

UNIVERSITY OF CALIFORNIA

Los Angeles

**Part I: A Virtual Node Method for Elliptic  
Interface Problems**  
**Part II: Local and Global Theory of  
Aggregation Equations with Nonlinear Diffusion**

A dissertation submitted in partial satisfaction  
of the requirements for the degree  
Doctor of Philosophy in Mathematics

by

**Jacob Bedrossian**

2011



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2011

*To all my family and friends*

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ABSTRACT OF THE DISSERTATION

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In Part I, the author presents an accurate and efficient method for solving elliptic interface problems or elliptic problems in irregular domains. Such problems occur in a wide variety of applications in physics and engineering and are regarded as computationally difficult, particularly when the interface or boundaries are moving. The work is in collaboration with James H. von Brecht, Siwei Zhu, Eftychios Sifakis and Joseph Teran and appears also in the publication [19]. We introduce a second order virtual node method for approximating elliptic interface problems on a uniform Cartesian grid. The use of a regular Cartesian grid simplifies the implementation and permits straightforward Lagrange multiplier spaces while retaining second order accuracy in  $L^\infty$  in numerical experiments. Our approach uses duplicated Cartesian bilinear elements along the interface to introduce additional “virtual” nodes that accurately account for the lack of regularity across the surface.

Part II discusses the work undertaken by the author and his collaborators Nancy Rodríguez and Andrea Bertozzi on the class of aggregation equations with nonlinear diffusion, which represent a generalization of the classical parabolic-elliptic Patlak-Keller-Segel system for chemotaxis. These models represent the competition between nonlocal self-attraction and diffusion. Local theory, such as existence, uniqueness and continuation is first discussed. A suitable notion of  $L^1$ -criticality is introduced for inhomogeneous problems and the sharp critical mass is identified: uniform bounds  $L^\infty$  are derived for solutions to subcritical problems and critical problems with less than critical mass and finite time blow-up is derived for a class of supercritical problems and critical problems with larger than critical mass. Global, dissipating solutions are constructed under certain, reasonably general hypotheses and the asymptotic profiles are shown to agree with the self-similar Barenblatt solutions to the homogeneous diffusion equations.

Part I

# **A Virtual Node Method for Elliptic Interface Problems**

# CHAPTER 1

## Introduction

### 1.1 Introduction

Interface problems have a wide variety of applications in physics and engineering, and naturally arise when two dissimilar materials interact across a thin interface. Common examples include immiscible, incompressible fluids in contact and phase change problems. Despite being common, they remain notoriously difficult to tackle, as the geometry of the domain and regularity of coefficients affect even linear PDE in a subtle and highly nonlinear fashion. Real world applications are generally far beyond the reach of most analytic tools, and numerical simulation is the only way to approach such problems in practice. However, these problems pose serious difficulties for traditional numerical methods, which generally do not cope well with irregular geometries and coefficients with low regularity. In this part of the dissertation we detail the work undertaken by the author and his collaborators James H. von Brecht, Siwei Zhu, Eftychios Sifakis and Joseph Teran on the design of a higher order accurate numerical method for elliptic problems in the presence of interfaces and irregular boundaries in two dimensions, published here [19].



The elliptic interface problem

$$-\nabla \cdot (\beta(\mathbf{x}) \nabla u(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x} \in D \setminus \Gamma \quad (1.1)$$

$$[u] = a(\mathbf{x}) \quad \mathbf{x} \in \Gamma \quad (1.2)$$

$$[\beta(\mathbf{x}) \nabla u \cdot \mathbf{n}] = b(\mathbf{x}) \quad \mathbf{x} \in \Gamma \quad (1.3)$$

$$u = p(\mathbf{x}) \quad \mathbf{x} \in \partial D_d \quad (1.4)$$

$$\beta(\mathbf{x}) \nabla u \cdot \mathbf{n} = q(\mathbf{x}) \quad \mathbf{x} \in \partial D_n \quad (1.5)$$

could arise either in quasistatic problems or in the discretization of time-dependent problems. The interface  $\Gamma$  is generally a co-dimension one closed curve that divides the domain into an interior  $D^-$  and an exterior region  $D^+$  such that  $D = D^+ \cup D^- \cup \Gamma \subset \mathbb{R}^2$  (see Figure 1.1). The scalar coefficient  $\beta$  and the source term  $f$  can exhibit discontinuities across  $\Gamma$ , but have smooth restrictions  $\beta^+, f^+$  to  $D^+$  and  $\beta^-, f^-$  to  $D^-$ . We let  $\mathbf{n}(\mathbf{x})$  denote the outward unit normal to  $D^-$  at a point  $\mathbf{x} \in \Gamma$ , and define  $[v](\mathbf{x}) := v^+(\mathbf{x}) - v^-(\mathbf{x}) := \lim_{\epsilon \rightarrow 0^+} v(\mathbf{x} + \epsilon \mathbf{n}(\mathbf{x})) - \lim_{\epsilon \rightarrow 0^+} v(\mathbf{x} - \epsilon \mathbf{n}(\mathbf{x}))$  as the “jump” of the quantity  $v$  across the interface  $\Gamma$ . The relevant physics generally determine the jumps in the solution (1.2) and in the flux (1.3), as well as the boundary conditions on  $\partial D$ . Unless stated otherwise, we assume the curves  $\Gamma, \partial D$  are smooth.

Due to irregular geometry of the interface in many physical phenomena, a natural approach to the numerical approximation is the finite element method (FEM) with unstructured meshes that conform to the geometry of  $\Gamma$  [8, 43, 130, 105, 79, 63]. However, meshing complex interface geometries can prove difficult and time-consuming when the interface frequently changes shape (especially in 3D). Also, many numerical methods, such as standard finite difference schemes and geometric multigrid methods, do not naturally apply to unstructured meshes. These concerns motivated the development of “embedded” (or, “immersed”) methods that approximate solutions to (1.1,1.2,1.3) on Cartesian

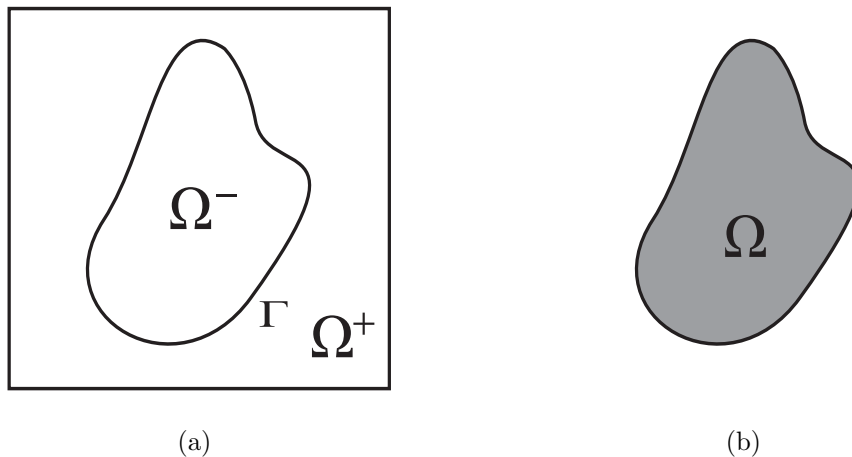


Figure 1.1: Graphical depiction of the problems (1-5). The image on the left depicts the relevant domains for interface problems, and the image on the right depicts the domain for embedded boundary problems.

grids or structured meshes that do not conform to the interface. Despite advances in this direction, embedded methods that retain higher order accuracy in  $L^\infty$  typically introduce relatively difficult linear algebra problems and complex implementations that sometimes require significant effort to adapt to general applications.

We introduce a second order virtual node method for approximating the elliptic interface problem (1.1,1.2,1.3) with irregular embedded Neumann and Dirichlet boundaries on a uniform Cartesian grid. We use a regular Cartesian grid because it simplifies the implementation, permits straightforward Lagrange multiplier spaces and achieves higher order accuracy in  $L^\infty$ . Our approach uses duplicated Cartesian bilinear elements along the interface to introduce additional “virtual” nodes that accurately account for the lack of regularity. From a theory perspective, the method is a mixed FEM, as it uses Lagrange multipliers to enforce embedded Dirichlet conditions and the jump conditions (1.2) weakly. However,

as we shall see, we have mitigated most of the associated negative stereotypes, in particular, our choice of Lagrange multiplier space will admit a symmetric positive definite discretization and the use of regular Cartesian elements capable of treating irregular geometry eliminates meshing and deformed elements. For this reason, in the exposition we will not overly emphasize the FEM viewpoint. For the special case of smooth coefficients  $\beta$ , we present a novel ‘discontinuity removal’ technique to allow the use of the standard 5-point difference stencil everywhere in the domain. This is unlike the numerous FEM approaches that use similar virtual node representations on unstructured meshes [96, 97, 188, 78, 186, 164, 14, 180], as although some finite difference methods possess the notable advantage of discontinuity removal [144], to our knowledge a technique that retains the original system matrix has previously been largely unexplored in the FEM frameworks. It will be apparent that our discontinuity removal technique is not limited to our choice of Cartesian elements, and should be applicable in other contexts and unstructured meshes. In all cases, our method yields the standard 5-point difference stencil away from the boundaries and interfaces. Numerical experiments indicate second order accuracy in  $L^\infty$ . A notable quality of this method is that the higher-order accuracy is essentially attained simply by a geometric refinement of the standard Cartesian bilinear element discretization, making the method intuitive to implement and understand conceptually, which may not be true of other higher order elliptic interface methods.

## 1.2 Existing Numerical Methods

The Immersed Interfaced Method (IIM) is perhaps the most popular finite difference method for approximating (1.1,1.2,1.3) to second order accuracy. LeVeque and Li first proposed the IIM for approximating elliptic interface problems in

[136] and the term now applies to a widely researched and extensively applied class of finite difference methods [137, 196, 133, 211, 210, 145, 134]. See [144] and the references therein for a complete exposition of the method and its numerous applications, and [16] for justification of the general IIM approach. Using generalized Taylor expansions, the original IIM adaptively modifies the stencil to obtain  $\mathcal{O}(h)$  truncation error along the interface. For smooth  $\beta$ , this reduces to the standard 5-point finite difference stencil, but otherwise results in a non-symmetric discretization that follows from locally solving constrained optimization problems that enforce a discrete maximum principle [143]. The IIM also generally requires the evaluation of higher-order jump conditions and surface derivatives along the interface. This can lead to difficulty in implementation, especially in 3D [70, 211, 144, 210]. The piecewise-polynomial interface method of [62] is a notable new approach to the IIM that does not require the derivation of additional jump conditions and accurately treats complex interfaces. The works of [206, 24, 141, 1, 144] describe other various attempts to improve the efficiency and reduce the complexity of the IIM.

Extrapolation based finite difference schemes such as [151, 64, 216, 89, 88, 110] introduce fictitious points along coordinate axes and use the known jump conditions to determine their values. The Ghost Fluid Method (GFM) of [151] exemplifies such methods. For two and three dimensional problems, the GFM neglects the tangential flux terms  $[\beta \nabla u \cdot \tau]$  in determining the fictitious values, resulting in a symmetric positive definite but first order [152] method. Various approaches attain higher order accuracy by accounting for the tangential flux in the finite difference framework, often sacrificing simplicity and symmetry of discretization in the process. For instance, the Coupling Interface Method (CIM) proposed in [64] extends the GFM to higher dimensions by using a second order extension at most grid points, but reverting to a first order method at grid points

where the second order extension cannot apply. The method couples jump conditions in different directions to express the tangential derivatives, and the use of one-sided differences results in a non-symmetric discretization. Similarly, the Matched Interface and Boundary (MIB) method [216] uses higher order extrapolations of the solution matched with higher order one-sided discretizations of the jump conditions to determine the values at fictitious points. The MIB method accounts for non-zero  $[\beta \nabla u \cdot \tau]$  by differentiating the given jump conditions using one-sided interpolations. This widens the stencil in several directions that depend on the local geometry, and results in a non-symmetric discretization. The work of [215] extended the MIB to handle high curvature geometry, and [213] provides a 3D version. In [104] Hou and Liu also use techniques seemingly inspired by the analysis of the original GFM approach done in [152]. They develop a second order variational GFM by altering finite element interpolating functions to capture the jump conditions in the solution. Their approach is remarkably robust to non-smooth interface geometry, but results in a non-symmetric discretization in the general case. The recent works of [167, 172] treated the cases of Robin and Neumann boundary conditions by altering the 5-point stencil along the boundary using a finite volume like approach. This results in an symmetric positive definite discretization.

Ideas similar to the extrapolation based finite difference schemes have also seen extensive use in FEM, for instance in the fictitious domain methods for embedded boundary problems [90, 173, 165, 78, 4, 212, 138, 77, 129] or the ‘extended finite element methods’ (XFEM) [21, 163, 68, 162, 108, 92, 161] [200]<sup>1</sup>. Fictitious domain methods handle embedded boundaries by including every element that intersects the interface into the discretization. This naturally introduces “virtual nodes” (or “ghost nodes”) into the resulting discretization. The XFEM “enriches”

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<sup>1</sup>See [15] for corrections to IIM convergence estimates

the standard finite element basis with additional discontinuous basis functions, thereby introducing new degrees of freedom. These basis functions exist only at the nodes of elements that intersect the interface, and usually are the standard basis elements multiplied by a generalized Heaviside function. The methods of [96, 97, 188, 78, 98, 14, 180] introduce a related virtual node concept to provide the additional degrees of freedom required to represent the discontinuities. The most straightforward implementation of this virtual node concept [97, 188, 78] yields a representation equivalent to the standard Heaviside enrichment of the XFEM. However, this approach generalizes to the slightly richer representations of [186, 164, 180] that attain more geometric detail, particularly when dealing with coarse grids and non-smooth interfaces. Moreover, virtual node representations are considered more geometrically intuitive and easier to incorporate into existing FEM code [78, 188, 180] than traditional Heaviside enrichment.

The solution spaces of these FEM approaches generally do not satisfy the embedded boundary or interface conditions. Thus, these methods impose linear constraints with either penalty methods or Lagrange multipliers to enforce the conditions in some weak sense. For example, see [90, 173, 165, 78] and the references therein. When using Lagrange multipliers, the Ladyzhenskaya-Babuška-Brezzi inf-sup conditions place stringent limitations on the types of constraints that will retain optimal convergence rates of the approximation spaces [9, 178, 138, 61, 161, 165]. Such inf-sup restrictions generally limit the strength of the Lagrange multiplier space relative to the solution approximation space. For certain elements, designing the proper approximation spaces is a non-trivial task [161, 108]. Moreover, the use of Lagrange multipliers requires the solution of an indefinite saddle point system that can potentially introduce significant cost. Applying stabilization through a consistent penalty method, such as Nitsche's method, presents an alternative approach [98, 78, 77, 165, 97]. However, these

can have adverse effects on conditioning and require the determination of the stabilization parameters. Instead of using Lagrange multipliers or stabilization, the methods of [128, 83, 142, 146, 104] alter the basis functions to either satisfy the constraints directly, or simplify the process of doing so. In this regard, such methods represent the finite element analogues of the IIM, especially [104, 142, 146]. The discontinuous Galerkin method of [138] and the broken nonconforming element method of [129] are also interesting alternatives that may be rather closely related to virtual node methods on further investigation, as the increase in degrees of freedom capable of capturing the geometric irregularities is the principle aim here as well.

The method of [109] offers a finite volume approach to embedded boundary problems. Like some fictitious domain methods, XFEM and our virtual node method, this method uses partially empty cells along the boundary. However, the one-sided quadratic interpolations used to compute the fluxes along the boundary yield a non-symmetric system. See [183] for a more recent 3D version applied to Poisson's equation and the heat equation. In [168], Oevermann and Klein proposed a second order finite volume method for interface problems, and simplified and extended their method to 3D in [169]. In an approach similar to ours, any Cartesian cell that intersects the interface yields a distinct bilinear (or trilinear) representation of the solution. The jump conditions are then built into the difference stencil by locally solving constrained overdetermined systems. An asymptotic technique resolves the problem of vanishing cell volumes, though it requires specific treatment for each possible cell geometry. The resulting system is non-symmetric for the general case of  $[\beta] \neq 0$ .

When  $[\beta] \neq 0$  the majority of these second-order methods do not retain a symmetric positive definite stencil. While the FEM approaches that use stabi-

lization do retain a symmetric positive definite discretization [78], generally the FEM that use Lagrange multipliers, such as [68], result in a symmetric indefinite discretization. Although we use Lagrange multipliers, we present a simple method of reducing the indefinite system to a symmetric positive definite system using a null-space method (described below in §2.0.2). On the other hand, when the coefficient  $\beta$  is smooth across the interface, methods such as the original IIM achieve second order accuracy by only altering the right hand side of the system. For this case, we present a method that uses the virtual node framework that also retains the original left hand side (described below in §2.0.3.1).



## CHAPTER 2

### Description of the Method

Our method naturally handles both interfacial discontinuities and irregular domains embedded in a Cartesian grid. In fact, a slight modification of our approach to embedded boundary conditions yields our method for interfacial discontinuities. Furthermore, our treatment of embedded Dirichlet boundary conditions is just a slight modification of our treatment of embedded Neumann boundary conditions. Therefore, we first present our method for embedded Neumann boundary conditions followed by our method for Dirichlet boundary conditions and then finally present our approach to interfacial discontinuities.

#### 2.0.1 Embedded Neumann

Our approach to solving embedded Neumann problems is very similar to that proposed by Almgren et. al. in [4], as well as some XFEM approaches, e.g. [68]. The recent methods proposed in [167] and [172] are comparable in accuracy to our method and are straightforward to implement.

Similar to [4], we discretize the embedded Neumann problem,

$$-\nabla \cdot (\beta(\mathbf{x}) \nabla u(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x} \in D \quad (2.1)$$

$$\beta(\mathbf{x}) \nabla u \cdot \mathbf{n} = q(\mathbf{x}) \quad \mathbf{x} \in \partial D, \quad (2.2)$$

over a regular Cartesian grid (one that does not have to conform to  $\partial D$ ) using the energy minimization form of (2.1,2.2):

over all  $u \in \mathbf{H}^1(D)$ , minimize

$$E(u) = e(u) - (f, u)_D - (q, u)_{\partial D} = \int_D \frac{1}{2} \nabla u \cdot \beta \nabla u dx - \int_D f u dx - \int_{\partial D} q u dS. \quad (2.3)$$

We begin by embedding the domain  $D$  in a regular Cartesian grid  $\mathcal{G}^h$  with grid-spacing  $\Delta y = \Delta x = h$ . We include all Cartesian cells (or elements)  $c_k$  that intersect  $D$  in the discretization, and refer to this set  $\mathcal{C}^h = \{c_k \cap D \neq \emptyset\} \subset \mathcal{G}^h$  as the “computational domain” (see Figure 2.1). Also, we define the set of all cells that intersect the boundary as  $\mathcal{C}_{\partial D}^h = \{c_k \cap \partial D \neq \emptyset\} \subset \mathcal{C}^h$ . We define the solution space  $\mathbf{V}^h \subset \mathbf{H}^1(D)$  as the space of continuous functions that are bilinear over each cell  $c_k \in \mathcal{C}^h$ . This approximation includes some partially empty cells that intersect the boundary and introduces “virtual” grid nodes (and virtual degrees of freedom) that lie outside of the domain. See Figure 2.2 for a diagram labeling the degrees of freedom along a typical boundary. We refer to the portion of the cell that lies in the domain  $D$  as the “material” region, and use the term “material” nodes to describe grid nodes lying inside  $D$ . For  $u^h \in \mathbf{V}^h$ , we write  $u^h(\mathbf{x}) = \sum_{i=1}^n u_i N_i(\mathbf{x})$  for  $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$  where  $N_i(\mathbf{x})$  are the standard piecewise bilinear interpolation basis functions associated with the grid nodes. Here,  $n$  denotes the number of degrees of freedom in the discretization and corresponds to the number of grid nodes that compose the cells of  $\mathcal{C}^h$ .

Using the virtual node representation, we define a discrete energy  $E^h(u^h)$  over  $u^h \in \mathbf{V}^h$ . Although we could discretize the energy directly with the piecewise bilinear representation, this would result in a second-order 9-point stencil away from the interface (as in [4]). To retain the standard 5-point difference stencil away from the boundary we use different definitions of the energy over  $\mathcal{C}^h \setminus \mathcal{C}_{\partial D}^h$

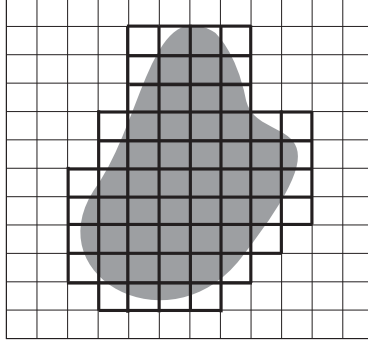


Figure 2.1: Embedding  $D$  in a Cartesian grid. The computational domain consists of all cells  $c_k \in \mathcal{G}^h$  that intersect  $D$ . Such cells are outlined in bold. This procedure introduces virtual degrees of freedom into the discretization, namely those nodes in the bold grid that do not lie in the shaded domain  $D$  itself.

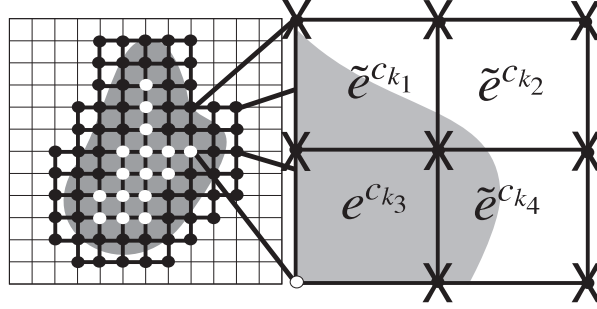
and  $\mathcal{C}_{\partial D}^h$ ,

$$E^h(u^h) = \sum_{c_k \in \mathcal{C}^h \setminus \mathcal{C}_{\partial D}^h} e^{c_k}(u^h) - (f, u^h)_{\partial D}^{c_k} + \sum_{c_k \in \mathcal{C}_{\partial D}^h} \tilde{e}^{c_k}(u^h) - (f, u^h)_{\partial D}^{c_k} - (q, u^h)_{\partial D}^{c_k}, \quad (2.4)$$

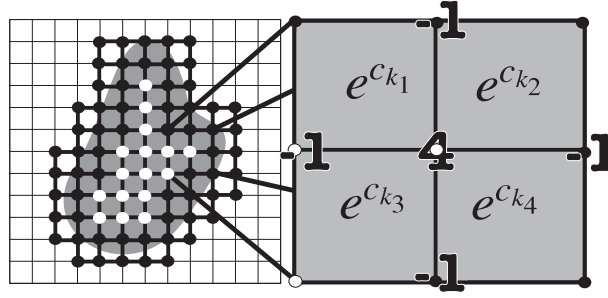
where the superscripts denote restriction to cell  $c_k$ . Over cells  $c_k \in \mathcal{C}^h \setminus \mathcal{C}_{\partial D}^h$  that do not intersect the boundary, we define  $e^{c_k}(u^h)$  as

$$e^{c_k}(u^h) = \frac{\bar{\beta}h^2}{4} \left\{ \left( \frac{u_{i+1,j} - u_{i,j}}{h} \right)^2 + \left( \frac{u_{i,j+1} - u_{i,j}}{h} \right)^2 \right\} + \frac{\bar{\beta}h^2}{4} \left\{ \left( \frac{u_{i+1,j+1} - u_{i+1,j}}{h} \right)^2 + \left( \frac{u_{i+1,j+1} - u_{i,j+1}}{h} \right)^2 \right\}. \quad (2.5)$$

Here  $\bar{\beta}$  denotes the cell average, and  $\{u_{p,q}\}$  denote the degrees of freedom at the four corners of the cell. If a cell  $c_k \in \mathcal{C}_{\partial D}^h$ , i.e. the cell intersects the boundary, then we use the Cartesian bilinear representation to define  $\tilde{e}^{c_k}(u^h)$ . If we let  $\{N_{p,q}\}$  denote the bilinear basis functions associated with the four corners of the



(a)



(b)

Figure 2.2: Illustration of the interior and boundary stencils. The black degrees of freedom have a modified stencil; the stencil is unaltered at the white degrees of freedom. On the left, the nodes marked with an **X** contribute a non-zero entry to the stencil for the center node via the cell-wise energies. The right figure depicts the 5-point stencil for the center node that results from the definition of  $e^{c_{k_i}}$ .

cell, this yields the discretization

$$\tilde{e}^{c_k}(u^h) = \frac{1}{2} \sum_{r,s,r',s' \in \{0,1\}} u_{i+r,j+s} u_{i+r',j+s'} \int_{c_k \cap D} \bar{\beta} \nabla N_{i+r,j+s} \cdot \nabla N_{i+r',j+s'} dx. \quad (2.6)$$

We evaluate the integrals analytically using the divergence theorem based on a polygonal representation of  $\partial D$  as in Almgren et. al. [4]. See Figure 2.3 and §3.1 where the procedure is described in more detail. The tilde denotes the different discretizations of the energy over cells that intersect the boundary. Notice we evaluate each integral only over the portion of the cell that lies within the domain. Similarly, as in [168, 169] we define the cell average  $\bar{\beta}$  as the average only over  $c_k \cap D$ . We discretize the other forms cell-wise as

$$(f, u^h)_D^{c_k} = \sum_{r,s \in \{0,1\}} u_{i+r,j+s} \int_{c_k \cap D} \bar{f} N_{i+r,j+s} dx \quad (2.7)$$

$$(q, u^h)_{\partial D}^{c_k} = \sum_{r,s \in \{0,1\}} u_{i+r,j+s} \int_{c_k \cap \partial D} \bar{q} N_{i+r,j+s} dS. \quad (2.8)$$

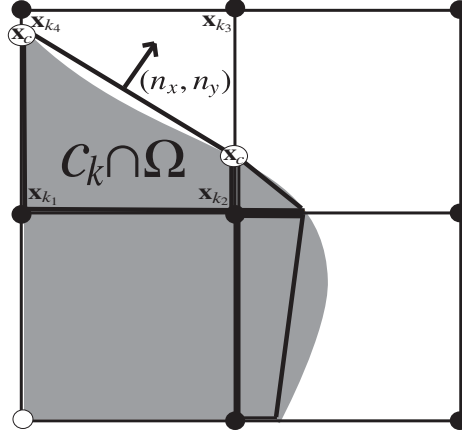
Here  $\bar{f}$  is the average source over  $c_k \cap D$  and  $\bar{q}$  is the average normal flux over  $c_k \cap \partial D$ . Again, we evaluate the integrals analytically, applying the divergence theorem where necessary. We minimize the discrete energy (2.4) by solving the linear system

$$A\vec{u} = \vec{f}, \quad (2.9)$$

$$A_{ij} = \frac{\partial^2}{\partial u_i \partial u_j} E^h(u^h), \quad (2.10)$$

$$f_i = \frac{\partial}{\partial u_i} ((f, u^h)_D + (q, u^h)_{\partial D}) \quad (2.11)$$

for the vector  $\vec{u}$ . We use the standard FEM term “stiffness matrix” to refer to the matrix  $A$ , and it is clear from the derivation that  $A$  is symmetric and positive semi-definite. With this approach, our definition of the energy (2.6) results in a slightly denser stencil near the boundary, as all four degrees of freedom in a



$$\begin{aligned}
& \int_{c_k \cap \Omega} \bar{\beta} \nabla N_i \cdot \nabla N_j \, d\mathbf{x} = \\
& \bar{\beta} \int_{c_k \cap \Omega} (\partial_y N_i)(\partial_y N_j) + (\partial_x N_i)(\partial_x N_j) \, d\mathbf{x} = \\
& \bar{\beta} \int_{c_k \cap \Omega} p_2(x) + q_2(y) \, dx dy = \\
& \bar{\beta} \int_{c_k \cap \Omega} \nabla \cdot (p_3(x), q_3(y)) \, dx dy = \\
& \bar{\beta} \int_{\partial(c_k \cap \Omega)} (p_3(x), q_3(y)) \cdot (n_x, n_y) \, dS
\end{aligned}$$

Figure 2.3: Polygonal representation of  $\partial D$ . We compute the modified stencil analytically by using the divergence theorem on the material region  $D \cap c_k$  in each cell. Here,  $p_n(x)$  and  $q_n(y)$  denote appropriate polynomials of order  $n$  in a single variable. Notice the relatively small area of the material region in the top, right cell. As this area approaches zero, the virtual node at the top, right of this cell introduces ill-conditioning into the stiffness matrix.

cell couple together if  $\partial D$  passes through that cell. See Figure 2.2 for a graphical depiction of the stencil definitions and the sparsity pattern of the stiffness matrix. In §3.1, the practical construction of  $A$  and  $\vec{f}$  is given in more detail.

We should note that the conditioning of the stiffness matrix may deteriorate when cells have very small material regions. This arises from the increasing irrelevance of virtual degrees of freedom (see the upper right node in Figure 2.3). The respective row and column in  $A$  and the corresponding entry in  $\vec{f}$  all approach zero simultaneously, however simple Jacobi preconditioning eliminated serious conditioning issues in our numerical experiments.

### 2.0.2 Embedded Dirichlet

In this section, we detail how a slight modification of our embedded Neumann approach allows us to solve embedded Dirichlet problems

$$-\nabla \cdot (\beta(\mathbf{x}) \nabla u(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x} \in D \quad (2.12)$$

$$u = p(\mathbf{x}) \quad \mathbf{x} \in \partial D, \quad (2.13)$$

within our virtual node framework. Although alternatives that are easier to implement exist for this particular problem, for instance [89], a straightforward combination of our embedded Neumann and embedded Dirichlet approaches yields our method for embedded interface problems. This results in a method that encapsulates all types of boundary conditions in a unified framework.

For the embedded Dirichlet case, we use the constrained minimization problem:

$$\text{over all } u \in \mathbf{H}^1(D), \text{ minimize}$$

$$E(u) = e(u) - (f, u)_D \text{ such that} \quad (2.14)$$

$$(u, \mu)_{\partial D} = (p, \mu)_{\partial D} \quad \forall \mu \in \mathbf{H}^{-1/2}(\partial D). \quad (2.15)$$

We discretize the energy (2.14) exactly as in the Neumann case, so the only difference comes in discretizing the constraints (2.15). We proceed by selecting a finite dimensional subspace  $\mathbf{\Lambda}^h \subset \mathbf{H}^{-1/2}(\partial D)$ , and enforce (2.15) for all  $\mu^h \in \mathbf{\Lambda}^h$ . Not all plausible choices will yield an acceptably accurate approximation, as in general  $(\mathbf{\Lambda}_h, \mathbf{V}_h)$  must satisfy an inf-sup stability criterion to retain the optimal convergence rates of the approximation spaces [178]. One suitable choice for  $\mathbf{\Lambda}^h$ , used for instance by the XFEM [200], defines  $\mu^h$  as piecewise-constant over the intersection of  $\partial\Omega$  with each Cartesian cell (see Figure 2.4). In other words, we define  $\mu^h \in \mathbf{\Lambda}^h$  as

$$\mu^h(\mathbf{x}) = \sum_{c_i \in \mathcal{C}_{\partial D}^h} \mu_i \chi_{c_i \cap \partial D}(\mathbf{x}),$$

where the sum ranges over all Cartesian cells  $c_i$  that intersect the boundary ( $c_i \in \mathcal{C}_{\partial D}^h$ ) and the characteristic functions  $\chi_{c_i \cap \partial D}$  are given by

$$\chi_{c_i \cap \partial D}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in c_i \cap \partial D \\ 0 & \mathbf{x} \notin c_i \cap \partial D. \end{cases}$$

With this choice of  $\mathbf{\Lambda}^h$ , satisfying (2.15) for all  $\mu^h$  yields a set of sparse linear constraints  $B\vec{u} = \vec{p}$  on the coefficient vector of the approximate solution  $u^h$ . Each row of the matrix  $B$  corresponds to a Cartesian cell  $c_i \in \mathcal{C}_{\partial D}^h$  (see Figure 2.4), and enforces the condition

$$\int_{c_i \cap \partial D} u^h(\mathbf{x}) dS = \int_{c_i \cap \partial D} p(\mathbf{x}) dS.$$

Therefore, if  $\mathcal{C}_{\partial D}^h = \{c_1, \dots, c_m\}$  and  $\vec{u} \in \mathbb{R}^n$ , then  $B \in \mathbb{R}^{m \times n}$  and

$$B_{ij} = \int_{c_i \cap \partial D} N_j(\mathbf{x}) dS$$



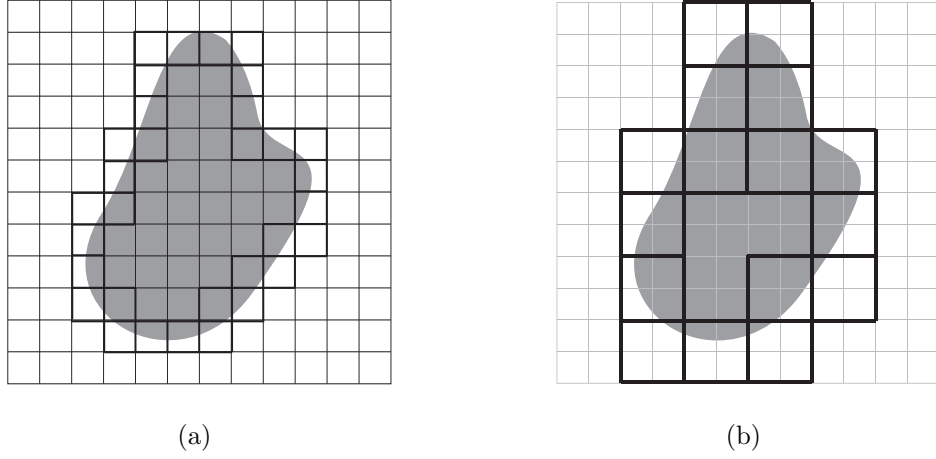


Figure 2.4: Lagrange Multiplier spaces. On the left: functions in  $\Lambda^h$  are piecewise constant over the intersection of the boldly outlined cells  $c_i \in \mathcal{C}_{\partial D}^h$  with the boundary  $\partial D$ . On the right: functions in  $\Lambda^{2h}$  are piecewise constant over the intersection of the coarser bold cells  $\hat{c}_i \in \mathcal{C}_{\partial D}^{2h}$  with the boundary  $\partial D$ . In the image on the right, the bold black lines mark the cells  $\hat{c}_i \in \mathcal{G}^{2h}$ .

for each Cartesian bilinear basis function  $N_j(\mathbf{x})$ . The corresponding entry in  $\vec{p}$  is

$$p_i = \int_{c_i \cap \partial D} p(\mathbf{x}) dS.$$

Again, we compute these integrals analytically. Discretizing (2.14,2.15) thus gives rise to the quadratic program:

$$\begin{aligned} & \text{minimize over } \vec{u} \in \mathbb{R}^n \\ & E^h(u^h) = e(u^h) - (f, u^h)_D = \frac{1}{2} \vec{u}^t A \vec{u} - \vec{f}^t \vec{u} \\ & \text{subject to } B \vec{u} = \vec{p}. \end{aligned} \tag{2.16}$$

The matrix  $A$  and the vector  $\vec{f}$  carry over exactly from the embedded Neumann case described in §2.0.1.

Unfortunately, solving this problem efficiently can require some care. While many approaches exist for solving minimization problems of the form (2.16) or the equivalent saddle-point system

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ \vec{\lambda} \end{pmatrix} = \begin{pmatrix} \vec{f} \\ \vec{p} \end{pmatrix},$$

we use a null-space method to retain a symmetric positive definite discretization. See [23] for a survey of alternative approaches. For any matrix  $Z$  whose columns span the null-space of  $B$ , and any vector  $\vec{c}$  satisfying  $B\vec{c} = \vec{p}$ ,

$$\vec{u} = \vec{c} + Z(Z^t A Z)^{-1} Z^t (\vec{f} - A\vec{c}) \tag{2.17}$$

uniquely solves (2.16). Therefore, given a null-basis  $Z$  and a particular solution  $\vec{c} \in \mathbb{R}^n$  satisfying  $B\vec{c} = \vec{p}$ , we solve the quadratic program (2.16) by solving the symmetric positive definite system  $Z^t A Z \vec{v} = Z^t (\vec{f} - A\vec{c})$ . The null-space of  $A$  is spanned by the vector  $(1, 1, \dots, 1)^t \in \mathbb{R}^n$  and the entries of  $B$  are all non-negative

so  $\ker(A) \cap \ker(B) = \{\vec{0}\}$ . Therefore,  $Z^t A Z > 0$  so we can use straightforward methods such as Conjugate Gradient to solve the symmetric positive definite linear algebra problem. However, obtaining  $Z$  through computational methods such as QR factorization or the SVD can prove costly, and moreover produce dense representations of  $Z$ .

A *fundamental basis* presents an alternative to numerical factorization [23]. The matrix  $B$  is full rank if and only if an ordering of the degrees of freedom exists so that  $B = (B_m | B_{n-m})$  for some  $m \times m$  non-singular matrix  $B_m$ . Any such ordering gives the corresponding fundamental basis

$$Z = \begin{pmatrix} -B_m^{-1} B_{n-m} \\ I_{n-m} \end{pmatrix}. \quad (2.18)$$

Clearly,  $BZ = 0$  and  $\vec{c} = \begin{pmatrix} B_m^{-1} \vec{p} \\ 0 \end{pmatrix}$  satisfies  $B\vec{c} = \vec{p}$ . Note this approach is nothing more than the straightforward elimination of degrees of freedom by writing them in terms of the constraints and other degrees of freedom. If we can solve systems of the form

$$B_m \vec{x} = \vec{d}, \quad (2.19)$$

efficiently, we can store the factors  $B_m, B_{n-m}, A$  sparsely and compute the action of  $Z^t A Z$  readily (e.g. for use in Conjugate Gradient). Regardless of the choice of  $B_m$ , the symmetric positive definite stencil defined by  $Z^t A Z$  coincides with the standard 5-point stencil for all degrees of freedom sufficiently far from the interface.

We now show that the rows and columns of the matrix  $B$  can be re-ordered to produce a non-singular, upper triangular matrix  $B_m$ . Specifically, ordering the cut-cells  $\{c_1, \dots, c_m\} = \mathcal{C}_{\partial D}^h$  lexicographically, and then selecting the lower-left node of the  $i^{\text{th}}$  cut-cell as the  $i^{\text{th}}$  degree of freedom (thus reordering the rows in

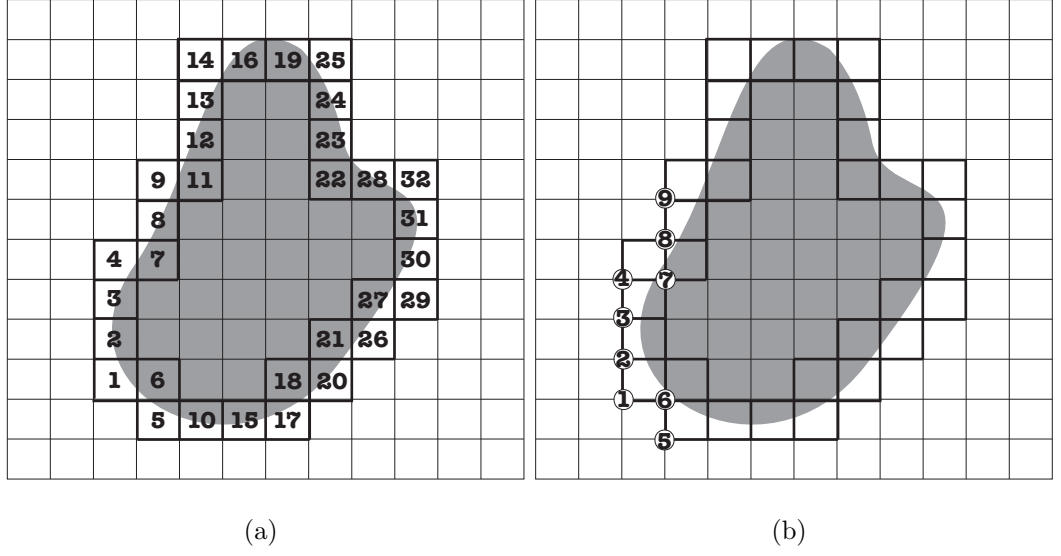


Figure 2.5: Upper triangular ordering. The cell-centered numbers in figure (a) indicate the ordering of the cells  $c_k \in \mathcal{C}_{\partial D}^h$ . The nodal numbers in figure (b) indicate the corresponding ordering for the first 9 degrees of freedom.

$A$ ,  $B$  and  $\vec{u}$ ), gives  $B = (B_m | B_{n-m})$  with  $B_m$  upper triangular and non-singular (see Figure 2.5). Unfortunately, despite the convenient triangular structure of  $B_m$ , prohibitively large numerical error persists when solving (2.19), even on relatively coarse grids. As the interface in a given cell recedes from the lower-left node of that cell (for instance, cells and nodes 1, 4, 5, 9, 11, 14, 17, 18, 19, 21, 22, 23, or 29 in Figure 2.5), the corresponding row in  $B_m$  has off-diagonal entries with substantially larger magnitude than the diagonal entry of that row. Generally,

enough rows of this type exist so that  $B_m$  behaves much like the matrix

$$C = \begin{pmatrix} 1 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 2 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 2 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Considering the linear system  $Cx = \vec{e}_m$  with  $\vec{e}_m = (0, \dots, 0, 1)^t$  illuminates the source of this error. Indeed, in this case  $\|C^{-1}\vec{e}_m\|_\infty$  grows like  $2^m$ . The forward substitutions with  $B_m^t$ , needed for  $Z^t$  multiplies, also exhibit this behavior. As  $m$  increases under grid refinement, these behaviors quickly (in some cases, anything finer than a 64x64 grid) dominate machine precision. In practice, scalar multiples of  $\vec{e}_m$  always appear in  $B_{n-m}$ , making such a  $B_m$  practically unusable in a null-space method. Moreover, this problem persists in all similar constructions of  $B_m$  (different orderings, node choices, etc).

For this reason, we use an alternative approximation to  $\mathbf{H}^{-1/2}(\partial D)$  that produces a different set of linear constraints. Our choice permits an ordering of  $B$  with a non-singular, diagonal sub-matrix  $B_m$ . If we enforce one constraint per cell as above, then in general there do not exist  $m$  degrees of freedom that each only participate in one constraint, so that no ordering could produce a diagonal matrix  $B_m$ . Motivated by this observation, we approximate  $\mathbf{H}^{-1/2}(\partial D)$  using  $\Lambda^{2h}$ , the space of Lagrange multipliers corresponding to the grid  $\mathcal{G}^{2h}$ . That is, for every  $\mu^h \in \Lambda^{2h}$ ,

$$\mu^h(\mathbf{x}) = \sum_{\hat{c}_k \in \mathcal{C}_{\partial D}^{2h}} \mu_k \chi_{\hat{c}_k \cap \partial D}(\mathbf{x}).$$

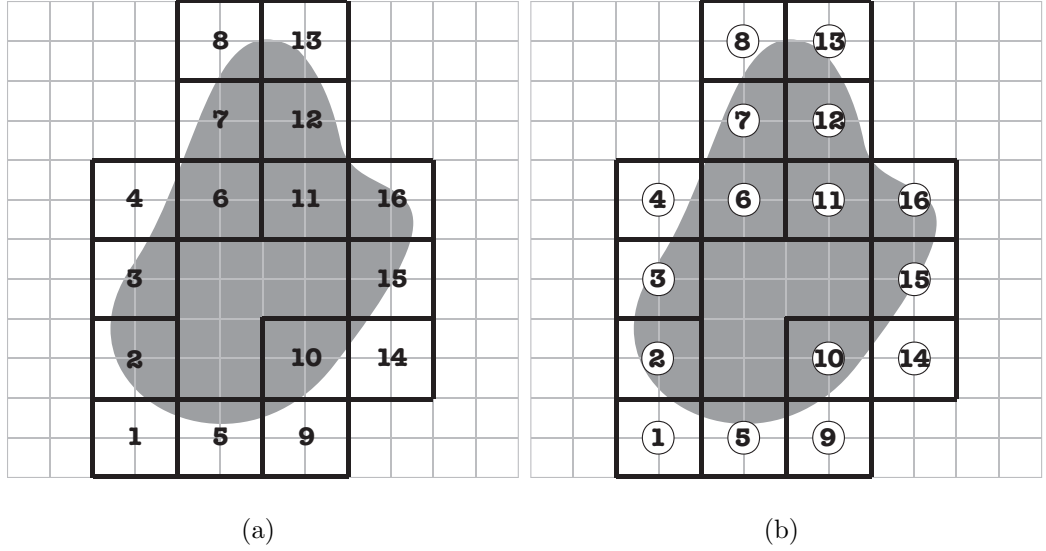


Figure 2.6: Diagonal ordering scheme. We enforce one constraint per coarse cell, enumerated in the image on the left. In the image on the right, we index the degrees of freedom at the centers of the coarse cells by the corresponding constraint indices. This gives a constraint matrix  $B$  with a diagonal sub-matrix  $B_m$ . Note that this gives a slightly denser  $B_{n-m}$ , since now as many as 9 degrees of freedom may contribute to a given row for embedded Dirichlet problems.

Thus, each row of  $B$  now enforces the condition

$$\int_{\hat{c}_k \cap \partial D} u^h(\mathbf{x}) dS = \int_{\hat{c}_k \cap \partial D} p(\mathbf{x}) dS,$$

for each of the cells  $\hat{c}_k \in \mathcal{C}_{\partial D}^{2h} = \{\hat{c}_1, \hat{c}_2, \dots, \hat{c}_m\}$  (see Figure 2.4) that intersect the boundary. In the figure, each of the cells in the grid  $\mathcal{G}^{2h}$  is the union of 4 cells in the grid  $\mathcal{G}^h$ , so that at the center of each cell  $\hat{c}_k \in \mathcal{G}^{2h}$  lies a degree of freedom  $u_k$  whose associated nodal basis function  $N_k$  vanishes outside the cell  $\hat{c}_k$ . Therefore for each cell  $\hat{c}_k \in \mathcal{C}_{\partial D}^{2h}$  we choose this central degree of freedom as the  $k^{\text{th}}$  in our reordering. As such,

$$\int_{\hat{c}_i \cap \partial D} N_k dS = 0, \quad \forall i \neq k, \quad 1 \leq i \leq m. \quad (2.20)$$

See Figure 2.6 for a pictorial description of this ordering. Clearly, this gives  $B = (B_m | B_{n-m})$  with  $B_m$  diagonal and non-singular. We then use the corresponding fundamental basis (2.18) to trivially reduce the saddle-point problem to the symmetric positive definite system  $Z^t A Z \vec{v} = Z^t(\vec{f} - A\vec{c})$  by applying the null-space method (2.17).

### 2.0.3 Embedded Interface

To handle the full elliptic interface problem (1.1,1.2,1.3), we combine our embedded Neumann and embedded Dirichlet approaches in a straightforward way. We consider the equivalent minimization form of the problem (1.1,1.2,1.3):

over all  $u \in \mathbf{V} = \{u : u^\pm \in \mathbf{H}^1(D^\pm)\}$ , minimize

$$E(u) = e(u) - (f, u)_D - (b, \bar{u})_\Gamma = \int_{D^+ \cup D^-} \frac{1}{2} \nabla u \cdot \beta \nabla u dx - \int_D f u dx - \int_\Gamma b \bar{u} dS \quad (2.21)$$

$$\text{such that } ([u], \mu)_\Gamma = (a, \mu)_\Gamma \quad \forall \mu \in \mathbf{H}^{-1/2}(\Gamma). \quad (2.22)$$

Here  $\bar{u}(\mathbf{x})|_{\Gamma} = (u^+ + u^-)/2$ . As before, we define discretizations of  $\mathbf{V}$  and  $\mathbf{H}^{-1/2}(\Gamma)$  and then solve the resulting discrete saddle-point problem. To define  $\mathbf{V}^h \subset \mathbf{V}$ , we separately discretize  $\mathbf{H}^1(D^+)$  and  $\mathbf{H}^1(D^-)$  using the same virtual node representation used to discretize the embedded Neumann problem. This will naturally introduce duplicate Cartesian cells that intersect the boundary, with independent copies associated with the interior and exterior discretizations (see Figure 2.7). This discretization results in the block diagonal stiffness matrix for the interface problem,

$$A = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix},$$

where  $A^+$  is the stiffness matrix associated with the embedded Neumann problem on  $D^+$  and  $A^-$  is the stiffness matrix associated with the embedded Neumann problem on  $D^-$ , as described in §2.0.1.

Similarly, along the interface we make the same choice of discrete Lagrange multiplier space as before, so that over every cell  $\hat{c}_k \in \mathcal{C}_{\Gamma}^{2h}$ ,

$$\int_{\hat{c}_k \cap \Gamma} [u^h] dS = \int_{\hat{c}_k \cap \Gamma} a dS.$$

This results in the block interface constraint matrix  $B = (B^+ | -B^-)$ , where  $B^{\pm}$  is respectively the constraint matrix associated with the embedded Dirichlet problem on the exterior or interior of the interface. In other words,  $B_{ij} = \int_{\hat{c}_i \cap \Gamma} \text{sign}(j) N_j(\mathbf{x}) dS$ , where  $\text{sign}(j) = 1$  if degree of freedom  $j$  is associated with  $u^{+,h}$  and  $\text{sign}(j) = -1$  if degree of freedom  $j$  is associated with  $u^{-,h}$ . These discretization choices give the saddle-point problem

$$\begin{pmatrix} A^+ & 0 & B^{+t} \\ 0 & A^- & -B^{-t} \\ B^+ & -B^- & 0 \end{pmatrix} \begin{pmatrix} \vec{u}^+ \\ \vec{u}^- \\ \vec{\lambda} \end{pmatrix} = \begin{pmatrix} \vec{f}^+ \\ \vec{f}^- \\ \vec{a} \end{pmatrix}, \quad (2.23)$$



where  $\vec{u}^+$  contains the degrees of freedom associated with the nodal values of the exterior discretization and  $\vec{u}^-$  contains the degrees of freedom associated with the nodal values of the interior discretization. We once again solve the saddle-point system using the null-space method described above in §2.0.2 by defining an ordering  $B = (B_m | B_{n-m})$  with  $B_m$  diagonal. Given any ordering for the constraints, we choose the virtual degree of freedom at the center of the  $i^{\text{th}}$  cell  $\hat{c}_i \in \mathcal{C}_\Gamma^{2h}$  as the  $i^{\text{th}}$  degree of freedom in our ordering. See Figures 2.6, 2.7 and section §2.0.2 for more information. There are always at least two degrees of freedom associated with the center node. We choose the virtual degree of freedom as this results in a system  $Z^t A Z$  with significantly better conditioning in practice.

### 2.0.3.1 Virtual node discontinuity removal

In general, our proposed method requires the solution of the symmetric positive definite system  $Z^t A Z$ . However, if the coefficient  $\beta$  is smooth, the IIM and similar methods achieve uniform second order accuracy without altering the original 5-point difference stencil. In this section, we demonstrate how the virtual node framework similarly allows the use of the 5-point difference stencil for continuous coefficients. For simplicity of exposition, we assume  $\beta(\mathbf{x}) \equiv 1$  for the rest of this section.

Suppose  $c(\mathbf{x}) \in \mathbf{V}$  is constructed to satisfy the jump conditions (1.2, 1.3) and  $u(\mathbf{x})$  is the exact solution. Then as  $[\beta] = 0$ , the difference  $w(\mathbf{x}) := u(\mathbf{x}) - c(\mathbf{x})$  satisfies  $[\beta \nabla w \cdot n] = \beta [\nabla w \cdot n] = 0$  and  $[w] = 0$ . Since  $w$  satisfies homogeneous jump conditions  $[\nabla w \cdot n] = 0$  and  $[w] = 0$ , we do not require virtual nodes to capture any discontinuities across  $\Gamma$ . In this manner, solving for  $w$  presents an appealing alternative as the presence of virtual nodes no longer adversely affects

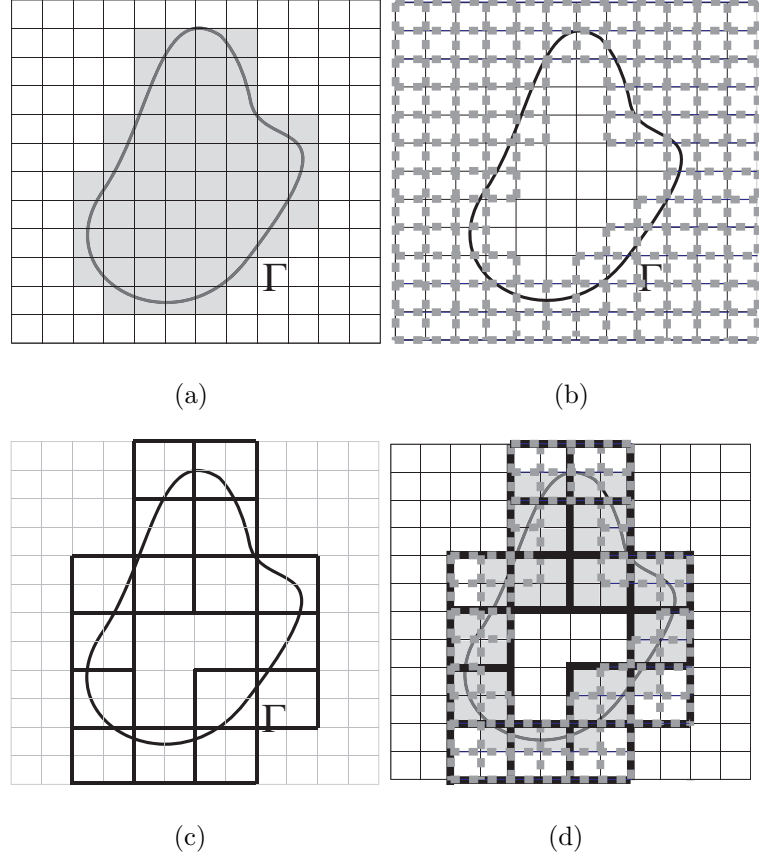


Figure 2.7: In (a), the degrees of freedom lying on shaded cells define  $u^{-,h}$ , and in (b), the degrees of freedom lying on dashed cells define  $u^{+,h}$ . Applying our embedded Neumann approach on the shaded and dashed grids defines the matrices  $A^-$  and  $A^+$ , respectively. Degrees of freedom associated with  $u^{-,h}$  and  $u^{+,h}$  are collocated along the interface. These representations couple together in the coarse cells outlined in Figure (c). Figure (d) depicts the overlapping domains of definition of  $u^{-,h}$  and  $u^{+,h}$  in the coarse cells.

the subsequent linear algebra problem. Therefore, when  $[\beta] = 0$  we recover an approximation to (1.2,1.3) by separately discretizing  $w$  and  $c$ , then setting  $u = w + c$ .

We discretize  $w$  over the unduplicated grid  $\mathcal{G}^h$  using  $\mathbf{H}^1(D)$  Cartesian piecewise bilinear elements. Consequently, if the grid  $\mathcal{G}^h$  contains  $r$  material degrees of freedom, then  $\vec{w} \in \mathbb{R}^r$  contains the coefficients in terms of the bilinear basis. We discretize  $u$  and  $c$  using the full virtual node basis  $\mathbf{V}^h$  as they possess lower regularity across  $\Gamma$ . With these choices, we can represent the coefficient vector  $\vec{u} \in \mathbb{R}^n$  ( $n > r$ ) of the approximate solution  $u^h$  in the basis of  $\mathbf{V}^h$  as  $\vec{u} = \vec{c} + T\vec{w}$ , where the matrix  $T \in \mathbb{R}^{n \times r}$  maps from the bilinear basis to the virtual node basis. We determine this change of basis by a simple identification of virtual and material nodes, as a function  $v^h \in \mathbf{V}^h$  satisfies homogeneous jump conditions if and only if the value of the function  $v^h$  at a virtual node always equals its value at the associated material node. Therefore,  $T$  maps the value at a given node in the original grid to every node, virtual or material, associated with the same location in the virtual node basis.

Although any ordering of degrees of freedom will suffice to construct  $T$ , for simplicity assume that

$$\vec{u} = (u_1, u_2, \dots, u_{n_v}, u_{n_v+1}, u_{n_v+2}, \dots, u_{2n_v}, u_{2n_v+1}, \dots, u_n)^t.$$

Here,  $\{u_k\}_{k=1}^{n_v}$  represent the  $n_v := n - r$  total coefficients of the virtual degrees of freedom;  $u_{n_v+k}$ ,  $1 \leq k \leq n_v$ , represents the coefficient of the real degree of freedom corresponding to the same physical node as  $u_k$ ; the remaining  $\{u_k\}_{k=2n_v+1}^n$  degrees of freedom do not lie on any cut-cells. Then

$$T = \begin{pmatrix} I_{n_v} & 0 \\ I_{n_v} & 0 \\ 0 & I_{n-2n_v} \end{pmatrix}. \quad (2.24)$$

More generally, each column of  $T$  corresponds to a material node in the grid, and each row of  $T$  corresponds to either a material node or a virtual node. Then the column of  $T$  corresponding to a material node  $\mathbf{x}_l$  simply has a one in the column corresponding to  $\mathbf{x}_l$ , a one in the column corresponding to any virtual node in the same physical location (i.e. coordinates) as  $\mathbf{x}_l$ , and zeros otherwise.

Determining  $w^h$  now proceeds in a manner analogous to the null-space method used to solve (2.16): we wish to minimize the energy over all vectors of the form  $\vec{u} = \vec{c} + T\vec{w}$ . For the following discussion, suppose we define the discrete energy (2.21) using the Cartesian bilinear representation everywhere in the domain. Then substituting the expression for  $\vec{u}$  into the energy (2.21) gives

$$E^h(\vec{u}) = \frac{1}{2}\vec{w}^t T^t A T \vec{w} - \vec{f}^t T \vec{w} + \vec{w}^t T^t A \vec{c} + \frac{1}{2}\vec{c}^t A \vec{c} - \vec{f}^t \vec{c}, \quad (2.25)$$

which defines an energy only over the original, material degrees of freedom  $\vec{w} \in \mathbb{R}^r$ . Differentiation with respect to  $w_i$  then leads to the linear system

$$T^t A T \vec{w} = T^t (\vec{f} - A \vec{c}) \quad (2.26)$$

$$\vec{u} = \vec{c} + T \vec{w}. \quad (2.27)$$

Remarkably, the matrix  $T^t A T$  is the straightforward discretization over the material degrees of freedom, i.e. a 9-point, second order approximation to the Laplacian. Moreover, as  $\vec{w}$  corresponds to the material nodal values on a regular grid, we may operate on it instead with the standard 5-point difference stencil  $\Delta^h$  and solve the system

$$\Delta^h \vec{w} = T^t (\vec{f} - A \vec{c}) \quad (2.28)$$

to provide an approximate solution at all of the relevant real degrees of freedom. This approach allows the application of efficient, black-box solvers for  $\Delta^h$  and only requires constructing the right hand side of (2.28). Thus, the lack of regularity in the problem no longer adversely affects the linear algebra.

In principle, many different constructions could result in a satisfactory particular solution  $c$ . To minimize the computational effort, we construct a  $c$  supported only along the interface. The time required to generate such a particular solution contributes negligibly to the overall computational cost. We assume that  $D^-$  does not intersect the computational boundary and construct a particular solution  $c$  that vanishes on the exterior region. That is,  $c|_{D^+} = 0$  so that  $c|_{\partial D} = 0$ ,  $[c] = -c^- = a$ ,  $\beta[\nabla c \cdot n] = -\beta \nabla c^- \cdot n = b$ . Therefore, we need only to define  $c^-$  over those interior material and interior virtual nodes along the interface. We performed this construction via bilinear least-squares extrapolation from the known behavior along the interface.

## CHAPTER 3

### Numerical Results

#### 3.1 Implementation

In this section we detail a sample implementation of our method with the interface represented as a level set  $\phi$ , where  $D = \{\phi < 0\}$  for irregular domain problems and  $D^- = \{\phi < 0\}$  for embedded interface problems. We describe the implementation for the embedded Dirichlet case  $D = \{\phi < 0\}$ , since the interface case is analogous.

First, we define the computational domain as those cells  $c_k = \{\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \mathbf{x}_{k_4}\}$  where  $\phi(\mathbf{x}_{k_i}) < 0$  for at least one node  $\mathbf{x}_{k_i}$ . If  $\phi(\mathbf{x}_{k_i}) < 0$  for all  $1 \leq i \leq 4$  then  $c_k$  lies in  $\mathcal{C}^h \setminus \mathcal{C}_{\partial D}^h$ . Otherwise, the cell  $c_k$  lies in  $\mathcal{C}_{\partial D}^h$ . That is,

$$\mathcal{C}^h = \{c_k = \{\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \mathbf{x}_{k_4}\} : \phi(\mathbf{x}_{k_i}) < 0 \text{ for at least one } i\}, \quad (3.1)$$

$$\mathcal{C}_{\partial D}^h = \{c_k \in \mathcal{C}^h : \phi(\mathbf{x}_{k_i}) > 0 \text{ for at least one } i\}, \quad (3.2)$$

$$\mathcal{C}^h \setminus \mathcal{C}_{\partial D}^h = \{c_k \in \mathcal{C}^h : \phi(\mathbf{x}_{k_i}) < 0, \forall i\}. \quad (3.3)$$

Next, we assemble the stiffness matrix  $A$ , the constraint matrix  $B$ , and the vectors  $\vec{f}$  and  $\vec{a}$  by looping over the cells  $c_k \in \mathcal{C}^h$ . The boundary contribution is described in Step 1 and the interior contribution is described in Step 2. Notice that if a node is not adjacent to any cell which is intersected by the boundary then the 5-point stencil is used.

---

**Step 1** Adding boundary contribution to  $A, \vec{f}$  and  $B$

---

```

for  $c_k \in \mathcal{C}_{\partial D}^h$  do
  for  $1 \leq i, j \leq 4$  do
     $A_{k_i k_j} + = \bar{\beta} \int_{D \cap c_k} \nabla N_{k_i} \cdot \nabla N_{k_j} d\mathbf{x}$  {See Figure 2.3 for integration details}
  end for
  for  $1 \leq i \leq 4$  do
     $f_{k_i} + = \bar{f} \int_{D \cap c_k} N_{k_i} d\mathbf{x}$  {See Figure 2.3 for integration details}
     $l \leftarrow$  index of the coarse cell containing  $c_k$  {See Figure 2.6}
     $B_{l k_i} + = \int_{c_k \cap \partial D} N_{k_i} dS$ 
     $a_l + = \bar{a} \int_{c_k \cap \partial D} N_{k_i} dS$ 
  end for
end for

```

---



---

**Step 2** Adding interior contribution to  $A$  and  $\vec{f}$

---

```

for  $c_k \in \mathcal{C}^h \setminus \mathcal{C}_{\partial D}^h$  do
  for  $1 \leq i \leq 4$  do
     $A_{k_i k_i} + = \bar{\beta}$ 
     $f_{k_i} + = .25 h^2 \bar{f}$ 
    for  $1 \leq j \leq 4$  do
      if  $\mathbf{x}_{k_i} \neq \mathbf{x}_{k_j}$  and are edge connected then
         $A_{k_i k_j} - = .5 \bar{\beta}$ 
      end if
    end for
  end for
end for

```

---

We compute the area integrals using a polygonal representation of  $D \cap c_k$  and the divergence theorem. The set of vertices of the polygon consists of all nodes with  $\phi < 0$ , as well as the two crossings  $\mathbf{x}_c$  on the edges of the cell (see Figure 2.3). Given a pair of nodes  $\mathbf{x}_{k_i}$  and  $\mathbf{x}_{k_j}$  with  $\phi(\mathbf{x}_{k_i})\phi(\mathbf{x}_{k_j}) < 0$ , we compute the edge crossing as

$$\theta = \frac{\phi(\mathbf{x}_{k_i})}{\phi(\mathbf{x}_{k_i}) - \phi(\mathbf{x}_{k_j})},$$

$$\mathbf{x}_c = \mathbf{x}_{k_j}\theta + \mathbf{x}_{k_i}(1 - \theta).$$

The divergence theorem converts the area integral of the second order polynomials  $\nabla N_{k_i} \cdot \nabla N_{k_j}$  over the irregular polygon into a line integral of third order polynomials over the polygonal boundary (see Figure 2.3). The integrals of the bilinear functions  $N_{k_i}$  over  $\partial D \cap c_k$  are line integrals over the segment joining the two edge crossings. The simple low-order polynomials are integrated over each segment analytically.

The averages  $\bar{f}, \bar{\beta}$  and  $\bar{a}$  are also required. When  $f$  is known at nodes, we use bilinear interpolation to represent it over a cell  $c_k$ ,  $f|_{c_k} = \sum_{i=1}^4 f_{k_i} N_{k_i}(\mathbf{x})$ . Then  $\bar{f}$  may be computed by integrating this bilinear function over the material region and dividing by the area. These integrals are computed as an area integral using the divergence theorem as above. Thus the average  $\bar{f}$  is,

$$\bar{f} = \frac{\sum_{i=1}^4 f_{k_i} \int_{D \cap c_k} N_{k_i} dx}{\text{Area}(D \cap c_k)} = \frac{\sum_{i=1}^4 f_{k_i} \int_{D \cap c_k} N_{k_i} dx}{\sum_{i=1}^4 \int_{D \cap c_k} N_{k_i} dx}. \quad (3.4)$$

With  $A$  and  $B$  in hand, we re-order the degrees of freedom so that  $B = (B_m | B_{n-m})$  with  $B_m$  diagonal and non-singular. This amounts to finding the index  $k_i$  of the degree of freedom at the center of the  $l^{th}$  coarse cell, then permuting the degrees of freedom with indices  $l$  and  $k_i$  (see Figure 2.6). Once we have re-ordered the degrees of freedom, the fundamental basis  $Z$  and reduced constraints  $\vec{c}$  can be easily computed (2.18). We then solve the system  $Z^t A Z \vec{v} = Z^t (\vec{f} - A \vec{c})$



iteratively, performing multiplications with  $Z, Z^t$  implicitly using the factors  $B_m$  and  $B_{n-m}$ , and lastly recover the solution  $\vec{u} = \vec{c} + Z\vec{v}$ .

## 3.2 Numerical Examples

This section presents a convergence test for each of the components of our method. We first demonstrate the expected second order accuracy for embedded Neumann and embedded Dirichlet problems in §3.2.1 and §3.2.2, respectively, and for interface problems in §3.2.3. In §3.2.3.2 and §3.2.3.3 we examine the performance of our method for the important special case when  $\beta$  exhibits a large jump across the interface. Lastly, in §3.2.4 we demonstrate the effectiveness of this discontinuity removal technique on a  $C^0$  Lipschitz segmented curve. The richer virtual node representation, as in Figure 4.1, allows us to achieve second order results for a non-smooth interface while still retaining the standard 5-point finite difference stencil.

We ran all of the examples on a sequence of  $N \times N$  grids, for  $80 \leq N \leq 800$ . Each grid ranges from  $-1 \leq x \leq 1$ ,  $-1 \leq y \leq 1$ . The error plots depict  $\log_{10} \|e\|_{L^\infty}$  versus  $\log_{10} N$ . The examples include both level set representations and Lagrangian representations of the interface. For interfaces that have more detail than the background grid can resolve, using a level set introduces non-negligible geometric regularization. See §4 for a discussion of the geometric precision of our method.

We used the SuiteSparse [69] numerics library with the Goto BLAS in the course of our research and to perform the extrapolations required for the discontinuity removal example 3.2.4.

### 3.2.1 Embedded Neumann

We demonstrate the method applied to the embedded Neumann problem

$$-\nabla \cdot \beta(\mathbf{x}) \nabla u = f, \quad \forall \mathbf{x} \in D,$$

$$\beta(\mathbf{x}) \nabla u \cdot \mathbf{n} = q(\mathbf{x}), \quad \forall \mathbf{x} \in \partial D_n.$$

Here  $\beta(\mathbf{x}) = 4 + x + y$ . We chose the parameters  $q$  and  $f$  using the exact solution

$$u = (x^3 - y^3) \cos(x + y).$$

The embedded Neumann boundary,  $\partial D_n$ , is given by the 5-pointed star with vertices

$$t_0 = .1243$$

$$r_i = .35 + .3(i \bmod 2)$$

$$X_i = r_i \cos\left(\frac{\pi i}{5} + t_0\right)$$

$$Y_i = r_i \sin\left(\frac{\pi i}{5} + t_0\right),$$

for  $1 \leq i \leq 10$ , represented as a Lagrangian curve. See Figure 3.1 for the error plot. A least squares regression estimates the order of accuracy as 1.95.

### 3.2.2 Embedded Dirichlet

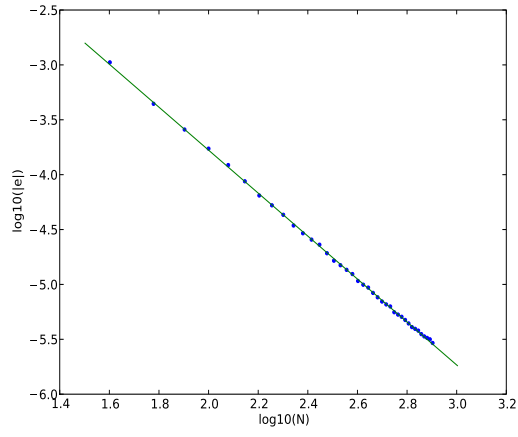
We demonstrate the method applied to the embedded Dirichlet problem

$$\Delta u = 0, \quad \forall \mathbf{x} \in D,$$

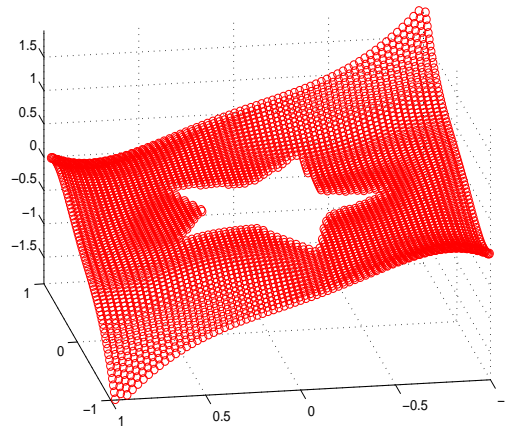
$$u = p(\mathbf{x}), \quad \forall \mathbf{x} \in \partial D_d = \partial D.$$

We chose the Dirichlet condition  $p$  using the chosen exact solution

$$u = x^2 - y^2.$$



(a) Estimated order: 1.95



(b)

Figure 3.1: Numerical results for Example 3.2.1

The embedded boundary  $\partial D$  is given by the curve

$$t_0 = .00132$$

$$r_0 = .02\sqrt{5}$$

$$r(t) = .5 + .2 \sin(5t)$$

$$X(\theta) = r_0 + r(\theta + t_0) \cos(\theta + t_0)$$

$$Y(\theta) = r_0 + r(\theta + t_0) \sin(\theta + t_0),$$

represented as a Lagrangian curve. See Figure 3.2 for the error plot. A least squares regression estimates the order of accuracy as 1.86.

### 3.2.3 Embedded Interface

#### 3.2.3.1 Embedded Interface Example 1

We demonstrate the method applied to the embedded interface problem

$$-\nabla \cdot (\beta(\mathbf{x}) \nabla u) = f(\mathbf{x}), \quad \forall \mathbf{x} \in D \setminus \Gamma,$$

$$[u] = a(\mathbf{x}),$$

$$[\beta(\mathbf{x}) \nabla u \cdot \mathbf{n}] = b(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma.$$

Here  $\beta(\mathbf{x}) = 4 + \sin(x + y)$  in the interior and  $\beta(\mathbf{x}) = 2 + x^2 + y^2$  in the exterior.

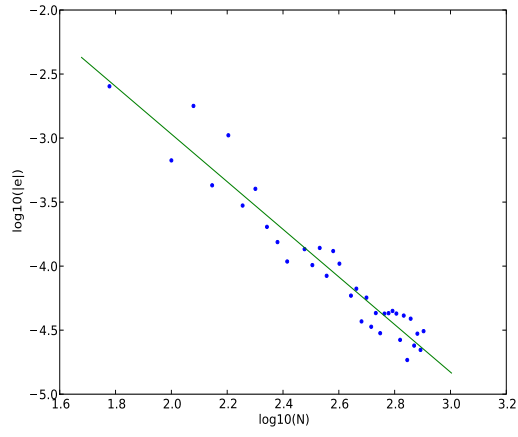
We chose the parameters  $a, b$  and  $f$  using the exact solution

$$u^- = \cos(y) \sin(x)$$

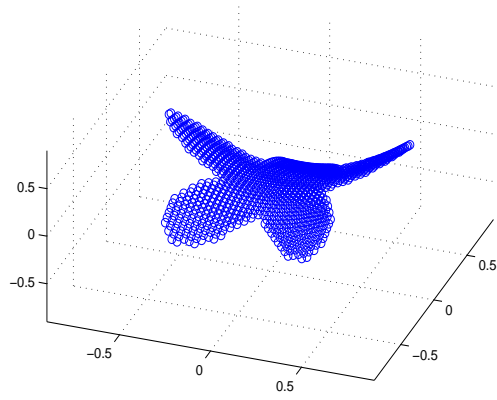
$$u^+ = 1 - x^2 - y^2.$$

The interface is given by the curve parametrized by

$$t_0 = .45234$$



(a) Estimated order: 1.86



(b)

Figure 3.2: Numerical results for Example 3.2.2

$$\theta(t) = t + \sin(4t)$$

$$r(t) = .60125 + .24012 \cos(4t + \pi/2)$$

$$X(t) = r(t + t_0) \cos(\theta(t + t_0))$$

$$Y(t) = r(t + t_0) \sin(\theta(t + t_0))$$

for  $0 \leq t \leq 2\pi$  represented with a level set. See Figure 3.3 for a plot of the error in the solution and in the gradient evaluated on the interface. A least squares regression estimates the order of accuracy of the solution as 1.92 and the order of accuracy of the gradient as .96. The gradient was evaluated point-wise at the mid-point  $\mathbf{x}_M$  of the interface segment in each cell by differentiating the bilinear basis elements, that is,  $\nabla u(\mathbf{x}_M) = \sum_{i=1}^4 u_i \nabla N_i(\mathbf{x}_M)$ .

### 3.2.3.2 Embedded Interface Example 2

In this example we examine the performance of the method in the case of when the coefficient  $\beta$  has a large jump across the interface. The following example was taken from [151]. We solve the interface problem

$$\nabla \cdot \beta(\mathbf{x}) \nabla u = f, \quad \forall \mathbf{x} \in D \setminus \Gamma,$$

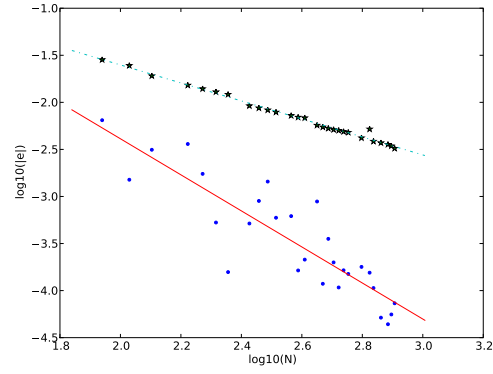
$$[u] = a(\mathbf{x}),$$

$$[\nabla u \cdot \mathbf{n}] = b(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma.$$

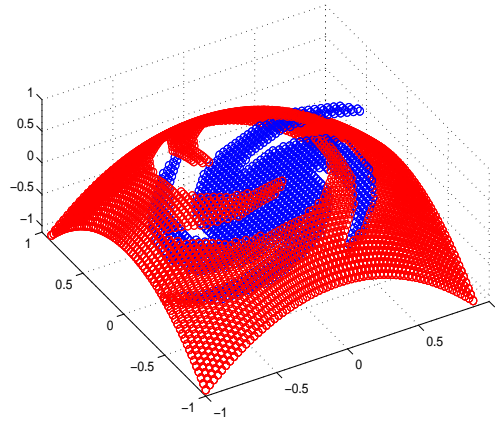
Here we take the coefficient to be piecewise constant,  $\beta(\mathbf{x}) = \beta^+$  in the exterior and  $\beta(\mathbf{x}) = \beta^-$  on the interior. We chose the parameters  $a, b$  and  $f$  using the exact solution

$$u^- = x^2 + y^2$$

$$u^+ = .1(x^2 + y^2)^2 - .01 \ln(2\sqrt{x^2 + y^2}).$$



(a) Estimated order: 1.92 for solution (solid line) and .96 for gradient along the interface (dotted line)



(b)

Figure 3.3: Numerical results for Example 3.2.3

The interface  $\Gamma$  is given by the curve used in example §3.2.2,

$$t_0 = .00132$$

$$r_0 = .02\sqrt{5}$$

$$r(t) = .5 + .2\sin(5t)$$

$$X(t) = r_0 + r(t + t_0)\cos(t + t_0)$$

$$Y(t) = r_0 + r(t + t_0)\sin(t + t_0),$$

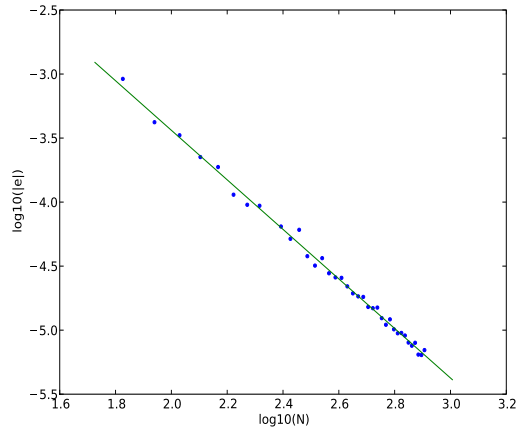
for  $0 \leq t \leq 2\pi$ . See Figures 3.4 and 3.5 for a plot of the error for three values of the ratio  $\beta^- : \beta^+ -$ ,  $1 : 10$ ,  $1 : 1000$  and  $1000 : 1$ . A least squares regression estimated the order of accuracies as 1.94 for  $1 : 10$ , 1.86 for  $1 : 1000$  and 1.77 for  $1000 : 1$ .

See Figure 3.6 for the number of Conjugate Gradient iterations, computer time in seconds and condition numbers of the linear systems before and after incomplete Cholesky preconditioning. Figure 3.6 compares the performance to the standard 5-point Laplacian on a square with no interface as a reference. All tests were run with grid resolution  $800 \times 800$  and to residual norm tolerance of  $10^{-12}$ . The linear system was solved using the PETSc Conjugate Gradient with the PETSc incomplete Cholesky preconditioner [12, 11, 13]. The code was run in serial on a 2.8 GHz laptop computer. All linear systems were normalized to have a constant diagonal before the preconditioner was applied. The high coefficient ratios incur a moderate cost but are still comparable to the standard 5-point discretization.

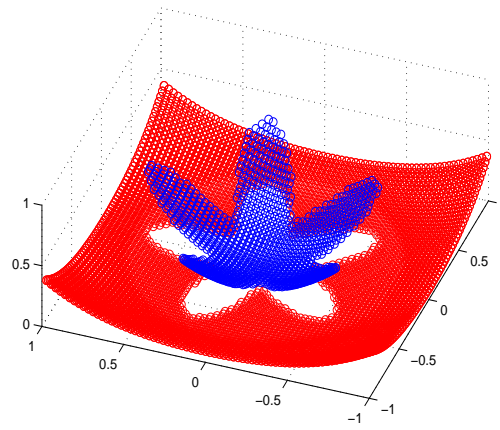
### 3.2.3.3 Embedded Interface Example 3

In this example we again examine the performance of the method in the case of when the coefficient  $\beta$  has a large jump across the interface. We solve the



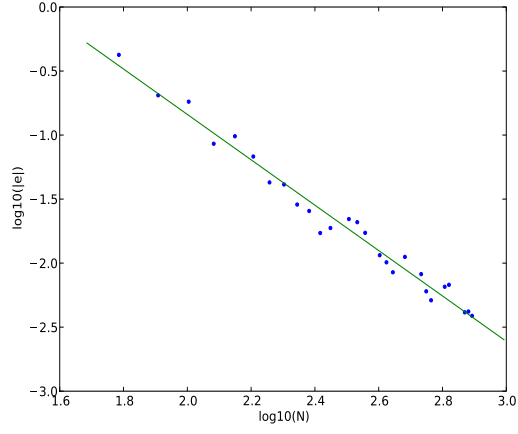


(a) Estimated order: 1.94,  $\beta^- : \beta^+ = 1 : 10$

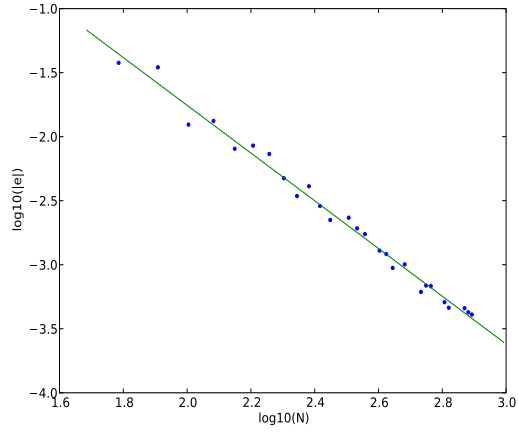


(b)

Figure 3.4: Numerical results for Example 3.2.3.2



(a) Estimated order: 1.86,  $\beta^- : \beta^+ = 1 : 1000$



(b) Estimated order: 1.77,  $\beta^- : \beta^+ = 1000 : 1$

Figure 3.5: Numerical results for Example 3.2.3.2

Case	$\frac{\sigma_{\max}}{\sigma_{\min}}$ (before IC)	$\frac{\sigma_{\max}}{\sigma_{\min}}$ (after IC)	PCG iter	Time(s)
1:1	$7.60x10^5$	$3.42x10^4$	1106	95.19
1000:1	$4.43x10^8$	$2.13x10^7$	1803	157.45
1:1000	$6.45x10^5$	$4.59x10^4$	1751	150.27
5-point stencil	$2.58x10^5$	$2.28x10^4$	723	60.23

Figure 3.6: Condition numbers of linear system and clock time of PCG at resolution  $800 \times 800$  for Example §3.2.3.2

interface problem

$$\nabla \cdot \beta(\mathbf{x}) \nabla u = f, \quad \forall \mathbf{x} \in D \setminus \Gamma,$$

$$[u] = a(\mathbf{x}),$$

$$[\nabla u \cdot n] = b(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma.$$

Here we take the coefficient to be piecewise constant,  $\beta(\mathbf{x}) = \beta^+$  in the exterior and  $\beta(\mathbf{x}) = \beta^-$  on the interior. We chose the parameters  $a, b$  and  $f$  using the exact solution

$$u^- = x^2 + y^2 + 1$$

$$u^+ = \cos(x + y)$$

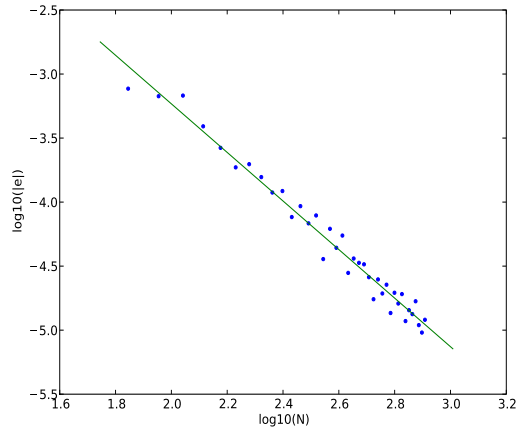
The interface  $\Gamma$  is given by the curve,

$$\theta_0 = .00132$$

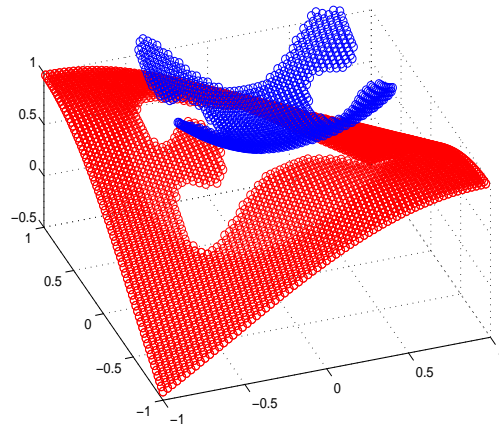
$$X(\theta) = .6 \cos(\theta + \theta_0) - .3 \cos(\theta - \theta_0)$$

$$Y(\theta) = .47 \sin(\theta + \theta_0) - .0047 \sin(3\theta - 3\theta_0) + .13 \sin(7\theta - 7\theta_0),$$

for  $0 \leq \theta \leq 2\pi$ . See Figures 3.7 and 3.8 for a plot of the error for three values of the ratio  $\beta^- : \beta^+$ ,  $1 : 10$ ,  $1 : 1000$  and  $1000 : 1$ . A least squares regression

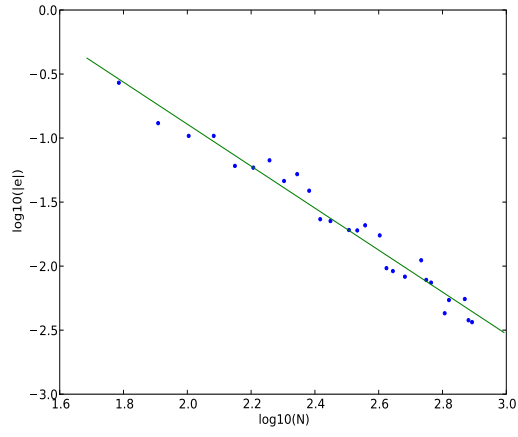


(a) Estimated order: 1.90

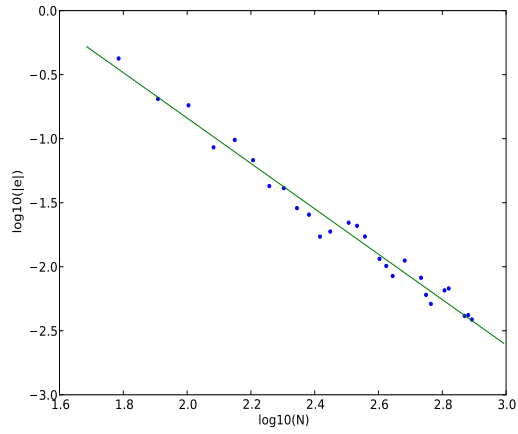


(b)

Figure 3.7: Numerical results for Example 3.2.3.3



(a) Estimated order: 1.64,  $\beta^- : \beta^+ = 1 : 1000$



(b) Estimated order: 1.77,  $\beta^- : \beta^+ = 1000 : 1$

Figure 3.8: Numerical results for Example 3.2.3.2

estimated the order of accuracies as 1.90 for 1 : 10, 1.64 for 1 : 1000 and 1.77 for 1000 : 1.

See Figure 3.9 for the number of Conjugate Gradient iterations, computer time in seconds and condition numbers of the linear systems before and after incomplete Cholesky preconditioning. Figure 3.9 compares the performance to the standard 5-point Laplacian on a square with no interface as a reference. All tests were run with grid resolution  $800 \times 800$  and to residual norm tolerance of  $10^{-12}$ . The linear system was solved using the PETSc Conjugate Gradient with the PETSc incomplete Cholesky preconditioner [12, 11, 13]. The code was run in serial on a 2.8 GHz laptop computer. All linear systems were normalized to have a constant diagonal before the preconditioner was applied. The high coefficient ratios incur a moderate cost but are still comparable to the standard 5-point discretization.

Case	$\frac{\sigma_{\max}}{\sigma_{\min}}$ (before IC)	$\frac{\sigma_{\max}}{\sigma_{\min}}$ (after IC)	PCG iter	Time(s)
1:1	$5.99x10^5$	$3.57x10^4$	1082	82.97
1000:1	$3.85x10^8$	$2.13x10^7$	1802	138.04
1:1000	$7.25x10^5$	$7.25x10^4$	1915	145.78
5-point stencil	$2.58x10^5$	$2.28x10^4$	723	60.23

Figure 3.9: Condition numbers of linear system and clock time of PCG at resolution  $800 \times 800$  for Example §3.2.3.3

### 3.2.4 Discontinuity Removal

We solve the interface problem

$$-\Delta u = f, \quad \forall \mathbf{x} \in D \setminus \Gamma,$$

$$[u] = a(\mathbf{x}),$$

$$[\nabla u \cdot n] = b(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma.$$

We chose the parameters  $a, b$  and  $f$  using the exact solution

$$u^- = \cos(y) \sin(x)$$

$$u^+ = 1 - x^2 - y^2.$$

The interface is the segmented 5-pointed star with vertices

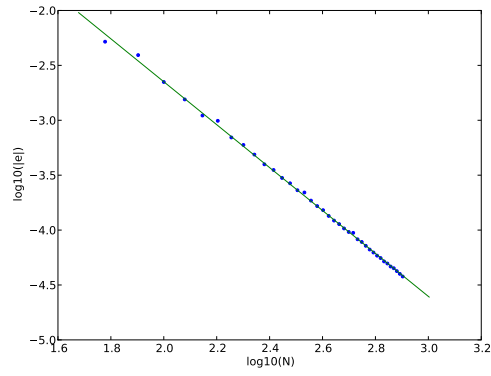
$$t_0 = .1243$$

$$r_i = .35 + .3(i \bmod 2)$$

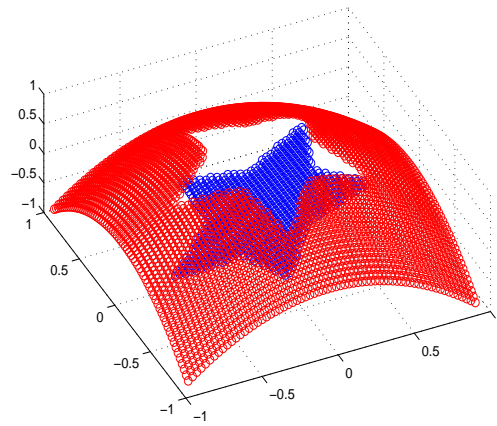
$$X_i = r_i \cos\left(\frac{\pi i}{5} + t_0\right)$$

$$Y_i = r_i \sin\left(\frac{\pi i}{5} + t_0\right),$$

for  $1 \leq i \leq 10$ , represented with a Lagrangian curve. See Figure 3.10 for the error plot. A least squares regression estimates the order of accuracy as 1.96.



(a) Estimated order: 1.96



(b)

Figure 3.10: Numerical results for Example 3.2.4



## CHAPTER 4

### Summary and Discussion

The proposed method uses a virtual node concept coupled with a Lagrange multiplier formulation to approximate the solution of the elliptic interface problem (1.1,1.2,1.3) and the related embedded Neumann and Dirichlet problems. Notably, the symmetric positive definite discretization and intuitive, geometric nature of the method make it relatively easy to implement. Numerical examples suggest second order convergence in  $L^\infty$ .

Certain other FEM approaches that also use virtual nodes [96, 97, 188, 78, 14, 180] parallel our work in spirit and implementation. Moreover, for typical geometric cases, virtual node representations yield the same space as those given by Heaviside enrichment XFEM approaches [188]. Such methods generally use lower-order triangular elements that do not permit the obvious choice of Lagrange multiplier space  $\mathbf{\Lambda}^h$  of one constraint per element [178]. Although the geometric processing required to apply our method is relatively non-trivial, we contend it is easier and more efficient than meshing with higher order finite elements, particularly for the 3D extension recently developed by J. Hellrung, L. Wang, E. Sifakis and J. Teran [99].

By design, our choice of Lagrange multiplier space eases the computational effort and memory limitations imposed by the saddle-point problem at the cost of accuracy, as the pairing  $(\mathbf{V}^h, \mathbf{\Lambda}^{2h})$  results in higher  $L^\infty$  error than the choice  $(\mathbf{V}^h, \mathbf{\Lambda}^h)$ . Our numerical experiments indicate that our choice does not sacri-

first second order convergence. Generally the approximations of  $\mathbf{H}^{-1/2}(\partial D)$  and  $\mathbf{H}^1(D)$  must satisfy an inf-sup condition uniformly in  $h$  for the solution to exhibit optimal convergence rates. See [178] for several characterizations of the relevant inf-sup conditions. However, as our numerical experiments indicate and the following argument demonstrates, our choice does not sacrifice such inf-sup stability. Assume the pairing  $(\mathbf{V}^h, \mathbf{\Lambda}^h)$  satisfies an inf-sup condition uniformly in  $h$ , that is, if there exist  $\gamma_0, h_0 > 0$  such that, for all  $h_0 \geq h > 0$ ,

$$\inf_{\mu^h \in \mathbf{\Lambda}^h} \sup_{v^h \in \mathbf{V}^h} \frac{\int_{\partial D} \mu^h T v^h dS}{\|\mu^h\|_{H^{-1/2}} \|v\|_{H^1}} := \inf_{\mu^h \in \mathbf{\Lambda}^h} \sup_{v^h \in \mathbf{V}^h} \alpha(\mu^h, v^h) \geq \gamma_0,$$

where  $T : \mathbf{H}^1(D) \rightarrow \mathbf{L}^2(\partial D)$  is the trace operator. Then whenever  $2h \leq h_0$ , as  $\mathbf{V}^{2h} \subset \mathbf{V}^h$

$$\gamma_0 \leq \inf_{\mu^h \in \mathbf{\Lambda}^{2h}} \sup_{v^h \in \mathbf{V}^{2h}} \alpha(\mu^h, v^h) \leq \inf_{\mu^h \in \mathbf{\Lambda}^{2h}} \sup_{v^h \in \mathbf{V}^h} \alpha(\mu^h, v^h), \quad (4.1)$$

so that our pairing  $(\mathbf{V}^h, \mathbf{\Lambda}^{2h})$  satisfies the same inf-sup condition uniformly in  $h$  as well. Moreover, the above argument holds if we begin with satisfactory constraints on any grid coarser than  $\mathcal{G}^h$  and then refine the corresponding space to obtain  $\mathbf{V}^h$ . In practice, we begin with the matrix  $B$  that results from using  $(\mathbf{V}^h, \mathbf{\Lambda}^h)$ . We then add together any constraints that lie in the same cell  $\hat{c}_k \in \mathcal{G}^{2h}$  to arrive at the constraints for the pairing  $(\mathbf{V}^h, \mathbf{\Lambda}^{2h})$ . The grid  $\mathcal{G}^{2h}$  merely serves as an easy means of determining which rows to sum to obtain a diagonal sub-matrix  $B_m$ . In theory, we could sum rows in some other fashion, so long as the resulting constraint corresponds to an inf-sup stable constraint from a coarser grid. Although it is not known if  $(\mathbf{V}^h, \mathbf{\Lambda}^h)$  satisfies an inf-sup condition, this analysis is an easy confirmation of the intuitive fact that coarsening constraints cannot adversely affect the stability of a discretization. Naturally, such coarsenings will adversely affect the accuracy, although our numerical tests indicate that higher order accuracy can be retained.

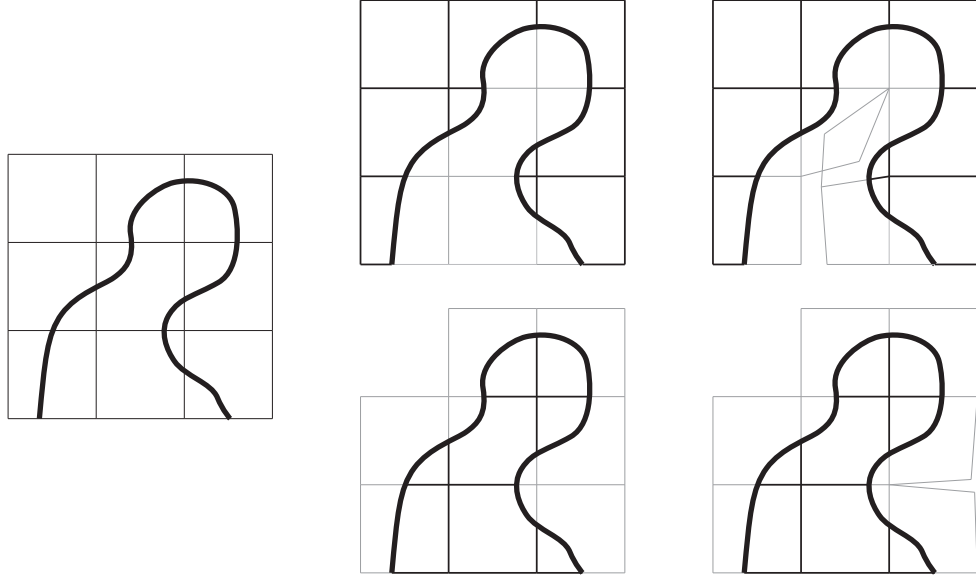


Figure 4.1: Virtual node representation. On the left, a portion of an interface passes through the unduplicated grid. The images in the center column show the result of applying the duplications schemes of [97, 188, 78]. This gives at most two degrees of freedom per original node. The right column shows the richer representation given by the virtual node algorithm. The cell in the center contains two disconnected interface segments. In this case, these segments lie in distinct cells after duplication. Including both cells in the same constraint in  $B$  degrades accuracy.

In our numerical examples we give results using both a level set representation of the interface and a segmented Lagrangian representation of the interface. In principle, our method does not rely upon a particular representation of  $\Gamma$ . However, for high curvature interfaces such as the examples of Chen and Strain [62], using a Lagrangian representation results in an interface with significantly more detail than the background grid can resolve. This can result in a grid cell  $c_k \in \mathcal{G}^h$  that contains two or more disconnected segments of the interface (see Figure 4.1). We found that, in this case, enforcing one constraint per cell results in unsatisfactory accuracy, and we have yet to attempt to resolve this issue. For smooth interfaces, this will always vanish under refinement, and using a level set representation generally prevents this phenomenon. However, for complex interfaces, the transfer to an under-resolved level set clearly involves non-negligible regularization. Moreover, our numerical experiments suggest we actually must guarantee none of the cells  $\hat{c}_k \in \mathcal{C}_\Gamma^{2h}$  contain disconnected interface segments in order to retain optimal accuracy in the pre-asymptotic regime. As they do not rely on Lagrange multipliers, this does not present a challenge to either embedded Neumann or our discontinuity removal technique. For instance, in example 3.2.4 we used the richer virtual node representation, illustrated in the right column of Figure 4.1, to appropriately handle the disconnected interface segments.

In our numerical examples, we solve the reduced saddle-point problem with a straightforward application of Conjugate Gradient with Jacobi preconditioning on examples §3.2.1, §3.2.2 and §3.2.4, and used PETSc Conjugate Gradient with incomplete Cholesky preconditioning [12, 11, 13] for examples §3.2.3, §3.2.3.2 and §3.2.3.3. Naturally, the use of our method in practical applications will require more efficient linear algebra solutions. Of course, in the discontinuity removal method of §2.0.3.1, optimal fast Poisson solvers may be applied. We also point out that we have not proposed a method for efficiently dealing with large jumps in  $\beta$

at the interface (see §3.2.3.2 and §3.2.3.3 for the performance of the method in this case). Such problems frequently arise in physics and engineering, for example, in computing the interaction between water and air. J. Hellrung, L. Wang, E. Sifakis and J. Teran [99] have recently developed a relatively straightforward geometric multigrid method to general interface and embedded Dirichlet problems, which provides an efficient 3D method and linear algebra solution, however further research will be required to handle large jumps in  $\beta$ .

As remarked above, the proposed method is, strictly speaking, a mixed FEM. Being a FEM, ideas from our method will mix with other FEM approaches and vice versa. As such, these ideas should be applicable to many kinds of elliptic problems with different kinds of discretizations. Moreover, the rigorous analysis of mixed FEM provides precedent and intuition for the success or failure of future directions, despite the fact that due to geometric irregularities, to our knowledge, little rigorous analysis has been done on similar methods, with the exception of the recent works [98, 129]. For example, the existing analysis suggests care must be taken when attempting to use virtual node methods based on Lagrange multipliers in conjunction with other mixed FEM. First, the use of partially-filled elements may change the stability properties of otherwise stable mixed elements. Second, the Lagrange multipliers on the boundaries and interfaces may interact negatively with the other existing constraints. For example, in computational incompressible fluid dynamics, the incompressibility constraints manifest as a large set of linear constraints on the discrete velocity, and these additional constraints may not interact well with virtual node methods. Preliminary investigations suggested that the virtual node method does not trivially extend to handle Cartesian, bilinear Stokes mixed finite elements, well-known to be second order accurate and inf-sup stable in the absence of partially filled elements [80]. Recent results from XFEM suggest that quadratic elements would

be sufficient [87], however these are significantly more expensive than bilinear elements. Perhaps one can hope to stabilize cheaper discretizations using projection methods [65, 4, 5, 20, 95, 124, 45, 93], which can be viewed as approximate LU/Schur factorizations of the of indefinite saddle-point systems [175, 194]. Although it seems that this discussion is negative, we stress that the theory of mixed FEM provides direction and intuition for how to move forward, whereas other embedded discretizations may not have such an established theory to provide guidance.

Part II

Local and Global Theory of  
Aggregation Equations with  
Nonlinear Diffusion

# CHAPTER 5

## Introduction

Nonlocal aggregation phenomena have been studied in a wide variety of biological applications such as migration patterns in ecological systems [40, 198, 160, 94, 46] and Patlak-Keller-Segel (PKS) models of chemotaxis [82, 174, 100, 114, 131]. Diffusion is generally included in these models to account for the dispersal of organisms. Classically, linear diffusion is used, however recently, there has been a widening interest in models with degenerate diffusion to include over-crowding effects [198, 46]. The parabolic-elliptic PKS is the most widely studied model for aggregation, where the nonlocal effects are modeled by convolution with the Newtonian or Bessel potential. On the other hand, in population dynamics, the nonlocal effects are generally modeled with smooth, fast-decaying kernels. However, all of these models are describing the same mathematical phenomenon: the competition between nonlocal self-attraction and diffusion. For this reason, we are interested in unifying and extending the local and global well-posedness theory of general aggregation models with nonlinear diffusion of the form,

$$\begin{cases} u_t + \nabla \cdot (u \nabla \mathcal{K} * u) = \Delta A(u) & \text{in } [0, T) \times D, \\ u(0, x) = u_0(x) \in L_+^1(D) \cap L^\infty(D), \end{cases} \quad (5.1)$$

where  $L_+^1(\mathbb{R}^d; \mu) := \{f \in L^1(\mathbb{R}^d; \mu) : f \geq 0\}$ . Always  $D \subset \mathbb{R}^d$  for  $d \geq 2$  which is either the entire space,  $D = \mathbb{R}^d$ , or a bounded, convex domain with a smooth boundary.

In this part of the dissertation we detail the work undertaken by the author on



this class of models. In collaboration with Rodríguez and Bertozzi, the local and global existence and uniqueness of solutions to (5.1) with degenerate diffusion is undertaken in [18] under the assumption of  $D$  being a bounded domain or  $d \geq 3$ . More recent work by Rodríguez and the author have since relaxed these assumptions to include  $\mathbb{R}^2$  and also provide simplified, alternative proof to local existence when  $D = \mathbb{R}^d$ . The primary purpose of these works is to unify the local and global existence and uniqueness theory of a large class of ‘reasonable’ models of the type (5.1). In [17], the author examined the topic of intermediate asymptotics, proving strong decay estimates for certain kinds of small data and showing that under some circumstances, dissipating solutions decay to the self-similar solutions of the homogeneous diffusion equations.

There is a natural notion of criticality associated with this problem, which corresponds to the balance between the aggregation and diffusion at the scaling limit of mass concentration. For problems with homogeneous kernels and power-law diffusion,  $\mathcal{K} = c|x|^{2-d}$  and  $A(u) = u^m$ , a simple scaling heuristic suggests that these forces are in balance if  $m = 2 - 2/d$  [36]. If  $m > 2 - 2/d$  then the problem is subcritical and the diffusion is dominant. On the other hand, if  $m < 2 - 2/d$  then the problem is supercritical and the aggregation is dominant. For the PKS with power-law diffusion, Sugiyama showed global existence for subcritical problems and that finite time blow up is possible for supercritical problems [193, 192, 191]. We extend this notion of criticality to general problems by observing that only the behavior of the solution at high concentrations will divide finite time blow up from global existence (see Definition 6). We show global well-posedness for subcritical problems and finite time blow up for certain supercritical problems.

If the problem is critical, it is well-known that in PKS there exists a critical mass, and solutions with larger mass can blow up in finite time [39, 107, 32, 76,

37, 49, 36, 191, 192, 47]. For linear diffusion, the same critical mass has been identified for the Bessel and Newtonian potentials [39, 49]; however for nonlinear diffusion, the critical mass has only been identified for the Newtonian potential [36]. In Chapter 7, we extend the free energy methods of [36, 76, 47, 37] to estimate the critical mass for a wide range of kernels and nonlinear diffusion, which include these known results. For a smaller class of problems, including standard PKS models, we show this estimate is sharp.

The problem (5.1) is formally a gradient flow with respect to the Euclidean Wasserstein distance for the *free energy*

$$\mathcal{F}(u(t)) = S(u(t)) - \mathcal{W}(u(t)), \quad (5.2)$$

where the *entropy*  $S(u(t))$  and the *interaction energy*  $\mathcal{W}(u(t))$  are given by

$$\begin{aligned} S(u(t)) &= \int \Phi(u(x, t)) dx, \\ \mathcal{W}(u(t)) &= \frac{1}{2} \int \int u(x, t) \mathcal{K}(x - y) u(y, t) dx dy. \end{aligned}$$

For the degenerate parabolic problems we consider, the *entropy density*  $\Phi(z)$  is a strictly convex function satisfying

$$\Phi''(z) = \frac{A'(z)}{z}, \quad \Phi'(1) = 0, \quad \Phi(0) = 0. \quad (5.3)$$

See [54] for more information on these kinds of entropies. Although there is a rich theory for gradient flows of this general type when the kernel is regular and  $\lambda$ -convex [159, 7, 52] the kernels we consider here are more singular and the notion of displacement convexity introduced in [159] no longer holds. For this reason, the rigorous results of the gradient flow theory are not generally applicable, however, certain aspects may be recovered, such as the use of steepest descent schemes [34, 35]. Moreover, the free energy (5.2) is still the important dissipated quantity in the global existence and finite time blow up arguments.

There is a vast literature of related works on models similar to (5.1). For literature on PKS we refer the reader to the review articles [103, 102]; see also [101, 72, 49] for parabolic-parabolic Keller-Segel systems. For the inviscid problem, see the recent works of [132, 26, 25, 27, 52]. For a study of these equations with fractional linear diffusion see [139, 140, 31]. When the diffusion is sufficiently nonlinear and the kernel is in  $L^1$ , (5.1) may be written as a regularized interface problem, a notion studied in [187]. Critical mass behavior is also a property of other related critical PDE, such as the marginal unstable thin film equation [209, 28] and critical semilinear dispersive equations [207, 121, 117, 119, 106].

In Chapter 6 the local theory is covered. The global existence and finite time blow up theory is discussed in Chapter 7, in particular, there the sharp critical mass is estimated for a range of problems. In Chapter 8, the intermediate asymptotics results are discussed. Finally, we discuss open problems and possible directions of future research in Chapter 9.

## 5.1 Notation

In what follows, We denote  $D_T := (0, T) \times D$ . We also denote  $\|u\|_p := \|u\|_{L^p(D)}$  where  $L^p$  is the standard Lebesgue space. We use the shorthand,

$$\{u > k\} := \{x \in D : u(x) > k\}.$$

If  $S \subset \mathbb{R}^d$  then  $|S|$  denotes the Lebesgue measure and  $\mathbf{1}_S$  denotes the standard characteristic function. In addition, we use  $\int f dx := \int_D f dx$ , and only indicate the domain of integration where it differs from  $D$ . We also denote the weak- $L^p$  space by  $L^{p,\infty}$  and the associated quasi-norm

$$\|f\|_{L^{p,\infty}} = \left( \sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) \right)^{1/p},$$

where  $\lambda_f(\alpha) = |\{f > \alpha\}|$  is the distribution function of  $f$ . Given an initial condition  $u(x, 0)$  we denote its mass by  $\int u(x, 0)dx = M$ . In formulas we use the notation  $C(p, k, M, \dots)$  to denote a generic constant, which may be different from line to line or even term to term in the same computation. In general, these constants will depend on more parameters than those listed, for instance those associated with the problem such as  $\mathcal{K}$  and the dimension but these dependencies are suppressed. We use the notation  $f \lesssim_{p,k,\dots} g$  to denote  $f \leq C(p, k, \dots)g$  where again, dependencies that are not relevant are suppressed. We will also use the notation  $f \approx_{p,k,\dots} g$  to denote  $g \lesssim_{p,k,\dots} f \lesssim_{p,k,\dots} g$ . We denote the  $N$ -th moments via

$$\mathcal{M}_N(u) = \int |x|^N u(x) dx.$$

The Fourier transform is defined as

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int e^{-ix \cdot \xi} f(x) dx$$

with also,

$$\mathcal{F}^{-1}[f](x) = \check{f}(x) = \frac{1}{2\pi} \int e^{ix \cdot \xi} f(\xi) d\xi.$$

On  $\mathbb{R}^d$  we define the homogeneous Sobolev spaces  $\dot{H}^s$ ,  $s \in \mathbb{R}$  in the usual way, as the closure of the Schwartz space under the norm

$$\|f\|_{\dot{H}^s}^2 := \int |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi.$$

## 5.2 Definitions and Assumptions

We consider either  $D = \mathbb{R}^d$  or  $D \subset \mathbb{R}^d$  smooth, bounded and convex with  $d \geq 2$ , in which case we impose no-flux conditions

$$(-\nabla A(u) + u \nabla \mathcal{K} * u) \cdot \nu = 0 \text{ on } \partial D \times [0, T], \quad (5.4)$$

where  $\nu$  is the outward unit normal to  $D$ .

We now make reasonable assumptions on the kernel which include important cases of interest, such as when  $\mathcal{K}$  is the fundamental solution of an elliptic PDE on  $\mathbb{R}^d$ . To this end we state the following definition.

**Definition 1** (Admissible Kernel). We say a kernel  $\mathcal{K}$  is *admissible* if  $\mathcal{K} \in W_{loc}^{1,1}$  and the following holds:

- (R)  $\mathcal{K} \in C^3 \setminus \{0\}$ .
- (KN)  $\mathcal{K}$  is radially symmetric,  $\mathcal{K}(x) = k(|x|)$  and  $k(|x|)$  is non-increasing.
- (MN)  $k''(r)$  and  $k'(r)/r$  are monotone on  $r \in (0, \delta)$  for some  $\delta > 0$ .
- (BD)  $|D^3\mathcal{K}(x)| \lesssim |x|^{-d-1}$ .

This definition ensures that the kernels we consider are radially symmetric, non-repulsive, reasonably well-behaved at the origin, and have second derivatives which define bounded distributions on  $L^p$  for  $1 < p < \infty$  (Lemma 2, Section §5.3). These conditions also imply that if  $\mathcal{K}$  is singular, the singularity is restricted to the origin. Note also, that the Newtonian and Bessel potentials are both admissible for all dimensions  $d \geq 2$ ; hence, the PKS and related models are included in our analysis. It will turn out that the Newtonian potential represents in some sense the most singular kernel treated by our analysis (Lemma 1, Section §5.3).

We now make precise what kind of nonlinear diffusion we are considering.

**Definition 2** (Admissible Diffusion Functions). We say that the function  $A(u)$  is an admissible diffusion function if:

- (D1)  $A \in C^1([0, \infty))$  with  $A'(z) > 0$  for  $z \in (0, \infty)$ .
- (D2)  $A'(z) > c$  for  $z > z_c$  for some  $c, z_c > 0$ .

**(D3)**  $A'(z) \leq C_A$  for  $z < z_A$  for some  $C_A > 0$  and  $z_A$ .

In certain cases we will further restrict **(D3)** to the strictly stronger condition

**(D4)**  $\int_0^1 A'(z)z^{-1}dz < \infty$ .

This latter condition requires the diffusion be nonlinear and degenerate, and the former allows the diffusion to behave linearly at small densities. In particular, the speed of propagation is infinite for **(D3)** but not for **(D4)**. The condition **(D1)** implies the diffusion can at most degenerate at zero and the condition **(D2)** implies that the diffusion does not disappear at high concentrations (sometimes referred to as *non-saturating*).

Following [29], the notions of weak solution are defined separately for bounded and unbounded domains.

**Definition 3** (Weak Solutions on Bounded Domains). Let  $A(u)$  and  $\mathcal{K}$  be admissible, and  $u_0(x) \in L^\infty(D)$  be non-negative. A non-negative function  $u : [0, T] \times D \rightarrow [0, \infty)$  is a weak solution to (5.1) if  $u \in L^\infty(D_T)$ ,  $A(u) \in L^2(0, T, H^1(D))$ ,  $u_t \in L^2(0, T, H^{-1}(D))$  and

$$\int_0^T \int u \phi_t \, dx dt = \int u_0(x) \phi(0, x) dx + \int_0^T \int (\nabla A(u) - u \nabla \mathcal{K} * u) \cdot \nabla \phi \, dx dt, \quad (5.5)$$

for all  $\phi \in C^\infty(\overline{D_T})$  such that  $\phi(T) = 0$ .

It follows that  $u \nabla \mathcal{K} * u \in L^\infty(0, T; L^2(D))$ ; therefore, definition 3 is equivalent to the following,

$$\langle u_t(t), \phi \rangle = \int (-\nabla A(u) + u \nabla \mathcal{K} * u) \cdot \nabla \phi \, dx, \quad (5.6)$$

for all test functions  $\phi \in H^1$  for almost all  $t \in [0, T]$ . Above  $\langle \cdot, \cdot \rangle$  denotes the standard dual pairing between  $H^{-1}$  and  $H^1$ . Similarly for  $\mathbb{R}^d$  we define the following notion of weak solution as in [29].

**Definition 4** (Weak Solution in  $\mathbb{R}^d$ ,  $d \geq 2$ ). Let  $A$  and  $\mathcal{K}$  be admissible, and  $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1_+(\mathbb{R}^d; (1 + |x|^2)dx)$ . If  $d \geq 3$ , a function  $u : [0, T] \times \mathbb{R}^d \rightarrow [0, \infty)$  is a weak solution of (5.1) if  $u \in L^\infty((0, T) \times \mathbb{R}^d) \cap L^\infty(0, T, L^1(\mathbb{R}^d; (1 + |x|^2)dx))$ ,  $A(u) \in L^2(0, T, \dot{H}^1(\mathbb{R}^d))$ ,  $u \nabla \mathcal{K} * u \in L^2(0, T; L^2(\mathbb{R}^2))$ ,  $u_t \in L^2(0, T, \dot{H}^{-1}(\mathbb{R}^d))$ , and for all test functions  $\phi \in \dot{H}^1(\mathbb{R}^d)$  for a.e  $t \in [0, T]$  (5.6) holds.

If  $d = 2$ , a function  $u : [0, T] \times \mathbb{R}^2 \rightarrow [0, \infty)$  is a weak solution of (5.1) if  $u \in L^\infty((0, T) \times \mathbb{R}^2) \cap L^\infty(0, T, L^1(\mathbb{R}^2; (1 + |x|^2)dx))$ ,  $\nabla A(u) \in L^2(0, T, L^2(\mathbb{R}^2))$ ,  $u \nabla \mathcal{K} * u \in L^2(0, T; L^2(\mathbb{R}^2))$ ,  $u_t \in L^2(0, T, \mathcal{V}^*(\mathbb{R}^2))$ , and for all test functions  $\phi(t) \in L^\infty(0, T; \mathcal{V})$  we have,

$$\int_0^T \langle u_t, \phi(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt = - \int_0^T \int (\nabla A(u) - u \nabla \mathcal{K} * u) \cdot \nabla \phi(t) dt,$$

where  $\mathcal{V} = \{f \in L^\infty(\mathbb{R}^2) : \nabla f \in L^2(\mathbb{R}^2)\}$ .

*Remark 1.* The additional complication in  $\mathbb{R}^2$  in the above definition is due to the fact that the norm  $\|\nabla f\|_2$  is not well-behaved in  $\mathbb{R}^2$ . Indeed, note that there exists sequence of Schwartz functions with  $\|\nabla f_n\|_2 = 1$  and  $f_n \rightarrow \infty$  pointwise a.e. (consider renormalizing  $f(x) = \log \log(1 + |x|^{-1}) \mathbf{1}_{|x| < 1}$  and scaling). The solution we use for this problem can be found in [25].

We show below (Theorem 4) that weak solutions satisfying Definition 3 or 4 are in fact unique. Moreover, we show the unique weak solution satisfies the energy dissipation inequality (Proposition 1),

$$\mathcal{F}(u(t)) + \int_0^t \int \frac{1}{u} |A'(u) \nabla u - u \nabla \mathcal{K} * u|^2 dx dt \leq \mathcal{F}(u_0(x)). \quad (5.7)$$

As (5.7) is important for the global theory, one could also refer to these solutions as free energy solutions, as is done in [36]. Uniqueness implies that there is essentially no distinction between free energy solutions in [36] and these weak solutions.

Since (5.1) conserves mass, the natural notion of criticality is with respect to the usual mass invariant scaling  $u_\lambda(x) = \lambda^d u(\lambda x)$ . A simple heuristic for understanding how this scaling plays a role in the global existence is seen by examining the case of power-law diffusion and homogeneous kernel,  $A(u) = u^m$  and  $\mathcal{K}(x) = |x|^{-d/p}$ . Under this mass invariant scaling, the free energy (5.2) becomes,

$$\begin{aligned}\mathcal{F}(u_\lambda) &= \lambda^{dm-d} S(u) - \lambda^{d/p} \mathcal{W}(u) \\ &= \lambda^{dm-d} \int \frac{u^m}{m-1} dx - \lambda^{d/p} \frac{1}{2} \int \int u(x)u(y) |x-y|^{-d/p} dx.\end{aligned}$$

As  $\lambda \rightarrow \infty$ , the entropy and the interaction energy are comparable if  $m = (p+1)/p$ . We should expect global existence if  $m > (p+1)/p$ , as the diffusion will dominate as  $u$  grows, and possibly finite time blow up if  $m < (p+1)/p$  as the aggregation will instead be increasingly dominant. We consider inhomogeneous kernels and general diffusion, however for the problem of global existence, only the behavior as  $u \rightarrow \infty$  will be important, in contrast to the problem of local existence. Noting that  $|x|^{-d/p}$  is, in some sense, the representative singular kernel in  $L^{p,\infty}$  leads to the following definition. Thus, a sort of limiting scale-invariance has been recovered in the limit which governs the criticality of the problem.

This critical exponent also appears indirectly in [150].

**Definition 5** (Critical Exponent). Let  $d \geq 3$  and  $\mathcal{K}$  be admissible such that  $\mathcal{K} \in L_{loc}^{p,\infty}$  for some  $d/(d-2) \leq p < \infty$ . Then the *critical exponent* associated to  $\mathcal{K}$  is given by

$$1 < m^* = \frac{p+1}{p} \leq 2 - 2/d.$$

If  $D^2\mathcal{K}(x) = \mathcal{O}(|x|^{-2})$  as  $x \rightarrow 0$ , then we take  $m^* = 1$ .

*Remark 2.* The case  $m^* = 1$  implies at worst a logarithmic singularity as  $x \rightarrow 0$  and if  $d = 2$  then all admissible kernels have  $m^* = 1$  by condition **(BD)**.



Now we define the notion of criticality. It is easier to define this notion in terms of the quantity  $A'(z)$ , as opposed to using  $\Phi(z)$  directly.

**Definition 6** (Criticality). We say that the problem is *subcritical* if

$$\liminf_{z \rightarrow \infty} \frac{A'(z)}{z^{m^*-1}} = \infty,$$

*critical* if

$$0 < \liminf_{z \rightarrow \infty} \frac{A'(z)}{z^{m^*-1}} < \infty,$$

and *supercritical* if

$$\liminf_{z \rightarrow \infty} \frac{A'(z)}{z^{m^*-1}} = 0.$$

Notice that in the case of power-law diffusion,  $A(u) = u^m$ , subcritical, critical and supercritical respectively correspond to  $m > m^*$ ,  $m = m^*$  and  $m < m^*$ . Moreover, in the case of the Newtonian or Bessel potential,  $m^* = 2 - 2/d$  and the critical diffusion exponent of the PKS models discussed in [192, 191, 36] is recovered.

### 5.3 Properties of Admissible Kernels

Definition 1 implies a number of useful characteristics which we state here and reserve the proofs for the Appendix 10.3. First, we have that every admissible kernel is at least as integrable as the Newtonian potential.

**Lemma 1.** *Let  $\mathcal{K}$  be admissible. Then  $\nabla \mathcal{K} \in L^{d/(d-1), \infty}$ . If  $d \geq 3$ , then  $\mathcal{K} \in L^{d/(d-2), \infty}$ .*

In general, the second derivatives of admissible kernels are not locally integrable, but we may still properly define  $D^2 \mathcal{K} * u$  as a linear operator which involves a Cauchy principal value integral. By Calderón-Zygmund theory (see

e.g. [Theorem 2.2 [189]]) we can conclude that this distribution is bounded on  $L^p$  for  $1 < p < \infty$ . The inequality also provides an estimate of the operator norms, which is of crucial importance to the proof of uniqueness.

**Lemma 2.** *Let  $\mathcal{K}$  be admissible and  $\vec{v} = \nabla \mathcal{K} * u$ . Then  $\forall p, 1 < p < \infty, \exists C(p)$  such that  $\|\nabla \vec{v}\|_p \leq C(p)\|u\|_p$  and  $C(p) \lesssim p$  for  $2 \leq p < \infty$ .*

One can further connect the integrability of the kernel with the integrability of the derivatives at the origin, which provides a natural extension of Lemma 2 through the Young's inequality for  $L^{p,\infty}$ .

**Lemma 3.** *Let  $d \geq 3$  and  $\mathcal{K}$  be admissible. Suppose  $\gamma$  is such that  $1 < \gamma < d/2$ . Then  $\mathcal{K} \in L_{loc}^{d/(d/\gamma-2),\infty}$  if and only if  $D^2\mathcal{K} \in L_{loc}^{\gamma,\infty}$ . The same holds for  $\nabla \mathcal{K} \in L_{loc}^{d/(d/\gamma-1),\infty}$ . In particular,  $m^* = 1 + 1/\gamma - 2/d$  for some  $1 < \gamma < d/2$  if and only if  $D^2\mathcal{K} \in L_{loc}^{\gamma,\infty}$ . Moreover,  $m^* = 1$  if and only if  $D^2\mathcal{K} \in L_{loc}^{d/2,\infty}$ .*

The following lemma clarifies the connection between the critical exponent and the interaction energy.

**Lemma 4** (Hardy-Littlewood-Sobolev inequality). *Consider the Hardy-Littlewood-Sobolev type inequality, for all  $f \in L^p, g \in L^q$  and  $\mathcal{K} \in L^{t,\infty}$  for  $1 < p, q, t < \infty$  satisfying  $1/p + 1/q + 1/t = 2$ ,*

$$\left| \int \int f(x)g(y)\mathcal{K}(x-y)dxdy \right| \lesssim \|f\|_p \|g\|_q \|\mathcal{K}\|_{L^{t,\infty}}. \quad (5.8)$$

*See [147]. In particular, if  $(p+1)/p = m^* > 1$ , then for all  $u \in L^1 \cap L^{m^*}$ ,*

$$\int u(x)u(y) |x-y|^{-d/p} dxdy \leq C_{m^*} \|u\|_1^{2-m^*} \|u\|_{m^*}^{m^*}. \quad (5.9)$$

*Here  $C_{m^*}$ , depending only on  $p$  and  $d$ , is taken to be the best constant for which (5.9) holds for all such  $u$ .*

*Remark 3.* It is not necessarily the case that  $C_{m^*}$  is easily related to the optimal constant in (5.8). It is shown in [36] that  $C_{2-2/d}$  is achieved for a fairly explicit family of extremals, but to our knowledge, extremals of (5.9) have not been constructed for other values of  $m^*$ .

If  $m^* = 1$  then we will need the logarithmic Hardy-Littlewood-Sobolev inequality, as in for instance [76, 37].

**Lemma 5** (Logarithmic Hardy-Littlewood-Sobolev inequality [50]). *Let  $d \geq 2$  and  $0 \leq f \in L^1$  be such that  $|\int f \ln f dx| < \infty$ . Then,*

$$-\int \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x)f(y) \ln |x-y| dx dy \leq \frac{\|f\|_1}{d} \int_{\mathbb{R}^d} f \ln f dx + C(\|f\|_1). \quad (5.10)$$

The case  $\mathbb{R}^2$  creates an additional difficulty due to the fact that by the definition of admissible, definition 1,  $\mathcal{K} \in BMO$ , but not in any  $L^{p,\infty}$  space. In particular,  $\mathcal{K}$  is permitted to grow logarithmically at infinity; indeed  $\mathcal{K} = -\log|x|$  is admissible in  $\mathbb{R}^2$ . This introduces a number of complications with the local well-posedness, since  $\mathcal{K} * f$  will be unbounded for general  $f \in L^1 \cap L^\infty$ , so more care must be taken than in  $d \geq 3$ . The most difficult steps to extend is the uniqueness of weak solutions and the energy dissipation inequality. We will need to recall that the dual of  $BMO$  is known to be the Hardy space  $\mathcal{H}^1$  [190]. For convenience, we define the Hardy space via this duality,

$$\|f\|_{\mathcal{H}^1} := \sup_{\mathcal{K} \in BMO, \|\mathcal{K}\|_{BMO}=1} \left| \int \mathcal{K}(x)f(x) dx \right|, \quad (5.11)$$

with,

$$\mathcal{H}^1 := \{f \in L^1 : \|f\|_{\mathcal{H}^1} < \infty\}.$$

We also have the analogue of Hölder's inequality [190]

$$\left| \int \mathcal{K} f dx \right| \leq \|\mathcal{K}\|_{BMO} \|f\|_{\mathcal{H}^1}, \quad (5.12)$$

which in particular implies  $\|\mathcal{K} * f\|_\infty \lesssim \|\mathcal{K}\|_{BMO} \|f\|_{\mathcal{H}^1}$ , which is the purpose of introducing the space  $\mathcal{H}^1$ . The following lemma provides useful, sufficient conditions on  $f \in L^1$  such that  $f \in \mathcal{H}^1$  and will be used to prove below to treat  $\mathbb{R}^2$ . The author acknowledges Jonas Azzam for his assistance in the proof, which is reserved for the Appendix 10.4.

**Lemma 6.** *Let  $f \in L^1 \cap L^p$  for some  $p > 1$  and satisfy  $\int f dx = 0$ ,  $\mathcal{M}_1(|f|) = \int |x| |f(x)| dx < \infty$ . Then  $f \in \mathcal{H}^1$  and*

$$\|f\|_{\mathcal{H}^1} \lesssim_{d,p} \|f\|_p + \mathcal{M}_1(|f|).$$

## CHAPTER 6

### Local Theory

Existence theory is complicated by the presence of degenerate diffusion and singular kernels such as the Newtonian potential. Bertozzi and Slepčev in [29] prove existence and uniqueness of models with general diffusion but restrict to non-singular kernels. Sugiyama [193] proved local existence for models with power-law diffusion and the Bessel potential for the kernel, but uniqueness of solutions was left open. We will provide two separate proofs of local existence. The first one applies on bounded domains and  $\mathbb{R}^d$ ,  $d \geq 3$  which is an extension of the work of [29]. The other is a more straightforward variant based on a fixed point argument which applies on  $\mathbb{R}^d$ ,  $d \geq 2$  for solutions with a bounded second moment. A similar argument was employed in [39] for the 2D critical PKS, but the presence of nonlinear diffusion and the interest in general kernels complicates several steps. Aside from simplicity, an advantage of this latter proof is that it supplies an obvious method for justifying formal computations involving homogeneous Sobolev inequalities, whereas the former proof constructs solutions on  $\mathbb{R}^d$  with sequences of solutions on bounded domains. Ultimately, both approaches are based on regularization: the degenerate diffusion is regularized to be uniformly parabolic and the kernel is regularized to be smooth. The a priori bounds are made independent of the regularization parameter, done here using an Alikakos iteration [3] developed in [125, 47], which eventually provide the requisite compactness necessary to extract convergent subsequences.

**Theorem 1** (Local Existence on Bounded Domains,  $d \geq 2$ ). *Let  $A(u)$  and  $\mathcal{K}(x)$  be admissible. Let  $u_0(x) \in L^\infty(D)$  be a non-negative initial condition, then (5.1) has a weak solution  $u(t)$  on  $[0, T] \times D$ , for some  $T > 0$ . Additionally,  $u \in C([0, T]; L^p(D))$  for  $p \in [1, \infty)$ .*

In dimensions  $d \geq 3$  we also construct local solutions on  $\mathbb{R}^d$  by taking the limit of solutions on bounded domains.

**Theorem 2** (Local Existence in  $\mathbb{R}^d$ ,  $d \geq 3$ ). *Let  $A(u)$  and  $\mathcal{K}(x)$  be admissible. Let  $u_0(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  be a non-negative initial condition, then (5.1) has a weak solution  $u$  on  $\mathbb{R}_T^d$ , for some  $T > 0$ . Additionally,  $u \in C([0, T]; L^p(\mathbb{R}^d))$  for all  $1 \leq p < \infty$ , the mass is conserved and  $|\mathcal{F}(u_0)| < \infty$ .*

Restricting to initial data with finite second moment not only simplifies the proof of local existence as it may now be done in a direct manner, but also allows treatment of  $\mathbb{R}^2$ . Although we show uniqueness below, the stability theory of these PDE seems to be not quite fully understood, so a proof which allows one to easily justify formal computations involving homogeneous Sobolev inequalities is desirable.

**Theorem 3** (Local Existence in  $\mathbb{R}^d$ ,  $d \geq 2$ ). *Let  $A(u)$  and  $\mathcal{K}(x)$  be admissible. Let  $u_0(x) \in \cap L_+^1(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ , then (5.1) has a weak solution  $u$  on  $(0, T) \times \mathbb{R}^d$ , for some  $T > 0$  which satisfies  $u(t) \in C([0, T]; L^1(\mathbb{R}^d; (1 + |x|^2)dx)) \cap L^\infty((0, T) \times \mathbb{R}^d)$ . Additionally the mass is conserved and  $|\mathcal{F}(u_0)| < \infty$ .*

As previously mentioned, the free energy is a dissipated quantity for weak solutions and is a key tool for the global theory.

**Proposition 1** (Energy Dissipation). *Under the assumptions of Theorems 1, 2 and 3, weak solutions to (5.1) satisfy the energy dissipation inequality (5.7) for almost all  $t \geq 0$ .*

The existing work on uniqueness of these problems included a priori regularity assumptions [126] or the use of entropy solutions [46] (see also [51]). The “Lagrangian” method introduced by Loeper in [154] estimates the difference of weak solutions in the Wasserstein distance and is very useful for inviscid problems and potentially solutions with linear diffusion [153, 27, 57]. This method relies very directly on the non-local velocity field satisfying the Osgood regularity condition and a certain  $L^2$  stability condition, the latter arising from the dependence of the velocity field on the transported quantity. In the presence of nonlinear diffusion, it seems more natural to approach uniqueness in  $\dot{H}^{-1}$ , where the diffusion is monotone (see [202]). This is the approach taken in [25, 29], which we extend to handle singular kernels such as the Newtonian potential, proving uniqueness of weak solutions with no additional assumptions, provided the domain is bounded or  $d \geq 3$  and with the assumption of bounded first moment if the domain is  $\mathbb{R}^2$ . The main difference is the use of more refined estimates to handle the lower regularity of  $\nabla K * u$ , similar to the traditional proof of uniqueness of solutions to the 2D Euler equations with bounded and integrable vorticity [214, 157] and a similar proof of the uniqueness of  $L^1 \cap L^\infty$  solutions to the Vlasov-Poisson equation [181]. Each of these proofs rely on the regularity provided by Calderón-Zygmund inequality [190], which takes the form

$$\|D^2\mathcal{K} * u\|_p \lesssim p\|u\|_p, \quad p \rightarrow \infty.$$

The “Eulerian”  $\dot{H}^{-1}$  method does not appear to be any more general than the Lagrangian method, as indeed, Calderón-Zygmund theory also implies  $\|D^2\mathcal{K} * u\|_{BMO} \lesssim \|u\|_\infty$ , which in turn implies the  $\nabla\mathcal{K} * u$  is log-Lipschitz (we could not locate a proof of this in the literature, however it is relatively straightforward, see Appendix 10.5).

Neither the Lagrangian or the Eulerian methods use the gradient flow struc-

ture and apply to time-reversible active scalars, for instance, Vlasov-Poisson equations [154, 181], incompressible Euler equations [154, 214] and the semi-geostrophic equations [153] to name a few. This is both an advantage in being more general, but also limits applicability to less regular problems; in fact, they appear pushed close to their limit already. The nonlinearity, and the associated stability criteria, seems to introduce a fundamental difficulty that is not there in the linear case. The DiPerna-Lions theory of renormalized solutions provides uniqueness for linear transport equations under very weak hypotheses, however, perhaps not coincidentally, these solutions do not have well understood regularity and stability properties [75, 6]. That said, Vishik [205] has shown uniqueness for the incompressible Euler equations with vorticity in certain Besov spaces which, for example, at least contain  $BMO$ . The methods employed therein are significantly more advanced than what is used to prove Theorem 4, relying on Littlewood-Paley and wavelet decompositions. However, the Eulerian velocity field in his work is still shown to satisfy the Osgood regularity condition, and therefore his method still cannot treat problems even nearly as singular to that approachable in linear theory. Given existing negative results (see e.g. [135, 182, 184]), it may not be possible to strengthen these results much further for time-reversible active scalars. On the other hand, taking advantage of the diffusion to prove stronger results should be possible in treating active scalars, for example uniqueness is known for the 2D Navier-Stokes equations with measure-valued initial vorticity [85]. The recent results of [52] on measure-valued solutions to the inviscid problem take advantage of the gradient flow structure and also successfully prove uniqueness to time-irreversible problems with velocity fields which do not satisfy the Osgood uniqueness criterion.

**Theorem 4** (Uniqueness). *Let  $D \subset \mathbb{R}^d$  for  $d \geq 2$  be bounded and convex, then weak solutions to (5.1) are unique. The conclusion also holds on  $\mathbb{R}^d$  for  $d \geq 3$ .*



Weak solutions with the additional property that  $|x|u(t) \in L^\infty(0, T; L^1_+(\mathbb{R}^2))$  are unique in  $\mathbb{R}^2$ .

*Remark 4.* The uniqueness result holds for any  $A$  which is non-decreasing. In particular, Theorem 4 covers cases including fast or strongly degenerate diffusion, or even no diffusion at all.

The final result on the local theory is the following continuation theorem which shows that concentration in the critical norm is necessary for finite time blow up. The proof is a key refinement of Alikakos iteration methods used to obtain a priori bounds in the local existence theory, and the proof is based on the work of [47, 36].

**Theorem 5** (Continuation). *Suppose  $u(t)$  is a weak solution to (5.1) with maximal time interval of existence  $[0, T_\star)$  which satisfies*

$$\lim_{k \rightarrow \infty} \sup_{t \in [0, T_\star)} \|(u - k)_+\|_{\frac{2-m}{2-m^\star}} = 0. \quad (6.1)$$

*Here  $m$  is such that  $1 \leq m \leq m^\star$  and  $\liminf_{z \rightarrow \infty} A'(z)z^{1-m} > 0$ . Then,  $T_\star = \infty$  and  $u(t) \in L^\infty(0, \infty; L^\infty(D))$ .*

*Remark 5.* Condition 6.1 is usually referred to as *uniform equi-integrability*.

*Remark 6.* Note that therefore, if  $T_\star < \infty$  then for all  $p > (2 - m)/(2 - m^\star)$ ,

$$\lim_{t \nearrow T_\star} \|u\|_p = \infty.$$

A crucial lemma in our results is the following Alikakos iteration lemma, due originally to Kowalczyk [125] and extended to  $d > 2$  and  $\mathbb{R}^d$  by Calvez and Carrillo in [47].

**Lemma 7** (Iteration Lemma [125, 47]). *Let  $0 < T \leq \infty$  and assume that there exists a  $c > 0$  and  $u_c > 0$  such that  $A'(u) > c$  for all  $u > u_c$ . Then if*

$\|\nabla \mathcal{K} * u\|_\infty \leq C_1$  on  $[0, T]$  then  $\|u\|_\infty \leq C_2(C_1) \max\{1, M, \|u_0\|_\infty\}$  on the same time interval.

## 6.1 Local Existence on Bounded Domains and $\mathbb{R}^d$ , $d \geq 3$

### 6.1.1 Local Existence in Bounded Domains

Let  $\tilde{A}(z)$  be a smooth function on  $\mathbb{R}^+$  such that  $\tilde{A}'(z) > \eta$  for some  $\eta > 0$ . In addition, let  $\vec{v}$  be a given smooth velocity field with bounded divergence. Classical theory gives a global smooth solution to the uniformly parabolic equation

$$u_t = \Delta \tilde{A}(u) - \nabla \cdot (u \vec{v}) \quad (6.2)$$

(see [148]). The solutions obey the global  $L^\infty$  bound

$$\|u\|_{L^\infty(D)} \leq \|u_0\|_{L^\infty(D)} e^{\|(\nabla \cdot \vec{v})_-\|_{L^\infty(D_T)} t}. \quad (6.3)$$

We take advantage of this theory to prove existence of weak solutions to (5.1) by regularizing the degenerate diffusion and the kernel. Consider the modified aggregation equation

$$u_t^\epsilon = \Delta A^\epsilon(u^\epsilon) - \nabla \cdot (u^\epsilon (\nabla \mathcal{J}_\epsilon \mathcal{K} * u^\epsilon)), \quad (6.4)$$

with corresponding no-flux boundary conditions (5.4). We define

$$A^\epsilon(z) = \int_0^z a'_\epsilon(z) dz, \quad (6.5)$$

where  $a'_\epsilon(z)$  is a smooth function, such that  $A'(z) + \epsilon \leq a'_\epsilon(z) \leq A'(z) + 2\epsilon$ , and the standard mollifier is denoted  $\mathcal{J}_\epsilon v$ . We first prove existence of solutions to the regularized equation (6.4), this is stated formally in the following proposition.

**Proposition 2** (Local Existence for the Regularized Aggregation Diffusion Equation). *Let  $\epsilon > 0$  be fixed and  $u_0(x) \in C^\infty(\bar{D})$ , then (6.4) has a classical solution  $u$  on  $D_T$  for all  $T > 0$ .*

We obtain the proof of Proposition 2 directly from Theorem 12 in [29]. The proof requires a bound on  $\|\nabla A^\epsilon\|_{L^2(D_T)}$ , for some  $T > 0$ . We state this lemma for completeness but reference the reader to [29] for a proof.

**Lemma 8** (Uniform Bound on Gradient of  $A(u)$ ). *Let  $\epsilon > 0$  be fixed and  $u^\epsilon \in L^\infty(D_T)$  be a solution to (6.4). There exists a constant*

$$C = C(T, \|\nabla \mathcal{J}_\epsilon \mathcal{K} * u^\epsilon\|_{L^\infty(D)}, \|u^\epsilon\|_\infty)$$

*such that:*

$$\|\nabla A^\epsilon(u^\epsilon)\|_{L^2(D_T)} \leq C. \quad (6.6)$$

*Remark 7.* The estimate given by (6.6) is independent of  $\epsilon$ .

Proposition 2 gives a family of solutions  $\{u^\epsilon\}_{\epsilon>0}$ . To prove local existence to the original problem (5.1) we first need some a priori estimates which are independent of  $\epsilon$ . Mainly, we obtain an independent-in- $\epsilon$  bound on the  $L^\infty$  norm of the solution and the velocity field. This is the main difference in the local existence theory from [29]. Due to the singularity of the kernels significantly more is required to obtain these a priori bounds, relying on the iteration Lemma 7.

**Lemma 9** ( $L^\infty$  Bound of Solution). *Let  $\{u^\epsilon\}_{\epsilon>0}$  be the classical solutions to (6.4) on  $D_T$ , with smooth, non-negative, and bounded initial data  $\mathcal{J}_\epsilon u_0$ . Then there exists  $C = C(\|u_0\|_1, \|u_0\|_\infty)$  and  $T = T(\|u_0\|_1, \|u_0\|_p)$  for any  $p > d$  such that for all  $\epsilon > 0$ ,*

$$\|u^\epsilon(t)\|_{L^\infty(D)} \leq C \quad (6.7)$$

*for all  $t \in [0, T]$ .*

*Proof.* For simplicity we drop the  $\epsilon$ . The first step is to obtain an interval for which the  $L^p$  norm of  $u$  is bounded. Following the work of [107] we define the function  $u_k^\epsilon = (u^\epsilon - k)_+$ , for  $k > 0$ . Due to conservation of mass the following inequality provides a bound for the  $L^p$  norm of  $u$  given a bound on the  $L^p$  norm of  $u_k$ ,

$$\|u\|_p^p \leq C(p)(k^{p-1}\|u\|_1 + \|u_k\|_p^p). \quad (6.8)$$

We look at the time evolution of  $\|u_k\|_p$  and make use of the parabolic regularization (6.24).

*Step 1:*

$$\begin{aligned} \frac{d}{dt} \|u_k\|_p^p &= p \int u_k^{p-1} \nabla \cdot (\nabla A^\epsilon(u) - u \nabla \mathcal{J}_\epsilon \mathcal{K} * u) dx \\ &= -p(p-1) \int A^{\epsilon'} \nabla u_k \cdot \nabla u dx - p(p-1) \int u u_k^{p-2} \nabla \mathcal{J}_\epsilon \mathcal{K} * u dx \\ &\leq -\frac{4(p-1)}{p} \int A'(u) \left| \nabla u_k^{p/2} \right|^2 dx + p(p-1) \int u_k^{p-1} \nabla u_k \cdot \nabla \mathcal{J}_\epsilon \mathcal{K} * u dx \\ &\quad + kp(p-1) \int u_k^{p-2} \nabla u_k \cdot \nabla \mathcal{J}_\epsilon \mathcal{K} * u dx, \end{aligned}$$

where we used the fact that for  $l > 0$

$$u(u_k)^l = (u_k)^{l+1} + k u_k^l. \quad (6.9)$$

Hence, integrating by parts once more gives

$$\begin{aligned}
\frac{d}{dt} \|u_k\|_p^p &\leq \frac{4(p-1)}{p} \int A'(u) \left| \nabla u_k^{p/2} \right|^2 dx \\
&\quad - (p-1) \int u_k^p \Delta \mathcal{J}_\epsilon \mathcal{K} * u dx - kp \int u_k^{p-1} \Delta \mathcal{J}_\epsilon \mathcal{K} * u dx \\
&\leq -C(p) \int A'(u) \left| \nabla u_k^{p/2} \right|^2 dx + C(p) \|u_k\|_{p+1}^p \|\Delta \mathcal{J}_\epsilon \mathcal{K} * u\|_{p+1} \\
&\quad + C(p)k \|u_k\|_p^{p-1} \|\Delta \mathcal{J}_\epsilon \mathcal{K} * u\|_p \\
&\leq -C(p) \int A'(u) \left| \nabla u_k^{p/2} \right|^2 dx + C(p) \left( \|u_k\|_{p+1}^{p+1} + \|u\|_{p+1}^{p+1} \right) \\
&\quad + C(p)k \left( \|u_k\|_p^p + \|u\|_p^p \right).
\end{aligned}$$

In the last inequality we use Lemma 2. Now, using (6.8) we obtain that

$$\begin{aligned}
\frac{d}{dt} \|u_k\|_p^p dx &\leq -C(p) \int A'(u) \left| \nabla u_k^{p/2} \right|^2 dx + C(p) \|u_k\|_{p+1}^{p+1} \\
&\quad + C(p, k) \|u_k\|_p^p + C(p, k, M).
\end{aligned}$$

An application of the Gagliardo-Nirenberg-Sobolev inequality gives that for any  $p$  such that  $d < 2(p+1)$  (see Lemma 25 in the Appendix):

$$\|u\|_{p+1}^{p+1} \lesssim \|u\|_p^{\alpha_2} \|u^{p/2}\|_{W^{1,2}}^{\alpha_1},$$

where  $\alpha_1 = d/p$ ,  $\alpha_2 = 2(p+1) - d$ . From the inequality  $a^r b^{(1-r)} \leq ra + (1-r)b$  (using that  $a = \delta \|u^{p/2}\|_{W^{1,2}}^2$  and  $r = \alpha_1/2$ ) we obtain

$$\|u\|_{p+1}^{p+1} \lesssim \frac{1}{\delta^{\beta_1}} \|u\|_p^{\beta_2} + r\delta^2 \|\nabla u^{p/2}\|_2^2 + r\delta^2 \|u\|_p^p.$$

Above  $\beta_1, \beta_2 > 1$ . For  $k$  large enough we have that  $A'(u) > c > 0$  over  $\{u > k\}$ ; hence, if we choose  $\delta$  small enough we obtain the final differential inequality:

$$\frac{d}{dt} \|u\|_p^p \lesssim C(p) \|u_k\|_p^{\beta_2} + C(p, k, r\delta) \|u_k\|_p^p + C(p, k, \|u_0\|_1). \quad (6.10)$$

The inequality (6.10) in turns gives a  $T_p = T(p) > 0$  such that  $\|u_k\|_p$  is bounded on  $[0, T_p]$ . Inequality (6.8) gives that  $\|u\|_p$  remains bounded on the same time

interval. Next we prove that the velocity field is bounded in  $L^\infty(D)$  on some time interval  $[0, T]$ . This then allows us invoke Lemma 7 and obtain the desired bound.

*Step 2:*

Since  $\nabla \mathcal{K} \in L^1_{loc}$  and  $\nabla \mathcal{K} \mathbf{1}_{\mathbb{R}^d \setminus B_1(0)} \in L^q$  for all  $q > d/(d-1)$  (by Lemma 1), we have for all  $p > d/(d-1)$ ,

$$\|\vec{v}\|_p = \|\nabla \mathcal{K} * u\|_p \leq \|\nabla \mathcal{K} \mathbf{1}_{B_1(0)}\|_1 \|u\|_p + \|\nabla \mathcal{K} \mathbf{1}_{\mathbb{R}^d \setminus B_1(0)}\|_p M.$$

By Lemma 2 we also have, for all  $p, 1 < p < \infty$ ,

$$\|\nabla \vec{v}\|_p = \|D^2 \mathcal{K} * u\|_p \lesssim \|u\|_p.$$

By Morrey's inequality we have  $\vec{v} \in L^\infty(D_T)$  by choosing some  $p > d$  and invoking step one, and Lemma 7 concludes the proof. Note that the bound depends on the geometry of the domain through the constant on the Gagliardo-Nirenberg-Sobolev inequality (Lemma 25). However, this constant is related to the regularity of the domain, and not directly to the diameter of the domain.  $\square$

In addition to the a priori estimates the proof of Theorem 1 requires precompactness of  $\{u^\epsilon\}_{\epsilon>0}$  in  $L^1(D_T)$ .

**Lemma 10** (Precompactness in  $L^1(\Omega_T)$ ). *The sequence of solutions obtained via Proposition 2,  $\{u_\epsilon\}_{\epsilon>0}$ , which exist on  $[0, T]$ , is precompact in  $L^1(D_T)$ .*

The proof of Lemma 10 follows exactly the work in [29]. We will show the sequence is precompact via the Riesz compactness criterion. This relies on the fact that  $\|A(u^\epsilon)\|_{L^2(0,T;H^1(D))} \leq C$  uniformly.

*Proof.* (**Theorem 1**) For a given  $\epsilon > 0$ , if we regularize the initial condition  $u_0^\epsilon(x) = \mathcal{J}_\epsilon u_0(x)$ , Proposition 2 gives a solution  $u^\epsilon$  to (6.4). Furthermore, the proof of Proposition 2 and Lemma 9 gave uniform-in- $\epsilon$  bounds on  $\|A^\epsilon(u)\|_{L^2(0,T,H^1(D))}$ ,  $\|u^\epsilon\|_{L^\infty(D_T)}$ , and  $\|u_t^\epsilon\|_{L^2(0,T,H^{-1}(D))}$ . By Lemma 9, all solutions exist on  $[0, T]$ , with  $T$  independent of  $\epsilon$ . Also, recalling that  $A^\epsilon(z) \geq A(z)$  and  $a'_\epsilon(z) \geq A'(z)$  gives that

$$\|A(u^\epsilon)\|_{L^2(0,T,H^1(D))} \leq C,$$

where  $C$  is independent of  $\epsilon$ . Since  $L^2(0, T, H^1(D))$  is weakly compact there exists a  $\rho$  such that some subsequence of  $\{u^\epsilon\}_{\epsilon>0}$  converges weakly, i.e.  $A(u^{\epsilon_j}) \rightharpoonup \rho$  in  $L^2(0, T, H^1(D))$ . Precompactness in  $L^1$  implies strong convergence of  $u^{\epsilon_j}$  to some  $u \in L^1(D_T)$ ; therefore,  $A(u) = \rho$ . In fact, the  $L^\infty(D_T)$  bound on  $u^{\epsilon_j}$  gives strong convergence in  $L^p(D_T)$ , for  $1 \leq p < \infty$ , via interpolation. Also, Young's inequality gives

$$\begin{aligned} \|u^{\epsilon_j} \nabla \mathcal{J}_{\epsilon_j} \mathcal{K} * u^{\epsilon_j} - u \nabla \mathcal{K} * u\|_{L^1(D_T)} &\leq \|u\|_{L^\infty(D_T)} \|\nabla \mathcal{J}_{\epsilon_j} \mathcal{K} * u^{\epsilon_j} - \nabla \mathcal{K} * u\|_{L^1(D_T)} \\ &\lesssim \|\nabla \mathcal{J}_{\epsilon_j} \mathcal{K} * u^{\epsilon_j}\|_{L^\infty(D_T)} \|u^{\epsilon_j} - u\|_{L^1(D_T)}. \end{aligned} \tag{6.11}$$

Therefore,

$$LHS(6.11) \lesssim \left( \|u\|_{L^\infty(D_T)} \|\nabla \mathcal{K}\|_{L^1_{loc}} + \|\nabla \mathcal{K} * u^{\epsilon_j}\|_{L^\infty(D_T)} \right) \|u^{\epsilon_j} - u\|_{L^1(D_T)}. \tag{6.12}$$

Therefore, by interpolation  $u$  satisfies (5.5). Furthermore, we obtain that  $u \in C([0, T]; H^{-1}(D))$ . To prove that  $u(t)$  is continuous with respect to the weak  $L^2$  topology one uses standard density arguments. Since  $D$  is a bounded,  $u$  is therefore also continuous in the weak  $L^1$  topology. To prove continuity in the strong  $L^2$  topology we define  $F(z) = \int_0^z A(s) ds$  and show that it is continuous in the strong  $L^1$  topology. Indeed, Lemma 22 in the Appendix, see [29] for a proof,

gives

$$\lim_{h \rightarrow 0} \left| \int (F(u(t)) - F(u(t+h))) dx \right| = \lim_{h \rightarrow 0} \int_t^{t+h} \langle u_\tau, A(\tau) \rangle d\tau. \quad (6.13)$$

Recall that  $\|A(u)\|_{L^\infty(D_T)} \leq A(\|u\|_{L^\infty(D_T)})$  and so  $A(u) \in L^2(0, T; H^{-1}(D))$ . Therefore, the left hand side of (6.13) goes to 0 as  $h \rightarrow 0$ . Now, we can invoke Lemma 23 in Appendix, [29], to obtain that  $u \in C([0, T]; L^2(D))$ . Using interpolation the  $L^\infty$  bound of  $u$  gives that  $u \in C([0, T]; L^p(D))$ , for  $1 \leq p < \infty$ .  $\square$

### 6.1.2 Local Existence in $\mathbb{R}^d$ , $d \geq 3$

Now we consider solutions to (5.1) in  $\mathbb{R}^d$  for  $d \geq 3$ . We obtain such solution by taking the limit of the solutions in balls centered on the origin with increasing radius  $n$ , denoted by  $B_n$ .

*Proof.* (**Theorem 2**) Let  $B_n$  be defined as above and consider a smooth truncation of the initial condition on  $B_n$ , i.e.  $u_0^n = \chi(R^{-1}x)u_0(x)$  where  $\chi \in C_c^\infty \cap L_+^1$  with  $\chi(x) = 1$  for  $|x| < 1/2$  and  $\chi(x) = 0$  for  $|x| \geq 1$ . By Theorem 1, we have a family of solutions  $\{u_n\}_{n>0}$  on  $B_n$  for all  $t \in [0, T]$ . Define a new sequence,  $\{\tilde{u}_n\}_{n>0}$ , where  $\tilde{u}_n$  is the zero extension of  $u_n$  convolved with a smooth mollifier (in order to smooth the potential jump discontinuity at the boundary of the ball). The previous work for bounded domains gives the uniform bounds

$$\|\tilde{u}_n\|_{L^\infty(\mathbb{R}_T^d)} \leq C_1, \quad (6.14)$$

$$\|\nabla A(\tilde{u}_n)\|_{L^2(\mathbb{R}_T^d)} \leq C_2. \quad (6.15)$$

The bounds may be taken independent of  $n$  since the constant in the Gagliardo-Nirenberg-Sobolev inequality, Lemma 25, does not depend directly on the diameter of the domain and may be taken uniform in  $n \rightarrow \infty$ .

Therefore, there exist  $u, w \in L^2(\mathbb{R}_T^d)$  for which  $\tilde{u}_n \rightharpoonup u$  and  $\nabla A(\tilde{u}_n) \rightharpoonup w$  in  $L^2(\mathbb{R}_T^d)$ . Furthermore, (6.14) implies  $\|u\|_{L^\infty(\mathbb{R}_T^d)} \leq C_1$ . Precompactness of  $\{\tilde{u}_n^\epsilon\}_{\epsilon>0}$



in  $L^1(B_n)$  for fixed  $n > 0$  and Theorem 2.33 in [2] gives that  $\{\tilde{u}_n\}_{n>0}$  is precompact in  $L^1_{loc}(\mathbb{R}^d_T)$ . Therefore, up to a subsequence, not renamed,  $\tilde{u}_n \rightarrow u$  in  $L^1_{loc}(\mathbb{R}^d_T)$ ; thus,  $w = \nabla A(u)$ . Also, the  $L^\infty$  bound gives that  $\tilde{u}_n \rightarrow u$  in  $L^p_{loc}(\mathbb{R}^d)$  for  $1 \leq p < \infty$ .

In addition, we have the estimate

$$\|\tilde{u}_n \nabla \mathcal{K} * \tilde{u}_n\|_{L^2(\mathbb{R}^d_T)} \leq \|\nabla \mathcal{K} * \tilde{u}_n\|_{L^\infty(\mathbb{R}^d_T)} \|\tilde{u}_n\|_{L^2(\mathbb{R}^d_T)}. \quad (6.16)$$

Therefore, we can extract a subsequence that converges weakly to some  $w_1 \in L^2(\mathbb{R}^d_T)$ . Since  $u \mathbf{1}_{B_n} \in L^\infty(0, T, L^1(\mathbb{R}^d))$  and  $u \mathbf{1}_{B_n} \nearrow u$  by monotone convergence  $u \in L^\infty(0, T, L^1(\mathbb{R}^d_T))$ . Once again, from the estimates performed in the bounded domains  $\tilde{u}_n \nabla \mathcal{K} * \tilde{u}_n \rightarrow u \nabla \mathcal{K} * u$  in  $L^1_{loc}(\mathbb{R}^d_T)$ . Therefore, we can identify  $w_1 = u \nabla \mathcal{K} * u$ .

We now show that  $u \in C([0, T]; L^1_{loc}(\mathbb{R}^d))$ , which we know to be true, implies that  $u \in C([0, T]; L^1(D))$ . Let  $t_n \rightarrow t \in [0, T]$  then for all  $R > 0$  we have,

$$\int |u(t_n) - u(t)| dx = \int_{B_R} |u(t_n) - u(t)| dx + \int_{\mathbb{R}^d \setminus B_R} |u(t_n) - u(t)| dx. \quad (6.17)$$

The first term on the right hand side of (6.17) can be bounded by  $\epsilon/2$ , provided  $n$  is chosen large enough, since  $u \in C([0, T]; L^1_{loc}(\mathbb{R}^d))$ . To bound the second term we first show that  $A(u) \in L^1(\mathbb{R}^d_T)$ . By **(D3)**, for  $k > 0$  there exists some  $0 < C_k < \infty$  such that if  $z < k$  then  $A(z) \leq Cz$ . Hence,

$$\begin{aligned} \int A(u) dx &= \int_{\{u < k\}} A(u) dx + \int_{\{u \geq k\}} A(u) dx \\ &\leq CM + A(\|u\|_\infty) \lambda_A(k) < \infty. \end{aligned}$$

Therefore,  $\|A(u)\|_{L^1(\mathbb{R}^d_T)} \leq C(M, \|u\|_\infty)T$ . Now, let  $w(x)$  be a smooth radially-symmetric cut-off function with  $w(x) = 0$  for  $|x| < 1/2$  and  $w(x) = 1$  for  $|x| \geq 1$ .

Then consider the quantity,  $M_R(t) = \int uw(x/R)dx$ . Then formally,

$$\frac{d}{dt}M_R(t) = \frac{1}{R} \int uv \cdot (\nabla w)(x/R)dx + \frac{1}{R^2} \int A(u)(\Delta w)(x/R)dx.$$

Estimating terms in  $L^\infty$  gives,

$$\frac{d}{dt}M_R(t) \lesssim \frac{\|v\|_\infty \|u\|_1}{R} + \frac{1}{R^2} \int A(u)dx.$$

Formally, then

$$M_R(t) \lesssim M_R(0) + M\|v\|_{L^1((0,t);L^\infty)}R^{-1} + \|A(u)\|_{L^1((0,t)\times\mathbb{R}^d)}R^{-2}. \quad (6.18)$$

Since  $A \in L^1((0,t) \times \mathbb{R}^d)$  and  $M_R(0) \rightarrow 0$  as  $R \rightarrow \infty$ , by choosing  $R$  sufficiently large, the last term of (6.17) can be bounded by  $\epsilon/2$ . Hence, implies that  $u \in C([0,T];L^1(\mathbb{R}^d))$ . Furthermore, via interpolation we obtain that  $u \in C([0,T];L^p(\mathbb{R}^d))$  for  $1 \leq p < \infty$ .

Conservation of mass can be proved similarly using a cut-off function  $w(x) = 1$  for  $|x| \leq 1/2$  and  $w(x) = 0$  for  $|x| \geq 1$ , see the proof of Theorem 15 in [29] for a similar proof.  $\square$

### 6.1.3 The Energy Dissipation Inequality

*Proof.* (**Proposition 1**) Define

$$h(u) = \int_1^u \frac{A'(s)}{s} ds,$$

then  $\Phi(u) = \int_0^u h(s)ds$ . The regularized entropy is defined similarly with  $a'_\epsilon(u)$ , as defined in (6.24), taking the place of  $A'(u)$ . Given a smooth solution  $u^\epsilon$  to (6.4) one can verify,

$$\mathcal{F}_\epsilon(u^\epsilon(t)) + \int_0^t \int \frac{1}{u^\epsilon} |a'_\epsilon(u^\epsilon) \nabla u^\epsilon - u^\epsilon \nabla \mathcal{J}_\epsilon \mathcal{K} * u^\epsilon|^2 dx d\tau = \mathcal{F}_\epsilon(u^\epsilon(0)). \quad (6.19)$$

Here  $\mathcal{F}_\epsilon(u(t))$  denotes the free energy with the regularized entropy and kernel. Once again we take the limit  $\epsilon$  approaches zero to obtain (5.7). We first show that the entropy converges.

*Step 1:* The parabolic regularization gives

$$\begin{aligned} h(z) + \epsilon \ln z &\leq h_\epsilon(z) \leq h(z) + 2\epsilon \ln z \quad \text{for } 1 \leq z, \\ h(z) + 2\epsilon \ln z &\leq h_\epsilon(z) \leq h'(z) + \epsilon \ln z \quad \text{for } z \leq 1. \end{aligned}$$

Therefore, writing  $\Phi(u) = \int_0^1 h(s)ds + \int_1^u h(s)ds$  one observes that

$$\Phi(u) - 2\epsilon \leq \Phi_\epsilon(u) \leq \Phi(u) + 2\epsilon(u \ln u)_+. \quad (6.20)$$

This will allow us to show convergence of the entropy. In fact,

$$\begin{aligned} \left| \int \Phi_\epsilon(u^\epsilon) - \Phi(u) dx \right| &\leq \int |\Phi_\epsilon(u^\epsilon) - \Phi(u^\epsilon)| dx + \int |\Phi(u^\epsilon) - \Phi(u)| dx \\ &\stackrel{(6.20)}{\leq} 2\epsilon \int (1 + u^\epsilon \ln u^\epsilon)_+ dx + \|\Phi\|_{C^1([0, \|u^\epsilon\|_\infty])} \int |u^\epsilon - u| dx \\ &\leq 2\epsilon (|D| + \|\ln u^\epsilon\|_\infty \|u_0^\epsilon\|_1) + C \|u^\epsilon - u\|_1. \end{aligned}$$

Conservation of mass, boundedness of smooth solutions, and precompactness in  $L^1_{loc}$  imply there exists a subsequence, such that as  $\epsilon_j \rightarrow 0$ ,

$$\int \Phi_{\epsilon_j}(u_j^\epsilon) dx \rightarrow \int \Phi(u) dx.$$

*Step 2:* To show convergence of the interaction energy we need that for a.e  $t \in (0, T)$

$$\int u^\epsilon(t) \mathcal{J}_\epsilon \mathcal{K} * u^\epsilon(t) dx \rightarrow \int u(t) \mathcal{K} * u(t) dx. \quad (6.21)$$

Since  $\mathcal{K} \in L^1_{loc}(D)$  we know that  $\|\mathcal{K} * u\|_{L^\infty}$  is bounded; hence, replacing  $\nabla \mathcal{K}$  with  $\mathcal{K}$  in (6.12) gives the desired result. Finally, we are left to deal with the

entropy production functional.

*Step 3:* From Lemma 10 in [54],

$$\int \frac{1}{u} |A'(u) \nabla u - u \nabla \mathcal{K} * u|^2 dx \leq \liminf_{\epsilon \rightarrow 0} \int \frac{1}{u^\epsilon} |a'_\epsilon(u^\epsilon) \nabla u^\epsilon - u^\epsilon \nabla \mathcal{J}_\epsilon \mathcal{K} * u^\epsilon|^2 dx. \quad (6.22)$$

We also note that this was proved in [29]. The proof of (6.22) relies on a result due to Otto in [170], refer to Lemma 24 in the Appendix. In our case,  $u^\epsilon \in L^1(D_T)$  and  $J_\epsilon = \nabla A^\epsilon(u^\epsilon) - u^\epsilon \nabla \mathcal{K} * u^\epsilon \in L^1_{loc}(D_T)$ . Furthermore, up to a sequence not renamed,  $u^\epsilon \rightharpoonup u \in L^2$  and  $J_\epsilon \rightharpoonup J$  in  $L^2$ , therefore, we can apply Lemma 24.

For the energy dissipation estimate in  $\mathbb{R}^d$  we again consider the family of solutions  $\{u_r\}$  to (5.1) on  $B_r$  (for simplicity let  $u_r$  denote the zero-extension of the solutions). Since  $u_n(0) \mathbf{1}_{B_n} \nearrow u(0)$  by monotone convergence we obtain that  $\mathcal{F}(u_n(0)) \rightarrow \mathcal{F}(u(0))$ . Noting that  $\mathcal{K} \in L^{d/(d-2)}$  allows us to make a modification to (6.16) and obtain that  $u_n \mathcal{K} * u_n \rightharpoonup u \mathcal{K} * u$  in  $L^2(\mathbb{R}^d_T)$ . Furthermore, (6.21) implies that  $u_n \mathcal{K} * u_n \rightarrow u \mathcal{K} * u$  in  $L^1_{loc}$ . We are left to verify the uniform integrability over all space. First note that Morrey's inequality implies

$$\begin{aligned} \|\mathcal{K} * \tilde{u}_n\|_\infty &\lesssim \|\nabla \mathcal{K} * u\|_\infty + \|\mathcal{K} * u_n\|_p \\ &\leq \|\nabla \mathcal{K} * u\|_\infty + \|\mathcal{K}\|_{L^{d/(d-2)}, \infty} \|u_n\|_{dp/(d+2p)}. \end{aligned}$$

Hence, taking  $p$  sufficiently large we obtain that  $\mathcal{K} * u_n$  is bounded in  $L^\infty(D_T)$ .

Therefore,

$$\int_{\mathbb{R}^d \setminus B_k} u_n \mathcal{K} * u_n dx \leq \|\mathcal{K} * u_n\|_\infty \int_{\mathbb{R}^d \setminus B_k} u_n dx.$$

This fact along with (6.18) gives that for any  $\epsilon > 0$  there exists a  $k_\epsilon$  sufficiently large such that for all  $k > k_\epsilon$

$$\int_{\mathbb{R}^d \setminus B_k} \tilde{u}_n \mathcal{K} * \tilde{u}_n dx \leq \epsilon.$$

This gives convergence of the interaction energy. The result follows from the weak lower semi-continuity of the entropy production functional and  $\int \Phi(u) dx$  in  $L^2$ .  $\square$

## 6.2 Local Existence on $\mathbb{R}^d$ , $d \geq 2$

As in the previous section, we will proceed by passing to the limit from a regularized, uniformly parabolic problem, (6.4). It is in the first step, the construction of global solutions to the regularized problems, where the simplification takes place. Indeed, consider the regularized aggregation-diffusion equation

$$u_t^\epsilon = \Delta A^\epsilon(u^\epsilon) - \nabla \cdot (u^\epsilon \nabla \mathcal{K}^\epsilon * u^\epsilon). \quad (6.23)$$

As above, we define,

$$A^\epsilon(z) = \int_0^z a'_\epsilon(z) dz, \quad (6.24)$$

where  $a'_\epsilon(z)$  is a smooth function, such that  $A'(z) + \epsilon \leq a'_\epsilon(z) \leq A'(z) + 2\epsilon$ . Let  $\eta(x) \in C_c^\infty(\mathbb{R}^d)$  with  $0 < \eta(x) \leq 1$  for  $|x| < 1$ ,  $\eta(x) = 0$  for  $|x| \geq 1$ , and  $\eta(x) \equiv 1$  for  $|x| \leq 1/2$ . We denote,

$$\mathcal{K}^\epsilon(x) := \begin{cases} \int \frac{1}{\epsilon^d \eta(x)^d} \eta\left(\frac{x-y}{\epsilon \eta(x)}\right) \mathcal{K}(y) dy & |x| < 1 \\ \mathcal{K}(x) & |x| \geq 1 \end{cases}$$

Hence,  $\mathcal{K}^\epsilon \in C^\infty$  but, note that importantly,  $\mathcal{K}^\epsilon(x) = \mathcal{K}(x)$  for all  $|x| \geq 1$ . Our main goal for this section is to prove

**Proposition 3** (Local Existence for the Regularized Aggregation Diffusion Equation). *Let  $\epsilon > 0$  be fixed and  $u_0(x) \in L^1_+(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ . Then (6.23) has a classical solution  $u^\epsilon$  on  $\mathbb{R}^d_T$  for all  $T > 0$  with  $u^\epsilon(0) = \mathcal{J}_\epsilon u_0$ .*

We begin with some preliminaries. Define the Hilbert space

$$V = \{v \in H^1 : \|v\|_V < \infty\}, \quad \|v\|_V = \sqrt{\langle v, v \rangle_V},$$

with the inner product defined via  $\langle u, v \rangle_V := \langle u, v \rangle_{H^1} + \int |x| uv dx$ . Note by the Rellich-Khondrashov compactness theorem,  $V$  is compactly embedded in  $L^2(\mathbb{R}^d)$ . We will construct a weak solution to (6.23) with the analogous definition of weak solution as in Definition 4.

We prove Proposition 3 using the Schauder fixed point theorem, see e.g. [22]. The necessary compactness for the application is obtained via the Aubin-Lions Lemma [185]. We first state and prove some a priori estimates that will be used in the proof of Proposition 3, many of which are the same or closely related to estimates proved elsewhere in this dissertation and so the proofs will simply be sketched.

**Lemma 11** (A priori bounds with linear advection). *For fixed  $\epsilon > 0$  let  $\tilde{u} \in L^2(0, T; L^2) \cap L^\infty(0, T; L^1)$ . Let  $u^\epsilon$  be the unique strong solution to*

$$\partial_t u^\epsilon = \Delta A^\epsilon(u^\epsilon) - \nabla \cdot (u^\epsilon \nabla \mathcal{K}^\epsilon * \tilde{u}) \quad (6.25)$$

*with initial data  $u^\epsilon_0 = \mathcal{J}_\epsilon u_0(x)$  with  $u_0 \in L^1_+(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ . Then,*

$$(i) \quad \|u^\epsilon\|_{L^2(0, T; L^2)} \leq T^{1/2} \|u^\epsilon\|_{L^\infty(0, T; L^2)} \leq T^{1/2} \|u_0\|_2 \exp(\|\Delta \mathcal{K}^\epsilon\|_2 \|\tilde{u}\|_{L^2(0, T; L^2)} T^{1/2})$$

$$(ii) \quad \|u^\epsilon(t)\|_\infty \leq \|u_0\|_\infty \exp(\|\Delta \mathcal{K}^\epsilon\|_2 \|\tilde{u}\|_{L^2(0, T; L^2)} T^{1/2}).$$

$$(iii) \quad \|\nabla A^\epsilon(u^\epsilon)\|_{L^2(0, T; L^2)}^2 \lesssim A^\epsilon(\|u^\epsilon\|_\infty) \|u^\epsilon\|_1 + \|\nabla \mathcal{K}^\epsilon\|_{L^\infty}^2 \|\tilde{u}\|_{L^\infty(0, T; L^1)}^2 \|u^\epsilon\|_{L^\infty(0, T; L^2)}^2.$$

$$(iv) \quad \|\nabla u^\epsilon\|_{L^2(0,T,L^2(\mathbb{R}^d))} \lesssim \epsilon^{-1} \|\nabla A^\epsilon(u^\epsilon)\|_{L^2(0,T;L^2)}.$$

$$(v) \quad \mathcal{M}_2(u^\epsilon(t)) \leq C(\mathcal{M}_2(u_0), \epsilon, \|u\|_{L^\infty(0,T;L^\infty)}, \|\tilde{u}\|_{L^\infty(0,T;L^1)}, \|u_0\|_1, T).$$

$$(vi) \quad \|\partial_t u^\epsilon\|_{L^2(0,T,H^{-1})} \leq C(\epsilon, \|u\|_{L^\infty(0,T;L^\infty)}, \|\tilde{u}\|_{L^2(0,T;L^2)}, \|u_0\|_1, T).$$

*Proof.* In what follows denote  $M := \|u^\epsilon(t)\|_1 = \|u_0\|_1$ . By (6.25) we obtain

$$\begin{aligned} \frac{d}{dt} \int (u^\epsilon)^2 dx &\leq - \int (u^\epsilon)^2 \Delta \mathcal{K}^\epsilon * \tilde{u} dx \\ &\leq \|\Delta \mathcal{K}^\epsilon\|_2 \|\tilde{u}\|_2 \int (u^\epsilon)^2 dx. \end{aligned}$$

Integrating implies,

$$\|u(t)\|_2 \leq \|u_0\|_2 \exp \left( \|\Delta \mathcal{K}^\epsilon\|_2 \|\tilde{u}\|_{L^1(0,T,L^2)} \right) \leq \|u_0\|_2 \exp \left( \|\Delta \mathcal{K}^\epsilon\|_2 \|\tilde{u}\|_{L^2(0,T,L^2)} T^{1/2} \right).$$

which gives (i). The bound (ii) follows similarly as above, by estimating the growth of  $\|u(t)\|_p$  and passing to the limit  $p \rightarrow \infty$ .

To continue, we require a bound on the  $L^1$  norm of  $A^\epsilon(u^\epsilon)$ . Condition **(D3)** implies that  $A^\epsilon(z) \leq (C_A + 2\epsilon)z$  for some  $C_A > 0$ . Hence by Chebyshev's inequality,

$$\begin{aligned} \int A^\epsilon(u) dx &= \int_{\{u < 1\}} A^\epsilon(u) dx + \int_{\{u \geq 1\}} A^\epsilon(u) dx \\ &\leq (C_A + 2\epsilon)M + A^\epsilon(\|u\|_\infty) \lambda_u(1) \leq (C_A + 2\epsilon + A^\epsilon(\|u\|_\infty))M. \end{aligned}$$

We now turn to the less trivial (iii). Let  $\eta_R(x) := \eta(xR^{-1})$  for some  $R > 0$ , where  $\eta$  is the smooth cut-off function defined above. Now take  $\tilde{A} = A^\epsilon(u^\epsilon)\eta_R$  as a test function in the definition of weak solution (Definition 4), which implies,

$$\begin{aligned} \int_0^T \left\langle u_t(t), \tilde{A} \right\rangle ds &= \int_0^T \int (-\nabla A^\epsilon(u) + u \nabla \mathcal{K}^\epsilon * \tilde{u}) \cdot \nabla \tilde{A} dx ds \\ &\leq - \int_0^T \int \nabla A^\epsilon(u^\epsilon) \cdot \nabla \tilde{A}(u^\epsilon) dx + \frac{1}{2} \int_0^T \int \left| \nabla \tilde{A}(u^\epsilon) \right|^2 dx dt \\ &\quad + \frac{1}{2} \|\nabla \mathcal{K}^\epsilon\|_{L^\infty}^2 \|\tilde{u}\|_{L^\infty(0,T;L^1)}^2 \int_0^T \int |u^\epsilon|^2 dx. \end{aligned}$$

Note also, we can apply the chain rule (Lemma 22 in Appendix 10.1) and get  $-\int_0^T \int \langle u_t^\epsilon(t), \tilde{A} \rangle dx ds \leq \liminf_{t \rightarrow 0} \int \int_0^{u^\epsilon} \tilde{A}(s) ds dx$ . Furthermore, since  $A$  is increasing, we have  $\int_0^u A^\epsilon(s) ds \leq A^\epsilon(u)u$ . Therefore,

$$\begin{aligned} \frac{1}{2} \int_0^T \int_{|x| \leq R/2} |\nabla A^\epsilon(u)|^2(x) dx &\leq A(\|u^\epsilon\|_{L^\infty}) \|u^\epsilon\|_{L^1} \\ &+ \frac{1}{2} \|\nabla \mathcal{K}^\epsilon\|_{L^\infty}^2 \|\tilde{u}\|_{L^\infty(0,T;L^1)}^2 \|u^\epsilon\|_{L^2(0,T;L^2)}^2 \\ &+ \int_0^T \int_{R/2 \leq |x| \leq R} -\nabla A^\epsilon \cdot \nabla(A^\epsilon \eta_R) dx dt \\ &+ \int_0^T \int_{R/2 \leq |x| \leq R} \frac{1}{2} |\nabla(A^\epsilon \eta_R)|^2 dx dt. \end{aligned}$$

We concentrate on the latter error terms. By straightforward computation,

$$\begin{aligned} \int_{R/2 \leq |x| \leq R} -\nabla A^\epsilon \cdot \nabla(A^\epsilon \eta_R) + \frac{1}{2} |\nabla(A^\epsilon \eta_R)|^2 dx &\leq \frac{1}{2} \int_{R/2 \leq |x| \leq R} |A^\epsilon(u^\epsilon)|^2 |\nabla \eta_R|^2 dx \\ &\lesssim R^{-2} A^\epsilon(\|u^\epsilon\|_\infty) \|A^\epsilon(u^\epsilon)\|_1. \end{aligned}$$

Therefore, since  $A^\epsilon(u^\epsilon) \in L^1 \cap L^\infty$ , we have for all  $T < \infty$ , by taking  $R \rightarrow \infty$ ,

$$\|\nabla A^\epsilon(u^\epsilon)\|_{L^2(0,T;L^2)}^2 \lesssim \tilde{A}(\|u^\epsilon\|_{L^\infty}) \|u^\epsilon\|_{L^1} + \|\nabla \mathcal{K}^\epsilon\|_{L^\infty}^2 \|\tilde{u}\|_{L^\infty(0,T;L^1)}^2 \|u^\epsilon\|_2^2.$$

This completes the proof of (iii). This bound, along with the fact that  $a'_\epsilon \geq \epsilon$ , gives us the bound in (iv). Note that this bound depends only the bound given by (iii) and  $\epsilon$ .

The bound on the second moment of  $u^\epsilon$ , (v), follows from the bound on the  $L^1$  norm of  $A^\epsilon(u)$ . Indeed,

$$\begin{aligned} \frac{d}{dt} \int |x|^2 u^\epsilon dx &\leq 2d \int A(u^\epsilon) dx + 2 \|\nabla \mathcal{K}^\epsilon * \tilde{u}\|_\infty \|u^\epsilon\|_{L^1}^{1/2} \left( \int |x|^2 u^\epsilon dx \right)^{1/2} \\ &\leq 2d \int A^\epsilon(u^\epsilon) dx + 2M^{1/2} \|\nabla \mathcal{K}^\epsilon\|_{L^\infty} \|\tilde{u}\|_{L^1} \left( \int |x|^2 u^\epsilon dx \right)^{1/2} \\ &\leq 2d \int A^\epsilon(u^\epsilon) dx + 2M^{1/2} \|\nabla \mathcal{K}^\epsilon\|_{L^\infty} \|\tilde{u}\|_{L^1} \left( 1 + \int |x|^2 u^\epsilon dx \right). \end{aligned}$$



An application of Grönwall's inequality then provides the quantitative result.

Let  $\phi \in C([0, T]; H^1)$ . Then by the definition of the weak solution and Cauchy-Schwarz,

$$\int_0^T \langle \partial_t u^\epsilon(t), \phi(t) \rangle^2 dt \leq \int_0^T \|\nabla A(u^\epsilon) - u^\epsilon \mathcal{J}_\epsilon \nabla \mathcal{K} * \tilde{u}\|_{L^2(\mathbb{R}^d)}^2 \|\nabla \phi(t)\|_{L^2(\mathbb{R}^d)}^2 dt.$$

The bounds previously obtained then imply that  $\|\partial_t u^\epsilon\|_{L^2(0, T; H^{-1})} \leq C(T, \tilde{u}, u_0)$ .  $\square$

Now that we have all the required a priori estimates we are ready to prove Proposition 3.

*Proof.* (**Proposition 3**) Define the compact and convex subset of  $L^2(0, T; L^2)$

$$\mathcal{S}_T = \{v \in L^2(0, T; V) : \|v_t\|_{L^2(0, T; H^{-1})} + \|v\|_{L^2(0, T; V)} \leq C_0, \|v\|_{L^\infty(0, T; L^1)} \leq \|u_0\|_1\},$$

for some  $C_0$  to be chosen below. Compactness in the  $L^2(0, T; L^2)$  topology follows from the Lions-Aubin lemma (since  $V \subset\subset L^2$ ) and the fact that  $\mathcal{S}_T$  is closed due to the weak compactness of  $V$  and  $L^2(0, T; H^{-1})$  and the lower semi-continuity of the  $L^1$  norm. Define  $M := \|u_0\|_1$ .

Define the map  $J : \mathcal{S}_T \rightarrow \mathcal{S}_T$  via the following procedure: given some  $\tilde{u} \in \mathcal{S}_T$  we define  $J\tilde{u} = u$ , where  $u$  is the solution to (6.25). Our goal is to now apply the Schauder fixed point theorem. First, we verify that  $J$  actually maps  $\mathcal{S}_T$  into itself. Let  $\tilde{u} \in \mathcal{S}_T$ . Note that as in previous sections,  $u$  is a global strong solution from standard regularity theory [148] which satisfies the easy a priori upper bound (ii). Furthermore, the bounds provided by Lemma 11 and conservation of mass,  $u \in \mathcal{S}_T$  for  $C_0$  chosen large enough and  $T$  chosen sufficiently small (depending on the mass, the  $L^\infty$  norm of the initial data and  $\epsilon$ ). Indeed, the bound (i) provides a bound on  $\|u\|_{L^2(0, T; L^2)}$ , the bound (iv) provides a bound on  $\|\nabla u\|_{L^2(0, T; L^2)}$ , the

bound (vi) provides a bound on  $\|\partial_t u\|_{L^2(0,T;H^{-1})}$ . Moreover, the bound (ii) and (v) along with,

$$\int |x| u^2(x) dx \leq \|u\|_\infty \left( \int |x|^2 u dx \right)^{1/2} \|u\|_1^{1/2},$$

provides an estimate on the first moment of the  $L^2$  norm of  $u$ . Hence,  $J : \mathcal{S}_T \rightarrow \mathcal{S}_T$ . We are left to show that  $J$  is a continuous map. We show  $J$  is continuous as a mapping from  $L^2(0, T; L^2)$  to  $C([0, T]; \dot{H}^{-1})$ , which by interpolation against uniform bounds in  $H^1$  provided by (iv), implies continuity in  $L^2(0, T; L^2(\mathbb{R}^d))$  (since  $\|f\|_2 \leq \|\nabla f\|_2^{1/2} \|f\|_{\dot{H}^{-1}}^{1/2}$  as can be easily seen from the Fourier transform). The reason for this approach, as opposed to working in  $L^2$  directly, is due to the presence of the nonlinear diffusion, which as can be seen from the proof of uniqueness below, interacts well with advection in  $\dot{H}^{-1}$ . Hence, let  $\{v_n\}_{n \geq 0} \subset \mathcal{S}_T$  be such that  $v_n \rightarrow v$  in  $L^2(0, T; L^2(\mathbb{R}^d))$ . We show that  $Jv_n \rightarrow Jv$  in  $C([0, T]; \dot{H}^{-1}(\mathbb{R}^d))$ . To this end, let  $\phi_n := -\mathcal{N} * (Jv_n - Jv)$  and denote  $u_n := Jv_n$  and  $u := Jv$ . It is very important to note that while the  $v_n$ 's may not have constant  $L^1$  norm,  $u_n$  and  $u$  in fact do. This allows us to apply the arguments used in the proof of uniqueness, Theorem 4 in §6.3.2, to deduce  $\|\phi_n(t)\|_\infty + \|\nabla \phi_n(t)\|_\infty + \|\nabla \phi_n(t)\|_2 \lesssim_{C_0, M, u_0} 1$ . By the regularity of  $\phi_n(t)$ ,

$$\frac{1}{2} \frac{d}{dt} \int |\nabla \phi_n(t)|^2 dx = \langle \nabla \phi_n(t), \partial_t \nabla \phi_n(t) \rangle = - \langle \partial_t u_n(t) - \partial_t u(t), \phi_n(t) \rangle.$$

Therefore, using  $\phi_n(t)$  in the definition of weak solution,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |\nabla \phi_n(t)|^2 dx &= \int (\nabla A^\epsilon(u_n) - \nabla A^\epsilon(u)) \cdot \nabla \phi_n dx \\ &\quad - \int (u_n - u)(\nabla \mathcal{K}^\epsilon * v) \cdot \nabla \phi_n dx - \int u(\nabla \mathcal{K}^\epsilon * (v_n - v)) \cdot \nabla \phi_n dx. \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

We drop the time dependence for notational simplicity. Since  $A^\epsilon$  is increasing,

we have the desired monotonicity of the diffusion,

$$I_1 = - \int (A^\epsilon(u_n) - A^\epsilon(u)) (u_n - u) dx \leq 0.$$

Using integration by parts as in the proof of Theorem 4 we have,

$$\begin{aligned} |I_2| &\lesssim \int |D^2 \mathcal{K}^\epsilon * v| |\nabla \phi_n|^2 dx \\ &\leq \|D^2 \mathcal{K}^\epsilon\|_2 \|v\|_2 \|\nabla \phi_n\|_2^2. \end{aligned}$$

Moreover,

$$\begin{aligned} |I_3| &\leq \|u\|_2 \|\nabla \mathcal{K}^\epsilon * (v_n - v)\|_\infty \|\nabla \phi_n\|_2 \\ &\lesssim \|\nabla \mathcal{K}^\epsilon\|_3 \|v_n - v\|_{3/2} \|\nabla \phi_n\|_2 \\ &\lesssim_\epsilon \|v_n - v\|_2^{2/3} \|v_n - v\|_1^{1/3} (1 + \|\nabla \phi_n\|_2^2). \end{aligned}$$

By the uniform bound on  $\|u\|_\infty$  (by part (ii) above), the uniform bound on the mass in  $\mathcal{S}_T$  and the regularization of  $\mathcal{K}$ ,

$$\frac{1}{2} \frac{d}{dt} \|\nabla \phi_n\|_2^2 \lesssim_\epsilon \|v_n - v\|_2^{2/3} + (1 + \|v\|_2^2 + \|v_n - v\|_2^2) \|\nabla \phi_n\|_2^2.$$

Integrating this implies for some  $C > 0$  depending on the uniform bounds and  $\epsilon$  (using  $\phi_n(0) = 0$ ),

$$\|\nabla \phi_n(t)\|_2^2 \leq \int_0^t \exp \left( C \int_s^t 1 + \|v(t')\|_2^2 + \|v_n(t') - v(t')\|_2^2 dt' \right) \|v_n(s) - v(s)\|_2^{2/3} ds.$$

Since  $t \leq T$  we have,

$$\|\nabla \phi_n(t)\|_2^2 \lesssim e^{CT + CC_0 + C\|v_n - v\|_{L^2(0,T;L^2)}^2} \int_0^t \|v_n(s) - v(s)\|_2^{2/3} ds.$$

By assumption,  $\|v_n(s) - v(s)\|_2 \rightarrow 0$  pointwise a.e. on  $(0, T)$  and  $\|v_n(s) - v(s)\|_2^{2/3} \leq 1 + \|v_n(s) - v(s)\|_2^2$ , so by the dominated convergence theorem we have  $\|\nabla \phi_n\|_2 \rightarrow 0$  uniformly on  $(0, T)$ . Therefore  $J$  is continuous on  $\mathcal{S}_T$ .

Finally, the Schauder fixed point theorem implies there exists a solution  $Ju = u$  with  $u \in \mathcal{S}_T$  for some  $T > 0$ . By the regularization of  $\mathcal{K}$  and uniform parabolicity, it should be straightforward to extend this solution (to the regularized system) indefinitely and to show that in fact  $u$  is a classical solution to (6.23).  $\square$

*Proof. (Theorem 3)* For all  $\epsilon > 0$ , let  $u^\epsilon$  be the solution to 6.23 provided by Proposition 3. Notice that the bounds provided by Lemma 11 are mostly not independent of  $\epsilon$ , and so some must be re-proved for the classical solutions  $u^\epsilon$  to be independent in  $\epsilon$ . For brevity, we simply sketch the proofs.

We may easily deduce the following a priori bound on the second moment:

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_2(u^\epsilon(t)) &= 2d \int A^\epsilon(u^\epsilon) dx - 2 \int \int u^\epsilon x \cdot \nabla \mathcal{K}^\epsilon * u^\epsilon dx \\ &\leq 2d \int A^\epsilon(u^\epsilon) dx + 2 \|\nabla \mathcal{K}^\epsilon * u^\epsilon\|_\infty \|u\|_1^{1/2} \mathcal{M}_2(u^\epsilon(t))^{1/2} \\ &\lesssim_{\|u\|_1} \int A^\epsilon(u^\epsilon) dx + (\|u^\epsilon\|_\infty + \|u\|_1) (1 + \mathcal{M}_2(u^\epsilon(t))). \end{aligned}$$

The first term is bounded in terms of  $\|u\|_1$  and  $\|u\|_\infty$  in the proof of Lemma 11. By Grönwall's inequality we therefore have a uniform in  $\epsilon$  a priori upper bound on the second moment, provided we have a uniform bound on the  $\|u^\epsilon\|_\infty$ . Similarly, one may alter the proof of (iii) in 11 to bound  $\|\nabla A\|_{L^2(0,T;L^2)}$  independent of  $\epsilon$  provided we have a uniform bound on  $\|u^\epsilon\|_\infty$ . This bound is provided by the Alikakos iteration argument, Lemma 9 in the previous section. At this point, using the precompactness provided by the uniform bound on the second moments and the  $\|\nabla A^\epsilon(u^\epsilon)\|_{L^2(0,T;L^2)}$  norms of  $\{u^\epsilon\}_{\epsilon>0}$ , we may follow the arguments of the previous section to extract a strongly convergent subsequence in  $L^1((0,T) \times \mathbb{R}^d)$ . Recall, tightness is supplied by the uniform bound on the second moments and equi-continuity by the bounds on  $\nabla A$ . Once the limit is extracted, similar arguments as in the previous section may be employed to upgrade the convergence

to  $C([0, T]; L^p)$  for all  $1 \leq p < \infty$ . The last remaining technical point is ensuring that the limit is indeed a weak solution of the unregularized system in the sense of Definition 4. In  $d \geq 3$  we may use the essentially the same proof as the previous section, however in  $d = 2$  we must also verify that  $u_t \in L^2(0, T; \mathcal{V}^*(\mathbb{R}^2))$  and justify the limiting procedure which allows us to conclude

$$\int_0^T \langle u_t, \phi(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} dt = \int_0^T \int (\nabla A(u) - u \nabla \mathcal{K} * u) \cdot \nabla \phi(t) dt,$$

for all  $\phi \in L^\infty(0, T; \mathcal{V}(\mathbb{R}^2))$ . This definition of weak solution is key to the proof of uniqueness in  $\mathbb{R}^2$ . By a standard boot-strap argument in parabolic regularity theory,  $u_t^\epsilon$  is smooth for all  $\epsilon > 0$  (clearly, not uniformly in  $\epsilon$ ). Let  $\phi \in C^1([0, T]; \mathcal{V}(\mathbb{R}^2))$ . Then, by  $u^\epsilon(t) \rightarrow u(t)$  uniformly in  $L^1$ ,

$$\begin{aligned} \int_0^T \langle u_t, \phi(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} &:= \int_0^T \int (u(t) - u_0) \phi_t(t) dx dt \\ &= \lim_{\epsilon \rightarrow 0} \int_0^T \int (u^\epsilon(t) - u_0^\epsilon) \phi_t(t) dx dt \\ &= - \lim_{\epsilon \rightarrow 0} \int_0^T \int (\nabla A^\epsilon(u^\epsilon) - u^\epsilon \nabla \mathcal{K}^\epsilon * u^\epsilon) \cdot \nabla \phi(t) dt \\ &= - \int_0^T \int (\nabla A(u) - u \nabla \mathcal{K} * u) \cdot \nabla \phi(t) dt. \end{aligned}$$

The latter convergence being already deduced via arguments similar to previous sections. By density, we then have that  $u$  is a weak solution the sense of Definition 4. □

Unfortunately, the energy dissipation inequality is not trivial to extend to the case  $\mathbb{R}^2$ , as  $\mathcal{K}$  is no longer bounded near infinity and we cannot follow the same argument as used in §6.1.3 above.

*Proof.* (**Proposition 1 for  $\mathbb{R}^2$** ) Let  $u(t)$  be the weak solution deduced in Theorem 3 above and  $\{u^\epsilon\}_{\epsilon > 0}$  be the solutions to the regularized system 6.23. We

may follow the steps of Proposition 1 in §6.1.3 above except for proving the convergence of the interaction energy. That is, we need to show

$$\int u^\epsilon(t) \mathcal{K}^\epsilon * u^\epsilon(t) dx - \int u(t) \mathcal{K} * u(t) dx \rightarrow 0.$$

To this end, write,

$$\begin{aligned} \int u^\epsilon(t) \mathcal{K}^\epsilon * u^\epsilon(t) dx - \int u(t) \mathcal{K} * u(t) dx &= \int u(\mathcal{K}^\epsilon - \mathcal{K}) * u + \int u \mathcal{K}^\epsilon * (u^\epsilon - u) dx \\ &\quad + \int (u^\epsilon - u) \mathcal{K}^\epsilon * u^\epsilon dx \\ &:= T1 + T2 + T3. \end{aligned}$$

Since the regularization  $\mathcal{K}^\epsilon(x) = \mathcal{K}(x)$  for all  $|x| > 1$  we have,

$$\begin{aligned} |T1| &\leq \|u(t)\|_1 \|(\mathcal{K}^\epsilon - \mathcal{K}) * u\|_\infty \\ &\leq \|u(t)\|_1 \|u(t)\|_\infty \|\mathcal{K}^\epsilon - \mathcal{K}\|_{L^1(B_1(0))} \rightarrow 0. \end{aligned}$$

Now consider  $T2$ . By the duality of  $BMO$  and  $\mathcal{H}^1$  and Lemma 6 in Appendix 10.4 we have,

$$\begin{aligned} |T2| &\leq \|u(t)\|_1 \|\mathcal{K}^\epsilon * (u^\epsilon - u)\|_\infty \\ &\lesssim \|\mathcal{K}^\epsilon\|_{BMO} \|u^\epsilon(t) - u(t)\|_{\mathcal{H}^1} \\ &\lesssim \|u^\epsilon(t) - u(t)\|_p + \int |x| |u^\epsilon(t) - u(t)| dx, \end{aligned}$$

for any  $1 < p < \infty$ . However,  $u^\epsilon \rightarrow u$  in  $C([0, T]; L^p)$  for all such  $p$ , so the first term is not an issue. To deal with the second term,

$$\int |x| (u^\epsilon - u) dx \leq \left( \int |x|^2 |u^\epsilon - u| dx \right)^{1/2} \left( \int |u^\epsilon - u| dx \right)^{1/2}.$$

However, since  $u^\epsilon \rightarrow u$  in  $C([0, T]; L^1)$  and both  $u^\epsilon$  and  $u$  have uniformly bounded second moments on  $[0, T]$  we have that  $T2 \rightarrow 0$ . The final term,  $T3$ , follows similarly. Hence the energy dissipation inequality holds in  $\mathbb{R}^2$ .  $\square$

### 6.3 Uniqueness

We now prove the uniqueness of weak solutions stated in Theorem 4.

*Proof.* (**Theorem 4**) The proof follows [25, 29] and estimates the difference of weak solutions in  $\dot{H}^{-1}$ , motivated by the fact that the nonlinear diffusion is monotone in this norm [202].

#### 6.3.1 Bounded Domains and $\mathbb{R}^d$ for $d \geq 3$

If  $D$  is bounded (with smooth boundary and convex) then we define  $\phi(t)$  as the zero mean strong solution of

$$\Delta\phi(t) = u(t) - v(t) \text{ in } D \quad (6.26)$$

$$\nabla\phi(t) \cdot \nu = 0, \text{ on } \partial D, \quad (6.27)$$

where  $\nu$  is the outward unit normal of  $D$ . If the domain is  $\mathbb{R}^d$  for  $d \geq 3$ , we let  $\phi(t) = -\mathcal{N} * (u - v)$  where  $\mathcal{N}$  is the Newtonian potential. In either case, by the integrability and boundedness of weak solutions  $u(t)$  and  $v(t)$  we can conclude  $\phi(t) \in L^\infty((0, T) \times D) \cap C([0, T]; \dot{H}^1(D))$ ,  $\nabla\phi(t) \in L^\infty(D_T) \cap L^2(D_T)$  and  $\phi_t$  solves,

$$\Delta\phi_t = \partial_t u - \partial_t v.$$

As can be seen easily from the Fourier transform we have  $\|u(t) - v(t)\|_{H^{-1}} \approx \|\nabla\phi(t)\|_2$ , and hence it suffices to show that  $\|\nabla\phi(t)\|_2 = 0$ . During the course of the proof, we integrate by parts on a variety of quantities. If the domain is bounded, then the boundary terms will vanish due to the no-flux conditions (5.4), (6.27). In  $\mathbb{R}^d$ , the computations are justified as  $\nabla\mathcal{K} * u, \nabla A(u), \nabla\mathcal{K} * v, \nabla A(v), \nabla\phi \in L^2(D_T)$ . We also remark that, strictly speaking, as pointed out

in [25], the strong  $\dot{H}^1$ -measurability of  $t \rightarrow \phi(t)$  should also be established. However, this follows from the formal computations below and an easy approximation argument (see [25]).

By the regularity of  $\phi(t)$  and the no-flux boundary conditions (6.27), (5.4) we have possibly up to a set of measure zero,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |\nabla \phi(t)|^2 dx &= \langle \nabla \phi(t), \partial_t \nabla \phi(t) \rangle \\ &= - \langle \partial_t u(t) - \partial_t v(t), \phi(t) \rangle. \end{aligned}$$

Therefore, using  $\phi(t)$  in the definition of weak solution and (6.27) we have,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |\nabla \phi(t)|^2 dx &= \int (\nabla A(u(t)) - \nabla A(v(t))) \cdot \nabla \phi(t) dx \\ &\quad - \int (u - v)(\nabla \mathcal{K} * u) \cdot \nabla \phi dx \\ &\quad - \int v(\nabla \mathcal{K} * (u - v)) \cdot \nabla \phi dx. \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

We drop the time dependence for notational simplicity. Since  $A$  is increasing, we have the desired monotonicity of the diffusion,

$$I_1 = - \int (A(u) - A(v))(u - v) dx \leq 0.$$

We now concentrate on bounding the advection terms.

We follow [29]. By integration by parts we have,

$$\begin{aligned} I_2 &= \sum_{i,j} \int \partial_i \phi (\partial_{ij} \mathcal{K} * u) \partial_j \phi dx \\ &\quad + \sum_{i,j} \int \partial_i \phi (\partial_j \mathcal{K} * u) \partial_{ij} \phi dx. \end{aligned} \tag{6.28}$$



If the domain is bounded, we may apply integration by parts,

$$\begin{aligned} \sum_{i,j} \int \partial_i \phi (\partial_j \mathcal{K} * u) \partial_{ij} \phi dx &= - \sum_{i,j} \int \partial_{ij} \phi \partial_j \mathcal{K} * u \partial_i \phi dx - \sum_{i,j} \int \partial_i \phi (\partial_{jj} \mathcal{K} * u) \partial_i \phi dx \\ &\quad + \sum_{i,j} \int_{\partial D} |\partial_i \phi|^2 \partial_j \mathcal{K} * u \nu_j dS, \end{aligned}$$

where  $\nu$  is the unit outward normal to  $D$ . As in [29], we have  $\vec{v} \cdot \nu \leq 0$  on  $\partial D$  since  $D$  is convex and  $\mathcal{K}$  is radially decreasing, so that term is non-positive. If the domain were  $\mathbb{R}^d$ , such boundary terms would vanish. Therefore by integration by parts again we have,

$$\sum_{i,j} \int \partial_i \phi (\partial_j \mathcal{K} * u) \partial_{ij} \phi dx \leq -\frac{1}{2} \int (\Delta \mathcal{K} * u) |\nabla \phi|^2 dx,$$

which together with (6.28) implies,

$$I_2 \lesssim \int |D^2 \mathcal{K} * u| |\nabla \phi|^2 dx.$$

By Hölder's inequality, Lemma 2 and  $\nabla \phi \in L^\infty(D_T)$  for  $p \geq 2$ ,

$$\begin{aligned} \int |D^2 \mathcal{K} * u| |\nabla \phi|^2 dx &\leq \|D^2 \mathcal{K} * u\|_p \left( \int |\nabla \phi|^{2p/(p-1)} dx \right)^{(p-1)/p} \\ &\lesssim p \|u\|_p \|\nabla \phi\|_\infty^{2/p} \left( \int |\nabla \phi|^2 dx \right)^{(p-1)/p} \\ &\lesssim p \left( \int |\nabla \phi|^2 dx \right)^{(p-1)/p}, \end{aligned}$$

where the implicit constant depends only on the uniformly controlled  $L^p$  norms of  $u$  and  $v$ .

As for  $I_3$ , we compute as in [29]. By the computations in the proof of Lemma 2 we may justify integration by parts on the inside of the convolution, that is,

$$\left\| \sum_j \int \partial_i \mathcal{K}(x-y) \partial_{jj} \phi dx \right\|_2 \lesssim \|\nabla \phi\|_2.$$

which by Cauchy-Schwarz implies,

$$I_3 \lesssim \|v\|_\infty \|\nabla \phi\|_2^2.$$

Letting  $\eta(t) = \int |\nabla \phi(t)|^2 dx$ , we get the differential inequality,

$$\frac{d}{dt} \eta(t) \leq \hat{C} p \max(\eta(t)^{1-1/p}, \eta(t)),$$

where  $\hat{C}$  again depends only on the uniformly controlled  $L^p$  norms of  $u, v$ . The differential equality does not have a unique solution, but all of the solutions are absolutely continuous integral solutions bounded above by the maximal solution  $\bar{\eta}(t)$ . By continuity, for  $t < 1/\hat{C}$  the maximal solution is given by  $\bar{\eta}(t) = (\hat{C}t)^p$ , hence,

$$\eta(t) \leq \bar{\eta}(t) = (\hat{C}t)^p.$$

For  $t < 1/(2\hat{C})$  we then have

$$\eta(t) \leq \bar{\eta}(t) \leq 2^{-p},$$

and we take  $p \rightarrow \infty$  to deduce that for  $\forall t \in [0, 1/(2\hat{C}))$ ,  $\eta(t) = 0$ , therefore the solution is unique. This procedure may be iterated to prove uniqueness over the entire interval of existence since the time interval only depends on uniformly controlled norms.

### 6.3.2 Uniqueness in two dimensions

We continue as above, and but now use the bound on the first moment to eliminate the complications. As above, let  $\mathcal{N}$  be the fundamental solution to Laplace's equation and let  $\phi(t) = -\mathcal{N} * (u - v)$ . As above, by the uniform bounds on  $u - v$  and Young's inequality,  $\nabla \phi \in L^\infty((0, T) \times \mathbb{R}^d)$ . Moreover, by Lemma 6,  $\phi \in L^\infty((0, T) \times \mathbb{R}^d)$ , which additionally shows that  $\Delta \phi$  is a bounded distribution and satisfies  $\Delta \phi(t) = u(t) - v(t)$  as well as  $\Delta \phi_t(t) = u_t(t) - v_t(t)$  in the

sense of distributions. The theorem now follows as above for  $d \geq 3$ , provided that  $\nabla\phi \in L^\infty(0, T; L^2(\mathbb{R}^d))$ . We will prove this latter point using the Fourier transform. First note,

$$\widehat{\nabla\phi}(t, \xi) \sim \frac{\xi}{|\xi|^2} (\hat{u}(t, \xi) - \hat{v}(t, \xi)).$$

By Plancherel's identity, it suffices to show  $\widehat{\nabla\phi}(t)$  is uniformly bounded in  $L_\xi^2(\mathbb{R}^2)$ .

Therefore, consider

$$\int \left| \widehat{\nabla\phi}(t, \xi) \right|^2 d\xi \sim \int \frac{1}{|\xi|^2} |\hat{u}(t, \xi) - \hat{v}(t, \xi)|^2 d\xi.$$

Again by Plancherel's identity,  $\hat{u}(t, \xi) - \hat{v}(t, \xi) \in C([0, T]; L_\xi^2(\mathbb{R}^d))$ , therefore we need only restrict attention to  $|\xi| \leq 1$ . Since  $\int u - v dx = 0$  and  $u - v \in L^1$ , therefore  $\hat{u} - \hat{v}$  is continuous (uniformly in time) and  $\hat{u}(0) - \hat{v}(0) = 0$ . Now, the required integrability will follow from some additional regularity of the Fourier transform. In fact, the uniform bound of the first moment implies the Fourier transform is uniformly Lipschitz continuous: let  $\xi_1, \xi_2 \in \mathbb{R}^2$ , then by the mean value theorem,

$$\begin{aligned} |\hat{u}(\xi_1) - \hat{u}(\xi_2)| &\lesssim \left| \int u(x) (e^{-ix \cdot \xi_1} - e^{-ix \cdot \xi_2}) dx \right| \\ &\leq \int u(x) |1 - e^{ix \cdot (\xi_1 - \xi_2)}| dx \\ &\leq |\xi_1 - \xi_2| \int u(x) |x| dx. \end{aligned}$$

Similarly for  $v$ . Therefore, we have  $\sup_{t \in (0, T)} \|\nabla\phi\|_2 < \infty$ . Strictly speaking, as pointed out in [25], the strong  $\mathcal{V}$ -measurability of  $t \rightarrow \phi(t)$  should also be established. This follows from an easy approximation argument using step functions and arguments very similar to those just employed.

Now that  $\phi(t)$  may be used in the definition of weak solution, the rest of the proof of Theorem 4 follows as above.  $\square$

## 6.4 Continuation Theorem

Continuation of weak solutions, Theorem 5, is a straightforward consequence of the local existence theory and the following lemma, which follows substantially the recent work in [36, 125, 39]. This lemma provides a more precise version of Lemma 9 and has a similar proof.

**Lemma 12.** *Let  $\{u^\epsilon\}_{\epsilon>0}$  be the classical solutions to (6.4) on  $D_T$ , with non-negative initial data  $\mathcal{J}_\epsilon u_0$ . Suppose there exists  $T_0$ ,  $0 < T_0 \leq \infty$ , such that*

$$\sup_{\epsilon>0} \lim_{k \rightarrow \infty} \sup_{t \in (0, T_0)} \|(u^\epsilon - k)_+\|_{\frac{2-m}{2-m^*}} = 0, \quad (6.29)$$

where  $m$  is such that  $1 \leq m \leq m^*$  and  $\liminf_{z \rightarrow \infty} A'(z)z^{1-m} > 0$ . Then there exists  $C = C(M, \|u_0\|_\infty)$  such that for all  $\epsilon > 0$ ,

$$\sup_{t \in (0, T_0)} \|u^\epsilon(t)\|_\infty \leq C.$$

In particular, if  $T_0 = \infty$ , then  $\{u^\epsilon\}_{\epsilon>0}$  are uniformly bounded for all time, and therefore the weak solution  $u(t)$ , is uniformly bounded for all time.

*Proof.* (**Lemma 12**) Let  $\bar{q} = (2 - m)/(2 - m^*) \geq 1$ . It will be convenient to define  $\gamma$ ,  $1 \leq \gamma \leq d/2$  such that  $m^* = 1 + 1/\gamma - 2/d$ . We first bound intermediate  $L^p$  norms over the same interval,  $(0, T_0)$ . Then we use Morrey's inequality and Lemma 7 to finish the proof.

*Step 1:*

We have two cases to consider,  $m^* = 2 - 2/d$  and  $m^* < 2 - 2/d$ , which occurs if  $D^2 K \in L_{loc}^{\gamma, \infty}$  for  $\gamma > 1$  (Lemma 3). In the former we show that for any  $p \in (\bar{q}, \infty)$  we have  $u^\epsilon(t)$  uniformly bounded in  $L^\infty(0, T_0; L^p)$ . In the latter case

we only show that for  $\bar{q} < p \leq \gamma/(\gamma - 1)$  we have  $u^\epsilon(t)$  uniformly bounded in  $L^\infty(0, T_0; L^p)$ . In either case, this is sufficient to apply Lemma 7 and conclude the proof.

Let  $k > 0$  be some constant to be determined later and let  $u_k = (u - k)_+$ . We have dropped the  $\epsilon$  and time dependence for notational convenience. By conservation of mass and (6.8), it suffices to control  $\|u_k\|_p$  for any  $k > 0$ . Thus, using the parabolic regularization, (6.24), and (6.8) we obtain

$$\frac{d}{dt} \|u_k\|_p^p \leq -p(p-1) \int u_k^{p-2} A'(u) |\nabla u|^2 dx + p(p-1) \int (u_k^{p-1} + k u_k^{p-2}) \nabla u \cdot \mathcal{J}_\epsilon \nabla \mathcal{K} * u dx.$$

Then,

$$\frac{d}{dt} \|u_k\|_p^p \leq -4(p-1) \int A'(u) \left| \nabla u_k^{p/2} \right|^2 dx - \int ((p-1)u_k^p + k p u_k^{p-1}) \mathcal{J}_\epsilon \Delta \mathcal{K} * u dx. \quad (6.30)$$

Since the constants are not relevant, we treat the cases together only noting minor differences when they appear. If  $m = 2 - 2/d$  we may use Hölder's inequality and then Lemma 2 to obtain a bound on the first term from the advection:

$$\left| \int u_k^p \mathcal{J}_\epsilon \Delta \mathcal{K} * u dx \right| \lesssim_{p, \mathcal{K}} \|u_k\|_{p+1}^p \|u\|_{p+1}.$$

On the other hand, if  $\gamma > 1$  we have from the generalized Hardy-Littlewood-Sobolev inequality (5.8) (Lemma (4)),

$$\left| \int u_k^p \mathcal{J}_\epsilon \Delta \mathcal{K} * u dx \right| \lesssim_{p, \mathcal{K}} \|u_k\|_{\alpha p}^p \|u\|_t + C(M) \|u_k\|_p^p,$$

with the scaling condition  $1/\alpha + 1/t + 1/\gamma = 2$ . Choosing  $t = \alpha p$  implies that

$$\frac{1}{\alpha} = \frac{2 - 1/\gamma}{1 + 1/p}. \quad (6.31)$$

Notice that from our choice of  $p$  then  $1 \leq 1/p + 1/\gamma$ ; thus,  $1/\alpha \leq 1$ . Note that in the case when  $m = 2 - 2/d$  then  $t = \alpha p = p + 1$ . Thus we estimate the advection

terms,

$$\begin{aligned}
\left| \int u_k^p \mathcal{J}_\epsilon \Delta \mathcal{K} * u dx \right| &\lesssim_{p, \mathcal{K}} \|u_k\|_{\alpha p}^p \|u\|_{\alpha p} + C(M) \|u_k\|_p^p \\
&\lesssim \|u_k\|_{\alpha p}^{p+1} + \|u\|_{\alpha p}^{p+1} + C(M) \|u_k\|_p^p \\
&\stackrel{(6.8)}{\lesssim} \|u_k\|_{\alpha p}^{p+1} + C(M) \|u_k\|_p^p + C(k, M). \tag{6.32}
\end{aligned}$$

The lower order terms in the advection can be controlled using Hölder's inequality and Lemma 2,

$$\begin{aligned}
\left| \int u_k^{p-1} \mathcal{J}_\epsilon \Delta \mathcal{K} * u dx \right| &\lesssim_p \|u_k\|_p^{p-1} \|u\|_p \\
&\leq \|u_k\|_p^p + \|u\|_p^p \\
&\stackrel{(6.8)}{\lesssim} \|u_k\|_p^p + C(k, M). \tag{6.33}
\end{aligned}$$

We now aim to compare the dissipation term in (6.30) with the estimates (6.32) and (6.33). We use the Gagliardo-Nirenberg-Sobolev inequality (Lemma 25),

$$\|u_k\|_{\alpha p} \lesssim \|u_k\|_{\bar{q}}^{\alpha_2} \|u_k\|_{W^{1,2}}^{\alpha_1 (p+m-1)/2} \tag{6.34}$$

with

$$\alpha_1 = \frac{2d}{p} \left( \frac{(p - \bar{q}/\alpha)}{\bar{q}(2-d) + dp + d(m-1)} \right),$$

and

$$\alpha_2 = 1 - \alpha_1(p + m - 1)/2 > 0.$$

By the definition of  $\bar{q}$  and (6.31) we have that,

$$\alpha_1(p + 1)/2 = 1, \tag{6.35}$$

which implies,

$$\|u_k\|_{\alpha p}^{p+1} \lesssim \|u_k\|_{\bar{q}}^{\alpha_2(p+1)} \left( \int u_k^{m-1} \left| \nabla u_k^{p/2} \right|^2 dx + \int u_k^{p+m-1} dx \right). \tag{6.36}$$

If  $d = 2$  then necessarily  $m = m^* = 1$  and this inequality will be sufficient. However, for  $d \geq 3$ , more work must be done. Define,

$$I = \int u_k^{m-1} \left| \nabla u_k^{p/2} \right|^2 dx.$$

Then, for  $\beta_1 \leq \alpha_1$  and  $(p+m-1)\beta_1/2 < 1$ ,

$$\beta_1 = \frac{2d(1 - \bar{q}/(p+m-1))}{\bar{q}(2-d) + dp + d(m-1)},$$

and  $\beta_2 = 1 - \beta_1(p+m-1)/2 > 0$ , we have the following by Lemma 25,

$$\begin{aligned} \int u_k^{p+m-1} dx &\lesssim \|u_k\|_{\bar{q}}^{(p+m-1)\beta_2} \left( I + \int u_k^{p+m-1} dx \right)^{(p+m-1)\beta_1/2} \\ &\lesssim \|u_k\|_{\bar{q}}^{(p+m-1)\beta_2} \left( I^{(p+m-1)\beta_1/2} + \left( \int u_k^{p+m-1} dx \right)^{(p+m-1)\beta_1/2} \right). \end{aligned}$$

Therefore, by weighted Young's inequality for products,

$$\int u_k^{p+m-1} dx \lesssim \|u_k\|_{\bar{q}}^{(p+m-1)\beta_2} (1 + I) + \|u_k\|_{\bar{q}}^{\gamma_0}, \quad (6.37)$$

for some  $\gamma_0 > 0$ , the exact value of which is not relevant. Putting (6.36) and (6.37) together implies,

$$\|u_k\|_{\alpha p}^{p+1} \lesssim \mathcal{P}(\|u_k\|_{\bar{q}})I + C(\|u_k\|_{\bar{q}}), \quad (6.38)$$

where  $\mathcal{P}(z)$  denotes a polynomial such that  $\mathcal{P}(z) \rightarrow 0$  as  $z \rightarrow 0$ . By definition of  $m$ ,  $\exists \delta > 0$  such that for  $k$  sufficiently large then  $u > k$  implies  $A'(u) > \delta u^{m-1}$ . Therefore, combining (6.30) with (6.38), (6.32) and (6.33) implies,

$$\begin{aligned} \frac{d}{dt} \|u_k\|_p^p &\leq -C(p)\delta \int u_k^{m-1} \left| \nabla u_k^{p/2} \right|^2 dx + C(p) \|u_k\|_{\alpha p}^{p+1} \\ &\quad + C(M, p) \|u_k\|_p^p + C(k, M, p) \\ &\leq -\frac{C(p)\delta}{\mathcal{P}(\|u_k\|_{\bar{q}})} \|u_k\|_{\alpha p}^{p+1} + C(p) \|u_k\|_{\alpha p}^{p+1} \\ &\quad + C(M, p) \|u_k\|_p^p + C(k, M, p, \|u_k\|_{\bar{q}}). \end{aligned}$$

By interpolation against  $L^1$ , conservation of mass and  $\alpha \geq 1$  we have

$$\|u_k\|_p^p \lesssim_M 1 + \|u_k\|_{p\alpha}^{p+1}.$$

Therefore, by assumption (6.29) we may choose  $k$  sufficiently large such that there exists some  $\eta > 0$  which satisfies the following for all  $t \in (0, T_0)$ ,

$$\frac{d}{dt} \|u_k\|_p^p \leq -\eta \|u_k\|_p^p + C(k, M, p, \|u_k\|_{\bar{q}}).$$

It follows that  $\|u_k\|_p$  is bounded uniformly on  $(0, T_0)$ .

*Step 2:*

The control of these  $L^p$  norms will enable us to invoke Lemma 7 and conclude  $u^\epsilon(t)$  is bounded uniformly in  $L^\infty(D_{T_0})$ . Since  $\nabla \mathcal{K} \in L_{loc}^1$  and  $\nabla \mathcal{K} \mathbf{1}_{\mathbb{R}^d \setminus B_1(0)} \in L^q$  for all  $q > d/(d-1)$  (by Lemma 1), we have for any  $q > d/(d-1)$

$$\|\vec{v}\|_q = \|\nabla \mathcal{K} * u\|_q \leq \|\nabla \mathcal{K} \mathbf{1}_{B_1(0)}\|_1 \|u\|_q + \|\nabla \mathcal{K} \mathbf{1}_{\mathbb{R}^d \setminus B_1(0)}\|_q M.$$

If  $\gamma > 1$ , then we may choose  $q \in (d/(d-1), \gamma/(\gamma-1)]$ , since in this case necessarily  $d \geq 3$ . Otherwise we may choose  $q > d/(d-1)$  arbitrarily. Then, step one implies  $\vec{v} \in L^\infty((0, T_0); L^q)$ . If  $\gamma > 1$  then, noting that Definition 1 implies  $D^2 \mathcal{K} \mathbf{1}_{\mathbb{R}^d \setminus B_1(0)} \in L^q$  for all  $q > 1$ ,

$$\|\nabla \vec{v}\|_{d+1} = \|D^2 \mathcal{K} * u\|_{d+1} \leq \|D^2 \mathcal{K} \mathbf{1}_{B_1(0)}\|_{L^{\gamma, \infty}} \|u\|_p + \|\nabla \mathcal{K} \mathbf{1}_{\mathbb{R}^d \setminus B_1(0)}\|_{d+1} M,$$

for  $p = \gamma(d+1)/(d(\gamma-1) + 2\gamma - 1)$ . Note that

$$1 < p = \frac{\gamma(d+1)}{d(\gamma-1) + 2\gamma - 1} \leq \frac{\gamma}{\gamma-1}.$$

On the other hand, if  $m^* = 2 - 2/d$  then the above proof shows that  $u^\epsilon(t)$  is bounded uniformly in  $L^\infty((0, T_0); L^p)$  for all  $p < \infty$ . Therefore, by Lemma 2 we



have  $\|\nabla \vec{v}\|_p \lesssim \|u\|_p \lesssim 1$ , for all  $1 < p < \infty$ . In either case, this is sufficient to apply Morrey's inequality and conclude that  $\|\vec{v}\|_\infty$  is uniformly bounded on  $(0, T_0)$ . By Lemma 7 we then have that  $u^\epsilon$  is uniformly bounded in  $L^\infty(D_{T_0})$  and we have proved the lemma. As in Lemma 9, the uniform bounds depend on the domain but not its diameter.  $\square$

*Remark 8.* The proof of this lemma directly implies global well-posedness in the subcritical case since (6.29) is only necessary in the critical and supercritical cases. Moreover, in the critical case, one may prove directly that there exists some  $M_0$  such that if  $M < M_0$  the solution is global. However,  $M_0$  will generally depend on the constant of the Gagliardo-Nirenberg-Sobolev inequality, as in [192, 193, 107]. As discussed in the recent works of [36, 39], the use of a continuation theorem will allow for a more accurate estimate of the critical mass through the use of the free energy.

## CHAPTER 7

### Global Existence and Finite Time Blow Up

For the case  $m^* = 2 - 2/d$ , Blanchet et al. [36] identified the critical mass for the problem with the Newtonian potential,  $\mathcal{K} = c_d |x|^{d-2}$ , and  $A(u) = u^m$ . The authors show that if  $M < M_c$  then the solution exists globally and if  $M > M_c$  then the solution may blow up in finite time. There  $M_c$  is identified as

$$M_c = \left( \frac{2}{(m^* - 1)C_{m^*}c_d} \right)^{1/(2-m^*)},$$

where  $C_{m^*}$  is the best constant in the Hardy-Littlewood-Sobolev inequality given below in Lemma 4. It is natural to ask the same question for more general cases. In this work we generalize these results to include inhomogeneous kernels and general nonlinear diffusion. First, we state the generalization of the finite time blow up results.

**Theorem 6** (Finite Time Blow Up for Critical Problems:  $m^* > 1$ ). *Let  $D$  either be bounded and convex with a smooth boundary or  $D = \mathbb{R}^d$ . Let  $\mathcal{K}$  and  $A(u)$  be admissible and satisfy*

**(B1)**  $\mathcal{K}(x) = c|x|^{-d/p} + o(|x|^{-d/p})$  as  $x \rightarrow 0$  for some  $c > 0$  and  $d/(d-2) \leq p < \infty$ .

**(B2)**  $x \cdot \nabla \mathcal{K}(x) \leq -(d/p)\mathcal{K}(x) + C_1$  for all  $x \in \mathbb{R}^d$ , for some  $C_1 \geq 0$ .

**(B3)**  $A'(z) = m\bar{A}z^{m-1} + o(z^{m-1})$  as  $z \rightarrow \infty$  for some  $m > 1, \bar{A} > 0$ .

**(B4)**  $A(z) \leq (m - 1)\Phi(z)$  for all  $z > R$ , for some  $R > 0$ .

Assume further that **(D4)** holds. Suppose the problem is critical, that is  $m = m^*$ . Then the critical mass  $M_c$  satisfies

$$M_c = \left( \frac{2\bar{A}}{(m^* - 1)C_{m^*c}} \right)^{1/(2-m^*)},$$

and for all  $M > M_c$  there exists a solution to (5.1) which blows up in finite time with  $\|u_0\|_1 = M$ .

**Theorem 7** (Finite Time Blow Up for Supercritical Problems). *Let  $D$  be as in Theorem 6. Let  $\mathcal{K}$  satisfy **(B1)** and **(B2)** in Theorem 6 and  $A(u)$  satisfy **(B3)** and **(B4)** in Theorem 6 with  $1 < m < m^*$  as well as **(D4)**. Then for all  $M > 0$  there exists a solution which blows up in finite time with  $\|u_0\|_1 = M$ .*

The Newtonian and Bessel potentials both satisfy these conditions with  $C_1 = 0$  (Lemma 2.2, [191]), and so the results apply to PKS with degenerate diffusion. Due to the decay of admissible kernels (Definition 1) condition **(B2)** should only impose a significant restriction on the behavior of  $\mathcal{K}$  at the origin. Power-law diffusion satisfies conditions **(B3)** and **(B4)**; however, **(B4)** is also restrictive, for example,  $A(u) = u^m - u$  for  $u$  large does not satisfy the condition. The accompanying global existence theorem is significantly more inclusive than the blow up theorems, both in the kinds of kernels and nonlinear diffusion considered. As in Theorem 6, the estimate of the critical mass only depends on the leading order term of an asymptotic expansion of the kernel at the origin and the growth of the entropy at infinity. The approach used here and in [36, 39] relies on using the energy dissipation inequality (5.7) and the continuation theorem (Theorem 5). The third key component is an inequality which relates the interaction energy  $\mathcal{W}(u)$  to the entropy  $S(u)$ . For  $m^* > 1$  this is the Hardy-Littlewood-Sobolev

inequality given in Lemma 4. In this case, the estimate of the critical mass is given by (7.1).

**Theorem 8** (Global Well-Posedness for  $m^* > 1$ ). *Suppose  $m^* > 1$ . Then we have the following:*

- (i) *If the problem is subcritical, then the solution exists globally (i.e.  $T_* = \infty$ ) and is uniformly bounded in the sense  $u \in L^\infty((0, \infty) \times D)$ .*
- (ii) *If the problem is critical then there exists a critical mass  $M_c > 0$  such that if  $\|u_0\|_1 = M < M_c$ , then the solution exists globally. If additionally **(D4)** holds or  $D$  is bounded, then  $u(t)$  is uniformly bounded in the sense  $u \in L^\infty((0, \infty) \times D)$ . The critical mass is estimated below in (7.1).*

**Proposition 4** (Critical Mass For  $m^* > 1$ ). *If  $\mathcal{K} = c|x|^{-d/p} + o(|x|^{-d/p})$  as  $x \rightarrow 0$  for some  $c \geq 0$  and  $p$ ,  $d/(d-2) \leq p < \infty$ , then  $M_c$  satisfies,*

$$\lim_{z \rightarrow \infty} \frac{\Phi(z)}{z^{m^*}} - \frac{C_{m^*}}{2} c M_c^{2-m^*} = 0. \quad (7.1)$$

*If  $c = 0$  or  $\lim_{z \rightarrow \infty} \Phi(z)z^{-m^*} = \infty$  then we define  $M_c = \infty$ .*

*Remark 9.* By Lemma 27, if  $\mathcal{K} \in L_{loc}^{p,\infty}$  then  $\exists \delta, C > 0$  such that  $\forall x, |x| < \delta$ ,  $\mathcal{K}(x) \leq C|x|^{-d/p}$ . Then, if the kernel does not admit an asymptotic expansion as in Proposition 4, the critical mass  $M_c$  can be estimated by,

$$\lim_{z \rightarrow \infty} \frac{\Phi(z)}{z^{m^*}} - \frac{C_{m^*}}{2} C M_c^{2-m^*} = 0.$$

*Remark 10.* Note,  $\lim_{z \rightarrow \infty} \Phi(z)z^{-m^*}$  is always well-defined but is not necessarily finite unless

$$\limsup_{z \rightarrow \infty} A'(z)z^{1-m^*} < \infty.$$

If the problem is critical then necessarily  $\lim_{z \rightarrow \infty} \Phi(z)z^{-m^*} > 0$  so there always exists a positive mass which satisfies (7.1). Moreover, if the problem is subcritical then necessarily  $\lim_{z \rightarrow \infty} \Phi(z)z^{-m^*} = \infty$ .

The case  $m^\star = 1$  is analogous to the classical PKS problem in 2D, where linear diffusion is critical. For the 2D PKS, the critical mass is given by  $M_c = 8\pi$  for both the Newtonian and Bessel potentials [39, 49]. In this work we treat the  $m^\star = 1$  case for  $d \geq 2$ , recovering the critical mass of the classical PKS. The case  $d \geq 3$  and  $m^\star = 1$  is approached in [113], but the optimal critical mass is not identified. Our estimate is given below in (7.2). As above, the critical mass only depends on the asymptotic expansion of the kernel at the origin and the growth of the entropy at infinity. We first state the analogue of Theorem 6.

**Theorem 9** (Finite Time Blow Up for Critical Problems  $m^\star = 1$ ). *Let  $D$  be a smooth, bounded and convex domain or  $\mathbb{R}^d$  with  $d \geq 2$ . Suppose  $\mathcal{K}$  satisfies*

**(C1)**  $\mathcal{K}(x) = -c \ln |x| + o(\ln |x|)$  as  $x \rightarrow 0$  for some  $c > 0$ .

**(C2)**  $x \cdot \nabla \mathcal{K}(x) \leq -c + C|x|$  for all  $x \in \mathbb{R}^d$ , for some  $C \geq 0$ .

**(C3)**  $A(z) \leq \bar{A}z$  for some  $\bar{A} > 0$ .

*Then the critical mass  $M_c$  satisfies*

$$M_c = \frac{2d\bar{A}}{c},$$

*and for all  $M > M_c$  there exists a solution which blows up in finite time with  $\|u_0\|_1 = M$ .*

The corresponding global existence theorem includes more general kernels and nonlinear diffusion. The proof is similar to Theorem 8, except that the logarithmic Hardy-Littlewood-Sobolev inequality (Lemma 5) is used in place of the Hardy-Littlewood-Sobolev inequality.

**Theorem 10** (Global Well-Posedness for  $m^\star = 1$ ). *Suppose  $m^\star = 1$  and  $d \geq 2$ , let  $D$  be bounded, smooth and convex or  $D = \mathbb{R}^d$  and  $|x|^2 u_0 \in L^1(\mathbb{R}^d)$ . Then we have the following:*

(i) If the problem is subcritical, then the solution exists globally and  $u(t)$  is uniformly bounded in the sense  $u \in L^\infty((0, \infty) \times D)$ .

(ii) If the problem is critical then there exists a critical mass,  $M_c > 0$ , such that if  $\|u_0\|_1 = M < M_c$ , then the solution exists globally. If  $D$  is bounded, then  $u(t)$  is uniformly bounded in the sense  $u \in L^\infty((0, \infty) \times D)$ . The critical mass is estimated below in (7.2).

**Proposition 5** (Critical Mass for  $m^* = 1$ ). If  $\mathcal{K}(x) = -c \ln |x| + o(\ln |x|)$  as  $x \rightarrow 0$  for some  $c \geq 0$ , then  $M_c$  satisfies,

$$\lim_{z \rightarrow \infty} \frac{\Phi(z)}{z \ln z} - \frac{c}{2d} M_c = 0. \quad (7.2)$$

If  $c = 0$  or  $\lim_{z \rightarrow \infty} \Phi(z)(z \ln z)^{-1} = \infty$  then we define  $M_c = \infty$ .

*Remark 11.* By **(BD)** and **(MN)**,  $\exists \delta, C > 0$  such that  $\forall x, |x| < \delta$ ,  $\mathcal{K}(x) \leq -C \ln x$ . Therefore, if the kernel does not have the asymptotic expansion required in Proposition 5 then the critical mass  $M_c$  may be estimated as,

$$\lim_{z \rightarrow \infty} \frac{\Phi(z)}{z \ln z} - \frac{C}{2d} M_c = 0.$$

*Remark 12.* These theorems include many known global existence and finite time blow up results in the literature including [192, 191, 192, 29, 36, 125, 47]. Our main contributions to the existing theory is the unification of these results and the estimate of the critical mass for inhomogeneous kernels and general nonlinear diffusion. In the case of the Newtonian potential Blanchet et al. showed in [36] that solutions at the critical mass also exist globally. See [76, 32, 37] for the corresponding result for classical 2D PKS.

## 7.1 Control on the Entropy from Below

In the case of degenerate diffusion, **(D4)**, we may uniformly bound the entropy from below.

**Lemma 13.** *Let  $u(x, t)$  be a weak solution to (5.1). Then,*

$$\int \Phi(u(t))dx \gtrsim -M.$$

*Proof.* Let  $h(z) = \int_1^z A'(s)s^{-1}ds$ . By Definition 2, **(D4)**, for  $z \leq 1$ ,

$$h(z) \geq -C > -\infty.$$

Therefore,

$$\begin{aligned} \int \Phi(u)dx &= \int \int_0^u h(z)dzdx \geq \int \mathbf{1}_{\{u \leq 1\}} \int_0^u h(z)dz + \mathbf{1}_{\{u \geq 1\}} \int_0^1 h(z)dzdx \\ &\geq - \int \mathbf{1}_{\{u \leq 1\}} Cu - \mathbf{1}_{\{u \geq 1\}} Cdx \\ &\geq -2C\|u\|_1. \end{aligned}$$

where the last line followed from Chebyshev's inequality.  $\square$

If only **(D3)** holds, only weaker control is available since the nonlinear entropy  $\int \Phi(u)dx$  and the linear entropy  $\int u \log u dx$  are no longer uniformly bounded from below. However, the following standard lemma ensures that the decay is bounded from below by controllable quantities.

**Lemma 14** (Entropy Lower Bound). *Let  $A$  be admissible,  $u \in L_+^1(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d)$  and  $\mathcal{M}_2 = \int |x|^2 u(x)dx$ . Then,*

$$\int u \log u dx \geq -C(M) - \mathcal{M}_2, \tag{7.3}$$

and

$$\int \Phi(u)dx \geq -C(M) - C_A \mathcal{M}_2. \tag{7.4}$$

*Proof.* Following [39], by Jensen's inequality for probability measures,

$$\begin{aligned} \int u(x) \log u(x) dx + \int |x|^2 u dx &= \int u \log \left( \frac{u(x)}{e^{-|x|^2}} \right) dx \\ &= \int \frac{\sqrt{\pi}}{\sqrt{\pi}} \frac{u(x)}{e^{-|x|^2}} \log \left( \frac{u(x)}{e^{-|x|^2}} \right) e^{-|x|^2} dx. \\ &\geq \sqrt{\pi} X \log X = -C(M) \end{aligned}$$

with

$$X = \int u e^{|x|^2} \frac{e^{-|x|^2}}{\sqrt{\pi}} dx = \frac{M}{\sqrt{\pi}}.$$

Therefore,

$$\int u \log u dx \geq -C(M) - \mathcal{M}_2.$$

By **(D3)**, for some  $\delta > 0$ ,  $A'(z) \leq C_A z$  for  $z < \delta$ . Let  $h(z) = \int_1^z A'(s) s^{-1} ds$  and note that  $\int \Phi(u) dx = \int_{\mathbb{R}^d} \int_0^u h(z) dz dx$ . For  $z < 1$  we have,

$$\begin{aligned} h(z) &= - \int_z^1 \frac{A'(s)}{s} ds \\ &\geq - \int_\delta^1 \frac{A'(s)}{s} ds - C_A \int_{\min(z, \delta)}^\delta \frac{1}{s} ds \\ &\geq -C + C_A \log z. \end{aligned}$$

Therefore, since  $\log z$  is integrable at zero, we have the following by Chebyshev's inequality,

$$\begin{aligned} \int \Phi(u) dx &= \int \int_0^u h(z) dz dx \\ &\geq \int \mathbf{1}_{\{u < 1\}} \int_0^u h(z) dz + \int \mathbf{1}_{\{u > 1\}} \int_0^1 h(z) dz dx \\ &\geq \int \mathbf{1}_{\{u < 1\}} (C_A(u \log u - u) - C u) - C \mathbf{1}_{\{u > 1\}} dx \\ &\geq C_A \int_{\{u < 1\}} u \log u dx - C(M). \end{aligned}$$

Therefore by (7.3),

$$\int \Phi(u) dx \geq -C(M) - C_A \mathcal{M}_\infty.$$

□



## 7.2 Global Existence for $m^* > 1$

*Proof.* (**Theorem 8**) We only prove the second the assertion under the hypotheses of Proposition 4, as the subcritical case follows similarly. By the energy dissipation inequality (5.7) we have for all time  $0 \leq t < T_*$ ,

$$S(u(t)) - \mathcal{W}(u(t)) \leq \mathcal{F}(u_0) := F_0. \quad (7.5)$$

We drop the time dependence of  $u(t)$  for notational simplicity. By the assumption on  $\mathcal{K}$ ,  $\forall \epsilon > 0$ ,  $\exists \delta > 0$  such that  $|\mathcal{K}(x)| \leq (c + \epsilon) |x|^{-d/p}$  for  $|x| < \delta$ . By Lemma 4 we have,

$$\int \Phi(u) dx - \frac{1}{2} C_{m^*} M^{2-m^*} (c + \epsilon) \|u\|_{m^*}^{m^*} \leq F_0 + \frac{1}{2} \|\mathcal{K}|_{B_\delta(0)}\|_\infty M^2,$$

By (7.1) and  $M < M_c$ , there exists  $\epsilon > 0$  small enough and  $\alpha, k > 0$  such that

$$\Phi(z) z^{-m^*} - \frac{1}{2} C_{m^*} M^{2-m^*} (c + \epsilon) \geq \alpha > 0, \text{ for all } z > k. \quad (7.6)$$

By Lemma 13 we have,

$$\begin{aligned} \int_{\{u>k\}} u^{m^*} \left( \Phi(u) u^{m^*} - \frac{1}{2} C_{m^*} M^{2-m^*} (c + \epsilon) \right) dx \\ - \frac{1}{2} \int_{\{u<k\}} C_{m^*} M^{2-m^*} (c + \epsilon) u^{m^*} dx \leq F_0 + C(\delta, M), \end{aligned}$$

and by (7.6),

$$\alpha \int_{\{u>k\}} u^{m^*} dx - \frac{1}{2} C_{m^*} M^{2-m^*} (c + \epsilon) \int_{\{u<k\}} u^{m^*} dx \leq F_0 + C(M, \delta).$$

By mass conservation we have that  $\|u\|_{m^*}$  is a priori bounded independent of time and Theorem 5 and Lemma 12 implies global existence and uniform boundedness.  $\square$

### 7.3 Global Existence for $m^* = 1$

The proof of Theorem 10 follows similarly, but requires the logarithmic Hardy-Littlewood-Sobolev inequality (Lemma 5) as opposed to Lemma 4. We will first prove the result on bounded domains.

*Proof.* ( **Theorem 10**) We will again use Theorem 5 and prove

$$\sup_{t \in (0, \infty)} \int (u \ln u)_+ dx < \infty.$$

#### 7.3.1 Bounded Domains

By the energy dissipation inequality (5.7) we again have (7.5). By the assumptions of Proposition 5, for all  $\epsilon > 0$  there exists  $\delta > 0$  such that,

$$\int \Phi(u) dx + (c + \epsilon) \frac{1}{2} \int \int_{|x-y| < \delta} u(x) u(y) \ln |x-y| dx dy \leq C(F_0, \delta, M).$$

By  $D$  bounded, the logarithmic Hardy-Littlewood-Sobolev inequality (5.10) implies,

$$\int \Phi(u) dx - (c + \epsilon) \frac{M}{2d} \int u \ln u dx \leq C(F_0, \delta, M, \text{diam} D).$$

Choosing  $k > 0$  large and recalling Lemma 13 implies

$$\int_{\{u > k\}} u \ln u \left( \frac{\Phi(u)}{u \ln u} - (c + \epsilon) \frac{M}{2d} \right) dx - (c + \epsilon) \int_{\{u < k\}} u \ln u dx \leq C(F_0, \delta, M, \text{diam} D).$$

As in the proof of Theorem 8, by conservation of mass, (7.2) and  $M < M_c$ , we may choose  $\epsilon > 0$  small enough and  $k$  large enough such that

$$\int_{\{u > k\}} u \ln u dx \leq C(F_0, M, \text{diam} D).$$

### 7.3.2 Unbounded Domains

On  $\mathbb{R}^d$ ,  $\int \int u(x)u(y) \log |x - y| dx dy$  is no longer uniformly controlled for large values of  $x - y$ , which causes a problem when using the logarithmic HLS. We deal with this issue by producing a bound on the second moment.

The following lemma establishes a uniform bound on the second moments for critical problems with  $m^\star = 1$ .

**Lemma 15** (Second Moment Estimate for  $m^\star = 1$ ). *Let  $A(z) \leq Cz$  for some  $C > 0$  and  $\mathcal{K}$  be admissible with  $m^\star = 1$ . Then,*

$$\mathcal{M}_2(t) \leq \mathcal{M}_2(0) + M(C_1 + C_2 M)t$$

for some constants  $C_i > 0$ .

*Proof.* We argue formally, noting that the computations can easily be made rigorous with standard arguments. Computing the time evolution of the second moment,

$$\frac{d}{dt} \mathcal{M}_2(t) = 2d \int A(u) dx + \int \int u(x)u(y)(x - y) \cdot \nabla \mathcal{K}(x - y) dx dy.$$

By assumption  $\int A(u) dx \lesssim M$ . By admissibility and  $m^\star = 1$  we have,

$$|(x - y) \cdot \nabla \mathcal{K}(x - y)| \leq C.$$

Therefore by integration, the lemma follows. □

*Remark 13.* By Definition 6, **(D3)** and  $A(z) \in C^1(\mathbb{R}^+)$ , any critical problem with  $m^\star = 1$  will satisfy

$$A'(z) \leq C,$$

for some  $C > 0$  and therefore  $A(z) \lesssim z$ .

We may now resume the proof of global existence for unbounded domains. The subcritical case is settled easily using a variant of the argument used in the proof of the continuation theorem (see Kowalczyk [125]) so we restrict our attention to assertion (ii).

Again by energy dissipation (Proposition 1),

$$\int \Phi(u)dx - \frac{1}{2} \int u\mathcal{K} * udx \leq \mathcal{F}(u_0) := F_0.$$

By the asymptotic expansion of the kernel assumed in Proposition 4 and **(BD)**, we have that for all  $\epsilon > 0$ ,  $\exists \delta, R > 0$  such that,

$$\begin{aligned} \int \Phi(u)dx + \frac{c + \epsilon}{2} \int \int_{|x-y| < \delta} u(x)u(y) \log |x-y| dx dy \\ \leq F_0 + \frac{1}{2} \int \int_{\delta < |x-y| < R} u(x)u(y)\mathcal{K}(|x-y|)dx dy \\ - C \int \int_{|x-y| > R} u(x)u(y) \log |x-y| dx dy. \end{aligned}$$

Note that for  $R > 0$  sufficiently large,

$$\begin{aligned} \int \int_{|x-y| > R} u(x)u(y) |\log |x-y|| dx dy &\leq \frac{\log R}{R} \int \int_{|x-y| > R} u(x)u(y) |x-y| dx dy \\ &\lesssim M^{3/2} \mathcal{M}_2(t)^{1/2}, \end{aligned}$$

where  $I(t) := \int |x|^2 u(t, x)dx$  is the second moment. Therefore,

$$\begin{aligned} \int \Phi(u)dx + (c + \epsilon) \frac{1}{2} \int \int_{|x-y| < \delta} u(x)u(y) \log |x-y| dx dy \\ \leq F_0 + C(M, \delta, R) + C(M) \mathcal{M}_2(t)^{1/2}. \end{aligned}$$

By the logarithmic HLS inequality (Lemma 5),

$$\begin{aligned} \int \Phi(u)dx - (c + \epsilon) \frac{M}{2d} \int u \log u dx - (c + \epsilon) \frac{1}{2} \int \int_{|x-y| > \delta} u(x)u(y) \log |x-y| dx dy \\ \leq F_0 + C(M, \delta, R) + C(M) \mathcal{M}_2(t)^{1/2}. \end{aligned}$$

Thus, choosing  $k \geq 1$  and arguing as above,

$$\begin{aligned} \int_{\{u>k\}} u \log u \left( \frac{\Phi(u)}{u \log u} - (c + \epsilon) \frac{M}{2d} \right) dx + \int_{u < k} \Phi(u) dx \\ \leq F_0 + C(M, \delta, R) + C(M) \mathcal{M}_2(t)^{1/2}. \end{aligned}$$

By Lemma 14,

$$\int_{\{u>k\}} u \log u \left( \frac{\Phi(u)}{u \log u} - (c + \epsilon) \frac{M}{2d} \right) dx \leq F_0 + C(M, \delta, R) + C(M) \mathcal{M}_2(t).$$

By Lemma 15 (see Remark 13),  $I(t) \lesssim 1 + t$  for all  $t < \infty$ . Since  $M < M_c$  as defined in Proposition (5), it is possible to choose  $k$  large enough and  $\epsilon$  small enough such that  $\int_{\{u>k\}} u \log u dx$  is uniformly bounded on any finite time interval. Arguing as above, this implies the solution exists globally.  $\square$

## 7.4 Finite Time Blow Up

In this section we prove Theorem 7 and Theorem 6. We prove Theorem 7 as it is somewhat easier, though the technique is the same as that used to prove Theorem 6.

### 7.4.1 Supercritical Case: Theorem 7

For Theorem 7 we state the following lemma, which provides insight into the nature of the supercritical cases. The proof and motivation follows [36].

**Lemma 16.** *Define  $\mathcal{Y}_{\mathcal{M}} = \{u \in L^1 \cap L^{m^*} : u \geq 0, \|u\|_1 = M\}$ . Suppose  $\mathcal{K}$  satisfies **(B1)** and  $A(u)$  satisfies **(B3)** for some  $m > 1, \bar{A} > 0$ . Suppose further that the problem is supercritical, that is,  $m < m^*$ . Then  $\inf_{\mathcal{Y}_{\mathcal{M}}} \mathcal{F} = -\infty$ . Moreover, there exists an infimizing sequence with vanishing second moments which converges to the Dirac delta mass in the sense of measures.*

*Proof.* Let  $0 < \theta < 1$ ,  $\alpha = d/p$ . Then by Lemma 4 there exists  $h^*$  such that,

$$\theta C_{m^*} \leq \frac{|\int \int h^*(x)h^*(y) |x-y|^{-\alpha} dx dy|}{\|h^*\|_1^{2-m^*} \|h^*\|_{m^*}^{m^*}} \leq C_{m^*}. \quad (7.7)$$

We may assume without loss of generality that  $h^* \geq 0$ , since replacing  $h^*$  by  $|h^*|$  will only increase the value of the convolution. By density, we may take  $h^* \in C_c^\infty$  and therefore with a finite second moment.

Let  $\mu = \|h^*\|_1^{1/d} M^{-1/d}$ ,  $\lambda > 0$  and  $h_\lambda(x) = \lambda^d h^*(\lambda \mu x)$ . First note, by **(B3)**,  $\forall \epsilon > 0, \exists R > 0$  such that,

$$\begin{aligned} \int \Phi(h_\lambda) dx &= \int \int_0^{h_\lambda} \int_1^s \frac{A'(z)}{z} dz ds dx \\ &\leq \int \int_0^{h_\lambda} \int_R^{\max(s,R)} (m\bar{A} + \epsilon) z^{m-2} dz + \int_1^R \frac{A'(z)}{z} dz ds dx \\ &\leq \frac{\bar{A} + \epsilon}{m-1} \|h_\lambda\|_m^m + C(R) \|h_\lambda\|_1. \end{aligned} \quad (7.8)$$

By **(B1)** and  $h^* \in C_c^\infty$ ,  $\forall \epsilon > 0, \exists \lambda > 0$  sufficiently large such that,

$$-\mathcal{W}(t) \leq -(c - \epsilon) \frac{\mu^{-2d+\alpha} \lambda^\alpha}{2} \int \int h^*(x)h^*(y) |x-y|^{-\alpha} dx dy. \quad (7.9)$$

Combining (7.9),(7.8) with (7.7) and Lemma 4, we have for  $\lambda, R$  sufficiently large,

$$\begin{aligned} \mathcal{F}(h_\lambda) &\leq \frac{\lambda^{dm-d} M}{(m-1) \|h^*\|_1} (\bar{A} + \epsilon) \|h^*\|_m^m \\ &\quad - \lambda^\alpha (\theta - \epsilon) \frac{C_{m^*}}{2} \left( \frac{\|h^*\|_1}{M} \right)^{-2+\alpha/d} \|h^*\|_1^{2-m^*} \|h^*\|_{m^*}^{m^*} + C(R) \mu^{-d} \|h^*\|_1. \end{aligned}$$

By supercriticality, we have  $\alpha = dm^* - d > dm - d$ , and so for  $\epsilon < \theta$ , we take  $\lambda \rightarrow \infty$  to conclude that for all values of the mass  $M > 0$  we have  $\inf_{\mathcal{Y}_M} \mathcal{F} = -\infty$ . Moreover, since  $h^* \in C_c^\infty$ , the second moment of  $h_\lambda$  goes to zero and  $h_\lambda$  converges to the Dirac delta mass in the sense of measures.  $\square$

*Proof.* (**Theorem 7**) We may justify the formal computations for weak solutions using the regularized problems and taking the limit but we do not include such

details. We treat both bounded and unbounded domains together pointing out the differences when they appear. Let

$$\mathcal{M}_2(t) = \int |x|^2 u(x, t) dx.$$

If the domain is bounded then by (5.4),

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_2(t) &= 2d \int A(u) dx \\ &\quad + 2 \int \int u(x) u(y) x \cdot \nabla \mathcal{K}(x - y) dx dy - \int_{\partial D} A(u) x \cdot \nu dS \\ &= 2d \int A(u) dx + \int \int (x - y) \cdot \nabla \mathcal{K}(x - y) u(x) u(y) dx dy \\ &\quad - \int_{\partial D} A(u) x \cdot \nu(x) dS, \end{aligned} \tag{7.10}$$

where the second integral was obtained by symmetrizing in  $x$  and  $y$ , the time dependence was dropped for notational simplicity and  $\nu(x)$  denotes the outward unit normal of  $D$  at  $x \in \partial D$ . By translation invariance and convexity of  $D$ , we may assume that  $x \cdot \nu(x) \geq 0$ . For the rest of the proof we may treat bounded domains and  $D = \mathbb{R}^d$  together, since for each,

$$\frac{d}{dt} \mathcal{M}_2(t) \leq 2d \int A(u) dx + 2 \int \int u(x) u(y) x \cdot \nabla \mathcal{K}(x - y) dx dy.$$

We use **(B2)** on  $\mathcal{K}$ , to obtain

$$\frac{d}{dt} \mathcal{M}_2(t) \leq 2d \int A(u) dx - 2d/p \mathcal{W}(u) + C_1 M^2.$$

By **(D4)**, **(B4)** and Lemma 13,

$$\begin{aligned} \int A(u) dx &= \int_{\{u < R\}} A(u) dx + \int_{\{u > R\}} A(u) dx \\ &\leq C(M) + (m - 1) \int_{\{u > R\}} \Phi(u) dx \\ &\leq C(M) + (m - 1) \int \Phi(u) dx. \end{aligned}$$

Using that  $2d(m-1) < 2d(m^*-1) = 2d/p$  we have,

$$\frac{d}{dt}\mathcal{M}_2(t) \leq 2d(m-1)\mathcal{F}(u) + C(M, C_1).$$

We use the energy dissipation inequality (5.7) to bound the first term,

$$\frac{d}{dt}\mathcal{M}_2(t) \leq 2d(m-1)\mathcal{F}(u_0) + C(M, C_1).$$

From this differential inequality, the second moment will be zero in finite time and the the solution blows up in finite time if

$$\mathcal{F}(u_0) < -\frac{C(M, C_1)}{2d(m-1)}.$$

By Lemma 16, we may always find initial data with any given mass  $M > 0$  such that this is true, since there exists infimizing sequences with vanishing second moments. The final assertion follows from Theorem 5. Indeed, we have

$$T_\star \leq -\frac{I(0)}{2d(m-1)\mathcal{F}(u_0) + C(M, C_1)}.$$

□

#### 7.4.2 Critical Case: Theorems 6 and 9

The proof of Theorem 6 follows the proof of Theorem 7.

**Lemma 17.** *Define  $\mathcal{Y}_M = \{u \in L^1 \cap L^\infty : u \geq 0, \|u\|_1 = M\}$ . Suppose  $\mathcal{K}$  satisfies **(B1)** and  $A(u)$  satisfies **(B3)** for  $m > 1$  and  $\bar{A} > 0$ . Suppose further that the problem is critical, that is,  $m = m^*$  and let  $M_c$  satisfy (7.1). If  $M$  satisfies  $M > M_c$ , then  $\inf_{\mathcal{Y}_M} \mathcal{F} = -\infty$ . Moreover, there exists an infimizing sequence with vanishing second moments which converges to the Dirac delta mass in the sense of measures.*



*Proof.* We may proceed as in the proof of Lemma 16, but instead choose  $\theta \in ((M_c/M)^{2-m^*}, 1)$ . Let  $\alpha = d/p$ . By optimality of  $C_{m^*}$ , as before there exists  $h^*$  such that,

$$\theta C_{m^*} \leq \frac{|\int \int h^*(x)h^*(y) |x-y|^{-\alpha} dx dy|}{\|h^*\|_1^{2-m^*} \|h^*\|_{m^*}^{m^*}} \leq C_{m^*}. \quad (7.11)$$

As above, we assume  $h^* \geq 0$  and  $h^* \in C_c^\infty$ .

Let  $\mu = \|h^*\|_1^{1/d} M^{-1/d}$ ,  $\lambda > 0$  and  $h_\lambda(x) = \lambda^d h^*(\lambda \mu x)$ . By **(B1)** and **(B3)**,  $\forall \epsilon > 0$  there exists a  $\lambda$  and  $R$  sufficiently large such that by  $h^* \in C_c^\infty$ ,

$$\begin{aligned} \mathcal{F}(h_\lambda) &\leq \frac{\lambda^{dm-d} M}{(m^* - 1) \|h^*\|_1} (\bar{A} + \epsilon) \|h^*\|_{m^*}^{m^*} + C(R) \mu^{-d} \|h^*\|_1 \\ &\quad - \frac{(\theta - \epsilon) C_{m^*}}{2} \left( \frac{\|h^*\|_1}{M} \right)^{-2+\alpha/d} \lambda^\alpha \|h^*\|_1^{2-m^*} \|h^*\|_{m^*}^{m^*} \end{aligned}$$

However in this case  $\alpha = dm - d$  and  $m = m^*$ , therefore by (7.11) and Lemma 4,

$$\mathcal{F}(h_\lambda) \leq \lambda^{dm^*-d} \|h^*\|_{m^*}^{m^*} \left[ \frac{M(\bar{A} + \epsilon)}{(m^* - 1) \|h^*\|_1} - \frac{(\theta - \epsilon) C_{m^*}}{2} \left( \frac{\|h^*\|_1}{M} \right)^{-2+\alpha/d} \|h^*\|_1^{2-m^*} \right].$$

Then,

$$\mathcal{F}(h_\lambda) \leq \lambda^{dm^*-d} \frac{\|h^*\|_{m^*}^{m^*}}{\|h^*\|_1} \left[ \frac{M(\bar{A} + \epsilon)}{(m^* - 1)} - \frac{(\theta - \epsilon)}{2} C_{m^*} M^{2-\alpha/d} \right].$$

Then since  $\bar{A}/(m^* - 1) = C_{m^*} M_c^{2-m^*}/2$  and  $\alpha/d - 1 = 2 - m^*$  we have,

$$\mathcal{F}(h_\lambda) \leq \lambda^{dm^*-d} \frac{\|h^*\|_m^m}{2 \|h^*\|_1} C_{m^*} M^{2-\alpha/d} \left[ \left( 1 + \frac{\epsilon}{\bar{A}} \right) \left( \frac{M_c}{M} \right)^{2-m^*} - (\theta - \epsilon) \right].$$

Since  $\theta > (M_c/M)^{2-m^*}$  we may take  $\epsilon$  sufficiently small and  $\lambda \rightarrow \infty$  to conclude that  $\inf_{\mathcal{Y}_M} \mathcal{F} = -\infty$ . As before,  $h_\lambda$  converges to the Dirac delta mass in the sense of measures.  $\square$

*Proof.* (Theorem 6) The theorem follows from a Virial identity as in Theorem 7.  $\square$

*Proof.* (Theorem 9) As in Theorem 7 we have by **(C2)**, **(C3)** and if  $D$  is bounded, the convexity of the domain,

$$\begin{aligned}\frac{d}{dt}\mathcal{M}_2(t) &\leq 2d\bar{A} \int A(u)dx + \int \int u(x)u(y)(x-y) \cdot \nabla \mathcal{K}(x-y)dx dy \\ &\leq 2dM \left( \bar{A} - \frac{cM}{2d} \right) + C_1 M^{3/2} \mathcal{M}_2(t)^{1/2}.\end{aligned}$$

Clearly, if  $M > M_c$  then  $\mathcal{M}_2 \rightarrow 0$  in finite time if  $\mathcal{M}_2(0)$  is sufficiently small.  $\square$

## CHAPTER 8

### Intermediate Asymptotics

In the previous chapter, the regime of most interest was the highly concentrated regime near blow up, and the goal was to determine whether or not the diffusion can overpower and halt finite time aggregation. In this chapter, our goal is to consider a different regime, namely the regime in which the solution is dissipating. We mostly follow the results of the preprint [17] by the author.

We will restrict to the special case

$$\begin{cases} u_t + \nabla \cdot (u \nabla \mathcal{K} * u) = \Delta u^m, & m \geq 1, \\ u(0, x) = u_0(x) \in L^1_+(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d), \end{cases} \quad (8.1)$$

In particular, we are interested in determining when solutions to (8.1) converge in  $L^1(\mathbb{R}^d)$  as  $t \rightarrow \infty$  to the self-similar spreading solutions of the diffusion equation

$$u_t = \Delta u^m. \quad (8.2)$$

All dissipating solutions are weak<sup>\*</sup> converging to zero as  $t \rightarrow \infty$ , but this kind of result implies that for  $1 \ll t < \infty$ , the dissipating solutions all look more or less like self-similar solutions of (8.2). For this reason, these results are often referred to as *intermediate asymptotics*.

We point out here that in this chapter, the case  $m = 2 - 2/d$  is referred to as critical and the case  $m < 2 - 2/d$  is referred to as supercritical. This is because the focus of this chapter is the long-term spreading behavior of solutions, and

achieving such a precise balance based on the singularity of the kernel (which is relatively unimportant for this analysis) is not necessary.

As strong nonlinearities vanish quickly near zero, scaling heuristics suggest that the nonlocal aggregation term should become irrelevant for small data in the critical and supercritical regime. This might seem slightly counter-intuitive at first, however the classical example from ODE,  $\partial_t u = -u + u^2$ , can immediately clarify what is going on. In fact, it will turn out that in many cases the aggregation wins at large length scales in subcritical cases, producing a confining effect which leads to the existence of stationary solutions (for example [149, 150]). We remark that analogous behavior is seen in the study of semilinear dispersive PDE with focusing nonlinearities [197], which in fact share many common features with (5.1) despite the many obvious differences. We will discuss the similarities and differences in more detail below in Chapter 9.

In this chapter, we use entropy dissipation methods [58, 199, 59, 54, 53, 30] to obtain several intermediate asymptotics results determining when solutions of (5.1) converge to self-similar solutions of (8.2). Entropy dissipation methods are well-suited for proving the convergence to equilibrium states of nonlinear Fokker-Plank-type equations for arbitrary data [58, 54]. Through a change of variables employed below, this also provides convergence to self-similarity of nonlinear homogeneous diffusion equations [59]. In contrast to these works, we employ such methods to prove a *small data* result, treating the nonlocal aggregation term as a perturbation. For this to work, sufficiently strong decay estimates on the solution must be obtained. Indeed, strong decay estimates imply the intermediate asymptotics results, and so we have chosen to state them separately in Theorem 11 below. Here, we obtain these estimates using iteration methods such those employed in Chapter 6. While nonlinear, these iteration methods are

essentially perturbative in nature and thus somewhat limited against arbitrary data. Analogous to related models, such as the nonlinear Schrödinger equations, it is likely a fully non-perturbative theory will need to be applied in order to treat large data, which is sometimes significantly more difficult (for example, see discussions in [197, 121]).

The first result, Theorem 12, covers the case  $\mathcal{K} \in W^{1,1}(\mathbb{R}^d)$ . Here, the nonlocal term can be considered to have a finite characteristic length-scale which becomes vanishingly small relative to the length-scale of the solution as it dissipates. A result similar to Theorem 12 for  $L^p$ ,  $1 < p < \infty$ , was proved for the special case of the Bessel potential in [155, 156] with the (soft) compactness method of [112] (see also [202]). In contrast to methods based on compactness, the entropy dissipation methods obtain quantitative convergence rates in  $L^1$ , which by interpolation against the decay estimates, provides quantitative estimates in all  $L^p$ ,  $1 \leq p < \infty$ . For supercritical problems, the convergence rate is shown to be the same as the optimal rates for (8.2) [58, 199, 59, 54, 202].

In general, if the kernel does not have critical scaling at large length-scales, the long-range effects should still become irrelevant as the solution dissipates. That is, we should expect results similar to the  $\mathcal{K} \in W^{1,1}(\mathbb{R}^d)$  case to hold, except when  $m = 2 - 2/d$  and  $\nabla \mathcal{K} \sim |x|^{1-d}$  as  $|x| \rightarrow \infty$ . Indeed, when  $\mathcal{K}$  is the Newtonian potential, there exists at least one self-similar spreading solution to (5.1) when  $m = 2 - 2/d$  [39, 36, 38, 48]. In the presence of linear diffusion, these are additionally known to be the global attractors [39, 38]. Theorem 13 below extends Theorem 12 to the general case of  $\mathcal{K} \notin W^{1,1}(\mathbb{R}^d)$ , where the decay of  $\mathcal{K}$  is characterized by  $\gamma \in [d - 1, d]$  such that  $|\nabla \mathcal{K}(x)| = \mathcal{O}(|x|^{-\gamma})$  as  $|x| \rightarrow \infty$ . We show that if  $\gamma > d - 1$ , then dissipating solutions converge to the self-similar spreading solutions of (8.2). However, in contrast to Theorem 12, the long-

range effects appear to degrade the convergence rate and Theorem 13 provides a quantitative estimate of this effect in terms of  $m$  and  $\gamma$ . It is not known whether the rates obtained in Theorem 13 are sharp. When  $\gamma = d - 1$ , the kernel behaves like the Newtonian potential on large length-scales, and the result is no longer expected to hold if  $m = 2 - 2/d$ . Indeed, we expect solutions to converge to the self-similar solutions of (5.1) constructed in [39, 36]. However, Theorem 13 asserts that in supercritical cases, self-similar solutions to (8.2) again govern the intermediate asymptotics. Thus, Theorem 13 provides intermediate asymptotics for Patlak-Keller-Segel models with linear diffusion in dimensions  $d \geq 3$ .

## 8.1 Main Results

The self-similar solutions to the diffusion equation (8.2) are well-known, see for instance [202] and [59]. In the linear case  $m = 1$ , the self-similar solution is simply the heat kernel,

$$\mathcal{U}(t, x; M) = \frac{M}{(4\pi t)^{d/2}} e^{\frac{-|x|^2}{4t}}. \quad (8.3)$$

In the case of degenerate diffusion  $m > 1$ , the self-similar solution is given by the Barenblatt solution,

$$\mathcal{U}(t, x; M) = t^{-\beta d} \left( C_1 - \frac{(m-1)\beta}{2m} |x|^2 t^{-2\beta} \right)_+^{\frac{1}{m-1}}, \quad (8.4)$$

where  $C_1$  is determined from the conservation of mass and

$$\beta = \frac{1}{d(m-1) + 2}. \quad (8.5)$$

It should be well noted that

$$\|\mathcal{U}(t; M)\|_p \lesssim t^{-d\beta(1-\frac{1}{p})},$$

and so to provide a real characterization of the convergence to self-similarity, quantitative estimates will be stated in terms of relative scale, as the reader shall see below.

The entropy dissipation methods of [58, 199, 59, 54] were used to determine the optimal rate of convergence in  $L^1(\mathbb{R}^d)$  to self-similarity for the homogeneous diffusion equations. That is, any solution  $u(t)$  of (8.2) satisfies

$$t^{d\beta(1-\frac{1}{p})} \|u(t) - \mathcal{U}(t; M)\|_p \lesssim (1+t)^{-\frac{2\beta}{p} \min(\frac{1}{2}, \frac{1}{m})}, \quad \forall p, 1 \leq p < \infty.$$

This rate should be contrasted with the rates obtained in Theorems 12 and 13, where it is shown that kernels with finite length-scales do not have much effect on the rate, but strong nonlocal effects might.

In order to emphasize the relationship between decay estimates and intermediate asymptotics, we state them separately. Results similar to (i) of Theorem 11 have been obtained in a variety of places, for example [176, 192, 38]. Our estimates are obtained in a closely related but different way than existing work. We first rescale into the self-similar variables of the diffusion equation as in [38], and then adapt the Alikakos [3] iteration techniques which were important for the local theory discussed above in Chapter 6. This approach to decay estimates has the advantage of naturally extending the existing methods used to obtain uniform bounds, and for a relatively mild increase in complexity, much stronger results are obtained. Here we use this advantage to also deduce a sufficient condition for decay estimates to hold in the critical case  $2 - 2/d$ , (ii) of Theorem 11 below. For critical problems, uniform equi-integrability in time is equivalent global uniform boundedness for solutions to (5.1) (Theorem 5 above, see also [47, 36]), and due to the similarities in the proof, we may state something analogous for decay estimates. Indeed, (8.7) is simply the requirement that the solution of the rescaled system remain uniformly equi-integrable. The proofs of Theorems 11, 12 and 13

are outlined in more detail in §8.2.1. Remarks on the limitations and possible extensions are made after the statements.

**Theorem 11** (Decay Estimates). *Let  $d \geq 2$ ,  $m \in [1, 2 - 2/d]$  and  $\mathcal{K}$  admissible. Let  $u_0 \in L^1_+(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ .*

(i) *There exists an  $\epsilon_0 > 0$  (independent of  $u_0$ ) such that if  $\|u_0\|_1 + \|u_0\|_{(2-m)d/2} < \epsilon_0$ , then the weak solution  $u(t)$  to (5.1) which satisfies  $u(0) = u_0$  is global and satisfies the decay estimate*

$$\|u(t)\|_\infty \lesssim (1+t)^{-d\beta}. \quad (8.6)$$

(ii) *If  $m = 2 - 2/d$  and  $u(t)$  is a global weak solution to (5.1) which satisfies*

$$\lim_{k \rightarrow \infty} \sup_{t \in [0, \infty)} \int \left( u(t, x) - k \left( \frac{t}{\beta} + 1 \right)^{d\beta} \right)_+ dx = 0, \quad (8.7)$$

*then  $u(t)$  satisfies the decay estimate (8.8).*

*Remark 14.* The decay estimate  $\|u(t)\|_\infty \lesssim (1+t)^{-d\beta}$  implies (8.7) but they are not a priori equivalent. Indeed, condition 8.7 is the requirement that  $\theta(\tau)$  (defined below in (8.22)) satisfy the equi-integrability condition (6.1) as  $\tau \rightarrow \infty$  however the decay estimate requires  $\theta(\tau)$  to be uniformly bounded in  $L^\infty$ .

Once the decay estimate (8.8) has been established, entropy-entropy dissipation methods can be adapted to deduce the following intermediate asymptotics theorems, as the decay estimate provides sufficient control of the nonlocal terms.

**Theorem 12** (Intermediate Asymptotics I: Finite Length-Scale). *Let  $d \geq 2$ ,  $m \in [1, 2 - 2/d]$  and  $\mathcal{K} \in W^{1,1}$  be admissible. Let  $f \in L^1_+(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ . Then there exists an  $\epsilon_0(\|f\|_1, \|f\|_{(2-m)d/2}) > 0$  such that for all  $\epsilon < \epsilon_0$ , if  $u_0 = \epsilon f$  then the weak solution  $u(t)$  to (8.1) is global and satisfies*

$$\|u(t)\|_\infty \lesssim (1+t)^{-d\beta}. \quad (8.8)$$



Moreover, if  $m < 2 - 2/d$ , then  $u(t)$  satisfies

$$t^{d\beta(1-\frac{1}{p})} \|u(t) - \mathcal{U}(t; M)\|_p \lesssim (1+t)^{-\frac{\beta}{p}}, \quad \forall p, 1 \leq p < \infty, \quad (8.9)$$

and if  $m = 2 - 2/d$ , then for all  $\delta > 0$ ,  $u(t)$  satisfies

$$t^{d\beta(1-\frac{1}{p})} \|u(t) - \mathcal{U}(t; M)\|_p \lesssim_\delta (1+t)^{-\frac{\beta}{p}(1-\delta)}, \quad \forall p, 1 \leq p < \infty. \quad (8.10)$$

Here  $\beta$  is defined in (8.5) and  $\mathcal{U}(x, t; M)$  is the self-similar solution to (8.2) with mass  $M = \epsilon \|f\|_1$  given in (8.3) or (8.4).

**Theorem 13** (Intermediate Asymptotics II: Infinite Length-Scales). *Let  $d \geq 2$  and  $\mathcal{K}$  be admissible with  $\nabla \mathcal{K}(x) = \mathcal{O}(|x|^{-\gamma})$  as  $|x| \rightarrow \infty$  for some  $\gamma \in [d-1, d]$ . If  $\gamma = d-1$  then suppose  $m \in [1, 2 - 2/d)$  and otherwise we may take  $m \in [1, 2 - 2/d]$ . Let  $f \in L^1_+(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ . Then there exists an  $\epsilon_0(\|f\|_1, \|f\|_{(2-m)d/2}) > 0$  such that for all  $\epsilon < \epsilon_0$ , if  $u_0 = \epsilon f$  then the weak solution  $u(t)$  to (8.1) is global and satisfies*

$$\|u(t)\|_\infty \lesssim (1+t)^{-d\beta}. \quad (8.11)$$

Moreover, for all  $\delta > 0$ ,  $u(t)$  satisfies

$$t^{d\beta(1-\frac{1}{p})} \|u(t) - \mathcal{U}(t; M)\|_p \lesssim_\delta (1+t)^{-\frac{\beta}{p} \min(1, 1+\gamma-\beta^{-1}-\delta)}, \quad \forall p, 1 \leq p < \infty. \quad (8.12)$$

Here  $\beta$  and  $\mathcal{U}(t, x; M)$  are as above.

*Remark 15.* Note that  $f \in L^1_+(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$  implies  $f \log f \in L^1(\mathbb{R}^d)$  by Jensen's inequality.

*Remark 16.* The convergence rate in (8.9) is optimal, as it matches that of the corresponding diffusion equation. Optimality is not known for (8.10) or (8.12), however we suspect that these rates are nearly optimal. Note that the convergence rate obtained in (8.12) reduces to (8.9) and (8.10) when  $\gamma = d$ . Moreover, if  $\gamma = d-1$ , then the convergence rate goes to zero as  $m \nearrow 2 - 2/d$ .

*Remark 17.* The results of the previous chapters suggest that if the kernel  $\mathcal{K}$  is less singular than the Newtonian potential at the origin, the  $L^{(2-m)d/2}$  norm could, in some cases, possibly be replaced by a weaker one.

*Remark 18.* We consider only the case of power-law diffusion, however, the estimates (8.8), (8.11) hold for (5.1) provided  $A'(z) \geq cz^{m-1}$  for some  $c > 0$ . Therefore, it is likely possible to apply the methods of [30, 53] to this more general case under some additional structural assumptions.

### 8.1.1 Entropy-Entropy Dissipation Methods

Entropy-entropy dissipation methods have recently become a very powerful framework for examining the global asymptotic behavior of certain dissipative systems. See [54] for a review of these methods in the context of Fokker-Plank equations and degenerate diffusion equations and [170, 55, 56] for the deep relationship with optimal transport. The methods also played a key role in the study of decay to equilibrium of the Boltzmann equations [204, 71].

In our context, we will use the entropy-entropy dissipation framework for the Fokker-Plank equation

$$\partial_\tau \theta = \nabla_\eta \cdot (\eta \theta) + \Delta_\eta \theta^m, \quad m \geq 1. \quad (8.13)$$

In the case  $m = 1$ , the associated entropy is given by,

$$H(\theta) = \int \theta \log \theta d\eta + \frac{1}{2} \int |\eta|^2 \theta d\eta, \quad (8.14)$$

and the *entropy production functional* or *Fisher information* by

$$I(\theta) = \int \theta |\nabla \log \theta + \eta|^2 d\eta. \quad (8.15)$$

In the nonlinear case  $m > 1$ , the corresponding quantities are,

$$H(\theta) = \frac{1}{m-1} \int \theta^m d\eta + \frac{1}{2} \int |\eta|^2 \theta d\eta, \quad (8.16)$$

and the entropy production functional,

$$I(\theta) = \int \theta \left| \frac{m}{m-1} \nabla \theta^{m-1} + \eta \right|^2 d\eta. \quad (8.17)$$

In the nonlinear case, these entropies were originally introduced for studying (8.13) in [166, 179]. For a given mass  $M$ , (8.16) has a unique non-trivial minimizer, which we refer to as the *ground state Barenblatt solution*  $\theta_M$ , since both (8.14) and (8.16) are displacement convex [159]. If we define the *relative entropy*

$$H(\theta|\theta_M) = H(\theta) - H(\theta_M), \quad (8.18)$$

then  $H(\theta|\theta_M) \geq 0$  with equality if and only if  $\theta = \theta_M$ . One can easily verify formally that if  $f(\tau, \eta)$  solves (8.13), then

$$\frac{d}{d\tau} H(f(\tau)) = -I(f(\tau)). \quad (8.19)$$

Since  $I(f) \geq 0$ , this implies  $H$  is a Lyapunov functional for (8.13) and that  $\int_0^\infty I(f(\tau)) d\tau < \infty$  for all solutions  $f$ . Moreover, one can also verify that  $I(f) = 0$  if and only if  $f = \theta_M$  from the Euler-Lagrange equation (see [54]), suggesting that at least along subsequences, all solutions converge to  $\theta_M$ . Combined with a priori bounds implying pre-compactness, this ‘soft’ argument works but cannot provide any information about the convergence rate. The classical work of Bakry and Émery [10] proceeds in the case  $m = 1$  by carrying out a computation to deduce

$$\frac{d}{dt} I(f(\tau)) \gtrsim I(f(\tau)).$$

Integrating this inequality and (8.19) implies  $I(f(0)) \gtrsim \int_0^\infty I(f(s)) ds = H(f(0)|\theta_M)$ .

In fact, this is the Gross logarithmic inequality [91] (see also [177]) and plays the role of an *entropy-entropy dissipation* inequality, as it relates the entropy dissipation to the relative entropy itself. The case  $m > 1$  has since been treated, providing the following Theorem. The explicit constant is important as it determines the rate of convergence; it is related to the convexity properties of the confining potential, in this case  $|x|^2/2$ .

**Theorem 14** (Generalized Gross Logarithmic Sobolev Inequality [59, 54, 177, 91]). *Let  $f \in L_+^1(\mathbb{R}^d)$  with  $\|f\|_1 = M$  and let  $\theta_M$  be the ground state Barenblatt solution with mass  $M$ . Then,*

$$H(f|\theta_M) \leq \frac{1}{2}I(f). \quad (8.20)$$

We see that for the Fokker-Plank equation (8.13), Theorem 14 implies

$$H(\theta(\tau)|\theta_M) \lesssim e^{-2\tau}.$$

A (generalized) Csiszar-Kullback inequality (for the case  $m = 1$ , this is actually an inequality of information theory [67, 127]) relates the relative entropy to the  $L^1$  norm.

**Theorem 15** (Generalized Csiszar-Kullback Inequality [54]). *Let  $f \in L_+^1(\mathbb{R}^d)$  with  $\|f\|_1 = M$  and let  $\theta_M$  be the ground state Barenblatt solution with mass  $M$ . Then,*

$$\|f - \theta_M\|_1 \lesssim H(f|\theta_M)^{\min(\frac{1}{2}, \frac{1}{m})}. \quad (8.21)$$

Note that since we are interested in  $1 \leq m \leq 2 - 2/d$ , we will only apply the inequality with exponent  $1/2$ .

Another viewpoint on entropy-entropy dissipation methods and the classical Bakry-Émery analysis is that of optimal transport, as detailed in [55, 56]. *Geodesic convexity* of a function  $E : M \rightarrow \mathbb{R}$ , where  $M$  is a Riemannian manifold, is the property that for any geodesic on  $M$ , parametrized by  $\phi : [0, 1] \rightarrow M$ , we have

$$E(\phi(t)) \leq (1 - t)E(\phi(0)) + tE(\phi(1)).$$

The notion of displacement convexity introduced by McCann [159] is precisely this notion, applied to the formal Riemannian manifold formed by the space of

absolutely continuous probability measures with finite second moment endowed with the optimal transport geodesics provided by Brenier’s theorem [44, 158]. The fact that PDE such as (8.13) could be interpreted, at least formally, as gradient flows with respect to this Riemannian structure first took form in [170] after the seminal work of Jordan, Kinderlehrer and Otto [111]. Rigorous meaning to this gradient flow concept has been examined in a number of works, for example [7, 56].

The intuition that (8.13) is a gradient flow for  $H$  suggests therefore that the entropy production functional is thus the derivative for  $H(\theta|\theta_M)$  along the geodesic connecting  $\theta$  and  $\theta_M$ . From this observation and some basic formal computations using convexity, one can construct an abstract framework for deducing quantitative estimates of the rate of convergence  $\theta \rightarrow \theta_M$ , developed in [171, 55, 56], referred to as the ‘HWI method’ as it is based on inequalities of the form

$$\begin{aligned} I(\theta) &\gtrsim H(\theta|\theta_M) \\ W_2(\theta, \theta_M) &\lesssim H(\theta|\theta_M)^{1/2}, \end{aligned}$$

where  $W_2$  denotes the Euclidean Wasserstein distance. More precise interpolation inequalities also arise [171, 55, 56]. The first inequality we saw above, as the Gross logarithmic Sobolev inequality. The latter inequality is a generalization of Talagrand’s inequality [195], examined in detail in [171, 55], and in the context of the HWI framework, is similar to the Csiszar-Kullback inequality.

## 8.2 Proof of Results

### 8.2.1 Outline

The proof of Theorems 12 and 13 involves several steps. As mentioned above, we use the entropy dissipation methods of [58, 59, 54] and in particular, the time-dependent rescaling used in [59]. All of the computations will be formal, they can be made rigorous for weak solutions either with a suitable parabolic regularization and passing to the limit, as in Chapter 6 or presumably also lifting to strictly positive solutions, as is common in the study of the porous media equation [202].

Following [59], we define  $\theta(\tau, \eta)$  such that

$$e^{-d\tau}\theta(\tau, \eta) = u(t, x), \quad (8.22)$$

with coordinates  $e^\tau\eta = x$  and  $\beta e^{\beta^{-1}\tau} - \beta = t$ , where  $\beta$  is given by (8.5). In what follows we denote  $\alpha := d\beta$ . In these coordinates, if  $u(t, x)$  solves (5.1) then  $\theta(\tau, \eta)$  solves,

$$\partial_\tau \theta = \nabla \cdot (\eta \theta) + \Delta \theta^m - e^{(1-\alpha-\beta)\beta^{-1}\tau} \nabla \cdot (\theta(e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta)). \quad (8.23)$$

Moreover,  $\mathcal{U}(t, x; M)$  is stationary in these coordinates, and is in fact the ground state Barenblatt profile  $\theta_M(\eta)$  discussed above in §8.1.1. That is (see [59]),

$$\mathcal{U}(t, x; M) = \left(1 + \frac{t}{\beta}\right)^{-d\beta} \theta_M \left( \left(1 + \frac{t}{\beta}\right)^{-\beta} x \right) = e^{-d\tau} \theta_M(\eta). \quad (8.24)$$

Therefore, the asymptotic convergence to self-similar profiles of solutions to (8.2) is equivalent to the convergence to the stationary profiles of (8.13). We will prove this convergence by combining strong decay estimates on (5.1) with the entropy-entropy dissipation methods for (8.13) detailed in §8.1.1.

A primary step to proving Theorems 12 and 13 is establishing that  $\theta(\tau, \eta) \in L_{\tau, \eta}^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ . Note that by the change of variables, this estimate is the decay estimates (8.8) and (8.11). This estimate is what allows us to treat the inhomogeneous non-local term in (8.23) as a vanishing perturbation of (8.13). The decay estimate  $\|u(t)\|_\infty \lesssim t^{-d\beta}$ , or equivalently,  $\|\theta(\tau)\|_\infty \lesssim 1$ , is easily obtained for (8.2) in the linear case and the classical Aronson-Bénilan estimate proves it in the case  $m > 1$  [202]. Clearly, no such analogues are available for (8.23). To prove Theorem 11, we adapt the Alikakos iterations of Chapter 6 (see also [125, 47, 36]) to (8.23) to prove a uniform bound in the rescaled variables. Dealing with time-dependent rescalings in (8.23) introduces several complications. Obtaining  $L^p$  estimates for the critical case  $m = 2 - 2/d$  is relatively straightforward due to the inherent scale-invariance of the relevant inequalities. In the supercritical case  $m < 2 - 2/d$ , the effect of the time-dependent rescaling in (8.23) is crucial for closing a key bootstrap/continuity argument necessary to control the solution uniformly in time. It is at this step that the method below diverges significantly from existing methods and is a key step to obtaining Theorem 11. The additional issue when  $\nabla \mathcal{K} \notin L^1$  is in obtaining  $L^\infty$  estimates, which requires measuring the rate at which  $e^{d\tau} \nabla \mathcal{K}(e^\tau \eta)$  blows up in  $L_{loc}^1$  as  $\tau \rightarrow \infty$ . This also arises later when we estimate how much the nonlocal term affects the entropy dissipation and is the source of the degraded convergence rates found in Theorem 13.

Once we have established  $\theta(\tau, \eta) \in L_{\tau, \eta}^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ , we prove that solutions to (8.23) converge to  $\theta_M$  and estimate the convergence rate in  $L^1$ . In fact, these are done together, as the quantitative estimate is direct and removes the need for compactness arguments.

To prove Theorems 12 and 13, the purpose of proving  $\theta(\tau, \eta) \in L_{\tau, \eta}^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$  is to control the growth of  $\|e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta\|_\infty$ . Ultimately, this provides a bound

essentially of the form,

$$\frac{d}{d\tau}H(\theta(\tau)) \leq -I(\theta(\tau)) + C(M, \|\theta\|_{L_{\tau,\eta}^\infty(\mathbb{R}^+ \times \mathbb{R}^d)})e^{-\gamma\tau},$$

for some  $\gamma > 0$  (in reality, it is not quite as clean). Theorem (14) then implies,

$$\frac{d}{d\tau}H(\theta(\tau)|\theta_M) \leq -2H(\theta(\tau)|\theta_M) + C(M, \|\theta\|_{L_{\tau,\eta}^\infty(\mathbb{R}^+ \times \mathbb{R}^d)})e^{-\gamma\tau}.$$

Integrating this and applying Theorem 15 implies,

$$\|\theta - \theta_M\|_1 \lesssim e^{-\frac{\tau}{2} \min(2,\gamma)},$$

which after rescaling and interpolation against the decay estimates (8.8),(8.11), will prove Theorems 12 and 13.

### 8.3 Preliminary Decay Estimates

Let  $\bar{q} = (2 - m)d/2$  and let  $\eta, \tau$  and  $\theta(\tau, \eta)$  be as defined in §8.2.1. As detailed above, we establish that  $\theta(\tau, \eta) \in L_{\tau,\eta}^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$  using Alikakos iteration [3] (see Chapter 6 and [107, 125, 36, 192, 193, 191]). The first step is to prove the following lemma which allows control over  $L^p$  norms with  $p < \infty$ . In what follows we denote  $\theta_0(\eta) := \theta(\eta, 0) = u(x, 0)$ .

**Lemma 18** (Control for  $L^p$ ,  $p < \infty$  for small data). *For all  $\bar{q} \leq p < \infty$ , there exists  $C_{\bar{q}} = C_{\bar{q}}(p, M)$  and  $C_M = C_M(p, \|\theta_0\|_{\bar{q}})$  such that if  $\|\theta_0\|_{\bar{q}} < C_{\bar{q}}$  and  $M < C_M$ , then  $\|\theta(\tau)\|_p \in L_\tau^\infty(\mathbb{R}^+)$ .*

*Proof.* Define

$$\mathcal{I} = \int \theta^{m-1} |\nabla \theta^{p/2}|^2 dx.$$



We estimate the time evolution of  $\|\theta\|_p$  using integration by parts, Hölder's inequality and Lemma 28 in the appendix,

$$\begin{aligned}
\frac{d}{d\tau}\|\theta\|_p^p &= -\frac{4mp}{(p+1)^2}\mathcal{I} + (p-1)e^{(1-\alpha-\beta)\beta^{-1}\tau} \int \theta^p \nabla \cdot (e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta) d\eta \\
&\quad + d(p-1)\|\theta\|_p^p \\
&\leq -C(p)\mathcal{I} + C(p)e^{(1-\alpha-\beta)\beta^{-1}\tau} \|\theta\|_{p+1}^p \|\nabla(e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta)\|_{p+1} + C(p)\|\theta\|_p^p \\
&\leq -C(p)\mathcal{I} + C(p)e^{(1-\alpha)\beta^{-1}\tau} \|\theta\|_{p+1}^{p+1} + C(p)\|\theta\|_p^p.
\end{aligned} \tag{8.25}$$

We bound the second term using the homogeneous Gagliardo-Nirenberg-Sobolev inequality (Lemma 25 in appendix),

$$\|\theta\|_{p+1}^{p+1} \lesssim \|\theta\|_{\bar{q}}^{\alpha_2(p+1)} \mathcal{I}^{\alpha_1(p+1)/2}, \tag{8.26}$$

where  $\alpha_2 = 1 - \alpha_1(p+m-1)/2$  and

$$\alpha_1 = \frac{2d(\bar{q} - p - 1)}{(p+1)(\bar{q}(d-2) - d(p+m-1))}.$$

By the definition of  $\bar{q}$  we have that,

$$\frac{\alpha_1(p+1)}{2} = \frac{d(\bar{q} - p - 1)}{\bar{q}(d-2) - d(p+m-1)} = 1.$$

We also estimate the second term in (8.25) using Lemma 25,

$$\|\theta\|_p^p \lesssim M^{\beta_2 p} \mathcal{I}^{\beta_1 p/2}, \tag{8.27}$$

where  $\beta_2 = 1 - \beta_1 p/2$  and,

$$\frac{\beta_1 p}{2} = \frac{d(p-1)}{2-d+d(p+m-1)} < 1,$$

by  $1-2/d < m$ . Then applying weighted Young's inequality we have from (8.26), (8.27) and (8.25),

$$\frac{d}{d\tau}\|\theta\|_p^p \leq \left( C_1(p)e^{(1-\alpha)\beta^{-1}\tau} \|\theta\|_{\bar{q}}^{\alpha_2(p+1)} - C_2(p) \right) \mathcal{I} + C_3(p)M^{\gamma(p)}, \tag{8.28}$$

for  $\gamma(p) = 2\beta_2 p / (2 - \beta_1 p) > 0$ . If  $m = 2 - 2/d$ , then  $\bar{q} = 1$  and  $1 - \alpha = 0$ , therefore by conservation of mass it is possible to choose  $M$  sufficiently small such that the first term in (8.28) is less than  $-\delta\mathcal{I}$  for some  $\delta > 0$ . If  $m < 2 - 2/d$ , then  $\bar{q} > 1$  and  $\|\theta\|_{\bar{q}}$  is no longer conserved. Here we must take advantage of  $1 - \alpha < 0$ . Note that (8.28) holds for  $p = \bar{q}$ ; therefore since  $1 - \alpha < 0$ , a continuity argument establishes that for  $\|\theta_0\|_{\bar{q}}$  and  $M$  sufficiently small,

$$\|\theta(\tau)\|_{\bar{q}}^{\bar{q}} \leq \|\theta_0\|_{\bar{q}}^{\bar{q}} + C_3(\bar{q})M^{\gamma(\bar{q})}\tau.$$

Indeed, for  $\|\theta_0\|_{\bar{q}}$  small, this holds for at least some time, and for  $M$  sufficiently small, this linear growth is such that the first term in (8.28) remains non-positive forever. Then by (8.28) for  $p > \bar{q}$ , if  $M$  and  $\|\theta_0\|_{\bar{q}}$  additionally satisfy

$$C_1(p)e^{(1-\alpha)\beta^{-1}\tau}(C_3(\bar{q})M^{\gamma(\bar{q})}\tau + \|\theta_0\|_{\bar{q}}^{\bar{q}\alpha_2(p+1)/\bar{q}} - C_2(p)) < -\delta,$$

for all  $\tau > 0$ , then the first term is less than  $-\delta\mathcal{I}$ . By  $1 - \alpha < 0$  we may always choose  $M$  and  $\|\theta_0\|_{\bar{q}}$  such that this is possible. Therefore, whether  $\bar{q} > 1$  or  $\bar{q} = 1$ , for small initial data in the suitable sense, we have

$$\frac{d}{d\tau}\|\theta\|_p^p \leq -\delta\mathcal{I} + C(M, p).$$

Using (8.27) and Young's inequality for products, we have a lower bound on  $\mathcal{I}$ ,

$$\|\theta\|_p^p - C(M) \leq \mathcal{I}.$$

This proves,

$$\frac{d}{d\tau}\|\theta\|_p^p \leq -\delta\|\theta\|_p^p + C(M, p),$$

which immediately concludes the lemma with  $\|\theta\|_p^p \leq \max(\|\theta_0\|_p^p, C(M, p)\delta^{-1})$ . □

We now turn to proving that (8.7) implies something analogous to Lemma 18. Let  $u(t)$  be as in (ii) of Theorem 11. One can verify that (8.7) is equivalent

to

$$\lim_{k \rightarrow \infty} \sup_{\tau \in [0, \infty)} \|(\theta(\tau) - k)_+\|_1 = 0, \quad (8.29)$$

which is precisely the condition of uniform equi-integrability which appears in the continuation theorem (Theorem 5 in Chapter 6 above (see also [47, 36])). We may refine Lemma 18 in the following fashion, adapting the techniques in Chapter 6, Theorem 5, and [47, 36] to this setting.

**Lemma 19** (Control for  $L^p$ ,  $p < \infty$  for equi-integrable solutions). *If  $\theta(\tau)$  satisfies (8.29) then we have  $\|\theta(\tau)\|_p \in L^\infty_\tau(\mathbb{R}^+)$  for all  $p < \infty$ .*

*Proof.* We proceed similar to the proof of Lemma 18, but now slightly refined to take advantage of (8.29). Since similar arguments appeared in Chapter 6, we sketch a proof and highlight mainly the differences that appear due to the rescaling in (8.23). Define  $\theta_k(\tau, \eta) := (\theta(\tau, \eta) - k)_+$  and

$$\mathcal{I} = \int \theta_k^{m-1} \left| \nabla \theta_k^{p/2} \right|^2 dx.$$

The  $L^p$  norms of  $\theta$  and  $\theta_k$  are related through the following inequality for  $1 \leq p < \infty$ ,

$$\|\theta\|_p^p \lesssim_p \|\theta_k\|_p^p + k^{p-1} \|\theta\|_1. \quad (8.30)$$

It is important to note that the implicit constant in (8.30) does not depend on  $k$ . Estimating the time evolution of  $\theta_k$  as in Lemma 18, using Lemma 28 and (8.30) implies,

$$\begin{aligned} \frac{d}{d\tau} \|\theta_k\|_p^p &= -C(p)\mathcal{I} - \int ((p-1)\theta_k^p + kp\theta_k^{p-1}) \nabla \cdot (e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta_k) d\eta \\ &\leq -C(p)\mathcal{I} + C(p)\|\theta_k\|_{p+1}^{p+1} + C(p, k)\|\theta_k\|_p^p + C(k, p, M). \end{aligned}$$

Using the Gagliardo-Nirenberg-Sobolev inequality (Lemma 25) implies,

$$\frac{d}{d\tau} \|\theta_k\|_p^p \leq -\frac{C(p)}{\|\theta_k\|_1^{\alpha_2}} \|\theta_k\|_{p+1}^{p+1} + C(p)\|\theta_k\|_{p+1}^{p+1} + C(p, k)\|\theta_k\|_p^p + C(k, p, M),$$

where  $\alpha_2 = 1 - \alpha_1(p + m - 1)/2 > 0$  and

$$\alpha_1 = \frac{2d(1 - 1/(p + 1))}{2 - d + dp + d(m - 1)}.$$

Note  $\|\theta_k\|_p \leq M^{1/p^2} \|\theta_k\|_{p+1}^{(p^2-1)/p^2}$ , which by weighted Young's inequality implies,

$$\frac{d}{d\tau} \|\theta_k\|_p^p \leq -\frac{C(p)}{\|\theta_k\|_1^{\alpha_2}} \|\theta_k\|_{p+1}^{p+1} + C(p) \|\theta_k\|_{p+1}^{p+1} + C(k, p, M).$$

Using (8.29) we may make the leading order terms as negative as we want and interpolating  $L^p$  against  $L^1$  and  $L^{p+1}$  again implies there is a  $\delta > 0$  such that if  $k$  is sufficiently large we have,

$$\frac{d}{d\tau} \|\theta_k\|_p^p \leq -\delta \|\theta_k\|_p^p + C(k, p, M).$$

By (8.30) and conservation of mass, this concludes the proof of Lemma 19.  $\square$

## 8.4 Finite Length-Scales

We begin by proving Theorem 11 for the case  $\nabla \mathcal{K} \in L^1$ . Alikakos iteration [3] is a standard method for using a result such as Lemma 18 to imply a result of the following form.

**Lemma 20** (Control of  $L^\infty$  for small data). *Let  $\nabla \mathcal{K} \in L^1$ . Then there exists  $C_{\bar{q}} = C_{\bar{q}}(M)$  and  $C_M = C_M(\|\theta_0\|_{\bar{q}})$  such that if  $\|\theta_0\|_{\bar{q}} < C_{\bar{q}}$  and  $M < C_M$ , then  $\|\theta(\tau)\|_\infty \in L_\tau^\infty(\mathbb{R}^+)$ .*

*Proof.* Standard iteration implies  $\|\theta(\tau)\|_\infty \in L_\tau^\infty(\mathbb{R}^+)$ , provided

$$\vec{v} := e^{(1-\alpha-\beta)\beta^{-1}\tau} e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta \in L_{\tau,\eta}^\infty(\mathbb{R}^+ \times \mathbb{R}^d).$$

This follows by Kowalczyk's iteration lemma, Lemma 7 in Chapter 6.

Fix  $p > d$ . Then by Lemma 18, for sufficiently small  $M$  and  $\|\theta_0\|_{\bar{q}}, \|\theta(\tau)\|_p \in L^\infty_\tau(\mathbb{R}^+)$ . Therefore by Lemma 28 in the appendix,

$$\|\nabla \vec{v}\|_p = \|e^{(1-\alpha-\beta)\beta^{-1}\tau} \nabla (e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta)\|_p \lesssim e^{(1-\alpha)\beta^{-1}\tau} \|\theta\|_p \lesssim e^{(1-\alpha)\beta^{-1}\tau}.$$

Moreover, by  $\nabla \mathcal{K} \in L^1(\mathbb{R}^d)$ ,

$$\|\vec{v}\|_p \leq e^{(1-\alpha-\beta)\beta^{-1}\tau} \|\theta\|_p \lesssim e^{(1-\alpha-\beta)\beta^{-1}\tau}.$$

Since  $1 - \alpha \leq 0$ , Morrey's inequality implies  $\vec{v} \in L^\infty_{\tau,\eta}(\mathbb{R}^+ \times \mathbb{R}^d)$  and the lemma follows.  $\square$

By Lemma 20 and the definition of  $\tau$ ,

$$\|u(t)\|_{L^\infty_x(\mathbb{R}^d)} = e^{-d\tau} \|\theta\|_{L^\infty_\eta(\mathbb{R}^d)} \lesssim (1+t)^{-d\beta},$$

establishing (8.8). A similar argument using Lemma 19 in place of Lemma 18 implies

**Lemma 21.** *Theorem 11 holds if  $\nabla \mathcal{K} \in L^1$ .*

Now we turn to Theorem 12.

*Proof.* (Theorem 12: **Intermediate Asymptotics I**) Now that the requisite decay estimate has been established, we proceed by estimating the decay of the relative entropy (8.18). By Young's inequality,  $\nabla \mathcal{K} \in L^1(\mathbb{R}^d)$  and 8.8,

$$\|e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta\|_\infty \leq \|\nabla \mathcal{K}\|_1 \|\theta\|_\infty \lesssim 1. \quad (8.31)$$

We first settle the case  $m > 1$ . By a standard computation, (8.31) and Cauchy-Schwarz, for all  $\delta > 0$ ,

$$\begin{aligned} \frac{d}{d\tau} H(\theta|\theta_M) &= -I(\theta) + e^{(1-\alpha-\beta)\beta^{-1}\tau} \int \nabla \left( \frac{1}{m-1} \theta^m + \frac{1}{2} |\eta|^2 \right) \cdot \theta e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta d\eta \\ &\leq -I(\theta) + e^{(1-\alpha-\beta)\beta^{-1}\tau} I(\theta)^{1/2} \left( \int \theta |e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta|^2 d\eta \right)^{1/2} \\ &\leq -(1 - e^{-2\delta\tau}) I(\theta) + C e^{(2-2\alpha-2\beta)\beta^{-1}\tau + 2\delta\tau}. \end{aligned}$$

Let  $\gamma(\delta) := (2\alpha + 2\beta - 2)\beta^{-1} - 2\delta > 0$ . By the generalized Gross Logarithmic Sobolev inequality, Theorem 14, we therefore have,

$$\frac{d}{d\tau} H(\theta(\tau)|\theta_M) \leq -2(1 - e^{-2\delta\tau})H(\theta|\theta_M) + Ce^{-\gamma\tau}. \quad (8.32)$$

Solving the differential inequality (8.32) implies,

$$H(\theta(\tau)|\theta_M) \lesssim e^{-\tau \min(2, \gamma(\delta))}.$$

Now by the generalized Csiszar-Kullback inequality, Theorem 15,

$$\|\theta(\tau) - \theta_M\|_1 \lesssim e^{-\frac{\tau}{2} \min(2, \gamma(\delta))}.$$

Re-writing in terms of  $x$  and  $t$  and using (8.24),

$$\|u(t) - \mathcal{U}(t; M)\|_1 \lesssim (1+t)^{-\frac{\beta}{2}\tau \min(2, \gamma(\delta))}.$$

If  $m < 2 - 2/d$ , it can be verified that  $\delta > 0$  may always be chosen small enough such that  $2 < \gamma(\delta)$ . If instead  $m = 2 - 2/d$ , then  $2d + 2 - 2\beta^{-1} = 2$ . This establishes (8.10) in the case  $p = 1$ . Interpolation against (8.8) completes the proof.

We now settle the case  $m = 1$ . The time evolution of the relative entropy is similar to above. By (8.31) and Cauchy-Schwarz, for all  $\delta > 0$ ,

$$\begin{aligned} \frac{d}{d\tau} H(\theta(\tau)|\theta_M) &= -I(\theta) + e^{(1-\alpha-\beta)\beta^{-1}\tau} \int \nabla \left( \log \theta + \frac{1}{2} |\eta|^2 \right) \cdot \theta e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta d\eta \\ &\leq -I(\theta) + e^{(1-\alpha-\beta)\beta^{-1}\tau} I(\theta)^{1/2} \left( \int \theta |e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta|^2 d\eta \right)^{1/2} \\ &\leq (1 - e^{-\delta\tau}) I(\theta) + Ce^{(2-2\alpha-2\beta)\beta^{-1}\tau + \delta\tau}. \end{aligned}$$

The rest of the proof follows similarly to the case  $m > 1$  using Theorems 14 and 15. This concludes the proof of Theorem 12.  $\square$

*Remark 19.* A generalization of Talagrand's inequality [55] shows that  $\theta \rightarrow \theta_M$  also in the Euclidean Wasserstein distance.

## 8.5 Infinite Length-Scales

We now turn to the proofs of Theorem 11 and Theorem 13 in the case  $\nabla\mathcal{K} \notin L^1$ . In order to properly extend the work of the previous section, we must estimate the quantities  $\|e^{d\tau}\nabla\mathcal{K}(e^\tau\cdot)*\theta\|_p$  appearing in (8.31) and the proof of Lemma 20. However,  $\nabla\mathcal{K} \notin L^1(\mathbb{R}^d)$  and Young's inequality is not sufficient; in fact we will not bound  $\|e^{d\tau}\nabla\mathcal{K}(e^\tau\cdot)*\theta\|_p$  uniformly in time but instead bound the rate at which it grows. We separately estimate the growth of the quantities  $\|\lambda^d\nabla\mathcal{K}(\lambda\cdot)\mathbf{1}_{B_1(0)}\|_1$  and  $\|\lambda^d\nabla\mathcal{K}(\lambda\cdot)\mathbf{1}_{\mathbb{R}^d\setminus B_1(0)}\|_p$  as  $\lambda \rightarrow \infty$ . Using  $|\nabla\mathcal{K}(x)| \lesssim |x|^{-\gamma}$  for sufficiently large  $|x|$ , if  $\gamma < d$ , then for large  $\lambda$ ,

$$\begin{aligned} \int \lambda^d |\nabla\mathcal{K}(\lambda y)| \mathbf{1}_{B_1(0)}(|y|) dy &= \int_{|y| \leq \lambda} |\nabla\mathcal{K}(y)| dy \\ &= \int_{S^{d-1}} \int_0^\lambda |\nabla\mathcal{K}(\rho\omega)| r \rho^{d-1} d\rho d\omega \\ &\lesssim 1 + \lambda^{d-\gamma}. \end{aligned} \tag{8.33}$$

Similarly, if  $\gamma = d$ , then for large  $\lambda$ ,

$$\int \lambda^d |\nabla\mathcal{K}(\lambda y)| \mathbf{1}_{B_1(0)}(|y|) dy \lesssim 1 + \log \lambda. \tag{8.34}$$

If  $d/(d-1) < q < \infty$ , since  $\gamma \geq d-1$ , for  $\lambda$  sufficiently large we have,

$$\begin{aligned} \int \lambda^{qd} |\nabla\mathcal{K}(\lambda y)|^q \mathbf{1}_{\mathbb{R}^d\setminus B_1(0)}(|y|) dy &= \int_{|y| \geq \lambda} \lambda^{qd-d} |\nabla\mathcal{K}(y)|^q dy \\ &= \lambda^{qd-d} \int_{S^{d-1}} \int_\lambda^\infty |\nabla\mathcal{K}(\rho\omega)|^q \rho^{d-1} d\rho d\omega \\ &\lesssim \lambda^{q(d-\gamma)}. \end{aligned} \tag{8.35}$$

Similarly,

$$\sup_{|x| \geq 1} |\lambda^d \nabla\mathcal{K}(\lambda x)| \lesssim 1 + \lambda^{d-\gamma}. \tag{8.36}$$

We may now complete the general proof of Theorem 11.

*Proof.* (**Theorem 11**) We first complete the proof of (i). Lemma 20 extends to the case  $\nabla \mathcal{K} \notin L^1$  provided we can bound  $\vec{v} := e^{(1-\alpha-\beta)\beta^{-1}\tau} e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) * \theta$  in  $L_\eta^\infty(\mathbb{R}^d)$  uniformly in time. Indeed, fix  $p > d$ . Then for  $M$  and  $\|\theta_0\|_{\bar{q}}$  sufficiently small, we have by Lemma 18,  $\|\theta(\tau)\|_p \in L_\tau^\infty(\mathbb{R}^+)$ . By Lemma 28,

$$\|\nabla \vec{v}\|_p \lesssim e^{(1-\alpha)\beta^{-1}\tau} \|\theta\|_p \lesssim e^{(1-\alpha)\beta^{-1}\tau}.$$

Let  $q$  be such that  $d/(d-1) < q \leq p$ , which implies  $\|\theta(\tau)\|_q \lesssim 1$ . If  $\gamma < d$  then by Young's inequality,

$$\begin{aligned} \|\vec{v}\|_q &\leq e^{(1-\alpha-\beta)\beta^{-1}\tau} \left( \|e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) \mathbf{1}_{B_1(0)} * \theta\|_q + \|e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) \mathbf{1}_{\mathbb{R}^d \setminus B_1(0)} * \theta\|_q \right) \\ &\leq e^{(1-\alpha-\beta)\beta^{-1}\tau} \left( \|e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) \mathbf{1}_{B_1(0)}\|_1 \|\theta\|_q + \|e^{d\tau} \nabla \mathcal{K}(e^\tau \cdot) \mathbf{1}_{\mathbb{R}^d \setminus B_1(0)}\|_q M \right). \end{aligned}$$

Since  $\|\theta(\tau)\|_q \lesssim 1$ , by (8.33) and (8.35) we have,

$$\begin{aligned} \|\vec{v}\|_q &\lesssim e^{(1-\alpha-\beta)\beta^{-1}\tau} (1 + e^{(d-\gamma)\tau}) \\ &\lesssim e^{(1-\alpha-\beta)\beta^{-1}\tau} + e^{(1-\beta-\gamma\beta)\beta^{-1}\tau}. \end{aligned}$$

Since  $1 - \beta - \gamma\beta \leq 0$  and  $1 - \alpha \leq 0$ , by Morrey's inequality we may conclude  $\vec{v} \in L_{\tau,\eta}^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ . Similarly if  $\gamma = d$ , then by the same reasoning as above, (8.34) and (8.35) imply,

$$\begin{aligned} \|\vec{v}\|_q &\lesssim e^{(1-\alpha-\beta)\beta^{-1}\tau} (1 + \tau + e^{(d-\gamma)\tau}) \\ &\lesssim e^{(1-\alpha-\beta)\beta^{-1}\tau} (1 + \tau) + e^{(1-\beta-\gamma\beta)\beta^{-1}\tau}. \end{aligned}$$

Since  $1 - \alpha - \beta < 0$ , we may conclude also in this case that  $\vec{v} \in L_{\tau,\eta}^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ . Therefore Lemma 20 applies with the hypotheses of Theorem 13. Re-writing in terms of  $x$  and  $t$ , this implies (8.8). A similar proof with Lemma 19 in place of Lemma 18 also proves (ii)  $\square$

We now prove Theorem 13.



*Proof.* (Theorem 13: **Intermediate Asymptotics II**) To complete the proof of Theorem 13, we estimate the decay of the relative entropy (8.18). The proof of Theorem 12 used the estimate (8.31). Here we use the bound  $\|\theta(\tau)\|_\infty \lesssim 1$  (8.36) and (8.33) to imply, if  $\gamma < d$ ,

$$\begin{aligned} \|e^{d\tau}\nabla\mathcal{K}(e^\tau\cdot)*\theta\|_\infty &\leq (\|e^{d\tau}\nabla\mathcal{K}(e^\tau\cdot)\mathbf{1}_{B_1(0)}*\theta\|_\infty + \|e^{d\tau}\nabla\mathcal{K}(e^\tau\cdot)\mathbf{1}_{\mathbb{R}^d\setminus B_1(0)}*\theta\|_\infty) \\ &\lesssim (1 + e^{(d-\gamma)\tau}) (\|\theta\|_\infty + M) \end{aligned} \quad (8.37)$$

$$\lesssim 1 + e^{(d-\gamma)\tau} \lesssim e^{(d-\gamma)\tau}. \quad (8.38)$$

Similarly, if  $\gamma = d$  then, for all  $\delta > 0$ ,

$$\|e^{d\tau}\nabla\mathcal{K}(e^\tau\cdot)*\theta\|_\infty \lesssim 1 + \tau \lesssim_\delta e^{\delta\tau}.$$

The growth of (8.38) in time is the source of the degraded convergence rate observed in (8.12). As noted above, this is a manifestation of slow decay in the kernel, which causes growth of  $e^{d\tau}\nabla\mathcal{K}(e^\tau\cdot)$  in  $L^1_{loc}$ . Indeed, computing the decay of the relative entropy (with linear or nonlinear diffusion) as above with (8.38),

$$\begin{aligned} \frac{d}{d\tau}H(\theta(\tau)|\theta_M) &\leq -I(\theta) + e^{(1-\alpha-\beta)\beta^{-1}\tau}I(\theta)^{1/2} \left( \int \theta |e^{d\tau}\nabla\mathcal{K}(e^\tau\cdot)*\theta|^2 d\eta \right)^{1/2} \\ &\leq (1 - e^{-2\delta\tau})I(\theta) + Ce^{(2(1-\alpha-\beta)\beta^{-1}+2(d-\gamma)+2\delta)\tau}. \end{aligned}$$

As before, Theorems 14 and 15 imply,

$$\|\theta(\tau) - \theta_M\|_1 \lesssim e^{-\tau \min(1, 1+\gamma-\beta^{-1}-\delta)}.$$

Re-writing in terms of  $x$  and  $t$  and interpolating against (8.8) completes the proof. The corresponding argument follows also for  $\gamma = d$ , absorbing the mild growth of  $\|e^{d\tau}\mathcal{K}(e^\tau\cdot)*\theta\|_\infty$  into the  $\delta$  already introduced.  $\square$

## CHAPTER 9

### Discussion and Open Problems

#### 9.1 What Remains

In Part II of this dissertation, the author has described recent advancements made by himself and his collaborators in the study of a general class of PDE (5.1) which model the competition between non-local self-attraction and possibly nonlinear diffusion. Despite the progress, many key questions of course remain.

Dissipation results do not yet hold up to the critical mass in critical scaling problems, except in scale-invariant cases with degenerate diffusion. In these cases, the self-similar change of variables preserves the energy dissipation inequality capable of deducing global bounds using the continuation theorem 5 and methods from Chapter 7. The difficulty in other cases is that the Alikakos iteration techniques (or related De Giorgi methods [176] and semilinear methods [38]) do not really provide a non-perturbative treatment of (5.1), but the energy dissipation inequality is not strong enough to directly deduce the decay estimates. Aside from critical problems, there are gaps which are not covered by the existing results or the results of Chapter 8 and the global minimizers of the free energy constructed in [149, 150]. That is, dissipation for small data is not known to hold but either global minimizers to the free energy truly do not exist or are not known to exist. In subcritical problems, a natural question to ask is whether or not solutions converge to these “ground-state” minimizers. Results in certain

special cases based on comparison principles have recently been obtained by Kim and Yao [123].

Another problem remaining is the behavior of solutions with precisely critical mass. In scale-invariant problems, these threshold solutions are known to exist globally [37, 36, 35] and in the case of linear diffusion the global qualitative behavior is becoming well understood. The key difficulty here is that the free energy is no longer coercive and does not provide a priori boundedness of the entropy at the critical mass. One can easily see this directly from the HLS inequalities. Hence more advanced techniques must be used, such as the Weinstein-type [208] concentration compactness arguments utilized in [36] and the surprising application of entropy-entropy dissipation methods for fast diffusion equations used in [35].

## 9.2 Other Critical PDE

The phenomenon of criticality, and the associated behavior such as critical thresholds, is pervasive in the study of PDE. For example, similar behavior is well-known in many models: reaction-diffusion equations, semilinear dispersive PDE with focusing nonlinearities, unstable thin film equations and many others. The prototypical example of a dispersive PDE with a focusing nonlinearity are the well-known focusing nonlinear Schrödinger equations (NLS)

$$i\phi_t + \Delta\phi = -|\phi|^p\phi, \quad (9.1)$$

which is formally a Hamiltonian for the mixed-sign energy

$$E(\phi) = \frac{1}{2} \int |\nabla\phi|^2 dx - \frac{1}{p+2} \int |\phi|^{p+2} dx.$$

While diffusion is very different from dispersion and the nonlinearity in (5.1) is of a very different form, these models do share some similarities. Being semilinear, the

NLS has local theory based on Strichartz estimates and the contraction mapping theorem [60, 197] which treats the NLS as a perturbation of the linear PDE. The theory of (5.1) has the Alikakos iteration techniques<sup>1</sup> which treats (5.1) as a perturbation of the nonlinear diffusion equation, and these two methods share essentially the same strengths and weaknesses. Where (5.1) has the Hardy-Littlewood-Sobolev inequalities to identify the critical mass, the NLS has the sharp Gagliardo-Nirenberg inequalities [207, 121]. In the context of the NLS, the analogue of the intermediate asymptotics results of Chapter 8 is the concept of scattering [197]. The regimes in which (9.1) is known to scatter (see e.g. [197, 121]) is similar to the dichotomy between dissipation and the existence of stationary solutions detailed in Chapter 8 and those constructed by Lions [149, 150]. Aside from directly providing mathematical methods, such as the arguments of [36] (the arguments in [37] are also a related form of concentration compactness), the comparatively more advanced NLS literature provides valuable intuition and suggestions for possible research directions.

### 9.3 Looking Forward

Despite a great deal of advances, there are many aspects of the theory of PDE such as (5.1) which are still being developed. In particular, we feel that in order to advance the theory of (5.1) significantly from the current point will require the development of new techniques (5.1). I briefly discuss three such directions.

Roughly speaking, the theory of the aggregation-diffusion equations is in approximately the same state as the study of the NLS (9.1) before the introduction of Bourgain's induction on energy methodology [42, 41]. This idea led to a num-

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<sup>1</sup>in the case of linear diffusion, one can re-write the local theory of (5.1) to look very much like that of the NLS, see for instance [38].

ber of results and refinements (see for instance [118, 116, 117, 203, 66, 120, 122] and the review of minimal counterexample arguments by Killip and Viřan [121]), ultimately producing many strong decay estimates and solutions to problems regarding global asymptotic behavior for critical NLS and other semilinear dispersive PDE. To briefly summarize, the methods systematically rule out minimal, or nearly minimal, counter-examples to these global bounds, which are better behaved than general solutions and can be characterized using concentration compactness [121]. This methodology is not tied to dispersive PDE, and has been recently used to obtain alternative proofs of the global regularity of solutions to the 3D Navier-Stokes equations which are bounded in  $\dot{H}^{1/2}$  or  $L^3$  [84, 115, 86]. These tools have proved extremely powerful for dealing with critical problems, and could potentially be applied to critical aggregation-diffusion equations, for instance to deduce global decay estimates for those critical PKS models which currently lack such bounds. As remarked above, simpler concentration compactness arguments from the NLS have already been applied to the study of PKS [37, 36].

Induction on energy/minimal counter-example methods have proved useful for conservative problems such as the NLS. However, the aggregation-diffusion equations are formally gradient flows, which formally at least, is a significantly stronger property. Not only does this introduce an additional controlled quantity (the free energy dissipation, see Proposition (1)), but this means there is a potential to apply the entropy-entropy dissipation methods discussed in Chapter 8 in §8.1.1. This has been slow in progress due to the fact that the free energy  $\mathcal{F}$  is not displacement convex, a key property for known methods to apply, however there have been developments in this direction nonetheless [34, 48, 35].

Due to the parabolic nature, aggregation-diffusion equations have the poten-

tial to be studied from the viewpoint of comparison principles. Although (5.1) does not satisfy any useful maximum principles, comparison arguments can be made on the level of the mass concentration function

$$M(t, r) = \int_{|x| \leq r} u(t, x) dx.$$

For certain kernels, mass comparison principles may hold between weak solutions [123], however, even when no mass comparison principles hold, there is still a possibility of using comparison principle arguments. Combined with symmetrization techniques, these principles are of independent interest and may be used to deduce qualitative, long term properties of solutions [73, 74] (see [201, 202] for such methods applied to other linear and nonlinear elliptic and parabolic PDE). The mass concentration also played a key role in [33] which examined radially symmetric threshold solutions. The recent work of Kim and Yao [123] uses mass comparison principles to prove convergence to stationary solutions, intermediate asymptotics results, and symmetrization inequalities comparing the relative concentration of radially symmetric solutions to that of non-radially symmetric solutions.

# CHAPTER 10

## Appendix

### 10.1 Technical Lemmas

**Lemma 22.** *Let  $F$  be a convex  $C^1$  function and  $f = F'$ . Assume that  $f(u) \in L^2(0, T, H^1(D))$ ,  $u \in H^1(0, T, H^{-1}(D))$  and  $F(u) \in L^\infty(0, T, L^1(D))$ . Then for almost all  $0 \leq s, \tau, \leq T$  the following holds:*

$$\int (F(u(x, \tau)) - F(u(x, s))) \, dx = \int_s^\tau \langle u_t, f(u(t)) \rangle \, dt.$$

**Lemma 23.** *Let  $F(u, t) \in C^2([0, \infty), [0, \infty))$  be a convex function such that  $F(0) = 0$  and  $F'' > 0$  on  $(0, \infty)$ . Let  $f_n$ , for  $n = 1, 2, \dots$ , and  $f$  be a non-negative function on  $D$  bounded from above by  $M > 0$ . Furthermore, assume that  $f_n \rightharpoonup f$  in  $L^1(D)$  and  $F(f_n) \rightarrow F(f)$  in  $L^1(D)$ , then  $\|f_n - f\|_{L^2(D)} \rightarrow 0$  as  $n \rightarrow \infty$ .*

**Lemma 24** (Weak Lower-semicontinuity). *Let  $\rho_\epsilon$  be non-negative  $L^1_{loc}(D_T)$  and  $f_\epsilon$  a vector valued function in  $L^1_{loc}(D_T)$  such that  $\forall \phi \in C^\infty_c(\overline{D_T})$  and  $\xi \in C^\infty_c(\overline{D_T}, \mathbb{R}^d)$*

$$\begin{aligned} \int_{D_T} \rho_\epsilon \phi \, dx \, dt &\rightarrow \int_{D_T} \rho \phi \, dx \, dt \\ \int_{D_T} f_\epsilon \cdot \xi \, dx \, dt &\rightarrow \int_{D_T} f \cdot \xi \, dx \, dt. \end{aligned}$$

*Then*

$$\int_{D_T} \frac{1}{\rho} |f|^2 \, dx \, dt \leq \liminf_{\epsilon \rightarrow 0} \int_{D_T} \frac{1}{\rho_\epsilon} |f_\epsilon|^2 \, dx \, dt$$

## 10.2 Gagliardo-Nirenberg-Sobolev Inequalities

Gagliardo-Nirenberg-Sobolev inequalities are the main tool for obtaining  $L^p$  estimates of PKS models and are used in many works, for instance [125, 36, 192, 107]. The following inequality follows by interpolation and the classical Gagliardo-Nirenberg-Sobolev inequality.

**Lemma 25** (Gagliardo-Nirenberg-Sobolev). *Let  $d \geq 2$  and  $D \subset \mathbb{R}^d$  satisfy the cone condition (see e.g. [2]). Let  $f : D \rightarrow \mathbb{R}$  satisfy  $f \in L^p \cap L^q$  and  $\nabla f^k \in L^r$ . Moreover let  $1 \leq p \leq rk \leq dk$ ,  $k < q < rkd/(d-r)$  and*

$$\frac{1}{r} - \frac{k}{q} - \frac{s}{d} < 0. \quad (10.1)$$

*Then there exists a constant  $C_{GNS}$  which depends on  $s, p, q, r, d$  and the dimensions of the cone for which  $D$  satisfies the cone condition such that*

$$\|f\|_{L^q} \leq C_{GNS} \|f\|_{L^p}^{\alpha_2} \|f^k\|_{W^{s,r}}^{\alpha_1}, \quad (10.2)$$

where  $0 < \alpha_i$  satisfy

$$1 = \alpha_1 k + \alpha_2, \quad (10.3)$$

and

$$\frac{1}{q} - \frac{1}{p} = \alpha_1 \left( \frac{-s}{d} + \frac{1}{r} - \frac{k}{p} \right). \quad (10.4)$$

*Remark 20.* If  $D = \mathbb{R}^d$ , then the homogeneous version of this inequality also holds, with the  $W^{s,r}$  norm replaced by  $\dot{W}^{s,r}$ .

*Proof.* We may assume that  $f$  is Schwartz then argue by density. Let  $\beta$  satisfy  $\max(q, rk) < \beta < rkd/(d-r)$ . First note by the Gagliardo-Nirenberg-Sobolev inequality, [Theorem 5.8, [2]], we have

$$\begin{aligned} \|f^k\|_{\beta/k} &\lesssim_{\beta,k,r,s} \|f^k\|_r^{1-\theta} \|f^k\|_{W^{s,r}}^\theta \\ &\leq \|f^k\|_{p/k}^{(1-\theta)(1-\mu)} \|f^k\|_{\beta/k}^{(1-\theta)\mu} \|f^k\|_{W^{s,r}}^\theta, \end{aligned}$$



for  $\mu \in (0, 1)$  determined by interpolation and  $\theta = s^{-1}(d/r - dk/\beta) \in (0, 1)$ . Moreover, the implicit constant does not depend directly on the size of the domain. Therefore,

$$\|f^k\|_{\beta/k} \lesssim \|f\|_p^{(1-\theta)(1-\mu)/(1-\mu(1-\theta))} \|f^k\|_{W^{s,r}}^{\theta/(1-\mu(1-\theta))}.$$

Now, where  $\lambda \in (0, 1)$  determined by interpolation,

$$\begin{aligned} \|f\|_q &\leq \|f\|_p^{(1-\lambda)} \|f^k\|_{\beta/k}^{\lambda/k} \\ &\lesssim \|f\|_p^{(1-\lambda)+(1-\theta)(1-\mu)/(1-\mu(1-\theta))} \|f^k\|_{W^{s,r}}^{\lambda\theta/(k-k\mu(1-\theta))}. \end{aligned}$$

□

### 10.3 Properties of Admissible Kernels

We now prove Lemmas 1,2 and 3. We begin with the following characterizations of  $L^{p,\infty}$ .

**Lemma 26.** *Let  $F(x) = f(|x|) \in L_{loc}^1 \cap C^0 \setminus \{0\}$  be monotone in a neighborhood of the origin. If  $r^{-d/p} = o(f(r))$  as  $r \rightarrow 0$ , then  $F \notin L_{loc}^{p,\infty}$ .*

*Proof.* Since we have assumed  $f$  to be monotone in a neighborhood of the origin, without loss of generality we prove the assertions assuming  $f \geq 0$  on that neighborhood, since corresponding work may be done if  $f$  is negative. For any  $\alpha > 0$ , by monotonicity, we have a unique  $r(\alpha)$  such that  $f(r) > \alpha, \forall r < r(\alpha)$ . We thus have that  $\lambda_f(\alpha) = \omega_d r(\alpha)^d$ , where  $\omega_d$  is the volume of the unit sphere in  $\mathbb{R}^d$ . By the growth condition on  $f$  and continuity we also have that for  $\alpha$  sufficiently large,

$$\frac{1}{\epsilon} r(\alpha)^{-d/p} \leq f(r(\alpha)) = \alpha.$$

Now,

$$\alpha^p \lambda_f(\alpha) = \omega_d \alpha^p r(\alpha)^d.$$

Hence, by (10.3) we have  $\forall \epsilon > 0$  there is a neighborhood of infinity such that,

$$\omega_d \alpha^p r(\alpha)^d \gtrsim \epsilon^{-p}.$$

We take  $\epsilon \rightarrow 0$  to deduce that  $F \notin L^{p,\infty}$ .

□

**Lemma 27.** *Let  $F(x) = f(|x|) \in L_{loc}^1 \cap C^0 \setminus \{0\}$  be monotone in a neighborhood of the origin. Then  $f \in L_{loc}^{p,\infty}$  if and only if  $f = \mathcal{O}(r^{-d/p})$  as  $r \rightarrow 0$ .*

*Proof.* Since we have assumed  $f$  to be monotone in a neighborhood of the origin, without loss of generality we prove the assertions assuming  $f \geq 0$  on that neighborhood.

First assume that  $f \neq \mathcal{O}(r^{-d/p})$  as  $r \rightarrow 0$ , which implies that for all  $\delta_0 > 0$  and every  $C > 0$  there exists an  $r_C < \delta_0$  such that

$$f(r_C) > Cr_C^{-d/p}.$$

We now show that in a neighborhood of the origin, the function  $f(r) - Cr^{-d/p}$  is strictly positive for  $r < r_C$ . Suppose not. Since both  $f, r^{-d/p}$  are monotone, there exists  $r_0$  such that  $f(r) < Cr^{-d/p}$  for  $r < r_0$ . However, this contradicts  $f \neq \mathcal{O}(r^{-d/p})$  as  $r \rightarrow 0$ . Thus, we have that

$$f(r) > Cr^{-d/p}$$

in a neighborhood of the origin ( $r < r_C$ ). Since for all  $C > 0$  we can find a corresponding  $r_C$ , this is equivalent to  $r^{-d/p} = o(f(r))$ , and by Lemma 26 we

have that  $f \notin L^{p,\infty}$ .

On the other hand, if  $f = \mathcal{O}(r^{-d/p})$  as  $r \rightarrow 0$  there exists  $\delta > 0$  and  $C > 0$  such that for all  $r < \delta$ ,

$$f(r) \leq Cr^{-d/p}. \quad (10.5)$$

By monotonicity, for all  $\alpha > 0$  there is a unique  $r(\alpha) \in [0, \delta]$  such that

$$f(r) > \alpha, \text{ for } r < r(\alpha), \quad (10.6)$$

where we take  $r(\alpha) = 0$  if  $f(r) < \alpha$  over the entire neighborhood. By (10.5) and (10.6), we have, necessarily that  $r(\alpha) \lesssim \alpha^{-p/d}$ . Therefore,

$$\alpha^p \lambda_f(\alpha) = \alpha^p \omega_d r(\alpha)^d \lesssim 1,$$

which implies  $f \mathbf{1}_{B_1(0)} \in L^{p,\infty}$ . □

*Remark 21.* Similar statements may be made about the decay of  $F(x)$  at infinity.

*Proof.* (**Lemma 1**) By the fundamental theorem of calculus and condition **(BD)**,

$$\begin{aligned} |\partial_{x_i} \partial_{x_j} \mathcal{K}(x)| &\leq \int_1^\infty |\partial_r \partial_{x_i} \partial_{x_j} \mathcal{K}(rx)| dr \\ &\lesssim |x|^{-d}. \end{aligned}$$

Similarly, this argument also implies  $|\nabla \mathcal{K}| \lesssim |x|^{1-d}$ , which in turn implies  $\nabla \mathcal{K} \in L^{d/(d-1),\infty}$ . If  $d > 2$  then we can carry out this argument another time and show that  $|\mathcal{K}| \lesssim |x|^{2-d}$ . Moreover, in  $d = 2$  we see that  $\mathcal{K}$  could have, at worst, logarithmic singularities at zero and infinity. □

*Proof.* (**Lemma 2**) We compute second derivatives of the kernel  $\mathcal{K}$  in the sense of distributions. Let  $\phi \in C_c^\infty$ , then by the dominated convergence theorem,

$$\begin{aligned} \int \partial_{x_i} \mathcal{K} \partial_{x_j} \phi dx &= \lim_{\epsilon \rightarrow 0} \int_{|x| \geq \epsilon} \partial_{x_i} \mathcal{K} \partial_{x_j} \phi dx \\ &= - \lim_{\epsilon \rightarrow 0} \int_{|x|=\epsilon} \partial_{x_j} \mathcal{K}(x) \frac{x_j}{|x|} \phi(x) dS - \text{PV} \int \partial_{x_i x_j} \mathcal{K} \phi dx. \end{aligned}$$

By  $\nabla \mathcal{K} \in L^{d/(d-1), \infty}$  and Lemma 27, we have  $\nabla \mathcal{K} = \mathcal{O}(|x|^{1-d})$  as  $x \rightarrow 0$ . Therefore for  $\epsilon$  sufficiently small, there exists  $C > 0$  such that,

$$\begin{aligned} \left| \int_{|x|=\epsilon} \partial_{x_j} \mathcal{K}(x) \frac{x_j}{|x|} \phi(x) dS \right| &\leq C \int_{|x|=\epsilon} |x|^{1-d} |\phi(x)| dS \\ &= C \int_{|x|=1} |\epsilon x|^{1-d} |\phi(\epsilon x)| \epsilon^{d-1} dS = C |\phi(0)|. \end{aligned}$$

Similarly, we may define  $D^2 \mathcal{K} * \phi$  and we have,

$$\|D^2 \mathcal{K} * \phi\|_p \leq C \|\phi\|_p + \|\text{PV} \int \partial_{x_i x_j} \mathcal{K}(y) \phi(x-y) dy\|_p.$$

Therefore, the first term can be extended to a bounded operator on  $L^p$  for  $1 \leq p \leq \infty$  by density. The admissibility conditions **(R)**, **(BD)** and **(KN)** are sufficient to apply the Calderón-Zygmund inequality [Theorem 2.2 [189]], which implies that the principal value integral in the second term is a bounded linear operator on  $L^p$  for all  $1 < p < \infty$ . Moreover the proof provides an estimate of the operator norms,

$$\|\text{PV} \int \partial_{x_i x_j} \mathcal{K}(y) u(x-y) dy\|_p \lesssim \begin{cases} \frac{1}{p-1} \|u\|_p & 1 < p < 2 \\ p \|u\|_p & 2 \leq p < \infty. \end{cases}$$

□

*Proof.* (**Lemma 3**) The assertion that  $D^2 \mathcal{K} \in L_{loc}^{\gamma, \infty}$  implies  $\mathcal{K} \in L_{loc}^{d/(d/\gamma-2), \infty}$  follows similarly as in Lemma 1.

Now we prove the reverse implication. Let  $\mathcal{K} \in L_{loc}^{d/(d/\gamma-2), \infty}$ . We show that  $D^2\mathcal{K} = \mathcal{O}(r^{-d/\gamma})$  as  $r \rightarrow 0$ . Assume for contradiction that  $D^2\mathcal{K} \neq \mathcal{O}(r^{-d/\gamma})$  as  $r \rightarrow 0$ . This implies that  $k'' \neq \mathcal{O}(r^{-d/\gamma})$  or that  $k'(r)r^{-1} \neq \mathcal{O}(r^{-d/\gamma})$  as  $r \rightarrow 0$ . These two possibilities are essentially the same, so just assume that  $k'' \neq \mathcal{O}(r^{-d/\gamma})$ . By monotonicity arguments used in the proof of Lemma 27, this in turn implies  $r^{-d/\gamma} = o(k'')$ . However, this means that for all  $\epsilon$ , there exists a  $\delta(\epsilon) > 0$  such that for  $r \in (0, \delta(\epsilon))$  we have,

$$\begin{aligned} k(r) - k(\delta(\epsilon)) &= - \int_{\delta(\epsilon)}^r k'(s) ds = \int_{\delta(\epsilon)}^r \int_{\delta(\epsilon)}^s k''(t) dt ds + (r - \delta(\epsilon))k'(\delta(\epsilon)) \\ &\gtrsim \epsilon^{-1} r^{2-d/\gamma} + 1, \end{aligned}$$

which contradicts the fact that  $k(r) = \mathcal{O}(r^{2-d/\gamma})$  as  $r \rightarrow 0$  by Lemma 27.

The assertion regarding  $\nabla\mathcal{K}$  is proved in the same fashion. □

The following lemma verifies that the distributions defined by the second derivatives of admissible kernels behave as expected under mass-invariant scalings.

**Lemma 28.** *Let  $\mathcal{K}$  be admissible. Then  $\forall p, 1 < p < \infty, u \in L^p$  and  $t > 0$ , we have*

$$\|\nabla (t^d \nabla \mathcal{K}(t \cdot) * u)\|_p \lesssim_p t \|u\|_p. \quad (10.7)$$

*Proof.* We take the second derivative in the sense of distributions. Let  $\phi \in C_c^\infty$ ,

then by the dominated convergence theorem,

$$\begin{aligned}
\int t^d \partial_{x_i} \mathcal{K}(tx) \partial_{x_j} \phi(x) dx &= \lim_{\epsilon \rightarrow 0} \int_{|x| \geq \epsilon} t^d \partial_{x_i} \mathcal{K}(tx) \partial_{x_j} \phi(x) dx \\
&= -t \lim_{\epsilon \rightarrow 0} \int_{|x|=\epsilon} t^{d-1} \partial_{x_j} \mathcal{K}(tx) \frac{x_j}{|x|} \phi(x) dS \\
&\quad - t \text{PV} \int t^d \partial_{x_i, x_j} \mathcal{K}(tx) \phi(x) dx.
\end{aligned}$$

By  $\nabla \mathcal{K} \in L^{d/(d-1), \infty}$ , we have  $\nabla \mathcal{K} = \mathcal{O}(|x|^{1-d})$  as  $x \rightarrow 0$ . Therefore for  $\epsilon$  sufficiently small, there exists  $C > 0$  such that,

$$\begin{aligned}
\left| t \int_{|x|=\epsilon} t^{d-1} \partial_{x_i} \mathcal{K}(tx) \frac{x_j}{|x|} \phi(x) dS \right| &\leq Ct \int_{|x|=\epsilon} |x|^{1-d} |\phi(x)| dS \\
&= Ct \int_{|x|=1} |\epsilon x|^{1-d} |\phi(\epsilon x)| \epsilon^{d-1} dS = Ct |\phi(0)|.
\end{aligned}$$

The admissibility conditions **(R)**, **(BD)** and **(KN)** are sufficient to apply the Calderón-Zygmund inequality [Theorem 2.2 [189]], which implies that the principal value integral in the second term is a bounded linear operator on  $L^p$  for all  $1 < p < \infty$ . The operator norms, which are the implicit constants in (10.7), only depend on the bound in **(BD)** and on the cancellation-type condition

$$\int_{|x| > 2|y|} |K(x-y) - K(x)| dx \leq B.$$

Both of these conditions are clearly left invariant under the rescaling in (10.7) and this concludes the proof.  $\square$

## 10.4 Hardy Space Lemma

In this Appendix we prove Lemma 6, which again, we acknowledge Jonas Azzam for his assistance in the proof. To restate:

*Let  $f \in L^1 \cap L^p$  for some  $p > 1$  and satisfy  $\int f dx = 0$ ,  $\mathcal{M}_1 = \int |x| |f(x)| dx < \infty$ .*

*Then  $f \in \mathcal{H}^1$  and*

$$\|f\|_{\mathcal{H}^1} \lesssim_{d,p} \|f\|_p + \mathcal{M}_1.$$

*Proof.* We will use duality to characterize  $\mathcal{H}^1$  via (5.11). Therefore, let  $\|\mathcal{K}\|_{BMO} = 1$ . In what follows, we denote  $\mathcal{K}_R := \frac{1}{|B_R(0)|} \int_{|x| \leq R} \mathcal{K}(x) dx$ . By the mean-zero condition on  $f$ ,

$$\begin{aligned} \left| \int \mathcal{K} f dx \right| &\leq \int_{|x| \leq 1} |\mathcal{K} - \mathcal{K}_1| |f| dx + \int_{|x| > 1} |\mathcal{K} - \mathcal{K}_1| |f| dx. \\ &:= T1 + T2. \end{aligned}$$

By Hölder and  $p > 1$  with  $\|\mathcal{K}\|_{BMO} = 1$ , we have

$$\begin{aligned} T1 &\leq \|f\|_p \left( \int_{|x| \leq 1} |\mathcal{K} - \mathcal{K}_1|^{p'} dx \right)^{1/p'} \\ &\lesssim_{p'} \|f\|_p. \end{aligned}$$

The second inequality can be found in the proof of the John-Nirenberg inequality in [190]. We now deal with the second term. Define the dyadic annuli  $A_n := \{x \in \mathbb{R}^d : 2^n < |x| < 2^{n+1}\}$  for  $n \geq 0$ . Let  $E_n := \{x \in A_n : |\mathcal{K} - \mathcal{K}_1| > 2^n\}$ . By definition we have,

$$T2 = \sum_{n \geq 0} \int_{E_n} |\mathcal{K} - \mathcal{K}_1| |f| dx + \int_{A_n \setminus E_n} |\mathcal{K} - \mathcal{K}_1| |f| dx.$$

The second term can be estimated via,

$$\begin{aligned} \int_{A_n \setminus E_n} |\mathcal{K} - \mathcal{K}_1| |f| dx &\leq 2^n \int_{A_n} |f| dx \\ &\leq \int_{A_n} |x| |f(x)| dx. \end{aligned} \tag{10.8}$$

Since  $\mathcal{K} \in BMO$ , we have  $|\mathcal{K}_{2^{n+1}} - \mathcal{K}_{2^n}| \lesssim 1$ , and therefore  $|\mathcal{K}_{2^n} - \mathcal{K}_1| \lesssim n$ .

Applying this to the first term implies,

$$\begin{aligned}
\int_{E_n} |\mathcal{K} - \mathcal{K}_1| |f| dx &\leq \left( \int_{E_n} |\mathcal{K} - \mathcal{K}_1|^{p'} dx \right)^{1/p'} \|f\|_p \\
&= \left( p' \int_{2^n}^\infty |\{\mathcal{K} - \mathcal{K}_1\} > \lambda\} \cap A_n| \lambda^{p'-1} d\lambda \right)^{1/p'} \|f\|_p \\
&\leq \left( p' \int_{2^n}^\infty |\{\mathcal{K} - \mathcal{K}_{2^n}\} > \lambda - Cn\} \cap A_n| \lambda^{p'-1} d\lambda \right)^{1/p'} \|f\|_p,
\end{aligned}$$

for some  $C > 0$ . Using the John-Nirenberg inequality [190]

$$\begin{aligned}
\int_{E_n} |\mathcal{K} - \mathcal{K}_1| |f| dx &\lesssim \left( |A_n| \int_{2^n}^\infty e^{-\lambda+cn} \lambda^{p'-1} d\lambda \right)^{1/p'} \|f\|_p \\
&\lesssim \left( 2^{nd} e^{cn} \int_{2^n}^\infty e^{-\lambda} \lambda^{p'-1} d\lambda \right)^{1/p'} \|f\|_p.
\end{aligned}$$

Therefore, for large  $n$  we have,

$$\int_{E_n} |\mathcal{K} - \mathcal{K}_1| |f| dx \lesssim 2^{(d+p'-1)n/p'} e^{cn/p'} e^{-2n/p'} \|f\|_p. \quad (10.9)$$

Summing (10.8) and (10.9) over  $n$  then implies

$$T2 \lesssim_p \|f\|_p + \mathcal{M}_1.$$

□

## 10.5 End-Point Morrey's Inequality

We acknowledge Jonas Azzam for assistance in proving this inequality.

**Proposition 6.** *Let  $v \in L_{loc}^1(\mathbb{R}^d)$  such that  $\nabla v \in BMO(\mathbb{R}^d)$ . Then for all  $x, y \in \mathbb{R}^d$  with  $|x - y| < 1$  we have,*

$$|v(x) - v(y)| \lesssim_d |x - y| |\log |x - y|| \|\nabla v\|_{BMO}.$$



*Proof.* We first follow the proof of Morrey's inequality in Evans [81]. Let  $x, y \in \mathbb{R}^d$  and let  $r = |x - y|$ . Define  $W := B(x, r) \cap B(y, r)$ . By the triangle inequality,

$$|v(x) - v(y)| \leq \int_W |v(x) - v(z)| dz + \int_W |v(y) - v(z)| dz.$$

Moreover,

$$\int_W |v(x) - v(z)| dz \leq \frac{C}{|B(x, r)|} \int |v(x) - v(z)| dz,$$

where  $C$  depends on the ratio between  $|B(x, r)|$  and  $|W|$ , which is fixed in  $r$ . The averaging argument in Evans' proof of Morrey's inequality implies,

$$\int_{B(x, r)} |v(x) - v(z)| dz \leq C \int_{B(x, r)} \frac{|\nabla v(z)|}{|z - x|^{d-1}} dz.$$

We are concerned with controlling this integral for  $r \ll 1$ . Without loss of generality we may assume  $x = 0$ . For notational simplicity, define  $\nabla v(y) = f(y)$ ,  $f_A := \int_A f dx$  and  $B_k := B_{2^{-k}}$ . Without loss of generality we can replace  $f$  by  $f - f_{B_1}$ , as this changes the derivative of  $v$  by a constant of  $\mathcal{O}(1)$  and would result in a negligible  $\mathcal{O}(r)$  error term in the final estimate. By  $f \in BMO$ , for all  $k \in \mathbb{N}$  we have,

$$\int_{B_k} |f - f_{B_k}| dx \leq c 2^{-dk} \|f\|_{BMO}.$$

Now let us estimate the oscillation in the means between different length scales,

$$\begin{aligned} |f_{B_k} - f_{B_{k-1}}| &= \left| \int_{B_k} f - f_{B_{k-1}} dx \right| \\ &\leq \int_{B_k} |f - f_{B_{k-1}}| dx \\ &\leq c \int_{B_{k-1}} |f - f_{B_{k-1}}| dx \\ &\leq c \|f\|_{BMO}, \end{aligned}$$

which implies  $|f_{B_k} - f_{B_1}| \leq ck \|f\|_{BMO}$ . We come to the main estimate, which

breaks the integral into successive length-scales,

$$\begin{aligned}
\int_{|z|<r} \frac{|f - f_{B_1}|}{|z|^{n-1}} dz &\sim \sum_{k \geq \log_2 r}^{\infty} \int_{|z| \sim 2^{-k}} \frac{|f - f_{B_1}|}{2^{-k(n-1)}} dz \\
&\leq \sum_{k \geq \log_2 r}^{\infty} \int_{|z| \leq 2^{-k}} \frac{|f - f_{B_1}|}{2^{-k(n-1)}} dz \\
&\leq c \|f\|_{BMO} \sum_{k \geq \log_2 r}^{\infty} \frac{k 2^{-kn}}{2^{-k(n-1)}} = c \|f\|_{BMO} \sum_{k \geq \log_2 r} k 2^{-k}.
\end{aligned}$$

It remains to sum the series. The elementary computation is as follows,

$$\sum_{k \geq \log_2 r} k x^{k-1} \frac{d}{dx} \sum_{k \geq \log_2 r} x^k = \frac{d}{dx} \frac{x^{\log_2 r}}{1-x}.$$

This finally implies,

$$\int_{B(x,r)} \frac{\nabla v(z)}{|z-x|^{n-1}} dz \lesssim \|\nabla v\|_{BMO} (r |\log r| - r),$$

the latter term being negligible for  $r \ll 1$ . □

*Remark 22.* Following the proof of Morrey's inequality in [81], this also implies that if  $v \in L^p$  for any  $1 \leq p < \infty$ , then  $v \in L^\infty$ .

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