligned}4\beta\int_{\mathbb{R}^2}\rho\left(\mathbf{x}\cdot\nabla\mathcal{B}_\eta * \rho\right)dx &\leq -\frac{2\beta}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho(x)\mathbf{x}\cdot\frac{(\mathbf{x}-\mathbf{y})}{|x-y|^2}g_\eta(x-y)\rho(y)dydx\\ &\leq \frac{\beta}{\pi}\iint\rho(x)[1 - g_\eta(x-y)]\rho(y)d\mathcal{A} - \frac{\beta}{\pi}\iint\rho(x)\rho(y)d\mathcal{A}\\ &= -\frac{\beta}{\pi}M_\rho^2(t) + \frac{\beta}{\pi}\iint\rho(x)[1 - g_\eta(x-y)]\rho(y)d\mathcal{A}.\end{aligned}$$

Observe that $g_\eta(z)$ is a positive, radially symmetric, decreasing function with maximum of one. This implies that $0 \le (1 - g_\eta(z)) \le 1$. Now, consider the derivative of $(1 - g_\eta(r))$ with respect to $r = |z|$:

$$\begin{aligned}\frac{d}{dr}(1 - g_\eta(r)) &\le \frac{r}{2\eta}\int_0^\infty \frac{1}{s}\, e^{-s-\frac{r^2}{4\eta s}} ds \\ &\le \frac{2\pi}{\eta} r\mathcal{B}_1(\frac{r}{\sqrt{\eta}}) \\ &\le \frac{2\pi}{\sqrt{\eta}} \sup_{\tilde{r}\in(0,1)} (\tilde{r}\mathcal{B}_1(\tilde{r})),\end{aligned}$$

where, $\tilde{r} = \frac{r}{\sqrt{\eta}}$ for $0 \le r \le \sqrt{\eta}$. If $C = \frac{2\pi}{\sqrt{\eta}} \max(\sup_{\tilde{r}\in(0,1)} \{\tilde{r}\mathcal{B}_1(\tilde{r})\}, 1)$ then $(1 - g_\eta(z)) \le C\,|z|$. Hence, we have:

$$\begin{aligned}4\beta\int_{\mathbb{R}^2} \rho\,(\mathbf{x}\cdot\nabla\mathcal{B}_\eta * \rho)\, dx &\le -\frac{\beta}{\pi}M_\rho^2(t) + \frac{2\beta}{\pi}CM_\rho(t)\int_{\mathbb{R}^2}\rho(x,t)\,|x|\,dx \\ \text{C.S.} &\le -\frac{\beta}{\pi}M_\rho^2(t) + \frac{2\beta}{\pi}C\,(M_\rho(t))^{3/2}\sqrt{I(t)}. \qquad (3.15)\end{aligned}$$

Substituting (3.14) and (3.15) into (3.13) gives:

$$\begin{aligned}\frac{dI}{dt} &\le \left(4 - \frac{\beta}{\pi}M_\rho(t)\right)M_\rho(t) + \frac{2\beta}{\pi}CM_\rho^{3/2}(t)\sqrt{I(t)} + a_1M_\rho^{1/2}(t)\sqrt{I(t)} + I_{\overline{B}} \\ &\le \left(4 - \frac{\beta}{\pi}M_\rho(t)\right)M_\rho(t) + \left(\frac{2\beta}{\pi}CM_\rho^{3/2}(t) + a_1M_\rho^{1/2}(t)\right)\sqrt{I(t)} + I_{\overline{B}} \\ &\le \left(4 - \frac{\beta}{\pi}M_\rho^{min}\right)M_\rho^{min} + \underbrace{\left(\frac{2\beta}{\pi}C\left(M_\rho^{max}\right)^{3/2} + a_1\left(M_\rho^{max}\right)^{1/2}\right)}_{K}\sqrt{I(t)} + I_{\overline{B}}.\end{aligned}$$

In the last inequality we use the fact that the initial conditions are chosen so that $\beta M_\rho^{\min} > 4\pi$. Integrating on $[0, t)$ gives the integral inequality:

$$I(t) \le I(0) + \int_0^t g(I(s))ds, \qquad (3.16)$$

where $g(I(t)) = \left(4 - \frac{\beta}{\pi}M_\rho^{min}\right)M_\rho^{min} + K\sqrt{I(t)} + I_{\overline{B}}$. The function $g(I)$ is continuous, increasing and such that $g(I(t^*)) = 0$ for $t^* > 0$ such that:

$$I(t^*) = \frac{1}{K^2}\left[\left(\frac{\beta}{\pi}M_\rho^{min} - 4\right)M_\rho^{min} - I_{\overline{B}}\right]^2,$$

where K is defined in the theorem. Since $I(0) \leq I(t^*)$ by continuity of g there exists a $\tilde{t} > 0$ such that $\int_0^{\tilde{t}} g(I(s))ds < 0$. Hence, $I(\tilde{t}) < I(0)$. Repeating this process will eventually give that $I(t) = 0$ for some positive t which proves the result.

□

From Theorem 7 we conjecture that a logarithmic sensitivity function is more suitable than a linear sensitivity function. Moreover, from the maximum principle of the attractiveness value a lower bound on $f(A)$ is implicit in the original model. Hence, setting $f(A) = A$ is only eliminating the upper bound on $f(A)$. This would only help prevent blow-up. The remaining difference between the two models is less obvious to analyze. We conjecture that the nonlinear $A\rho$ aids blow-up more than $\beta\rho$. This is because we expect, and indeed we observe numerically, that A and ρ grow and decay together. Hence, we have that $A\rho \approx \rho^2$ which would aid blow-up more so than $\beta\rho$ would.

3.3.3 Exploring Blow-up of a Modified Residential Burglaries Model in 1D

Although we see blow-up in the modified model for large enough mass of the initial criminal density in two dimensions, a similar type of blow-up in finite time of the model (3.9) cannot occur in one dimension. This is due to change of properties of the Bessel Kernel in one dimension. In fact, a simple computation shows that the second moment will always be bounded below by something positive. For simplicity of notation we take $\eta = 1$, in this case in one-dimension we have that $\mathcal{B}(x) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{t^{1/2}} e^{-\frac{|x|^2}{4t} - t} dt$ and $\partial_x \mathcal{B}(x) = -\frac{x}{\sqrt{\pi}} \int_0^\infty \frac{1}{4t^{3/2}} e^{-\frac{|x|^2}{4t} - t} dt = -\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{|x|^2}{4s^2} - s^2} ds$. In contrast to the previous section we now seek a bound from below for the second moment.

$$
\begin{aligned}
\frac{dI}{dt} &= 2M_\rho(t) + 4\beta \int_{\mathbb{R}^2} \rho \left(x \partial_x \mathcal{B} * \rho \right) dx + 4 \int_{\mathbb{R}^2} \rho \left(x \partial_x \mathcal{B} * A^o(x) \right) dx - \int_{\mathbb{R}^2} f(A) \left| x \right|^2 \rho dx + I_{\overline{B}} \\
&\geq 2M_\rho^{min} + I_{\overline{B}} + 4\beta \int_{\mathbb{R}^2} \rho \left(x \partial_x \mathcal{B} * \rho \right) dx + 4 \int_{\mathbb{R}^2} \rho \left(x \partial_x \mathcal{B} * A^o(x) \right) dx - f_{max} I(t) \\
&\geq 2M_\rho^{min} + I_{\overline{B}} - f_{max} I(t) - \left[4 \left| \mathcal{B}_x \right|_\infty \left(\beta \left(M_\rho^{max} \right)^{3/2} + \left\| A^o \right\|_1 M_\rho^{max} \right) \right] I(t)^{1/2} \\
&\geq C_1 - C_2 I(t),
\end{aligned}
$$

where, $C_1 = 2M_\rho^{min} + I_{\overline{B}} - \delta \left[4 \left| \mathcal{B}_x \right|_\infty \left(\beta \left(M_\rho^{max} \right)^{3/2} + \left\| A^o \right\|_1 M_\rho^{max} \right) \right]^2$ and $C_2 = \frac{1}{\delta} - f_{max}$. We choose δ small enough such that $C_1 > 0$. This implies that $I(t) \geq e^{-C_2 t} \left(I(0) - C_1/C_2 \right) + C_1/C_2$, which has a bound from below for all time. Hence, if there is blow-up in finite time we cannot show it via this method. This agrees with preliminary numerical results which show finite time blow-up in two-dimensions but not in one-dimension.

Part II

Biological Aggregation and Dispersal

CHAPTER 4

The Aggregation Diffusion Equation with Degenerate Diffusion

4.1 Introduction and Motivation

Nonlocal aggregation phenomena have been studied in a wide variety of biological applications such as migration patterns in ecological systems [23, 103, 84, 52, 25] and Patlak-Keller-Segel (PKS) models of chemotaxis [47, 87, 54, 65, 71]. Diffusion is generally included in these models to account for the dispersal of organisms. Classically, linear diffusion is used, however recently, there has been a widening interest in models with degenerate diffusion to include over-crowding effects [103, 25]. The parabolic-elliptic PKS is the most widely studied model for aggregation, where the nonlocal effects are modeled by convolution with the Newtonian or Bessel potential. On the other hand, in population dynamics, the nonlocal effects are generally modeled with smooth, fast-decaying kernels. However, all of these models are describing the same mathematical phenomenon: the competition between nonlocal aggregation and diffusion. For this reason, we are interested in unifying and extending the local and global well-posedness theory of general aggregation models with degenerate diffusion of the form

$$u_t + \nabla \cdot (u\vec{v}) = \Delta A(u) \quad \text{in } [0,T) \times D, \tag{4.1a}$$

$$\vec{v} = \nabla \mathcal{K} * u. \tag{4.1b}$$

Mathematical works most relevant this paper are those with degenerate diffusion [10, 98, 99, 100, 20, 70, 69, 27] and those from the classical PKS literature [59, 44, 22, 21]. See also [64].

Existence theory is complicated by the presence of degenerate diffusion and singular kernels such as the Newtonian potential. Bertozzi and Slepčev in [10] prove existence and uniqueness of models with general diffusion but restrict to non-singular kernels. Sugiyama [100] proved local existence for models with power-law diffusion and the Bessel potential for the kernel, but uniqueness of solutions was left open. We extend the work of [10] to prove the local existence of (4.1) with degenerate diffusion and singular kernels including the Bessel and Newtonian potentials. The existing work on uniqueness of these problems included a priori regularity assumptions [70] or the use of entropy solutions [25] (see also [30]). The Lagrangian method introduced by Loeper in [79] estimates the difference of weak solutions in the Wasserstein distance and is very useful for inviscid problems or problems with linear diffusion [78, 8, 33]. In the presence of nonlinear diffusion, it seems more natural to approach uniqueness in H^{-1}, where the diffusion is monotone (see [105]). This is the approach taken in [6, 10], which we extend to handle singular kernels such as the Newtonian potential, proving uniqueness of weak solutions with no additional assumptions, provided the domain is bounded or $d \geq 3$.The main difference is the use of more refined estimates to handle the lower regularity of $\nabla K * u$, similar to the traditional proof of uniqueness of $L^1 \cap L^\infty$-vorticity solutions to the 2D Euler equations [111, 81] and a similar proof of the uniqueness of $L^1 \cap L^\infty$ solutions to the Vlasov-Poisson equation [89].

There is a natural notion of criticality associated with this problem, which roughly corresponds to the balance between the aggregation and diffusion. For problems with homogeneous kernels and power-law diffusion, $\mathcal{K} = c\,|x|^{2-d}$ and

$A(u) = u^m$, a simple scaling heuristic suggests that these forces are in balance if $m = 2 - 2/d$ [20]. If $m > 2 - 2/d$ then the problem is subcritical and the diffusion is dominant. On the other hand, if $m < 2 - 2/d$ then the problem is supercritical and the aggregation is dominant. For the PKS with power-law diffusion, Sugiyama showed global existence for subcritical problems and that finite time blow-up is possible for supercritical problems [100, 99, 98]. We extend this notion of criticality to general problems by observing that only the behavior of the solution at high concentrations will divide finite time blow-up from global existence (see Definition 7). We show global well-posedness for subcritical problems and finite time blow-up for certain supercritical problems.

If the problem is critical, it is well-known that in PKS there exists a critical mass, and solutions with larger mass can blow-up in finite time [22, 59, 13, 44, 21, 28, 20, 98, 99, 27]. For linear diffusion, the same critical mass has been identified for the Bessel and Newtonian potentials [22, 28]; however for nonlinear diffusion, the critical mass has only been identified for the Newtonian potential [20]. In this paper we extend the free energy methods of [20, 44, 27, 21] to estimate the critical mass for a wide range of kernels and nonlinear diffusion, which include these known results. For a smaller class of problems, including standard PKS models, we show this estimate is sharp.

The problem (4.1) is formally a gradient flow with respect to the Euclidean Wasserstein distance for the *free energy*

$$\mathcal{F}(u(t)) = S(u(t)) - \mathcal{W}(u(t)), \tag{4.2}$$

where the *entropy* $S(u(t))$ and the *interaction energy* $\mathcal{W}(u(t))$ are given by

$$S(u(t)) = \int \Phi(u(x,t))dx,$$
$$\mathcal{W}(u(t)) = \frac{1}{2}\int\int u(x,t)\mathcal{K}(x-y)u(y,t)dxdy.$$

For the degenerate parabolic problems we consider, the *entropy density* $\Phi(z)$ is a strictly convex function satisfying

$$\Phi''(z) = \frac{A'(z)}{z}, \quad \Phi'(1) = 0, \quad \Phi(0) = 0. \tag{4.3}$$

See [32] for more information on these kinds of entropies. Although there is a rich theory for gradient flows of this general type when the kernel is regular and λ-convex [83, 3, 31] the kernels we consider here are more singular and the notion of displacement convexity introduced in [83] no longer holds. For this reason, the rigorous results of the gradient flow theory are not directly applicable, however, certain aspects may be recovered, such as the use of steepest descent schemes [18, 19]. Moreover, the free energy (4.2) is still the important dissipated quantity in the global existence and finite time blow-up arguments. The free energy has been used by many authors for the same purpose, see for instance [98, 22, 27, 20, 7, 21]. For the remainder of the paper we only consider initial data with finite free energy, although the local existence arguments may hold in more generality.

There is a vast literature of related works on models similar to (4.1). For literature on PKS we refer the reader to the review articles [58, 57]; see also [56, 42, 28] for parabolic-parabolic Keller-Segel systems. For the inviscid problem, see the recent works of [72, 7, 6, 8, 31]. For a study of these equations with fractional linear diffusion see [73, 74, 12]. When the diffusion is sufficiently nonlinear and the kernel is in L^1, (4.1) may be written as a regularized interface problem, a notion studied in [95]. Critical mass behavior is also a property of other related critical PDE, such as the marginal unstable thin film equation [109, 9] and critical nonlinear Schrödinger equations [107, 67].

Outline: This chapter was work done in collaboration with Jacob Bedrossian and Andrea Bertozzi and was published in [5]. In §4.2 we discuss the definitions and notation that will be used in the remainder of the chapter. Note that this might differ from the notation used in Part I of this work. We also summarize the results in this section. Following, in §4.3 we prove the uniqueness of weak solutions. We remark that this section was done exclusively by Jacob Bedrossian. The local existence is proved in §4.4. A continuation argument is proved in §4.5, which connects the local theory to the global theory. The global theory is discussed in §4.6.

4.2 Definitions and Notation

We consider either $D = \mathbb{R}^d$ with $d \geq 3$ or D smooth, bounded and convex with $d \geq 2$, in which case we impose no-flux conditions

$$(-\nabla A(u) + u\nabla\mathcal{K} * u) \cdot \nu = 0 \text{ on } \partial D \times [0, T), \tag{4.4}$$

where ν is the outward unit normal to D. We neglect the case $D = \mathbb{R}^2$ for technicalities introduced by the logarithmic potential.

We denote $D_T := (0, T) \times D$. We also denote $\|u\|_p := \|u\|_{L^p(D)}$ where L^p is the standard Lebesgue space. We denote the set $\{u > k\} := \{x \in D : u(x) > k\}$, if $S \subset \mathbb{R}^d$ then $|S|$ denotes the Lebesgue measure and $\mathbf{1}_S$ denotes the standard characteristic function. In addition, we use $\int f dx := \int_D f dx$, and only indicate the domain of integration where it differs from D. We also denote the weak L^p space by $L^{p,\infty}$ and the associated quasi-norm

$$\|f\|_{L^{p,\infty}} = \left(\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) \right)^{1/p},$$

where $\lambda_f(\alpha) = |\{f > \alpha\}|$ is the distribution function of f. Given an initial condition $u(x, 0)$ we denote its mass by $\int u(x, 0) dx = M$. In formulas we use the notation $C(p, k, M, ..)$ to denote a generic constant, which may be different from line to line or even term to term in the same computation. In general, these constants will depend on more parameters than those listed, for instance those associated with the problem such as $\mathcal{K}$ and the dimension but these dependencies are suppressed. We use the notation $f \lesssim_{p,k,...} g$ to denote $f \leq C(p,k,..)g$ where again, dependencies that are not relevant are suppressed.

We now make reasonable assumptions on the kernel which include important cases of interest, such as when $\mathcal{K}$ is the fundamental solution of an elliptic PDE. To this end we state the following definition.

Definition 2 (Admissible Kernel)**.** *We say a kernel* $\mathcal{K}$ *is* admissible *if* $\mathcal{K} \in W^{1,1}_{loc}$ *and the following holds:*

(R) $\mathcal{K} \in C^3 \setminus \{0\}$.

(KN) $\mathcal{K}$ *is radially symmetric,* $\mathcal{K}(x) = k(|x|)$ *and* $k(|x|)$ *is non-increasing.*

(MN) $k''(r)$ *and* $k'(r)/r$ *are monotone on* $r \in (0, \delta)$ *for some* $\delta > 0$.

(BD) $|D^3\mathcal{K}(x)| \lesssim |x|^{-d-1}$.

This definition ensures that the kernels we consider are radially symmetric, non-repulsive, reasonably well-behaved at the origin, and have second derivatives which define bounded distributions on L^p for $1 < p < \infty$ (see Section §4.2.1). These conditions imply that if $\mathcal{K}$ is singular, the singularity is restricted to the origin. Note also, that the Newtonian and Bessel potentials are both admissible for all dimensions $d \geq 2$; hence, the PKS and related models are included in our analysis.

We now make precise what kind of nonlinear diffusion we are considering.

Definition 3 (Admissible Diffusion Functions)**.** *We say that the function $A(u)$ is an admissible diffusion function if:*

(D1) $A \in C^1([0,\infty))$ *with* $A'(z) > 0$ *for* $z \in (0,\infty)$.

(D2) $A'(z) > c$ *for* $z > z_c$ *for some* $c, z_c > 0$.

(D3) $\int_0^1 A'(z)z^{-1}dz < \infty$.

This definition includes power-law diffusion $A(u) = u^m$ for $m > 1$. Note that **(D3)** requires the diffusion to be degenerate at $u = 0$, however it is permitted to behave linearly at infinity. Furthermore, on bounded domains condition **(D3)** can be relaxed without any significant modification to the methods. Following [10], the notions of weak solution are defined separately for bounded and unbounded domains.

Definition 4 (Weak Solutions on Bounded Domains)**.** *Let $A(u)$ and $\mathcal{K}$ be admissible, and $u_0(x) \in L^\infty(D)$ be non-negative. A non-negative function $u : [0,T] \times D \to [0,\infty)$ is a weak solution to* (4.1) *if $u \in L^\infty(D_T)$, $A(u) \in L^2(0,T,H^1(D))$, $u_t \in L^2(0,T,H^{-1}(D))$ and*

$$\int_0^T \int u\phi_t \, dxdt = \int u_0(x)\phi(0,x)dx + \int_0^T \int (\nabla A(u) - u\nabla\mathcal{K} * u) \cdot \nabla\phi \, dxdt, \tag{4.5}$$

for all $\phi \in C^\infty(\overline{D_T})$ such that $\phi(T) = 0$.

It follows that $u\nabla\mathcal{K} * u \in L^2(D_T)$; therefore, definition 4 is equivalent to the following,

$$\langle u_t(t), \phi \rangle = \int (-\nabla A(u) + u\nabla\mathcal{K} * u) \cdot \nabla\phi \, dx, \tag{4.6}$$

for all test functions $\phi \in H^1$ for almost all $t \in [0,T]$. Above $\langle \cdot, \cdot \rangle$ denotes the standard dual pairing between H^1 and H^{-1}. Similarly for $\mathbb{R}^d$ we define the following notion of weak solution as in [10].

Definition 5 (Weak Solution in $\mathbb{R}^d$, $d \geq 3$)**.** *Let A and $\mathcal{K}$ be admissible, and $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be non-negative. A function $u : [0,T] \times \mathbb{R}^d \to [0,\infty)$ is a weak solution of* (4.1) *if $u \in L^\infty((0,T) \times \mathbb{R}^d) \cap L^\infty(0,T,L^1(\mathbb{R}^d))$, $A(u) \in L^2(0,T,\dot{H}^1(\mathbb{R}^d))$, $u\nabla K * u \in L^2(D_T)$, $u_t \in L^2(0,T,\dot{H}^{-1}(\mathbb{R}^d))$, and for all test functions $\phi \in \dot{H}^1(\mathbb{R}^d)$ for a.e $t \in [0,T]$* (4.6) *holds.*

We show below (Theorem 10) that weak solutions satisfying Definition 4 or 5 are in fact unique. Moreover, we show the unique weak solution satisfies the energy dissipation inequality (Proposition 4),

$$\mathcal{F}(u(t)) + \int_0^t \int \frac{1}{u} \left|A'(u)\nabla u - u\nabla\mathcal{K} * u\right|^2 dxdt \leq \mathcal{F}(u_0(x)). \tag{4.7}$$

As (4.7) is important for the global theory, one could also refer to these solutions as free energy solutions, as is done in [20]. Uniqueness implies that there is no distinction between free energy solutions in [20] and weak solutions.

Since (4.1) conserves mass, the natural notion of criticality is with respect to the usual mass invariant scaling $u_\lambda(x) = \lambda^d u(\lambda x)$. A simple heuristic for understanding how this scaling plays a role in the global existence is seen by examining the case of power-law diffusion and homogeneous kernel, $A(u) = u^m$ and $\mathcal{K}(x) = |x|^{-d/p}$. Under this mass invariant scaling the free energy (4.2) becomes,

$$\mathcal{F}(u_\lambda) = \lambda^{dm-d} S(u) - \lambda^{d/p}\mathcal{W}(u).$$

As $\lambda \to \infty$, the entropy and the interaction energy are comparable if $m = (p+1)/p$. We should expect global existence if $m > (p+1)/p$, as the diffusion will dominate as u grows, and possibly finite time blow-up if $m < (p+1)/p$ as the aggregation will instead be increasingly dominant. We consider inhomogeneous kernels and general diffusion, however for the problem of global existence, only the behavior as $u \to \infty$ will be important, in contrast to the problem of local existence. Noting that $|x|^{-d/p}$ is, in some sense, the representative singular kernel in $L^{p,\infty}$ leads to the following definition. This critical exponent also appears indirectly in [77].

Definition 6 (Critical Exponent)**.** *Let $d \geq 3$ and $\mathcal{K}$ be admissible such that $\mathcal{K} \in L^{p,\infty}_{loc}$ for some $d/(d-2) \leq p < \infty$. Then the* critical exponent *associated to $\mathcal{K}$ is given by*

$$1 < m^\star = \frac{p+1}{p} \leq 2 - 2/d.$$

If $D^2\mathcal{K}(x) = \mathcal{O}(|x|^{-2})$ as $x \to 0$, then we take $m^\star = 1$.

Remark 4. *The case $m^\star = 1$ implies at worst a logarithmic singularity as $x \to 0$ and if $d = 2$ then all admissible kernels have $m^\star = 1$.*

Now we define the notion of criticality. It is easier to define this notion in terms of the quantity $A'(z)$, as opposed to using $\Phi(z)$ directly.

Definition 7 (Criticality)**.** *We say that the problem is* subcritical *if*

$$\liminf_{z\to\infty} \frac{A'(z)}{z^{m^\star-1}} = \infty,$$

critical *if*

$$0 < \liminf_{z\to\infty} \frac{A'(z)}{z^{m^\star-1}} < \infty,$$

and supercritical *if*

$$\liminf_{z\to\infty} \frac{A'(z)}{z^{m^\star-1}} = 0.$$

Notice that in the case of power-law diffusion, $A(u) = u^m$, subcritical, critical and supercritical respectively correspond to $m > m^\star, m = m^\star$ and $m < m^\star$. Moreover, in the case of the Newtonian or Bessel potential, $m^\star = 2-2/d$ and the critical diffusion exponent of the PKS models discussed in [99, 98, 20] is recovered.

The proof of local existence follows the work of Bertozzi and Slepčev [10], where (4.1) is approximated by a family of uniformly parabolic problems. The primary new difficulty, due to the singularity of the kernel, is obtaining uniform a priori L^∞ bounds, which is overcome here using the Alikakos iteration [2]. Solutions are first constructed on bounded domains.

Theorem 8 (Local Existence on Bounded Domains, $d \geq 2$). *Let $A(u)$ and $\mathcal{K}(x)$ be admissible. Let $u_0(x) \in L^\infty(D)$ be a non-negative initial condition, then* (4.1) *has a weak solution u on $[0,T] \times D$, for some $T > 0$. Additionally, $u \in C([0,T]; L^p(D))$ for $p \in [1,\infty)$.*

In dimensions $d \geq 3$ we also construct local solutions on $\mathbb{R}^d$ by taking the limit of solutions on bounded domains.

Theorem 9 (Local Existence in $\mathbb{R}^d$, $d \geq 3$). *Let $A(u)$ and $\mathcal{K}(x)$ be admissible. Let $u_0(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be a non-negative initial condition, then* (4.1) *has a weak solution u on $\mathbb{R}^d_T$, for some $T > 0$. Additionally, $u \in C([0,T]; L^p(\mathbb{R}^d))$ for all $1 \leq p < \infty$ and the mass is conserved.*

As previously mentioned, the free energy is a dissipated quantity for weak solutions and is a key tool for the global theory.

Proposition 4 (Energy Dissipation). *Weak solutions to* (4.1) *satisfy the energy dissipation inequality* (4.7) *for almost all $t \geq 0$.*

As in [10], uniqueness holds on bounded, convex domains in $d \geq 2$ or on $\mathbb{R}^d$ for $d \geq 3$. The proof also holds for more general diffusion (e.g. fast or strongly degenerate diffusion) or no diffusion at all.

Theorem 10 (Uniqueness). *Let $D \subset \mathbb{R}^d$ for $d \geq 2$ be bounded and convex, then weak solutions to* (4.1) *are unique. The conclusion also holds on $\mathbb{R}^d$ for $d \geq 3$.*

We also prove the following continuation theorem, which generalizes similar theorems used in for instance [22, 20]. The proof follows the well known approach of first bounding intermediate L^p norms and using Alikakos iteration [2] to conclude the solution is bounded in L^∞ (for instance, see [69, 100, 20, 39, 59, 22]).

Theorem 11 (Continuation)**.** *The weak solution to* (4.1) *has a maximal time interval of existence* $T_\star$ *and either* $T_\star = \infty$ *or* $T_\star < \infty$ *and*

$$\lim_{k\to\infty} \limsup_{t \nearrow T_\star} \|(u-k)_+\|_{\frac{2-m}{2-m^\star}} > 0. \tag{4.8}$$

Here m *is such that* $1 \leq m \leq m^\star$ *and* $\liminf_{z\to\infty} A'(z)z^{1-m} > 0$*. In particular, for all* $p > (2-m)/(2-m^\star)$,

$$\lim_{t \nearrow T_\star} \|u\|_p = \infty.$$

Remark 5. *Note that the order of the limits in Theorem 11 is important. In fact, if the ordered is reversed the limit is always zero.*

For the case $m^\star = 2 - 2/d$, Blanchet et al. [20] identified the critical mass for the problem with the Newtonian potential, $\mathcal{K} = c_d |x|^{d-2}$, and $A(u) = u^m$.The authors show that if $M < M_c$ then the solution exists globally and if $M > M_c$ then the solution may blow-up in finite time. There M_c is identified as

$$M_c = \left(\frac{2}{(m^\star - 1)C_{m^\star} c_d}\right)^{1/(2-m^\star)},$$

where $C_{m^\star}$ is the best constant in the Hardy-Littlewood-Sobolev inequality given below in Lemma 19. It is natural to ask the same question for more general cases. In this work we generalize these results to include inhomogeneous kernels and general nonlinear diffusion. First, we state the generalization of the finite time blow-up results.

Theorem 12 (Finite Time blow-up for Critical Problems: $m^\star > 1$)**.** *Let D either be bounded and convex with a smooth boundary or $D = \mathbb{R}^d$. Let $\mathcal{K}$ and $A(u)$ be admissible and satisfy*

(B1) $\mathcal{K}(x) = c\,|x|^{-d/p} + o(|x|^{-d/p})$ *as* $x \to 0$ *for some* $c > 0$ *and* $d/(d-2) \le p < \infty$.

(B2) $x \cdot \nabla\mathcal{K}(x) \le -(d/p)\mathcal{K}(x) + C_1$ *for all* $x \in \mathbb{R}^d$, *for some* $C_1 \ge 0$.

(B3) $A'(z) = m\overline{A}z^{m-1} + o(z^{m-1})$ *as* $z \to \infty$ *for some* $m > 1, \overline{A} > 0$.

(B4) $A(z) \le (m-1)\Phi(z)$ *for all* $z > R$, *for some* $R > 0$.

Suppose the problem is critical, that is $m = m^\star$. Then the critical mass M_c satisfies

$$M_c = \left(\frac{2\overline{A}}{(m^\star - 1)C_{m^\star}c}\right)^{1/(2-m^\star)},$$

and for all $M > M_c$ there exists a solution to (4.1) *which blows up in finite time with $\|u_0\|_1 = M$.*

Theorem 13 (Finite Time blow-up for Supercritical Problems)**.** *Let D be as in Theorem 12. Let $\mathcal{K}$ satisfy **(B1)** and **(B2)** in Theorem 12 and $A(u)$ satisfy **(B3)** and **(B4)** in Theorem 12 with $1 < m < m^\star$. Then for all $M > 0$ there exists a solution which blows up in finite time with $\|u_0\|_1 = M$.*

The Newtonian and Bessel potentials both satisfy these conditions with $C_1 = 0$ (Lemma 2.2, [98]), and so the results apply to PKS with degenerate diffusion. Due to the decay of admissible kernels (Definition 2) condition **(B2)** should only impose a significant restriction on the behavior of $\mathcal{K}$ at the origin. Power-law diffusion satisfies conditions **(B3)** and **(B4)**; however, **(B4)** is also restrictive, for example, $A(u) = u^m - u$ for u large does not satisfy the condition.

The accompanying global existence theorem is significantly more inclusive than the blow-up theorems, both in the kinds of kernels and nonlinear diffusion considered. As in Theorem 12, the estimate of the critical mass only depends on the leading order term of an asymptotic expansion of the kernel at the origin and the growth of the entropy at infinity. The approach used here and in [20, 22] relies on using the energy dissipation inequality (4.7) and the continuation theorem (Theorem 11). The third key component is an inequality which relates the interaction energy $\mathcal{W}(u)$ to the entropy $S(u)$. For $m^\star > 1$ this is the Hardy-Littlewood-Sobolev inequality given in Lemma 19. In this case, the estimate of the critical mass is given by (4.9).

Theorem 14 (Global Well-Posedness for $m^\star > 1$)**.** *Suppose $m^\star > 1$. Then we have the following:*

- *(i) If the problem is subcritical, then the solution exists globally (i.e. $T_\star = \infty$) and is uniformly bounded in the sense $u \in L^\infty((0,\infty) \times D)$.*
- *(ii) If the problem is critical then there exists a critical mass $M_c > 0$ such that if $\|u_0\|_1 = M < M_c$, then the solution exists globally and is uniformly bounded in the sense $u \in L^\infty((0,\infty) \times D)$. The critical mass is estimated below in (4.9).*

Proposition 5 (Critical Mass For $m^\star > 1$)**.** *If $\mathcal{K} = c\,|x|^{-d/p} + o(|x|^{-d/p})$ as $x \to 0$ for some $c \geq 0$ and p, $d/(d-2) \leq p < \infty$, then M_c satisfies,*

$$\lim_{z \to \infty} \frac{\Phi(z)}{z^{m^\star}} - \frac{C_{m^\star}}{2} c M_c^{2-m^\star} = 0. \tag{4.9}$$

If $c = 0$ or $\lim_{z\to\infty} \Phi(z) z^{-m^\star} = \infty$ then we define $M_c = \infty$.

Remark 6. *By Lemma 35, if $\mathcal{K} \in L^{p,\infty}_{loc}$ then $\exists\, \delta, C > 0$ such that $\forall x$, $|x| < \delta$, $\mathcal{K}(x) \leq C\,|x|^{-d/p}$. Then, if the kernel does not admit an asymptotic expansion as* *in Proposition 5, the critical mass M_c can be estimated by,*

$$\lim_{z\to\infty}\frac{\Phi(z)}{z^{m^\star}} - \frac{C_{m^\star}}{2}CM_c^{2-m^\star} = 0.$$

Remark 7. *Note,* $\lim_{z\to\infty}\Phi(z)z^{-m^\star}$ *is always well-defined but is not necessarily finite unless*

$$\limsup_{z\to\infty} A'(z)z^{1-m^\star} < \infty.$$

If the problem is critical then necessarily $\lim_{z\to\infty}\Phi(z)z^{-m^\star} > 0$ *so there always exists a positive mass which satisfies* (4.9)*. Moreover, if the problem is subcritical then necessarily* $\lim_{z\to\infty}\Phi(z)z^{-m^\star} = \infty$*.*

The case $m^\star = 1$ is analogous to the classical PKS problem in 2D, where linear diffusion is critical. For the 2D PKS, the critical mass is given by $M_c = 8\pi$ for both the Newtonian and Bessel potentials [22, 28]. In this work we treat the $m^\star = 1$ case for $d \geq 2$ on bounded domains, recovering the critical mass of the classical PKS, although **(D3)** technically requires the diffusion to be nonlinear and degenerate. The case $d \geq 3$ and $m^\star = 1$ is approached in [64], but the optimal critical mass is not identified. Our estimate is given below in (4.10). As above, the critical mass only depends on the asymptotic expansion of the kernel at the origin and the growth of the entropy at infinity. We first state the analogue of Theorem 12.

Theorem 15 (Finite Time blow-up for Critical Problems $m^\star = 1$)**.** *Let D be a smooth, bounded and convex domain and $d \geq 2$. Suppose $\mathcal{K}$ satisfies*

(C1) $\mathcal{K}(x) = -c\ln|x| + o(\ln|x|)$ *as $x \to 0$ for some $c > 0$.*

(C2) $x\cdot\nabla\mathcal{K}(x) \leq -c + C|x|$ *for all $x \in \mathbb{R}^d$, for some $C \geq 0$.*

(C3) $A(z) \leq \overline{A}z$ *for some $\overline{A} > 0$.*

Then the critical mass M_c satisfies

$$M_c = \frac{2d\overline{A}}{c},$$

and for all $M > M_c$ there exists a solution which blows up in finite time with $\|u_0\|_1 = M$.

The corresponding global existence theorem includes more general kernels and nonlinear diffusion. The proof is similar to Theorem 14, except that the logarithmic Hardy-Littlewood-Sobolev inequality (Lemma 20) is used in place of the Hardy-Littlewood-Sobolev inequality.

Theorem 16 (Global Well-Posedness for $m^\star = 1$ on Bounded Domains)**.** *Suppose $m^\star = 1$ and $d \geq 2$, let D be bounded, smooth and convex. Then we have the following:*

(i) If the problem is subcritical, then the solution exists globally and is uniformly bounded in the sense $u \in L^\infty((0,\infty) \times D)$.

(ii) If the problem is critical then there exists a critical mass, $M_c > 0$, such that if $\|u_0\|_1 = M < M_c$, then the solution exists globally and is uniformly bounded in the sense $u \in L^\infty((0,\infty) \times D)$. The critical mass is estimated below in (4.10)*.*

Proposition 6 (Critical Mass for $m^\star = 1$ on Bounded Domains)**.** *If $\mathcal{K}(x) = -c\ln|x| + o(\ln|x|)$ as $x \to 0$ for some $c \geq 0$, then M_c satisfies,*

$$\lim_{z\to\infty} \frac{\Phi(z)}{z\ln z} - \frac{c}{2d} M_c = 0. \tag{4.10}$$

If $c = 0$ or $\lim_{z\to\infty} \Phi(z)(z\ln z)^{-1} = \infty$ then we define $M_c = \infty$.

Remark 8. *By **(BD)** and **(MN)**, $\exists\,\delta, C > 0$ such that $\forall x$, $|x| < \delta$, $\mathcal{K}(x) \leq -C \ln x$. Therefore, if the kernel does not have the asymptotic expansion required in Proposition 6 then the critical mass M_c may be estimated as,*

$$\lim_{z \to \infty} \frac{\Phi(z)}{z \ln z} - \frac{C}{2d} M_c = 0.$$

Remark 9. *These theorems include many known global existence and finite time blow-up results in the literature including [99, 98, 99, 10, 20, 69, 27]. Our main contributions to the existing theory is the unification of these results and the estimate of the critical mass for inhomogeneous kernels and general nonlinear diffusion. In the case of the Newtonian potential Blanchet et al. showed in [20] that solutions at the critical mass also exist globally. See [44, 13, 21] for the corresponding result for classical 2D PKS.*

4.2.1 Properties of Admissible Kernels

Definition 2 implies a number of useful characteristics which we state here and reserve the proofs for the Appendix C.3. First, we have that every admissible kernel is at least as integrable as the Newtonian potential.

Lemma 16. *Let $\mathcal{K}$ be admissible. Then $\nabla \mathcal{K} \in L^{d/(d-1),\infty}$. If $d \geq 3$, then $\mathcal{K} \in L^{d/(d-2),\infty}$.*

In general, the second derivatives of admissible kernels are not locally integrable, but we may still properly define $D^2\mathcal{K} * u$ as a linear operator which involves a Cauchy principal value integral. By the Calderón-Zygmund inequality (see e.g. [Theorem 2.2 [96]]) we can conclude that this distribution is bounded on L^p for $1 < p < \infty$. The inequality also provides an estimate of the operator norms, which is of crucial importance to the proof of uniqueness.

Lemma 17. *Let $\mathcal{K}$ be admissible and $\vec{v} = \nabla\mathcal{K} * u$. Then $\forall p$, $1 < p < \infty$, $\exists C(p)$ such that $\|\nabla\vec{v}\|_p \leq C(p)\|u\|_p$ and $C(p) \lesssim p$ for $2 \leq p < \infty$.*

One can further connect the integrability of the kernel with the integrability of the derivatives at the origin, which provides a natural extension of Lemma 17 through the Young's inequality for $L^{p,\infty}$.

Lemma 18. *Let $d \geq 3$ and $\mathcal{K}$ be admissible. Suppose γ is such that $1 < \gamma < d/2$. Then $\mathcal{K} \in L_{loc}^{d/(d/\gamma-2),\infty}$ if and only if $D^2\mathcal{K} \in L_{loc}^{\gamma,\infty}$. The same holds for $\nabla\mathcal{K} \in L_{loc}^{d/(d/\gamma-1),\infty}$. In particular, $m^\star = 1 + 1/\gamma - 2/d$ for some $1 < \gamma < d/2$ if and only if $D^2\mathcal{K} \in L_{loc}^{\gamma,\infty}$. Moreover, $m^\star = 1$ if and only if $D^2\mathcal{K} \in L_{loc}^{d/2,\infty}$.*

The following lemma clarifies the connection between the critical exponent and the interaction energy.

Lemma 19. *Consider the Hardy-Littlewood-Sobolev type inequality, for all $f \in L^p$, $g \in L^q$ and $\mathcal{K} \in L^{t,\infty}$ for $1 < p, q, t < \infty$ satisfying $1/p + 1/q + 1/t = 2$,*

$$\left| \int\int f(x)g(y)\mathcal{K}(x-y)dxdy \right| \lesssim \|f\|_p \|g\|_q \|\mathcal{K}\|_{L^{t,\infty}}. \tag{4.11}$$

See [75]. In particular, if $(p+1)/p = m^\star > 1$, then for all $u \in L^1 \cap L^{m^\star}$,

$$\int u(x)u(y)|x-y|^{-d/p}dxdy \leq C_{m^\star}\|u\|_1^{2-m^\star}\|u\|_{m^\star}^{m^\star}. \tag{4.12}$$

Here $C_{m^\star}$, depending only on p and d, is taken to be the best constant for which (4.12) holds for all such u.

Remark 10. *It is not necessarily the case that $C_{m^\star}$ is easily related to the optimal constant in (4.11). It is shown in [20] that $C_{2-2/d}$ is acheived for a fairly explicit family of extremals, but to our knowledge, extremals of (4.12) have not been constructed for other values of $m^\star$.*

If $m^\star = 1$ then we will need the logarithmic Hardy-Littlewood-Sobolev inequality, as in for instance [44, 21].

Lemma 20 (Logarithmic Hardy-Littlewood-Sobolev inequality [29])**.** *Let $d \geq 2$ and $0 \leq f \in L^1$ be such that $\left|\int f \ln f dx\right| < \infty$. Then,*

$$-\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)f(y)\ln|x-y|\,dxdy \leq \frac{\|f\|_1}{d}\int_{\mathbb{R}^d} f\ln f dx + C(\|f\|_1). \qquad (4.13)$$

4.3 Uniqueness

We now prove the uniqueness of weak solutions stated in Theorem 10.

Proof. (**Theorem 10**) The proof follows [6, 10] and estimates the difference of weak solutions in H^{-1}, motivated by the fact that the nonlinear diffusion is monotone in this norm [105]. To this end, if the domain is bounded, we define $\phi(t)$ as the zero mean strong solution of

$$\Delta\phi(t) = u(t) - v(t) \text{ in } D \qquad (4.14)$$

$$\nabla\phi(t)\cdot\nu = 0, \text{ on } \partial D, \qquad (4.15)$$

where ν is the outward unit normal of D. If the domain is $\mathbb{R}^d$ for $d \geq 3$, we let $\phi(t) = -\mathcal{N} * (u - v)$ where $\mathcal{N}$ is the Newtonian potential. In either case, by the integrability and boundedness of weak solutions $u(t)$ and $v(t)$ we can conclude $\phi(t) \in L^\infty(D_T) \cap C([0,T]; \dot{H}^1)$, $\nabla\phi(t) \in L^\infty(D_T) \cap L^2(D_T)$ and ϕ_t solves,

$$\Delta\phi_t = \partial_t u - \partial_t v.$$

Then since $\|u(t) - v(t)\|_{H^{-1}} = \|\nabla\phi(t)\|_2$, we will show that $\|\nabla\phi(t)\|_2 = 0$. During the course of the proof, we integrate by parts on a variety of quantities. If the domain is bounded, then the boundary terms will vanish due to the no-flux conditions (4.4),(4.15). In $\mathbb{R}^d$, the computations are justified as

$\nabla\mathcal{K} * u, \nabla A(u), \nabla\mathcal{K} * v, \nabla A(v), \nabla\phi \in L^2(D_T)$.

By the regularity of $\phi(t)$ and the no-flux boundary conditions (4.15), (4.4) we have possibly up to a set of measure zero,

$$\frac{1}{2}\frac{d}{dt}\int |\nabla\phi(t)|^2\,dx =< \nabla\phi(t), \partial_t\nabla\phi(t) >= - < \partial_t u(t) - \partial_t v(t), \phi(t) > .$$

Therefore, using $\phi(t)$ in the definition of weak solution and (4.15) we have,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |\nabla\phi(t)|^2\,dx &= \int (\nabla A(u(t)) - \nabla A(v(t)))\cdot\nabla\phi(t)dx \\ &- \int (u-v)(\nabla\mathcal{K} * u)\cdot\nabla\phi dx - \int v(\nabla\mathcal{K} * (u-v))\cdot\nabla\phi dx. \\ &:= I_1 + I_2 + I_3.\end{aligned}$$

We drop the time dependence for notational simplicity. Since A is increasing, we have the desired monotonicity of the diffusion,

$$I_1 = -\int (A(u) - A(v))\,(u-v)dx \leq 0.$$

We now concentrate on bounding the advection terms.

We follow [10]. By integration by parts we have,

$$I_2 = \sum_{i,j}\int \partial_i\phi(\partial_{ij}\mathcal{K} * u)\partial_j\phi dx + \sum_{i,j}\int \partial_i\phi(\partial_j\mathcal{K} * u)\partial_{ij}\phi dx. \tag{4.16}$$

If the domain is bounded, we may apply integration by parts,

$$\begin{aligned}\sum_{i,j}\int\partial_i\phi(\partial_j\mathcal{K} * u)\partial_{ij}\phi dx &= -\sum_{i,j}\int\partial_{ij}\phi\partial_j\mathcal{K} * u\partial_i\phi dx - \sum_{i,j}\int \partial_i\phi(\partial_{jj}\mathcal{K} * u)\partial_i\phi dx \\ &+ \sum_{i,j}\int_{\partial D} |\partial_i\phi|^2\,\partial_j\mathcal{K} * u\nu_j dS,\end{aligned}$$

where ν is the unit outward normal to D. As in [10], we have $\nabla\mathcal{K} * u\cdot\nu \leq 0$ on ∂D since D is convex and $\mathcal{K}$ is radially decreasing, so that term is non-positive. If the domain were $\mathbb{R}^d$, such boundary terms would vanish. Therefore by integration by parts again we have,

$$\sum_{i,j} \int \partial_i \phi (\partial_j \mathcal{K} * u) \partial_{ij} \phi dx \leq -\frac{1}{2} \int (\Delta \mathcal{K} * u) |\nabla \phi|^2 \, dx,$$

which together with (4.16) implies,

$$I_2 \lesssim \int \left| D^2 \mathcal{K} * u \right| |\nabla \phi|^2 \, dx.$$

By Hölder's inequality, Lemma 17 and $\nabla \phi \in L^\infty(D_T)$ for $p \geq 2$,

$$\begin{aligned} \int \left| D^2 \mathcal{K} * u \right| |\nabla \phi|^2 \, dx &\leq \left\| D^2 \mathcal{K} * u \right\|_p \left(\int |\nabla \phi|^{2p/(p-1)} \, dx \right)^{(p-1)/p} \\ &\lesssim p \|u\|_p \|\nabla \phi\|_\infty^{2/p} \left(\int |\nabla \phi|^2 \, dx \right)^{(p-1)/p} \\ &\lesssim p \left(\int |\nabla \phi|^2 \, dx \right)^{(p-1)/p}, \end{aligned} \tag{4.17}$$

where the implicit constant depends only on the uniformly controlled L^p norms of u and v.

As for I_3, we compute as in [10]. By the computations in the proof of Lemma 17 we may justify integration by parts on the inside of the convolution, that is,

$$\left\| \sum_j \int \partial_i \mathcal{K}(x-y) \partial_{jj} \phi dx \right\|_2 \lesssim \|\nabla \phi\|_2 .$$

which by Cauchy-Schwarz implies,

$$I_3 \lesssim \|v\|_\infty \|\nabla \phi\|_2^2 . \tag{4.18}$$

Letting $\eta(t) = \int |\nabla \phi(t)|^2 \, dx$, (4.17) and (4.18) imply the differential inequality,

$$\frac{d}{dt} \eta(t) \leq \hat{C} p \max(\eta(t)^{1-1/p}, \eta(t)),$$

where $\hat{C}$ again depends only on the uniformly controlled L^p norms of u, v. The differential equality does not have a unique solution, but all of the solutions are absolutely continuous integral solutions bounded above by the maximal solution $\overline{\eta}(t)$. By continuity, for $t < 1/\hat{C}$ the maximal solution is given by $\overline{\eta}(t) = (\hat{C}t)^p$, hence,

$$\eta(t) \leq \overline{\eta}(t) = (\hat{C}t)^p.$$

For $t < 1/(2\hat{C})$ we then have

$$\eta(t) \leq \overline{\eta}(t) \leq 2^{-p},$$

and we take $p \to \infty$ to deduce that for $t \in [0, 1/(2\hat{C}))$, $\eta(t) = 0$, therefore the solution is unique. This procedure may be iterated to prove uniqueness over the entire interval of existence since the time interval only depends on uniformly controlled norms. □

4.4 Local Existence

4.4.1 Local Existence in Bounded Domains

Let $\tilde{A}(z)$ be a smooth function on $\mathbb{R}^+$ such that $\tilde{A}'(z) > \eta$ for some $\eta > 0$. In addition, let $\vec{v}$ be a given smooth velocity field with bounded divergence. Classical theory gives a global smooth solution to the uniformly parabolic equation

$$u_t = \Delta \tilde{A}(u) - \nabla \cdot (u\vec{v}) \tag{4.19}$$

(see [76]). The solutions obey the global L^∞ bound

$$\|u\|_{L^\infty(D)} \leq \|u_0\|_{L^\infty(D)} e^{\|(\nabla \cdot \vec{v})_-\|_{L^\infty(D_T)} t}. \tag{4.20}$$

We take advantage of this theory to prove existence of weak solutions to (4.1) by regularizing the degenerate diffusion and the kernel. Consider the modified aggregation equation

$$u_t^\epsilon = \Delta A^\epsilon(u^\epsilon) - \nabla \cdot \left(u^\epsilon \left(\nabla \mathcal{J}_\epsilon \mathcal{K} * u^\epsilon\right)\right), \tag{4.21}$$

with corresponding no-flux boundary conditions (4.4). We define

$$A^\epsilon(z) = \int_0^z a'_\epsilon(z)dz, \tag{4.22}$$

where $a'_\epsilon(z)$ is a smooth function, such that $A'(z) + \epsilon \leq a'_\epsilon(z) \leq A'(z) + 2\epsilon$, and the standard mollifier is denoted $\mathcal{J}_\epsilon v$. We first prove existence of solutions to the regularized equation (4.21), this is stated formally in the following proposition.

Proposition 7 (Local Existence for the Regularized Aggregation Diffusion Equation)**.** *Let $\epsilon > 0$ be fixed and $u_0(x) \in C^\infty(\overline{D})$, then* (4.21) *has a classical solution u on D_T for all $T > 0$.*

We obtain the proof of Proposition 7 directly from Theorem 12 in [10]. The proof requires a bound on $\|\nabla A^\epsilon\|_{L^2(D_T)}$, for some $T > 0$. We state this lemma for completeness but reference the reader to [10] for a proof.

Lemma 21 (Uniform Bound on Gradient of $A(u)$)**.** *Let $\epsilon > 0$ be fixed and $u^\epsilon \in L^\infty(D_T)$ be a solution to* (4.21)*. There exists a constant such that:*

$$\|\nabla A^\epsilon(u^\epsilon)\|_{L^2(D_T)} \leq C, \tag{4.23}$$

where, $C = C(T, \|\nabla \mathcal{J}_\epsilon \mathcal{K} * u^\epsilon\|_{L^\infty(D)}, \|u^\epsilon\|_\infty)$.

Remark 11. *The estimate given by* (4.23) *is independent of ϵ.*

Proposition 7 gives a family of solutions $\{u^\epsilon\}_{\epsilon>0}$. To prove local existence to the original problem (4.1) we first need some a priori estimates which are independent of ϵ. Mainly, we obtain an independent-in-ϵ bound on the L^∞ norm of the solution and the velocity field. This is the main difference in the local existence theory from [10]. Due to the singularity of the kernels significantly more is required to obtain these a priori bounds. We first state a lemma, due to Kowalczyk [69] and extended to $d > 2$ and $\mathbb{R}^d$ in [27]. The proof is based on the Alikakos iteration.

Lemma 22 (Iteration Lemma [69, 27]). *Let $0 < T \leq \infty$ and assume that there exists a $c > 0$ and $u_c > 0$ such that $A'(u) > c$ for all $u > u_c$. Then if $\|\nabla\mathcal{K} * u\|_\infty \leq C_1$ on $[0,T]$ then $\|u\|_\infty \leq C_2(C_1)\max\{1, M, \|u_0\|_\infty\}$ on the same time interval.*

Lemma 23 (L^∞ Bound of Solution). *Let $\{u^\epsilon\}_{\epsilon>0}$ be the classical solutions to* (4.21) *on D_T, with smooth, non-negative, and bounded initial data $\mathcal{J}_\epsilon u_0$. Then there exists $C = C(\|u_0\|_1, \|u_0\|_\infty)$ and $T = T(\|u_0\|_1, \|u_0\|_p)$ for any $p > d$ such that for all $\epsilon > 0$,*

$$\|u^\epsilon(t)\|_{L^\infty(D)} \leq C \tag{4.24}$$

for all $t \in [0,T]$.

Proof. For simplicity we drop the ϵ. The first step is to obtain an interval for which the L^p norm of u is bounded. Following the work of [59] we define the function $u_k^\epsilon = (u^\epsilon - k)_+$, for $k > 0$. Due to conservation of mass the following inequality provides a bound for the L^p norm of u given a bound on the L^p norm of u_k,

$$\|u\|_p^p \leq C(p)(k^{p-1}\|u\|_1 + \|u_k\|_p^p). \tag{4.25}$$

We look at the time evolution of $\|u_k\|_p$ and make use of the parabolic regularization (4.22).

Step 1:

$$\frac{d}{dt}\|u_k\|_p^p = p\int u_k^{p-1}\nabla\cdot(\nabla A^\epsilon(u) - u\nabla\mathcal{J}_\epsilon\mathcal{K}*u)\,dx$$
$$= -p(p-1)\int A^{\epsilon'}\nabla u_k\cdot\nabla u dx - p(p-1)\int u u_k^{p-2}\nabla\mathcal{J}_\epsilon\mathcal{K}*u\;dx.$$
$$\leq -\frac{4(p-1)}{p}\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx + p(p-1)\int u_k^{p-1}\nabla u_k\cdot\nabla\mathcal{J}_\epsilon\mathcal{K}*u\;dx$$
$$+ kp(p-1)\int u_k^{p-2}\nabla u_k\cdot\nabla\mathcal{J}_\epsilon\mathcal{K}*u\;dx,$$

where we used the fact that for $l > 0$

$$u(u_k)^l = (u_k)^{l+1} + ku_k^l. \tag{4.26}$$

Hence, integrating by parts once more gives

$$\frac{d}{dt}\|u_k\|_p^p \leq \frac{4(p-1)}{p}\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx - (p-1)\int u_k^p\Delta\mathcal{J}_\epsilon\mathcal{K}*udx$$
$$- kp\int u_k^{p-1}\Delta\mathcal{J}_\epsilon\mathcal{K}*udx$$
$$\leq -C(p)\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx + C(p)\|u_k\|_{p+1}^p\|\Delta\mathcal{J}_\epsilon\mathcal{K}*u\|_{p+1}$$
$$+ C(p)k\|u_k\|_p^{p-1}\|\Delta\mathcal{J}_\epsilon\mathcal{K}*u\|_p$$
$$\leq -C(p)\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx + C(p)\left(\|u_k\|_{p+1}^{p+1} + \|u\|_{p+1}^{p+1}\right)$$
$$+ C(p)k\left(\|u_k\|_p^p + \|u\|_p^p\right).$$

In the last inequality we use Lemma 17. Now, using (4.25) we obtain that

$$\frac{d}{dt}\|u_k\|_p^p dx \leq -C(p)\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx + C(p)\|u_k\|_{p+1}^{p+1}$$
$$+ C(p,k)\|u_k\|_p^p + C(p,k,M).$$

An application of the Gagliardo-Nirenberg-Sobolev inequality gives that for any p such that $d < 2(p+1)$ (see Lemma 33 in the Appendix):

$$\|u\|_{p+1}^{p+1} \lesssim \|u\|_p^{\alpha_2}\left\|u^{p/2}\right\|_{W^{1,2}}^{\alpha_1},$$

where $\alpha_1 = d/p,\ \alpha_2 = 2(p+1) - d$. From the inequality $a^r b^{(1-r)} \leq ra + (1-r)b$ (using that $a = \delta \left\| u^{p/2} \right\|_{W^{1,2}}^2$ and $r = \alpha_1/2$) we obtain

$$\|u\|_{p+1}^{p+1} \lesssim \frac{1}{\delta^{\beta_1}} \|u\|_p^{\beta_2} + r\delta^2 \left\| \nabla u^{p/2} \right\|_2^2 + r\delta^2 \|u\|_p^p .$$

Above $\beta_1,\ \beta_2 > 1$. For k large enough we have that $A'(u) > c > 0$ over $\{u > k\}$; hence, if we choose δ small enough we obtain the final differential inequality:

$$\frac{d}{dt} \|u\|_p^p \lesssim C(p) \|u_k\|_p^{\beta_2} + C(p, k, r\delta) \|u_k\|_p^p + C(p, k, \|u_0\|_1). \tag{4.27}$$

The inequality (4.27) in turns gives a $T_p = T(p) > 0$ such that $\|u_k\|_p$ is bounded on $[0, T_p]$. Inequality (4.25) gives that $\|u\|_p$ remains bounded on the same time interval. Next we prove that the velocity field is bounded in $L^\infty(D)$ on some time interval $[0, T]$. This then allows us invoke Lemma 22 and obtain the desired bound.

Step 2:

Since $\nabla \mathcal{K} \in L^1_{loc}$ and $\nabla \mathcal{K} \mathbf{1}_{\mathbb{R}^d \setminus B_1(0)} \in L^q$ for all $q > d/(d-1)$ (by Lemma 16), we have for all $p > d/(d-1)$,

$$\|\vec{v}\|_p = \|\nabla \mathcal{K} * u\|_p \leq \left\| \nabla \mathcal{K} \mathbf{1}_{B_1(0)} \right\|_1 \|u\|_p + \left\| \nabla \mathcal{K} \mathbf{1}_{\mathbb{R}^d \setminus B_1(0)} \right\|_p M.$$

By Lemma 17 we also have, for all $p, 1 < p < \infty$,

$$\|\nabla \vec{v}\|_p = \left\| D^2 \mathcal{K} * u \right\|_p \lesssim \|u\|_p .$$

By Morrey's inequality we have $\vec{v} \in L^\infty(D_T)$ by choosing some $p > d$ and invoking step one, and Lemma 22 concludes the proof. Note that the bound depends on the geometry of the domain through the constant on the Gagliardo-Nirenberg-Sobolev inequality (Lemma 33). However, this constant is related to the regularity of the domain, and not directly to the diameter of the domain. □

In addition to the a priori estimates the proof of Theorem 8 requires precompactness of $\{u^\epsilon\}_{\epsilon>0}$ in $L^1(D_T)$.

Lemma 24 (Precompactness in $L^1(\Omega_T)$). *The sequence of solutions obtained via Proposition 7, $\{u_\epsilon\}_{\epsilon>0}$, which exist on $[0,T]$, is precompact in $L^1(D_T)$.*

The proof of Lemma 24 follows exactly the work in [10]. The key is to prove that the sequence satisfies the Riesz-Frechet-Kolmogorov Criterion. This relies on the fact that $\|A(u^\epsilon)\|_{L^2(0,T;H^1(D))} \leq C$ uniformly.

Proof. (**Theorem 8**) For a given $\epsilon > 0$, if we regularize the initial condition $u_0^\epsilon(x) = \mathcal{J}_\epsilon u_0(x)$, Proposition 7 gives a solution u^ϵ to (4.21). Furthermore, the proof of Proposition 7 and Lemma 23 gave uniform-in-ϵ bounds on $\|A^\epsilon(u)\|_{L^2(0,T,H^1(D))}$, $\|u^\epsilon\|_{L^\infty(D_T)}$, and $\|u_t^\epsilon\|_{L^2(0,T,H^{-1}(D))}$. By Lemma 23, all solutions exist on $[0,T]$, with T independent of ϵ. Also, recalling that $A^\epsilon(z) \geq A(z)$ and $a_\epsilon'(z) \geq A'(z)$ gives that

$$\|A(u^\epsilon)\|_{L^2(0,T,H^1(D))} \leq C,$$

where C is independent of ϵ. Since $L^2(0,T,H^1(D))$ is weakly compact there exists a ρ such that some subsequence of $\{u^\epsilon\}_{\epsilon>0}$ converges weakly, i.e $A(u^{\epsilon_j}) \rightharpoonup \rho$ in $L^2(0,T,H^1(D))$. Precompactness in L^1 implies strong convergence of u^{ϵ_j} to some $u \in L^1(D_T)$; therefore, $A(u) = \rho$. In fact, the $L^\infty(D_T)$ bound on u^{ϵ_j} gives strong convergence in $L^p(D_T)$, for $1 \leq p < \infty$, via interpolation. Also, Young's inequality gives

$$\begin{aligned}\left\|u^{\epsilon_j}\nabla\mathcal{J}_{\epsilon_j}\mathcal{K} * u^{\epsilon_j} - u\nabla\mathcal{K} * u\right\|_{L^1(D_T)} &\leq \|u\|_{L^\infty(D_T)}\left\|\nabla\mathcal{J}_{\epsilon_j}\mathcal{K} * u^{\epsilon_j} - \nabla\mathcal{K} * u\right\|_{L^1(D_T)} \\ &\quad + \left\|\nabla\mathcal{J}_{\epsilon_j}\mathcal{K} * u^{\epsilon_j}\right\|_{L^\infty(D_T)}\|u^{\epsilon_j} - u\|_{L^1(D_T)} \\ &\lesssim \Big(\|u\|_{L^\infty(D_T)}\|\nabla\mathcal{K}\|_{L^1_{loc}} + \|\nabla\mathcal{K} * u^{\epsilon_j}\|_{L^\infty(D_T)}\Big)\times \\ &\quad \|u^{\epsilon_j} - u\|_{L^1(D_T)}. \end{aligned} \tag{4.28}$$

Therefore, by interpolation u satisfies (4.5). Furthermore, we obtain that $u \in C([0,T];H^{-1}(D))$. To prove that $u(t)$ is continuous with respect to the weak

L^2 topology one uses standard density arguments. Since D is a bounded, u is therefore also continuous in the weak L^1 topology. To prove continuity in the strong L^2 topology we define $F(z) = \int_0^z A(s)ds$ and show that it is continuous in the strong L^1 topology. Indeed, Lemma 30 in the Appendix, see [10] for a proof, gives

$$\lim_{h\to 0}\left|\int (F(u(t)) - F(u(t+h)))\,dx\right| = \lim_{h\to 0}\int_t^{t+h} < u_\tau, A(\tau) > \,d\tau. \tag{4.29}$$

Recall that $\|A(u)\|_{L^\infty(D_T)} \leq A(\|u\|_{L^\infty(D_T)})$ and so $A(u) \in L^2(0,T,H^{-1}(D))$. Therefore, the left hand side of (4.29) goes to 0 as $h \to 0$. Now, we can invoke Lemma 31 in Appendix, [10], to obtain that $u \in C([0,T]; L^2(D))$. Using interpolation the L^∞ bound of u gives that $u \in C([0,T]; L^p(D))$, for $1 \leq p < \infty$. □

4.4.2 Local Existence in $\mathbb{R}^d$

Now we consider solutions to (4.1) in $\mathbb{R}^d$ for $d \geq 3$. We obtain such solution by taking the limit of the solutions in balls centered on the origin with increasing radius n, denoted by B_n. Once again, following [10] we state the following definition.

Proof. (**Theorem 9**) Let B_n be defined as above and consider the truncation of the initial condition on B_n, i.e. $u_0^n = \mathbf{1}_{B_n} u_0$. By Theorem 8, we have a family of solutions $\{u_n\}_{n>0}$ on B_n for all $t \in [0,T]$. Define a new sequence, $\{\tilde{u}_n\}_{n>0}$, where $\tilde{u}_n$ is the zero extension of u_n. The previous work for bounded domains gives the uniform bounds

$$\|\tilde{u}_n\|_{L^\infty(\mathbb{R}^d_T)} \leq C_1, \tag{4.30}$$

$$\|\nabla A(\tilde{u}_n)\|_{L^2(\mathbb{R}^d_T)} \leq C_2. \tag{4.31}$$

The bounds may be taken independent of n since the constant in the Gagliardo-Nirenberg-Sobolev inequality, Lemma 33, does not depend directly on the diameter of the domain and may be taken uniform in $n \to \infty$.

Therefore, there exist $u, w \in L^2(\mathbb{R}^d_T)$ for which $\tilde{u}_n \rightharpoonup u$ and $\nabla A(\tilde{u}_n) \rightharpoonup w$ in $L^2(\mathbb{R}^d_T)$. Furthermore, (4.30) implies $\|u\|_{L^\infty(\mathbb{R}^d_T)} \leq C_1$. Precompactness of $\{\tilde{u}^\epsilon_n\}_{\epsilon>0}$ in $L^1(B_n)$ for fixed $n > 0$ and Theorem 2.33 in [1] gives that $\{\tilde{u}_n\}_{n>0}$ is precompact in $L^1_{loc}(\mathbb{R}^d_T)$. Therefore, up to a subsequence, not renamed, $\tilde{u}_n \to u$ in $L^1_{loc}(\mathbb{R}^d_T)$; thus, $w = \nabla A(u)$. Also, the L^∞ bound gives that $\tilde{u}_n \to u$ in $L^p_{loc}(\mathbb{R}^d)$ for $1 \leq p < \infty$.

In addition, we have the estimate

$$\|\tilde{u}_n \nabla \mathcal{K} * \tilde{u}_n\|_{L^2(\mathbb{R}^d_T)} \leq \|\nabla \mathcal{K} * \tilde{u}_n\|_{L^\infty(\mathbb{R}^d_T)} \|\tilde{u}_n\|_{L^2(\mathbb{R}^d_T)} . \tag{4.32}$$

Therefore, we can extract a subsequence that converges weakly to some $w_1 \in L^2(\mathbb{R}^d_T)$. Since $u\mathbf{1}_{B_n} \in L^\infty(0, T, L^1(\mathbb{R}^d))$ and $u\mathbf{1}_{B_n} \nearrow u$ by monotone convergence $u \in L^\infty(0, T, L^1(\mathbb{R}^d_T))$. Once again, from the estimates performed in the bounded domains $\tilde{u}_n \nabla \mathcal{K} * \tilde{u}_n \to u \nabla \mathcal{K} * u$ in $L^1_{loc}(\mathbb{R}^d_T)$. Therefore, we can identify $w_1 = u \nabla \mathcal{K} * u$.

We now show that $u \in C([0, T]; L^1_{loc}(\mathbb{R}^d))$, which we know to be true, implies that $u \in C([0, T]; L^1(D))$. Let $t_n \to t \in [0, T]$ then for all $R > 0$ we have,

$$\int |u(t_n) - u(t)|\, dx = \int_{B_R} |u(t_n) - u(t)|\, dx + \int_{\mathbb{R}^d \setminus B_R} |u(t_n) - u(t)|\, dx. \tag{4.33}$$

The first term on the right hand side of (4.33) can be bounded by $\epsilon/2$, provided n is chosen large enough, since $u \in C([0, T]; L^1_{loc}(\mathbb{R}^d))$. To bound the second term we first show that $A(u) \in L^1(\mathbb{R}^d_T)$. By **(D3)** we can deduce $\lim_{z \to 0} A(z) z^{-1} = 0$. Then, for $k > 0$ there exists some $0 < C_k < \infty$ such that if $z < k$ then $A(z) \leq Cz$.

Hence,

$$\int A(u)dx = \int_{\{u<k\}} A(u)dx + \int_{\{u\geq k\}} A(u)dx$$
$$\leq CM + A(\|u\|_\infty)\lambda_u(k) < \infty.$$

Therefore, $\|A(u)\|_{L^1(\mathbb{R}^d_T)} \leq C(M, \|u\|_\infty)T$. Now, let $w(x)$ be a smooth radially-symmetric cut-off function with $w(x) = 0$ for $|x| < 1/2$ and $w(x) = 1$ for $|x| \geq 1$. Then consider the quantity, $M_R(t) = \int uw(x/R)dx$. Then formally,

$$\frac{d}{dt}M_R(t) = \frac{1}{R}\int uv \cdot (\nabla w)(x/R)dx + \frac{1}{R^2}\int A(u)(\Delta w)(x/R)dx.$$

Estimating terms in L^∞ gives,

$$\frac{d}{dt}M_R(t) \lesssim \frac{\|v\|_\infty \|u\|_1}{R} + \frac{1}{R^2}\int A(u)dx.$$

Formally, then

$$M_R(t) \lesssim M_R(0) + M\|v\|_{L^1((0,t);L^\infty)} R^{-1} + \|A(u)\|_{L^1((0,t)\times\mathbb{R}^d)} R^{-2}. \tag{4.34}$$

Since $A \in L^1((0,t) \times \mathbb{R}^d)$ and $M_R(0) \to 0$ as $R \to \infty$, by choosing R sufficiently large, the last term of (4.33) can be bounded by $\epsilon/2$. Hence, implies that $u \in C([0,T]; L^1(\mathbb{R}^d))$. Furthermore, via interpolation we obtain that $u \in C([0,T]; L^p(\mathbb{R}^d))$ for $1 \leq p < \infty$.

Conservation of mass can be proved similarly using a cut-off function $w(x) = 1$ for $|x| \leq 1/2$ and $w(x) = 0$ for $|x| \geq 1$, see the proof of Theorem 15 in [10] for a similar proof. □

We are left to prove the energy dissipation inequality (4.7). As expected, the approach is to regularize the energy and take the limit in the regularizing parameters.

Proof. (**Proposition 4**) Define

$$h(u) = \int_1^u \frac{A'(s)}{s} ds,$$

then $\Phi(u) = \int_0^u h(s)ds$. The regularized entropy is defined similarly with $a'_\epsilon(u)$, as defined in (4.22), taking the place of $A'(u)$. Given a smooth solution u^ϵ to (4.21) one can verify,

$$\mathcal{F}_\epsilon(u^\epsilon(t)) + \int_0^t \int \frac{1}{u^\epsilon} \left| a'_\epsilon(u^\epsilon)\nabla u^\epsilon - u^\epsilon \nabla \mathcal{J}_\epsilon \mathcal{K} * u^\epsilon \right|^2 dx d\tau = \mathcal{F}_\epsilon(u^\epsilon(0)). \tag{4.35}$$

Here $\mathcal{F}_\epsilon(u(t))$ denotes the free energy with the regularized entropy and kernel. Once again we take the limit ϵ approaches zero to obtain (4.7). We first show that the entropy converges.

Step 1: The parabolic regularization gives

$$h(z) + \epsilon \ln z \leq h_\epsilon(z) \leq h(z) + 2\epsilon \ln z \quad \text{for } 1 \leq z,$$
$$h(z) + 2\epsilon \ln z \leq h_\epsilon(z) \leq h'(z) + \epsilon \ln z \quad \text{for } z \leq 1.$$

Therefore, writing $\Phi(u) = \int_0^1 h(s)ds + \int_1^u h(s)ds$ one observes that

$$\Phi(u) - 2\epsilon \leq \Phi_\epsilon(u) \leq \Phi(u) + 2\epsilon(u \ln u)_+. \tag{4.36}$$

This will allow us to show convergence of the entropy. In fact,

$$\begin{aligned}\left| \int \Phi_\epsilon(u^\epsilon) - \Phi(u) dx \right| &\leq \int \left| \Phi_\epsilon(u^\epsilon) - \Phi(u^\epsilon) \right| dx + \int \left| \Phi(u^\epsilon) - \Phi(u) \right| dx \\ &\overset{(4.36)}{\leq} 2\epsilon \int (1 + u^\epsilon \ln u^\epsilon)_+ dx + \|\Phi\|_{C^1([0,\|u^\epsilon\|_\infty])} \int |u^\epsilon - u| \, dx. \\ &\leq 2\epsilon \left(|D| + \|\ln u^\epsilon\|_\infty \|u_0^\epsilon\|_1 \right) + C \|u^\epsilon - u\|_1 .\end{aligned}$$

Conservation of mass, boundedness of smooth solutions, and precompactness in L^1_{loc} imply there exists a subsequence, such that as $\epsilon_j \to 0$,

$$\int \Phi_{\epsilon_j}(u_j^\epsilon) dx \to \int \Phi(u) dx.$$

Step 2: To show convergence of the interaction energy we need that for $a.e\ t \in (0,T)$

$$\int u^\epsilon(t)\mathcal{J}_\epsilon\mathcal{K} * u^\epsilon(t)dx \to \int u(t)\mathcal{K} * u(t)dx. \tag{4.37}$$

Since $\mathcal{K} \in L^1_{loc}(D)$ we know that $\|\mathcal{K} * u\|_{L^\infty}$ is bounded; hence, replacing $\nabla\mathcal{K}$ with $\mathcal{K}$ in (4.28) gives the desired result. Finally, we are left to deal with the entropy production functional.

Step 3: From Lemma 10 in [32],

$$\int \frac{1}{u}\left|A'(u)\nabla u - u\nabla\mathcal{K} * u\right|^2 dx \le \liminf_{\epsilon\to 0}\int \frac{1}{u^\epsilon}\left|a'_\epsilon(u^\epsilon)\nabla u^\epsilon - u^\epsilon\nabla\mathcal{J}_\epsilon\mathcal{K} * u^\epsilon\right|^2 dx. \tag{4.38}$$

We also note that this was proved in [10]. The proof of (4.38) relies on a result due to Otto in [86], refer to Lemma 32 in the Appendix. In our case, $u^\epsilon \in L^1(D_T)$ and $J_\epsilon = \nabla A^\epsilon(u^\epsilon) - u^\epsilon\nabla\mathcal{K} * u^\epsilon \in L^1_{loc}(D_T)$. Furthermore, up to a sequence not renamed, $u^\epsilon \rightharpoonup u \in L^2$ and $J_\epsilon \rightharpoonup J$ in L^2, therefore, we can apply Lemma 32.

For the energy dissipation estimate in $\mathbb{R}^d$ we again consider the family of solutions $\{u_r\}$ to (4.1) on B_r (for simplicity let u_r denote the zero-extension of the solutions). Since $u_n(0)\mathbf{1}_{B_n} \nearrow u(0)$ by monotone convergence we obtain that $\mathcal{F}(u_n(0)) \to \mathcal{F}(u(0))$. Noting that $\mathcal{K} \in L^{d/(d-2)}$ allows us to make a modification to (4.32) and obtain that $u_n\mathcal{K} * u_n \rightharpoonup u\mathcal{K} * u$ in $L^2(\mathbb{R}^d_T)$. Furthermore, (4.37) implies that $u_n\mathcal{K}*u_n \to u\mathcal{K}*u$ in L^1_{loc}. We are left to verify the uniform integrability over all space. First note that Morrey's inequality implies

$$\begin{aligned}\|\mathcal{K} * \tilde{u}_n\|_\infty &\lesssim \|\nabla\mathcal{K} * u\|_\infty + \|\mathcal{K} * u_n\|_p \\ &\le \|\nabla\mathcal{K} * u\|_\infty + \|\mathcal{K}\|_{L^{d/(d-2),\infty}}\|u_n\|_{dp/(d+2p)}.\end{aligned}$$

Hence, taking p sufficiently large we obtain that $\mathcal{K} * u_n$ is bounded in $L^\infty(D_T)$. Therefore,

$$\int_{\mathbb{R}^d \setminus B_k} u_n \mathcal{K} * u_n dx \leq \|\mathcal{K} * u_n\|_\infty \int_{\mathbb{R}^d \setminus B_k} u_n dx.$$

This fact along with (4.34) gives that for any $\epsilon > 0$ there exists a k_ϵ sufficiently large such that for all $k > k_\epsilon$

$$\int_{\mathbb{R}^d \setminus B_k} \tilde{u}_n \mathcal{K} * \tilde{u}_n dx \leq \epsilon.$$

This gives convergence of the interaction energy. The result follows from the weak lower semi-continuity of the entropy production functional and $\int \Phi(u)dx$ in L^2. $\square$

4.5 Continuation Theorem

Continuation of weak solutions, Theorem 11, is a straightforward consequence of the local existence theory and the following lemma, which follows substantially the recent work in [20, 69, 22]. This lemma provides a more precise version of Lemma 23 and has a similar proof.

Lemma 25. *Let $\{u^\epsilon\}_{\epsilon>0}$ be the classical solutions to* (4.21) *on D_T, with non-negative initial data $\mathcal{J}_\epsilon u_0$. Suppose there exists T_0, $0 < T_0 \leq \infty$, such that*

$$\sup_{\epsilon>0} \lim_{k\to\infty} \sup_{t\in(0,T_0)} \|(u^\epsilon - k)_+\|_{\frac{2-m}{2-m^\star}} = 0, \tag{4.39}$$

where m is such that $1 \leq m \leq m^\star$ and $\liminf_{z\to\infty} A'(z)z^{1-m} > 0$. Then there exists $C = C(M, \|u_0\|_\infty)$ such that for all $\epsilon > 0$,

$$\sup_{t\in(0,T_0)} \|u^\epsilon(t)\|_\infty \leq C.$$

In particular, if $T_0 = \infty$, then $\{u^\epsilon\}_{\epsilon>0}$ are uniformly bounded for all time, and therefore the weak solution $u(t)$, is uniformly bounded for all time.

Proof. (**Lemma 25**) Let $\overline{q} = (2-m)/(2-m^\star) \geq 1$. It will be convenient to define γ, $1 \leq \gamma \leq d/2$ such that $m^\star = 1 + 1/\gamma - 2/d$. We first bound intermediate L^p norms over the same interval, $(0, T_0)$. Then we use Morrey's inequality and Lemma 22 to finish the proof.

Step 1:

We have two cases to consider, $m^\star = 2 - 2/d$ and $m^\star < 2 - 2/d$, which occurs if $D^2 K \in L^{\gamma,\infty}_{loc}$ for $\gamma > 1$ (Lemma 18). In the former we show that for any $p \in (\overline{q}, \infty)$ we have $u^\epsilon(t)$ uniformly bounded in $L^\infty(0, T_0; L^p)$. In the latter case we only show that for $\overline{q} < p \leq \gamma/(\gamma-1)$ we have $u^\epsilon(t)$ uniformly bounded in $L^\infty(0, T_0; L^p)$. In either case, this is sufficient to apply Lemma 22 and conclude the proof.

Let $k > 0$ be some constant to be determined later and let $u_k = (u-k)_+$. We have dropped the ϵ and time dependence for notational convenience. By conservation of mass and (4.25), it suffices to control $\|u_k\|_p$ for any $k > 0$. Thus, using (4.25) we obtain

$$\frac{d}{dt}\|u_k\|_p^p \leq -p(p-1)\int u_k^{p-2} A'(u)\left|\nabla u\right|^2 dx + p(p-1)\int (u_k^{p-1} + k u_k^{p-2})\nabla u \cdot \mathcal{J}_\epsilon \nabla \mathcal{K} * u dx.$$

Then,

$$\frac{d}{dt}\|u_k\|_p^p \leq -4(p-1)\int A'(u)\left|\nabla u_k^{p/2}\right|^2 dx - \int ((p-1)u_k^p + kpu_k^{p-1})\mathcal{J}_\epsilon \Delta \mathcal{K} * u dx. \tag{4.40}$$

Since the constants are not relevant, we treat the cases together only noting minor differences when they appear. If $m = 2 - 2/d$ we may use Hölder's inequality and then Lemma 17 to obtain a bound on the first term from the advection:

$$\left| \int u_k^p \mathcal{J}_\epsilon \Delta \mathcal{K} * u dx \right| \lesssim_{p,\mathcal{K}} \|u_k\|_{p+1}^p \|u\|_{p+1}.$$

On the other hand, if $\gamma > 1$ we have from the generalized Hardy-Littlewood-Sobolev inequality (4.11) (Lemma (19)),

$$\left| \int u_k^p \mathcal{J}_\epsilon \Delta \mathcal{K} * u dx \right| \lesssim_{p,\mathcal{K}} \|u_k\|_{\alpha p}^p \|u\|_t + C(M) \|u_k\|_p^p,$$

with the scaling condition $1/\alpha + 1/t + 1/\gamma = 2$. Choosing $t = \alpha p$ implies that

$$\frac{1}{\alpha} = \frac{2 - 1/\gamma}{1 + 1/p}. \tag{4.41}$$

Notice that from our choice of p then $1 \leq 1/p + 1/\gamma$; thus, $1/\alpha \leq 1$. Note that in the case when $m = 2 - 2/d$ then $t = \alpha p = p + 1$. Thus we estimate the advection terms,

$$\begin{aligned} \left| \int u_k^p \mathcal{J}_\epsilon \Delta \mathcal{K} * u dx \right| &\lesssim_{p,\mathcal{K}} \|u_k\|_{\alpha p}^p \|u\|_{\alpha p} + C(M) \|u_k\|_p^p \\ &\lesssim \|u_k\|_{\alpha p}^{p+1} + \|u\|_{\alpha p}^{p+1} + C(M) \|u_k\|_p^p \\ &\overset{(4.25)}{\lesssim} \|u_k\|_{\alpha p}^{p+1} + C(M) \|u_k\|_p^p + C(k, M). \end{aligned} \tag{4.42}$$

The lower order terms in the advection can be controlled using Hölder's inequality and Lemma 17,

$$\begin{aligned} \left| \int u_k^{p-1} \mathcal{J}_\epsilon \Delta \mathcal{K} * u dx \right| &\lesssim_p \|u_k\|_p^{p-1} \|u\|_p \\ &\leq \|u_k\|_p^p + \|u\|_p^p \\ &\overset{(4.25)}{\lesssim} \|u_k\|_p^p + C(k, M). \end{aligned} \tag{4.43}$$

We now aim to compare the dissipation term in (4.40) with the estimates (4.42) and (4.43). We use the Gagliardo-Nirenberg-Sobolev inequality (Lemma 33),

$$\|u_k\|_{\alpha p} \lesssim \|u_k\|_{\bar{q}}^{\alpha_2} \left\| u_k^{(p+m-1)/2} \right\|_{W^{1,2}}^{\alpha_1} \tag{4.44}$$

with

$$\alpha_1 = \frac{2d}{p}\left(\frac{(p-\overline{q}/\alpha)}{\overline{q}(2-d)+dp+d(m-1)}\right),$$

and

$$\alpha_2 = 1-\alpha_1(p+m-1)/2 > 0.$$

By the definition of $\overline{q}$ and (4.41) we have that,

$$\alpha_1(p+1)/2 = 1, \tag{4.45}$$

which implies,

$$\|u_k\|_{\alpha p}^{p+1} \lesssim \|u_k\|_{\overline{q}}^{\alpha_2(p+1)}\left(\int u_k^{m-1}\left|\nabla u_k^{p/2}\right|^2 dx + \int u_k^{p+m-1}dx\right). \tag{4.46}$$

If $d = 2$ then necessarily $m = m^\star = 1$ and this inequality will be sufficient. However, for $d \geq 3$, more work must be done. Define,

$$I = \int u_k^{m-1}\left|\nabla u_k^{p/2}\right|^2 dx.$$

Then, for $\beta_1 \leq \alpha_1$ and $(p+m-1)\beta_1/2 < 1$,

$$\beta_1 = \frac{2d(1-\overline{q}/(p+m-1))}{\overline{q}(2-d)+dp+d(m-1)},$$

and $\beta_2 = 1-\beta_1(p+m-1)/2 > 0$, we have the following by Lemma 33,

$$\begin{aligned}\int u_k^{p+m-1}dx &\lesssim \|u_k\|_{\overline{q}}^{(p+m-1)\beta_2}\left(I+\int u_k^{p+m-1}dx\right)^{(p+m-1)\beta_1/2} \\ &\lesssim \|u_k\|_{\overline{q}}^{(p+m-1)\beta_2}\left(I^{(p+m-1)\beta_1/2}+\left(\int u_k^{p+m-1}dx\right)^{(p+m-1)\beta_1/2}\right).\end{aligned}$$

Therefore, by weighted Young's inequality for products,

$$\int u_k^{p+m-1}dx \lesssim \|u_k\|_{\overline{q}}^{(p+m-1)\beta_2}(1+I) + \|u_k\|_{\overline{q}}^{\gamma_0}, \tag{4.47}$$

for some $\gamma_0 > 0$, the exact value of which is not relevant. Putting (4.46) and (4.47) together implies,

$$\|u_k\|_{\alpha p}^{p+1} \lesssim \mathcal{P}(\|u_k\|_{\overline{q}})I + C(\|u_k\|_{\overline{q}}), \tag{4.48}$$

where $\mathcal{P}(z)$ denotes a polynomial such that $\mathcal{P}(z) \to 0$ as $z \to 0$. By definition of m, $\exists\ \delta > 0$ such that for k sufficiently large then $u > k$ implies $A'(u) > \delta u^{m-1}$.Therefore, combining (4.40) with (4.48),(4.42) and (4.43) implies,

$$\begin{aligned}\frac{d}{dt}\left\|u_k\right\|_p^p &\leq -C(p)\delta\int u_k^{m-1}\left|\nabla u_k^{p/2}\right|^2 dx + C(p)\left\|u_k\right\|_{\alpha p}^{p+1}\\ &+ C(M,p)\left\|u_k\right\|_p^p + C(k,M,p)\\ &\leq -\frac{C(p)\delta}{\mathcal{P}(\left\|u_k\right\|_{\overline{q}})}\left\|u_k\right\|_{\alpha p}^{p+1} + C(p)\left\|u_k\right\|_{\alpha p}^{p+1}\\ &+ C(M,p)\left\|u_k\right\|_p^p + C(k,M,p,\left\|u_k\right\|_{\overline{q}}).\end{aligned}$$

By interpolation against L^1, conservation of mass and $\alpha \geq 1$ we have

$$\left\|u_k\right\|_p^p \lesssim_M 1 + \left\|u_k\right\|_{p\alpha}^{p+1}.$$

Therefore, by assumption (4.39) we may choose k sufficiently large such that there exists some $\eta > 0$ which satisfies the following for all $t \in (0, T_0)$,

$$\frac{d}{dt}\left\|u_k\right\|_p^p \leq -\eta\left\|u_k\right\|_p^p + C(k,M,p,\left\|u_k\right\|_{\overline{q}}).$$

It follows that $\left\|u_k\right\|_p$ is bounded uniformly on $(0, T_0)$.

Step 2:

The control of these L^p norms will enable us to invoke Lemma 22 and conclude $u^\epsilon(t)$ is bounded uniformly in $L^\infty(D_{T_0})$. Since $\nabla\mathcal{K} \in L^1_{loc}$ and $\nabla\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_1(0)} \in L^q$ for all $q > d/(d-1)$ (by Lemma 16), we have for any $q > d/(d-1)$

$$\left\|\vec{v}\right\|_q = \left\|\nabla\mathcal{K} * u\right\|_q \leq \left\|\nabla\mathcal{K}\mathbf{1}_{B_1(0)}\right\|_1\left\|u\right\|_q + \left\|\nabla\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_1(0)}\right\|_q M.$$

If $\gamma > 1$, then we may choose $q \in (d/(d-1), \gamma/(\gamma-1)]$, since in this case necessarily $d \geq 3$. Otherwise we may choose $q > d/(d-1)$ arbitrarily. Then, step one implies $\vec{v} \in L^\infty((0,T_0); L^q)$. If $\gamma > 1$ then, noting that Definition 2 implies $D^2\mathcal{K}\mathbf{1}_{\mathbb{R}^d \setminus B_1(0)} \in L^q$ for all $q > 1$,

$$\|\nabla \vec{v}\|_{d+1} = \left\|D^2\mathcal{K} * u\right\|_{d+1} \leq \left\|D^2\mathcal{K}\mathbf{1}_{B_1(0)}\right\|_{L^{\gamma,\infty}} \|u\|_p + \left\|\nabla\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_1(0)}\right\|_{d+1} M,$$

for $p = \gamma(d+1)/(d(\gamma-1)+2\gamma-1)$. Note that

$$1 < p = \frac{\gamma(d+1)}{d(\gamma-1)+2\gamma-1} \leq \frac{\gamma}{\gamma-1}.$$

On the other hand, if $m^\star = 2 - 2/d$ then the above proof shows that $u^\epsilon(t)$ is bounded uniformly in $L^\infty((0,T_0); L^p)$ for all $p < \infty$. Therefore, by Lemma 17 we have $\|\nabla\vec{v}\|_p \lesssim \|u\|_p \lesssim 1$, for all $1 < p < \infty$. In either case, this is sufficient to apply Morrey's inequality and conclude that $\|\vec{v}\|_\infty$ is uniformly bounded on $(0,T_0)$. By Lemma 22 we then have that u^ϵ is uniformly bounded in $L^\infty(D_{T_0})$ and we have proved the lemma. As in Lemma 23, the uniform bounds depend on the domain but not it's diameter. □

Remark 12. *The proof of this lemma directly implies global well-posedness in the subcritical case since* (4.39) *is only necessary in the critical and supercritical cases. Moreover, in the critical case, one may prove directly that there exists some* M_0 *such that if* $M < M_0$ *the solution is global. However,* M_0 *will generally depend on the constant of the Gagliardo-Nirenberg-Sobolev inequality, as in [99, 100, 59]. As discussed in the recent works of [20, 22], the use of a continuation theorem will allow for a more accurate estimate of the critical mass through the use of the free energy.*

Proof. (**Theorem 11**) Suppose, for contradiction, that the weak solution cannot be continued past $T_\star < \infty$ and (4.8) fails. As the regularized problems are bounded, this implies the hypotheses of Lemma 25 are satisfied on $(0,T_\star)$, and therefore $\sup_{\epsilon>0}\sup_{t\in(0,T_\star)}\|u^\epsilon(t)\|_p \leq \eta$ as $t \nearrow T_\star$ for some $p > \overline{q}$ and $\eta > 0$.

By the proof of Lemma 23, for any $\eta > 0$ there exists a $\tau = \tau(\eta, M) > 0$ such that if $\|u_0\|_p < \eta$ then $\|u^\epsilon\|_p \leq C$ for all $\epsilon > 0$. Therefore, we may choose some $t_n < T_\star$ such that τ satisfies $t_n + \tau > T_\star$ and, by Theorems 8 and 9, we construct a solution $\tilde{u}(x,t)$ on the time interval $[t_n, t_n+\tau)$. By uniqueness, $\tilde{u}(x,t) = u(x,t)$ a.e. for $t \in [t_n, T_\star)$; hence, it is a genuine extension of the original solution $u(x,t)$. However, it exists on a longer time interval which is a contradiction. □

4.6 Global Existence

We now prove Theorem 14. We first note that the entropy is bounded below uniformly in time, which is a consequence of assumption **(D3)** of Definition 3.

Lemma 26. *Let $u(x,t)$ be a weak solution to* (4.1)*. Then,*

$$\int \Phi(u(t))dx \geq -CM.$$

Proof. Let $h(z) = \int_1^z A'(s)s^{-1}ds$. By Definition 3, **(D3)**, for $z \leq 1$,

$$h(z) \geq -C > -\infty.$$

Therefore,

$$\begin{aligned}\int \Phi(u)dx = \int\int_0^u h(z)dzdx &\geq \int \mathbf{1}_{\{u\leq 1\}} \int_0^u h(z)dz + \mathbf{1}_{\{u\geq 1\}} \int_0^1 h(z)dzdx. \\ &\geq -\int \mathbf{1}_{\{u\leq 1\}}Cu - \mathbf{1}_{\{u\geq 1\}}Cdx \\ &\geq -2C\|u\|_1.\end{aligned}$$

where the last line followed from Chebyshev's inequality. □

4.7 Theorem 14: $m^\star > 1$

Proof. (**Theorem 14**) We only prove the second assertion under the hypotheses of Proposition 5, as the subcritical case follows similarly. By the energy dissipation inequality (4.7) we have for all time $0 \leq t < T_\star$,

$$S(u(t)) - \mathcal{W}(u(t)) \leq \mathcal{F}(u_0) := F_0. \tag{4.49}$$

We drop the time dependence of $u(t)$ for notational simplicity. By the assumption on $\mathcal{K}$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|\mathcal{K}(x)| \leq (c+\epsilon)|x|^{-d/p}$ for $|x| < \delta$. By Lemma 19 we have,

$$\int \Phi(u)dx - \frac{1}{2}C_{m^\star} M^{2-m^\star}(c+\epsilon)\|u\|_{m^\star}^{m^\star} \leq F_0 + \frac{1}{2}\left\|\mathcal{K}|_{B_\delta(0)}\right\|_\infty M^2,$$

By (4.9) and $M < M_c$, there exists $\epsilon > 0$ small enough and $\alpha, k > 0$ such that

$$\Phi(z)z^{-m^\star} - \frac{1}{2}C_{m^\star}M^{2-m^\star}(c+\epsilon) \geq \alpha > 0, \text{ for all } z > k. \tag{4.50}$$

By Lemma 26 we have,

$$\int_{\{u>k\}} u^{m^\star}\left(\Phi(u)u^{m^\star} - \frac{1}{2}C_{m^\star}M^{2-m^\star}(c+\epsilon)\right)dx - \frac{1}{2}\int_{\{u<k\}} C_{m^\star}M^{2-m^\star}(c+\epsilon)\,u^{m^\star}dx \leq F_0 + C(\delta, M),$$

and by (4.50),

$$\alpha\int_{\{u>k\}} u^{m^\star}dx - \frac{1}{2}C_{m^\star}M^{2-m^\star}(c+\epsilon)\int_{\{u<k\}} u^{m^\star}dx \leq F_0 + C(M,\delta).$$

By mass conservation we have that $\|u\|_{m^\star}$ is a priori bounded independent of time and Theorem 11 and Lemma 25 implies global existence and uniform boundedness. □

4.8 Theorem 16: $m^\star = 1$

The proof of Theorem 16 follows similarly, but requires the logarithmic Hardy-Littlewood-Sobolev inequality (Lemma 20) as opposed to Lemma 19.

Proof. (**Theorem 16**)

We only prove the second assertion under the hypotheses of Proposition 6, as the subcritical case follows similarly. We will again use Theorem 11 and prove

$$\sup_{t\in(0,\infty)} \int (u\ln u)_+ dx < \infty.$$

By the energy dissipation inequality (4.7) we again have (4.49). By the assumptions of Proposition 6, for all $\epsilon > 0$ there exists $\delta > 0$ such that,

$$\int \Phi(u)dx + (c+\epsilon)\frac{1}{2}\int\int_{|x-y|<\delta} u(x)u(y)\ln|x-y|\,dxdy \leq C(F_0,\delta,M).$$

By D bounded, the logarithmic Hardy-Littlewood-Sobolev inequality (4.13) implies,

$$\int \Phi(u)dx - (c+\epsilon)\frac{M}{2d}\int u\ln u dx \leq C(F_0,\delta,M,\mathrm{diam}D).$$

Choosing $k > 0$ large and recalling Lemma 26 implies

$$\int_{\{u>k\}} u\ln u\left(\frac{\Phi(u)}{u\ln u} - (c+\epsilon)\frac{M}{2d}\right)dx - (c+\epsilon)\int_{\{u<k\}} u\ln u dx \leq C(F_0,\delta,M,\mathrm{diam}D).$$

As in the proof of Theorem 14, by conservation of mass, (4.10) and $M < M_c$, we may choose $\epsilon > 0$ small enough and k large enough such that

$$\int_{\{u>k\}} u\ln u dx \leq C(F_0,M,\mathrm{diam}D).$$

□

4.9 Finite Time blow-up

In this section we prove Theorem 13 and Theorem 12. We prove Theorem 13 as it is somewhat easier, though the technique is the same as that used to prove Theorem 12.

4.10 Supercritical Case: Theorem 13

For Theorem 13 we state the following lemma, which provides insight into the nature of the supercritical cases. The proof and motivation follows [20].

Lemma 27. *Define* $\mathcal{Y}_{\mathcal{M}} = \left\{u \in L^1 \cap L^{m^\star} : u \geq 0, \|u\|_1 = M\right\}$*. Suppose* $\mathcal{K}$ *satisfies* ***(B1)*** *and* $A(u)$ *satisfies* ***(B3)*** *for some* $m > 1, \overline{A} > 0$*. Suppose further that the problem is supercritical, that is,* $m < m^\star$*. Then* $\inf_{\mathcal{Y}_{\mathcal{M}}} \mathcal{F} = -\infty$*. Moreover, there exists an infimizing sequence with vanishing second moments which converges to the Dirac delta mass in the sense of measures.*

Proof. Let $0 < \theta < 1$, $\alpha = d/p$. Then by Lemma 19 there exists $h^\star$ such that,

$$\theta C_{m^\star} \leq \frac{\left|\int\int h^\star(x)h^\star(y)\,|x-y|^{-\alpha}\,dxdy\right|}{\|h^\star\|_1^{2-m^\star}\,\|h^\star\|_{m^\star}^{m^\star}} \leq C_{m^\star}. \tag{4.51}$$

We may assume without loss of generality that $h^\star \geq 0$, since replacing $h^\star$ by $|h^\star|$ will only increase the value of the convolution. By density, we may take $h^\star \in C_c^\infty$ and therefore with a finite second moment.

Let $\mu = \|h^\star\|_1^{1/d} M^{-1/d}$, $\lambda > 0$ and $h_\lambda(x) = \lambda^d h^\star(\lambda\mu x)$. First note, by **(B3)**,

$\forall \epsilon > 0$, $\exists\, R > 0$ such that,

$$\begin{aligned}\int \Phi(h_\lambda)dx &= \int \int_0^{h_\lambda} \int_1^s \frac{A'(z)}{z} dzdsdx \\ &\leq \int \int_0^{h_\lambda} \int_R^{\max(s,R)} (m\overline{A} + \epsilon) z^{m-2} dz + \int_1^R \frac{A'(z)}{z} dzdsdx \\ &\leq \frac{\overline{A} + \epsilon}{m-1} \left\| h_\lambda \right\|_m^m + C(R) \left\| h_\lambda \right\|_1 . \end{aligned} \tag{4.52}$$

By **(B1)** and $h^\star \in C_c^\infty$, $\forall\, \epsilon > 0$, $\exists \lambda > 0$ sufficiently large such that,

$$-\mathcal{W}(t) \leq -(c - \epsilon) \frac{\mu^{-2d+\alpha} \lambda^\alpha}{2} \int \int h^\star(x) h^\star(y) \left| x - y \right|^{-\alpha} dxdy. \tag{4.53}$$

Combining (4.53),(4.52) with (4.51) and Lemma 19, we have for λ, R sufficiently large,

$$\begin{aligned}\mathcal{F}(h_\lambda) \leq{}& \frac{\lambda^{dm-d} M}{(m-1) \left\| h^\star \right\|_1} (\overline{A} + \epsilon) \left\| h^\star \right\|_m^m - \lambda^\alpha (\theta - \epsilon) \frac{C_{m^\star}}{2} \left(\frac{\left\| h^\star \right\|_1}{M} \right)^{-2+\alpha/d} \left\| h^\star \right\|_1^{2-m^\star} \left\| h^\star \right\|_{m^\star}^{m^\star} \\ &+ C(R) \mu^{-d} \left\| h^\star \right\|_1 . \end{aligned}$$

By supercriticality, we have $\alpha = dm^\star - d > dm - d$, and so for $\epsilon < \theta$, we take $\lambda \to \infty$ to conclude that for all values of the mass $M > 0$ we have $\inf_{\mathcal{Y}_\mathcal{M}} \mathcal{F} = -\infty$. Moreover, since $h^\star \in C_c^\infty$, the second moment of h_λ goes to zero and h_λ converges to the Dirac delta mass in the sense of measures. □

Proof. (**Theorem 13**) We may justify the formal computations for weak solutions using the regularized problems and taking the limit but we do not include such details. We treat both bounded and unbounded domains together pointing out the differences when they appear. Let

$$I(t) = \int \left| x \right|^2 u(x,t) dx.$$

If the domain is bounded then by (4.4),

$$\frac{d}{dt}I(t) = 2d\int A(u)dx + 2\int\int u(x)u(y)x\cdot\nabla\mathcal{K}(x-y)dxdy - \int_{\partial D} A(u)x\cdot\nu dS$$
$$= 2d\int A(u)dx + \int\int (x-y)\cdot\nabla\mathcal{K}(x-y)u(x)u(y)dxdy - \int_{\partial D} A(u)x\cdot\nu(x)dS, \tag{4.54}$$

where the second integral was obtained by symmetrizing in x and y, the time dependence was dropped for notational simplicity and $\nu(x)$ denotes the outward unit normal of D at $x \in \partial D$. By translation invariance and convexity of D, we may assume without loss of generality that $x\cdot\nu(x) \geq 0$. For the rest of the proof we may treat bounded domains and $D = \mathbb{R}^d$ together, since for each,

$$\frac{d}{dt}I(t) \leq 2d\int A(u)dx + 2\int\int u(x)u(y)x\cdot\nabla\mathcal{K}(x-y)dxdy.$$

We use **(B2)** on $\mathcal{K}$, to obtain

$$\frac{d}{dt}I(t) \leq 2d\int A(u)dx - 2d/p\mathcal{W}(u) + C_1M^2.$$

By **(D3)**, **(B4)** and Lemma 26,

$$\int A(u)dx = \int_{\{u<R\}} A(u)dx + \int_{\{u>R\}} A(u)dx$$
$$\leq C(M) + (m-1)\int_{\{u>R\}}\Phi(u)dx$$
$$\leq C(M) + (m-1)\int\Phi(u)dx.$$

Using that $2d(m-1) < 2d(m^\star - 1) = 2d/p$ we have,

$$\frac{d}{dt}I(t) \leq 2d(m-1)\mathcal{F}(u) + C(M, C_1).$$

We use the energy dissipation inequality (4.7) to bound the first term,

$$\frac{d}{dt}I(t) \leq 2d(m-1)\mathcal{F}(u_0) + C(M, C_1).$$

From this differential inequality, the second moment will be zero in finite time and the the solution blows up in finite time if

$$\mathcal{F}(u_0) < -\frac{C(M, C_1)}{2d(m-1)}.$$

By Lemma 27, we may always find initial data with any given mass $M > 0$ such that this is true, since there exists infimizing sequences with vanishing second moments. The final assertion follows from Theorem 11. Indeed, we have

$$T_\star \leq \frac{I(0)}{2d(m-1)\mathcal{F}(u_0) + C(M, C_1)}.$$

□

4.11 Critical Case: Theorems 12 and 15

The proof of Theorem 12 follows the proof of Theorem 13.

Lemma 28. *Define $\mathcal{Y}_{\mathcal{M}} = \{u \in L^1 \cap L^\infty : u \geq 0, \|u\|_1 = M\}$. Suppose $\mathcal{K}$ satisfies **(B1)** and $A(u)$ satisfies **(B3)** for $m > 1$ and $\overline{A} > 0$. Suppose further that the problem is critical, that is, $m = m^\star$ and let M_c satisfy* (4.9)*. If M satisfies $M > M_c$, then $\inf_{\mathcal{Y}_{\mathcal{M}}} \mathcal{F} = -\infty$. Moreover, there exists an infimizing sequence with vanishing second moments which converges to the Dirac delta mass in the sense of measures.*

Proof. We may proceed as in the proof of Lemma 27, but instead choose $\theta \in \left((M_c/M)^{2-m^\star}, 1\right)$. Let $\alpha = d/p$. By optimality of $C_{m^\star}$, as before there exists $h^\star$ such that,

$$\theta C_{m^\star} \leq \frac{\left|\int\int h^\star(x)h^\star(y)\,|x-y|^{-\alpha}\,dxdy\right|}{\|h^\star\|_1^{2-m^\star}\,\|h^\star\|_{m^\star}^{m^\star}} \leq C_{m^\star}. \tag{4.55}$$

As above, we assume $h^\star \geq 0$ and $h^\star \in C_c^\infty$.

Let $\mu = \|h^\star\|_1^{1/d} M^{-1/d}$, $\lambda > 0$ and $h_\lambda(x) = \lambda^d h^\star(\lambda\mu x)$. By **(B1)** and **(B3)**, $\forall\, \epsilon > 0$ there exists a λ and R sufficiently large such that by $h^\star \in C_c^\infty$,

$$\begin{aligned}\mathcal{F}(h_\lambda) \leq & \frac{\lambda^{dm-d} M}{(m^\star - 1)\|h^\star\|_1}(\overline{A} + \epsilon)\|h^\star\|_{m^\star}^{m^\star} + C(R)\mu^{-d}\|h^\star\|_1 \\ & - \frac{(\theta - \epsilon) C_{m^\star}}{2}\left(\frac{\|h^\star\|_1}{M}\right)^{-2+\alpha/d} \lambda^\alpha \|h^\star\|_1^{2-m^\star}\|h^\star\|_{m^\star}^{m^\star}\end{aligned}$$

However, in this case $\alpha = dm - d$ and $m = m^\star$, therefore by (4.55) and Lemma 19,

$$\mathcal{F}(h_\lambda) \leq \lambda^{dm^\star - d}\|h^\star\|_{m^\star}^{m^\star}\left[\frac{M(\overline{A} + \epsilon)}{(m^\star - 1)\|h^\star\|_1} - \frac{(\theta - \epsilon) C_{m^\star}}{2}\left(\frac{\|h^\star\|_1}{M}\right)^{-2+\alpha/d}\|h^\star\|_1^{2-m^\star}\right].$$

Then,

$$\mathcal{F}(h_\lambda) \leq \lambda^{dm^\star - d}\frac{\|h^\star\|_{m^\star}^{m^\star}}{\|h^\star\|_1}\left[\frac{M(\overline{A} + \epsilon)}{(m^\star - 1)} - \frac{(\theta - \epsilon)}{2} C_{m^\star} M^{2-\alpha/d}\right].$$

Then since $\overline{A}/(m^\star - 1) = C_{m^\star} M_c^{2-m^\star}/2$ and $\alpha/d - 1 = 2 - m^\star$ we have,

$$\mathcal{F}(h_\lambda) \leq \lambda^{dm^\star - d}\frac{\|h^\star\|_m^m}{2\|h^\star\|_1} C_{m^\star} M^{2-\alpha/d}\left[\left(1 + \frac{\epsilon}{\overline{A}}\right)\left(\frac{M_c}{M}\right)^{2-m^\star} - (\theta - \epsilon)\right].$$

Since $\theta > (M_c/M)^{2-m^\star}$ we may take ϵ sufficiently small and $\lambda \to \infty$ to conclude that $\inf_{\mathcal{Y}_\mathcal{M}} \mathcal{F} = -\infty$. As before, h_λ converges to the Dirac delta mass in the sense of measures. □

Proof. (Theorem 12) The theorem follows from a Virial identity as in Theorem 13. □

Proof. (Theorem 15) As in Theorem 13 we have by **(C2)**, **(C3)** and if D is bounded, the convexity of the domain,

$$\begin{aligned}\frac{d}{dt} I(t) &\leq 2d\overline{A}\int A(u)dx + \int\int u(x)u(y)(x-y)\cdot\nabla\mathcal{K}(x-y)dxdy \\ &\leq 2dM\left(\overline{A} - \frac{cM}{2d}\right) + C_1 M^{3/2} I^{1/2}.\end{aligned}$$

Clearly, if $M > M_c$ then $I \to 0$ in finite time if $I(0)$ is sufficiently small. □

APPENDIX A

Appendix: Mathematical Theory

A.1 Newtonian Potential

One of the mathematical tools which will play a key role in much of this work is the Newtonian potential, $\mathcal{N}(x)$, which is the fundamental solution to Laplace's equation

$$\Delta u = 0,$$

in $\mathbb{R}^d$ for $d \geq 2$. Recognizing that Laplace's equation is invariant under rotations one can find that that

$$\mathcal{N}(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{n(n-1)\alpha_n} |x|^{2-n} & n \geq 3, \end{cases} \tag{A.1}$$

where α_n is the volume of the unit ball in $\mathbb{R}^n$, see for example [46]. The singularity of at the origin and the slow decay as $|x| \to \infty$ is an issue which we have to deal with through a lot of this work. By taking the derivatives directly to (A.1) we obtain that

$$|\nabla \mathcal{N}| \leq C |x|^{-n}$$
$$|\Delta \mathcal{N}| \leq C |x|^{-n-2}.$$

Because of the singularity mentioned $\mathcal{N}(x) \notin L^p(\mathbb{R}^n)$ for any p. However, $\mathcal{N}$, ∇N are locally integrable for $n \geq 3$. This is not true for $\Delta \mathcal{N}$. Therefore, it will be convenient to use weak L^p spaces ($L^{p,\infty}$).

A.2 Weak L^p Spaces

In the previous section we saw that the Newtonian potential does not have enough decay as $|x| \to \infty$ to be in any L^p spaces. This is just one example that might motivate generalizing L^p spaces. One generalization, known as Lorentz Spaces $L^{p,\infty}$, was introduced by George Lorentz [80]. Given a measure space (S, μ) these spaces are equipped with the semi-norm (triangle inequality fails)

$$\|f\|_{L^{p,\infty}} = \inf \{C \mid \lambda_f(t) \leq C^p/t^p \; \forall \, t > 0\}, \tag{A.2}$$

where $\lambda_f(t) = \mu\{x \in S : f(x) > t\}$. For $p > 1$ these spaces are Banach spaces and $L^p \subset L^{p,\infty}$ [50]. The Newtonian potential characterizes these weak L^p spaces.

A.3 Sobolev Spaces

Sobolev spaces can be extended for $s \in \mathbb{R}$ by defining the norm

$$\|f\|_{H^s} := \frac{1}{(2\pi)^{d/2}} \left\| (1 + |\xi|^2)^{1/2} \hat{f} \right\|_{L^2_\xi},$$

this definition is inspired by the Fourier Transform (see for example [101]). For $s \in \mathbb{N}$ this definition is equivalent to (2.3). Furthermore,

$$\|u\|_2 \leq \|u\|_{\dot{H}^1} \|u\|_{\dot{H}^{-1}}. \tag{A.3}$$

A.3.1 Sobolev Embeddings

Theorem 17 (Extended Sobolev Inequalities in Bounded Domains)**.** *Let Ω be a bounded domain with $\partial\Omega$ in C^m, and let u be any function in $W^{m,r}(\Omega) \cap L^p(\Omega)$, $1 \leq r, q \leq \infty$. For any integer j, $0 \leq j \leq m$, and for any number a in the interval $j/m \leq a \leq 1$, set*

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}.$$

If $m - j - n/r$ is a nonnegative integer, then

$$\left\|D^j u\right\|_{L^p} \leq C \left\|u\right\|_{W^{m,r}}^{a} \left\|u\right\|_{L^q}^{(1-a)}. \tag{A.4}$$

If $m - j - n/r$ is a nonnegative integer, then (A.4) *holds for $a = j/m$. The constant C depends only on Ω, r, q, m, j, a.*

Theorem 18 (Gagliardo-Nirenberg inequality in $\mathbb{R}^d$)**.** *Let $1 < p < q \leq \infty$ and $s > 0$ be such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{\theta s}{d}$$

for some $0 < \theta < 1$. Then for any $u \in W^{s,p}$ we have

$$\left\|u\right\|_q \lesssim_{d,p,q,s} \left\|u\right\|_p^{1-\theta} \left\|u\right\|_{\dot{W}^{s,p}}^{\theta}.$$

APPENDIX B

Appendix: Part I

B.1 Chapter 2 Additional Computations

B.1.1 Computations for Theorem 4

Computations for F_2

$$\|F_2(v_1) - F_2(v_2)\|_2 \leq \left\|J_\epsilon^2 \Delta(\rho_1 - \rho_2)\right\|_2 + 2\left\|J_\epsilon\left[\nabla \cdot \frac{\rho_1}{A_1} J_\epsilon \nabla A_1 - \nabla \cdot \frac{\rho_2}{A_2} J_\epsilon \nabla A_2\right]\right\|_2$$
$$+ \|\rho_1 A_1 - A_2 \rho_2\|_2 = S_1 + S_2 + S_3.$$

The terms S_1 and S_3 appeared in the inequality for F_1; therefore, we are only concerned with S_2:

$$\begin{aligned}
\frac{1}{2} S_2 &\lesssim \frac{1}{\epsilon^2}\left\|\frac{\rho_1}{A_1} J_\epsilon \nabla A_1 - \frac{\rho_2}{A_2} J_\epsilon \nabla A_2\right\|_1 \\
&\leq \frac{1}{\epsilon^2}\left(\left\|\frac{\rho_1}{A_1} J_\epsilon \nabla\left(A_1 - A_2\right)\right\|_1 + \left\|J_\epsilon \nabla A_2\left(\frac{\rho_1}{A_1} - \frac{\rho_2}{A_2}\right)\right\|_1\right) \\
&\lesssim \frac{1}{\epsilon^2}\left(\left|\frac{\rho_1}{A_1}\right|_\infty \left\|D\left\{J_\epsilon \nabla\left(A_1 - A_2\right)\right\}\right\|_0 + \left|J_\epsilon \nabla\left(A_1 - A_2\right)\right|_\infty \left\|D\left(\frac{\rho_1}{A_1}\right)\right\|_0\right) \\
&\quad + \frac{1}{\epsilon^2}\left(\left|J_\epsilon \nabla A_2\right|_\infty \left\|D\left(\frac{\rho_1}{A_1} - \frac{\rho_2}{A_2}\right)\right\|_0 + \left|\frac{\rho_1}{A_1} - \frac{\rho_2}{A_2}\right|_\infty \left\|D J_\epsilon \nabla A_1\right\|_0\right) \\
&= \frac{1}{\epsilon^2}\left(R_1 + R_2 + R_3 + R_4\right).
\end{aligned}$$

R_1 can be easily bounded, without any additional factors of $1/\epsilon$, by $\left|A_1^{-1}\right|_\infty |\rho_1|_\infty \|A_1 - A_2\|_2$. On the other hand, for R_2 we need to use (5) of *Lemma*

2.1 and (*1*) of Lemma 2.2. More precisely, we have:

$$R_2 \lesssim \frac{1}{\epsilon} \|A_1 - A_2\|_2 \left(\left|\frac{1}{A_1}\right|_\infty \|\rho_1\|_1 + \left|\frac{1}{A_1}\right|_\infty^2 \|A\|_1 |\rho_1|_\infty \right).$$

The next term requires more work, basically repeated applications of the Lemma 2.

$$\begin{aligned} R_3 &\lesssim \frac{1}{\epsilon} \|A_2\|_2 \left\| \frac{A_2\rho_1 - A_1\rho_2}{A_1 A_2} \right\|_1 \\ &\lesssim \frac{1}{\epsilon} \|A_2\|_2 \left\{ \left|\frac{1}{A_1A_2}\right|_\infty \|\rho_1 A_2 - \rho_2 A_1\|_1 + |\rho_1 A_2 - \rho_2 A_1|_\infty \left\| D\left(\frac{1}{A_1A_2}\right)\right\|_0 \right\} \\ &\lesssim \frac{1}{\epsilon} \|A_2\|_2 \left\{ \left|\frac{1}{A_1A_2}\right|_\infty \|\rho_1 A_2 - \rho_2 A_1\|_1 + |\rho_1 A_2 - \rho_2 A_1|_\infty \right\}. \end{aligned}$$

Since, $|v|_\infty \lesssim \|v\|_2$ and

$$\left\| D\left(\frac{1}{A_1A_2}\right)\right\|_0 \lesssim \left|\frac{1}{A_1}\right|_\infty \left|\frac{1}{A_2}\right|_\infty^2 \|\nabla A_2\|_0 + \left|\frac{1}{A_2}\right|_\infty \left|\frac{1}{A_1}\right|_\infty^2 \|\nabla A_1\|_0 ,$$

we have:

$$\begin{aligned} R_3 \lesssim \frac{1}{\epsilon} \|A_2\|_2 & \left(\left|\frac{1}{A_1}\right|_\infty \left|\frac{1}{A_2}\right|_\infty + \left|\frac{1}{A_1}\right|_\infty \left|\frac{1}{A_2}\right|_\infty^2 \|A_2\|_1 + \left|\frac{1}{A_2}\right|_\infty \left|\frac{1}{A_1}\right|_\infty^2 \|A_1\|_1 \right) \times \\ & \|\rho_1 A_2 - \rho_2 A_1\|_2 . \end{aligned}$$

Finally,

$$R_4 \leq \left|\frac{1}{A_1}\right|_\infty \left|\frac{1}{A_2}\right|_\infty \|A_1\|_2 |A_2\rho_1 - \rho_2 A_1|_\infty .$$

B.1.2 Computations for Higher-Order Energy Estimate Estimates

Claim 1:

$$\sum_{|\alpha|\leq m} \|D^\alpha u\|_0 \|D^\alpha (uv)\|_0 \lesssim (|\nabla u|_\infty + |u|_\infty + |v|_\infty) \|u\|_m^2 + (|\nabla u|_\infty + |u|_\infty) \|v\|_m^2 .$$

Proof.

$$\begin{aligned}\sum_{|\alpha|\le m}\|D^\alpha u\|_0\|D^\alpha(uv)\|_0 &\le \sum_{|\alpha|\le m}\|D^\alpha u\|_0\{\|D^\alpha(uv)-uD^\alpha v\|_0+|u|_\infty\|D^\alpha v\|_0\}\\ &\le \|u\|_m\left(\sum_{|\alpha|\le m}\|D^\alpha(uv)-uD^\alpha v\|_0+|u|_\infty\|v\|_m\right)\\ &\le c\|u\|_m\left\{|\nabla u|_\infty\left\|D^{m-1}v\right\|_0+\|D^m u\|_0|v|_\infty+|u|_\infty\|v\|_m\right\}\end{aligned}$$

This proves the claim. □

Lemma 29.

$$\left\|D^m\left(\frac{1}{A^\epsilon}\right)\right\|_0 \le \sum_{k=0}^{m-1}C_k\left|\frac{1}{A^\epsilon}\right|_\infty^{k+2}|\nabla A^\epsilon|_\infty^k\left\|D^{m-k}A^\epsilon\right\|_0, \tag{B.1}$$

where the C_k's are constants.

Proof. Using *(1)* of Lemma 2 and dropping the constants we get:

$$\begin{aligned}\left\|D^m\left(\frac{1}{A}\right)\right\|_0 &= \left\|D^{m-1}\left(\frac{\nabla A}{A^2}\right)\right\|_0\\ &\lesssim |\nabla A|_\infty\left\|D^{m-1}\left(\frac{1}{A^2}\right)\right\|_0+\left|\frac{1}{A}\right|_\infty^2\left\|D^{m-1}\nabla A\right\|_0\\ &\lesssim |\nabla A|_\infty\left(|\nabla A|_\infty\left\|D^{m-2}\left(\frac{1}{A^3}\right)\right\|_0+\left|\frac{1}{A}\right|_\infty^3\left\|D^{m-2}\nabla A\right\|_0\right)\\ &\quad+\left|\frac{1}{A}\right|_\infty^2\|D^m A\|_0\\ &\ \ \vdots\\ &\lesssim |\nabla A|_\infty^{m-1}\left\|D^1\left(\frac{1}{A^m}\right)\right\|_\infty+\sum_{k=0}^{m-2}\left|\frac{1}{A}\right|_\infty^{k+2}|\nabla A|_\infty^k\left\|D^{m-k}A\right\|_0\\ &\lesssim \sum_{k=0}^{m-1}\left|\frac{1}{A}\right|_\infty^{k+2}|\nabla A|_\infty^k\left\|D^{m-k}A\right\|_0.\end{aligned}$$

□

Proof. (**Lemma 4**)

$$\overbrace{\int D^\alpha(J_\epsilon\nabla\rho)\cdot D^\alpha\left(\frac{\rho}{A}J^\epsilon\nabla A\right)dx}^{I_\alpha} \le \|D^\alpha(J_\epsilon\nabla\rho)\|_0\left\|D^\alpha\left(\frac{\rho}{A}J^\epsilon\nabla A\right)\right\|_0$$

$$\le \|D^\alpha(J_\epsilon\nabla\rho)\|_0\left\|D^\alpha\left(\frac{\rho}{A}J^\epsilon\nabla A\right)-\frac{\rho}{A}D^\alpha(J_\epsilon\nabla A)\right\|_0$$

$$+\|D^\alpha(J_\epsilon\nabla\rho)\|_0\left|\frac{\rho}{A}\right|_\infty\|D^\alpha J_\epsilon\nabla A\|_0\,.$$

Summing over $|\alpha|\le m$ gives:

$$\sum_{|\alpha|\le m}I_\alpha \le \|J_\epsilon\nabla\rho\|_m\left(\sum_{|\alpha|\le m}\left\|D^\alpha\left(\frac{\rho}{A}J^\epsilon\nabla A\right)-\frac{\rho}{A}D^\alpha(J_\epsilon\nabla A)\right\|_0+\left|\frac{\rho}{A}\right|_\infty\|J_\epsilon\nabla A\|_m\right)$$

$$\lesssim \|J_\epsilon\nabla\rho\|_m\left(\left|\nabla\left(\frac{\rho}{A}\right)\right|_\infty\left\|D^{m-1}J_\epsilon\nabla A\right\|_0+|J_\epsilon\nabla A|_\infty\left\|D^m\frac{\rho}{A}\right\|_0+\left|\frac{\rho}{A}\right|_\infty\|J_\epsilon\nabla A\|_m\right).$$

We bound the first term $\left|\nabla(\frac{\rho}{A})\right|_\infty \le (C_1|\nabla\rho|_\infty+|\rho|_\infty|\nabla A|_\infty C_1^2)$. Therefore, the above inequality can be bounded by:

$$\sum_{|\alpha|\le m}I_\alpha \lesssim \|J_\epsilon\nabla\rho\|_m\left(C_1|\nabla\rho|_\infty+|\rho|_\infty|\nabla A|_\infty C_1^2\right)\|A\|_m$$

$$+\|J_\epsilon\nabla\rho\|_m\left(|J_\epsilon\nabla A|_\infty\left\|D^m\frac{\rho}{A}\right\|_0+\left|\frac{\rho}{A}\right|_\infty\|J^\epsilon\nabla A\|_m\right). \qquad \text{(B.2)}$$

The term $\left\|D^m\frac{\rho}{A}\right\|_0$ can be bounded by simpler terms using part *(1)* of Lemma 2. In particular,

$$\left\|D^m\frac{\rho}{A}\right\|_0 \le c\left(|\rho|_\infty\left\|D^m\frac{1}{A}\right\|_0+C_1\|D^m\rho\|_0\right). \qquad \text{(B.3)}$$

Here we make use of Lemma 29 by substituting (B.1) into (B.3). From (B.2) after applying a Cauchy inequality of the form $2ab\le\delta a^2+\frac{1}{\delta}b^2$ we get the desired result. □

B.1.3 Computations for L^2-Cauchy Sequence

$$
\begin{aligned}
R_1 &\lesssim \left(\left| \frac{1}{A^\epsilon} \right|_\infty \| D(\nabla \rho \cdot J_\epsilon \nabla A^\epsilon) \|_0 + |\nabla \rho^\epsilon|_\infty \, |\nabla J_\epsilon A^\epsilon|_\infty \left\| D\left(\frac{1}{A^\epsilon}\right) \right\|_0 \right) \\
&\lesssim \left| \frac{1}{A^\epsilon} \right|_\infty \left\{ |J_\epsilon \nabla A^\epsilon|_\infty \|\Delta \rho^\epsilon\|_0 + \left\| D^2 J_\epsilon A^\epsilon \right\|_0 |\nabla \rho^\epsilon|_\infty \right\} + \left| \frac{1}{A^\epsilon} \right|_\infty^2 |\nabla \rho^\epsilon|_\infty |\nabla J_\epsilon A^\epsilon|_\infty \|\nabla A^\epsilon\|_0 \\
&\le c \left(\left| \frac{1}{A^\epsilon} \right|_\infty + \left| \frac{1}{A^\epsilon} \right|_\infty^2 \right) \|A^\epsilon\|_3 \left(\|\rho^\epsilon\|_3 + \|A^\epsilon\|_3 \|\rho^\epsilon\|_3 \right)
\end{aligned}
$$

similarly,

$$
\begin{aligned}
R_2 &\le c \left(\left| \frac{1}{A^\epsilon} \right|_\infty \| D(\rho^\epsilon \Delta J_\epsilon A^\epsilon) \|_0 + |\rho^\epsilon \Delta J_\epsilon A^\epsilon|_\infty \left\| D\left(\frac{1}{A^\epsilon}\right) \right\|_0 \right) \\
&\le c \left| \frac{1}{A^\epsilon} \right|_\infty \left(\|\rho^\epsilon\|_3^2 + \|A^\epsilon\|_4^2 \right)
\end{aligned}
$$

Computations for Lemma 7

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int v^2 dx &= \int v \left[\Delta v - 2\nabla \cdot \left(\frac{\rho_1}{A_1}\nabla A_1 - \frac{\rho_2}{A_2}\nabla A_2 \right) - A_1\rho_1 + A_2\rho_2 \right] dx \\
\text{C.I} &\le \left\| \frac{\rho_1}{A_1}\nabla A_1 - \frac{\rho_2}{A_2}\nabla A_2 \right\|_0^2 - \int A_2 v^2 dx - \int \rho_1 u v dx \\
&\le \left\| \frac{\rho_1}{A_1}\nabla A_1 - \frac{\rho_2}{A_2}\nabla A_2 \right\|_0^2 + \frac{1}{2} |\rho_1|_\infty \|u\|_0^2 + \frac{1}{2} |\rho_1|_\infty \|v\|_0^2
\end{aligned}
$$

Unfortunately, the advection term leaves a term which still has to be dealt with:

$$
\begin{aligned}
\left\| \frac{\rho_1}{A_1}\nabla A_1 - \frac{\rho_2}{A_2}\nabla A_2 \right\|_0^2 &\le \left| \frac{\rho_1}{A_1} \right|_\infty^2 \|\nabla u\|_0^2 + |\nabla A_2|_\infty^2 \left\| \frac{A_2\rho_1 - A_1\rho_2}{A_1 A_2} \right\|_0^2 \\
&\le \left| \frac{\rho_1}{A_1} \right|_\infty^2 \|\nabla u\|_0^2 + |\nabla A_2|_\infty^2 \left| \frac{1}{A_1 A_2} \right|_\infty^2 \left(|\rho_1|_\infty^2 \|u\|_0^2 + |A_1|_\infty^2 \|v\|_0^2 \right).
\end{aligned}
$$

Making use of the fact that $|1/A|_\infty \le C_1$ gives the final result.

B.1.4 Sobolev Inequalities

Deriving Inequality (2.33)

Applying (A.4) for $p = 2$ gives:

$$\begin{aligned}
\|u\|_{L^2}^2 &= C\,\|u\|_{W^{1,2}}\,\|u\|_{L^1} \\
&\leq \epsilon\left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) + \frac{C}{\epsilon}\,\|u\|_{L^1}^2
\end{aligned}$$

APPENDIX C

Appendix: Part II

C.1 Auxiliary Lemmas

Lemma 30. *Let F be a convex C^1 function and $f = F'$. Assume that $f(u) \in L^2(0,T,H^1(D))$, $u \in H^1(0,T,H^{-1}(D))$ and $F(u) \in L^\infty(0,T,L^1(D))$. Then for almost all $0 \leq s, \tau, \leq T$ the following holds:*

$$\int (F(u(x,\tau)) - F(u(x,s)))\ dx = \int_s^\tau \langle u_t, f(u(t))\rangle\ dt.$$

Lemma 31. *Let $F(u,t) \in C^2([0,\infty),[0,\infty))$ be a convex function such that $F(0) = 0$ and $F'' > 0$ on $(0,\infty)$. Let f_n, for $n = 1,2,...,$ and f be a non-negative function on D bounded from above by $M > 0$. Furthermore, assume that $f_n \rightharpoonup f$ in $L^1(D)$ and $F(f_n) \to F(f)$ in $L^1(D)$, then $\|f_n - f\|_{L^2(D)} \to 0$ as $n \to 0$.*

Lemma 32 (Weak Lower-semicontinuity)**.** *Let ρ_ϵ be non-negative $L^1_{loc}(D_T)$ and f_ϵ a vector valued function in $L^1_{loc}(D_T)$ such that $\forall \phi \in C_c^\infty(\overline{D_T})and\xi \in C_c^\infty(\overline{D_T},\mathbb{R}^d)$*

$$\int_{D_T} \rho_\epsilon \phi dxdt \to \int_{D_T} \rho\phi dxdt$$
$$\int_{D_T} f_\epsilon \cdot \xi dxdt \to \int_{D_T} f \cdot \xi dxdt.$$

Then

$$\int_{D_T} \frac{1}{\rho} |f|^2\, dxdt \leq \liminf_{\epsilon\to 0} \int_{D_T} \frac{1}{\rho_\epsilon} |f_\epsilon|^2\, dxdt$$

C.2 Gagliardo-Nirenberg-Sobolev Inequality

Gagliardo-Nirenberg-Sobolev inequalities are the main tool for obtaining L^p estimates of PKS models and are used in many works, for instance [69, 20, 99, 59]. The following inequality follows by interpolation and the classical Gagliardo-Nirenberg-Sobolev inequality.

Lemma 33 (Inhomogeneous Gagliardo-Nirenberg-Sobolev)**.** *Let $d \geq 2$ and $D \subset \mathbb{R}^d$ satisfy the cone condition (see e.g. [1]). Let $f : D \to \mathbb{R}$ satisfy $f \in L^p \cap L^q$ and $\nabla f^k \in L^r$. Moreover let $1 \leq p \leq rk \leq dk$, $k < q < rkd/(d-r)$ and*

$$\frac{1}{r} - \frac{k}{q} - \frac{s}{d} < 0. \tag{C.1}$$

Then there exists a constant C_{GNS} which depends on s, p, q, r, d and the dimensions of the cone for which D satisfies the cone condition such that

$$\|f\|_{L^q} \leq C_{GNS} \|f\|_{L^p}^{\alpha_2} \left\|f^k\right\|_{W^{s,r}}^{\alpha_1}, \tag{C.2}$$

where $0 < \alpha_i$ satisfy

$$1 = \alpha_1 k + \alpha_2, \tag{C.3}$$

and

$$\frac{1}{q} - \frac{1}{p} = \alpha_1 (\frac{-s}{d} + \frac{1}{r} - \frac{k}{p}). \tag{C.4}$$

Proof. We may assume that f is Schwartz then argue by density. Let β satisfy $\max(q, rk) < \beta < rkd/(d-r)$. First note by the Gagliardo-Nirenberg-Sobolev inequality, [Theorem 5.8, [1]], we have

$$\begin{aligned}\left\|f^k\right\|_{\beta/k} &\lesssim_{\beta,k,r,s} \left\|f^k\right\|_r^{1-\theta} \left\|f^k\right\|_{W^{s,r}}^{\theta} \\ &\leq \left\|f^k\right\|_{p/k}^{(1-\theta)(1-\mu)} \left\|f^k\right\|_{\beta/k}^{(1-\theta)\mu} \left\|f^k\right\|_{W^{s,r}}^{\theta},\end{aligned}$$

for $\mu \in (0,1)$ determined by interpolation and $\theta = s^{-1}(d/r - dk/\beta) \in (0,1)$. Moreover, the implicit constant does not depend directly on the size of the domain. Therefore,

$$\left\|f^k\right\|_{\beta/k} \lesssim \|f\|_p^{(1-\theta)(1-\mu)/(1-\mu(1-\theta))} \left\|f^k\right\|_{W^{s,r}}^{\theta/(1-\mu(1-\theta))}.$$

Now, where $\lambda \in (0,1)$ determined by interpolation,

$$\begin{aligned}\|f\|_q &\leq \|f\|_p^{(1-\lambda)} \left\|f^k\right\|_{\beta/k}^{\lambda/k} \\ &\lesssim \|f\|_p^{(1-\lambda)+(1-\theta)(1-\mu)/(1-\mu(1-\theta))} \left\|f^k\right\|_{W^{s,r}}^{\lambda\theta/(k-k\mu(1-\theta))}.\end{aligned}$$

□

C.3 Admissible Kernels

We now prove Lemmas 16,17 and 18. We begin with the following characterizations of $L^{p,\infty}$.

Lemma 34. *Let $F(x) = f(|x|) \in L^1_{loc} \cap C^0 \setminus \{0\}$ be monotone in a neighborhood of the origin. If $r^{-d/p} = o(f(r))$ as $r \to 0$, then $F \notin L^{p,\infty}_{loc}$.*

Proof. Since we have assumed f to be monotone in a neighborhood of the origin, without loss of generality we prove the assertions assuming $f \geq 0$ on that neighborhood, since corresponding work may be done if f is negative. For any $\alpha > 0$, by monotonicity, we have a unique $r(\alpha)$ such that $f(r) > \alpha, \forall r < r(\alpha)$. We thus have that $\lambda_f(\alpha) = \omega_d r(\alpha)^d$, where ω_d is the volume of the unit sphere in $\mathbb{R}^d$. By the growth condition on f and continuity we also have that for α sufficiently large,

$$\frac{1}{\epsilon} r(\alpha)^{-d/p} \leq f(r(\alpha)) = \alpha.$$

Now,

$$\alpha^p \lambda_f(\alpha) = \omega_d \alpha^p r(\alpha)^d.$$

Hence, by (C.3) we have $\forall\, \epsilon > 0$ there is a neighborhood of infinity such that,

$$\omega_d \alpha^p r(\alpha)^d \gtrsim \epsilon^{-p}.$$

We take $\epsilon \to 0$ to deduce that $F \notin L^{p,\infty}$.

$\square$

Lemma 35. *Let $F(x) = f(|x|) \in L^1_{loc} \cap C^0 \setminus \{0\}$ be monotone in a neighborhood of the origin. Then $f \in L^{p,\infty}_{loc}$ if and only if $f = \mathcal{O}(r^{-d/p})$ as $r \to 0$.*

Proof. Since we have assumed f to be monotone in a neighborhood of the origin, without loss of generality we prove the assertions assuming $f \geq 0$ on that neighborhood.

First assume that $f \neq \mathcal{O}(r^{-d/p})$ as $r \to 0$, which implies that for all $\delta_0 > 0$ and every $C > 0$ there exists an $r_C < \delta_0$ such that

$$f(r_C) > C r_C^{-d/p}.$$

We now show that in a neighborhood of the origin, the function $f(r) - Cr^{-d/p}$ is strictly positive for $r < r_C$. Suppose not. Since both $f, r^{-d/p}$ are monotone, there exists r_0 such that $f(r) < Cr^{-d/p}$ for $r < r_0$. However, this contradicts $f \neq \mathcal{O}(r^{-d/\gamma})$ as $r \to 0$. Thus, we have that

$$f(r) > Cr^{-d/p}$$

in a neighborhood of the origin ($r < r_C$). Since for all $C > 0$ we can find a corresponding r_C, this is equivalent to $r^{-d/p} = o(f(r))$, and by Lemma 34 we have that $f \notin L^{p,\infty}$.

On the other hand, if $f = \mathcal{O}(r^{-d/p})$ as $r \to 0$ there exists $\delta > 0$ and $C > 0$ such that for all $r < \delta$,

$$f(r) \leq C r^{-d/p}. \tag{C.5}$$

By monotonicity, for all $\alpha > 0$ there is a unique $r(\alpha) \in [0, \delta]$ such that

$$f(r) > \alpha, \text{ for } r < r(\alpha), \tag{C.6}$$

where we take $r(\alpha) = 0$ if $f(r) < \alpha$ over the entire neighborhood. By (C.5) and (C.6), we have, necessarily that $r(\alpha) \lesssim \alpha^{-p/d}$. Therefore,

$$\alpha^p \lambda_f(\alpha) = \alpha^p \omega_d r(\alpha)^d \lesssim 1,$$

which implies $f \mathbf{1}_{B_1(0)} \in L^{p,\infty}$. □

Remark 13. *Similar statements may be made about the decay of $F(x)$ at infinity.*

Proof. (**Lemma 16**) By the fundamental theorem of calculus and condition **(BD)**,

$$\begin{aligned}\left|\partial_{x_i}\partial_{x_j}\mathcal{K}(x)\right| &\leq \int_1^\infty \left|\partial_r \partial_{x_i}\partial_{x_j}\mathcal{K}(rx)\right| dr \\ &\lesssim |x|^{-d}.\end{aligned}$$

Similarly, this argument also implies $|\nabla\mathcal{K}| \lesssim |x|^{1-d}$, which in turn implies $\nabla\mathcal{K} \in L^{d/(d-1),\infty}$. If $d > 2$ then we can carry out this argument another time and show that $|\mathcal{K}| \lesssim |x|^{2-d}$. Moreover, in $d = 2$ we see that $\mathcal{K}$ could have, at worst, logarithmic singularities at zero and infinity.

□

Proof. (**Lemma 17**) We compute second derivatives of the kernel $\mathcal{K}$ in the sense of distributions. Let $\phi \in C_c^\infty$, then by the dominated convergence theorem,

$$\int \partial_{x_i}\mathcal{K}\partial_{x_j}\phi dx = \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} \partial_{x_i}\mathcal{K}\partial_{x_j}\phi dx$$
$$= -\lim_{\epsilon\to 0}\int_{|x|=\epsilon} \partial_{x_j}\mathcal{K}(x)\frac{x_j}{|x|}\phi(x)dS - \text{PV}\int \partial_{x_i x_j}\mathcal{K}\phi dx.$$

By $\nabla\mathcal{K} \in L^{d/(d-1),\infty}$ and Lemma 35, we have $\nabla\mathcal{K} = \mathcal{O}(|x|^{1-d})$ as $x \to 0$. Therefore for ϵ sufficiently small, there exists $C > 0$ such that,

$$\left|\int_{|x|=\epsilon} \partial_{x_j}\mathcal{K}(x)\frac{x_j}{|x|}\phi(x)dS\right| \leq C\int_{|x|=\epsilon} |x|^{1-d}|\phi(x)|\,dS$$
$$= C\int_{|x|=1} |\epsilon x|^{1-d}|\phi(\epsilon x)|\,\epsilon^{d-1}dS = C\,|\phi(0)|\,.$$

Similarly, we may define $D^2\mathcal{K} * \phi$ and we have,

$$\left\|D^2\mathcal{K} * \phi\right\|_p \leq C\left\|\phi\right\|_p + \left\|\text{PV}\int \partial_{x_i x_j}\mathcal{K}(y)\phi(x-y)dy\right\|_p.$$

Therefore, the first term can be extended to a bounded operator on L^p for $1 \leq p \leq \infty$ by density. The admissibility conditions **(R)**,**(BD)** and **(KN)** are sufficient to apply the Calderón-Zygmund inequality [Theorem 2.2 [96]], which implies that the principal value integral in the second term is a bounded linear operator on L^p for all $1 < p < \infty$. Moreover the proof provides an estimate of the operator norms,

$$\left\|\text{PV}\int \partial_{x_i,x_j}\mathcal{K}(y)u(x-y)dy\right\|_p \lesssim \begin{cases} \frac{1}{p-1}\left\|u\right\|_p & 1 < p < 2 \\ p\left\|u\right\|_p & 2 \leq p < \infty. \end{cases}$$

□

Proof. (**Lemma 18**) The assertion that $D^2\mathcal{K} \in L^{\gamma,\infty}_{loc}$ implies $\mathcal{K} \in L^{d/(d/\gamma-2),\infty}_{loc}$ follows similarly as in Lemma 16.

Now we prove the reverse implication. Let $\mathcal{K} \in L_{loc}^{d/(d/\gamma-2),\infty}$. We show that $D^2\mathcal{K} = \mathcal{O}(r^{-d/\gamma})$ as $r \to 0$. Assume for contradiction that $D^2\mathcal{K} \neq \mathcal{O}(r^{-d/\gamma})$ as $r \to 0$. This implies that $k'' \neq \mathcal{O}(r^{-d/\gamma})$ or that $k'(r)r^{-1} \neq \mathcal{O}(r^{-d/\gamma})$ as $r \to 0$. These two possibilities are essentially the same, so just assume that $k'' \neq \mathcal{O}(r^{-d/\gamma})$. By monotonicity arguments used in the proof of Lemma 35, this in turn implies $r^{-d/\gamma} = o(k'')$. However, this means that for all ϵ, there exists a $\delta(\epsilon) > 0$ such that for $r \in (0, \delta(\epsilon))$ we have,t

$$\begin{aligned} k(r) - k(\delta(\epsilon)) = -\int_{\delta(\epsilon)}^{r} k'(s)ds = \int_{\delta(\epsilon)}^{r}\int_{\delta(\epsilon)}^{s} k''(t)dtds + (r-\delta(\epsilon))k'(\delta(\epsilon)) \\ \gtrsim \epsilon^{-1}r^{2-d/\gamma} + 1, \end{aligned}$$

which contradicts the fact that $k(r) = \mathcal{O}(r^{2-d/\gamma})$ as $r \to 0$ by Lemma 35.

The assertion regarding $\nabla\mathcal{K}$ is proved in the same fashion.

□

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