

# ON THE CONTINUITY OF IMAGES BY TRANSMISSION IMAGING

CHUNLIN WU\*

**Abstract.** *Transmission* imaging is an important imaging technique which is widely used in astronomy, medical diagnosis, and biology science, whose imaging principle is quite different from that of *reflection* imaging used in our everyday life. As well known, images by reflection imaging are usually modeled as discontinuous functions and even piecewise constant functions in most cases. Because of this, many successful image processing techniques are based on the discontinuities (edges) of the image, such as the popular total variation (TV) regularization. Very recently, many scholars studied the application of TV regularization to processing images generated by transmission imaging and piecewise constant images were widely used to test their algorithms. In this paper we show that almost all images by transmission imaging are actually continuous functions and consequently TV regularization for this kind of images can be greatly improved. For the convenience of description, we will consider transmission imaging with parallel line geometry of wave beam, which is a fundamental geometry of wave beam in transmission imaging and has been extensively applied in microscopes. In this kind of imaging, people take images of the physical scene from many different projection directions. We will prove that for almost every projection direction, the image generated by transmission imaging is a continuous function, even if the density function of the physical scene is discontinuous. The set of projection directions generating discontinuous images has measure zero. Hence, almost every image taken in practice is continuous (regardless of noise). If the density functions of the objects to be imaged are radial regardless of some coordinate shifts, then all the projection directions generate continuous images. As far as we know, this continuity property has not been published yet in the literature. As a straightforward application, we finally present a simple yet effective improvement of TV regularization approach for Poisson noise (which is the most significant noise in transmission imaging) removal. Numerical examples and comparisons verify our analysis and demonstrate the effectiveness of the improved model.

**Key words.** transmission imaging, reflection imaging, Radon transform, parallel line geometry, continuity, measure zero, radial function, Poisson noise, variational method

**AMS subject classifications.** 92C55, 90C90, 68U10

**1. Introduction.** Imaging is an important technique which translates a physical scene to lower dimensional (typically 2D) data for convenient observation and record. It has been applied to many fields, including our everyday life, medical diagnosis, exploring the universe, and biological structure analysis. Many imaging systems and instruments, such as various digital cameras, X-ray CT, telescopes, and microscopes, have been developed. Different imaging systems are based on different physical principles. Digital cameras used in our everyday life record the reflection part of the incoming light [18], whereas transmission electron microscopes generate images by counting the electrons having transmitted the scene [15, 10, 19]. We refer to these two kinds of imaging techniques by *reflection* imaging and *transmission* imaging in this paper for clearness.

As well known, images by reflection imaging are usually modeled as discontinuous functions and even piecewise constant functions in most cases. Consequently most of them have sparse gradients. Due to this property, many image processing techniques, such as the popular total variation (TV) regularization [23], have achieved great successes.

Although the central topic in transmission imaging is the reconstruction of the 3D objects from their 2D projection data (which are our so-called images), restoration of these data (before 3D reconstruction) is sometimes also important due to the involvement of noise and other degradations during the imaging procedure. A typical

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\*Department of Mathematics, National University of Singapore, Singapore.

problem is how to remove the Poisson noise (which is the dominant noise) in images by transmission imaging. Very recently, many scholars studied the application of TV regularization to this ill-posed problem [2, 8, 16, 11, 21, 25, 29, 32] and piecewise constant images were widely used in their numerical tests.

In this paper we show that almost all images by transmission imaging are actually continuous and consequently TV regularization for this kind of images can be greatly improved. For convenience of description and consistency of notation, we consider transmission imaging with parallel line geometry of wave beam, which is a fundamental beam geometry [15, 19] and has been extensively used in microscopes [10]. In transmission imaging (with parallel line geometry), people take images (also called projections in the literature) from many different projection directions in order to reconstruct the density functions of the imaged objects. Each projection direction corresponds to one image. We will prove that for almost every projection direction, the generated image is a continuous function, even if the density functions of the imaged objects are discontinuous (discontinuous density functions are very common). The set of projection directions generating discontinuous images has measure zero. If the density functions of the objects to be imaged are radial regardless of some coordinate shifts, then all the projection directions will generate continuous images. As well known, Radon Transform [22] is the essential mathematical tool to describe the imaging procedure. As far as we know, theoretical results on Radon Transform in the literature focus on the analysis of the imaging procedure as a mapping operator [14, 15, 19], e.g., the invertability of the operator. In addition, most analysis assume that the density function of the object to be imaged is a continuous or even Schwartz function all over the Euclidean space [14, 19, 22]. So far our analysis has not been appeared yet in the literature, although discontinuous density functions of objects are very common and thus important in applications. As a consequence of our results, most current digital image processing techniques which are based on the discontinuities (edges) of the image, may not be best for images generated via transmission imaging. The popular TV regularization for Poisson noise removal [2, 8, 16, 11, 21, 25, 29, 32] can be greatly improved. We will finally present a simple variational approach for Poisson noise removal by appropriately combining some existing regularization and fidelity terms, which improves the TV regularization based methods [2, 8, 16, 11, 21, 25, 29, 32] dramatically.

The paper is organized as follows. To better understand the background of our analysis, we will present in section 2, the principles of reflection imaging and transmission imaging and brief comparisons between the features of images generated by them. In section 3, we will focus on the continuity problem of images taken via transmission imaging. As a straightforward application of our analysis, a simple yet effective improvement of the most popular TV regularization for Poisson noise removal will be presented in section 4. The paper is concluded in section 5.

**2. Principles of reflection imaging and transmission imaging.** As indicated by the terms, reflection imaging relies on the reflection of the light wave on the surface of the objects [18], whereas transmission imaging is based on some radial wave (such as electron wave or X-ray) arriving at the image plane by transmitting the physical scene [15, 10, 19]. See Fig. 2.1. For simplicity, we here focus on the essentially different parts of these two imaging procedures and omit other common parts (such as focus and magnification subsystems). In the following we will explain the principles of these two imaging procedures in both physical and mathematical points of view with brief comparisons.

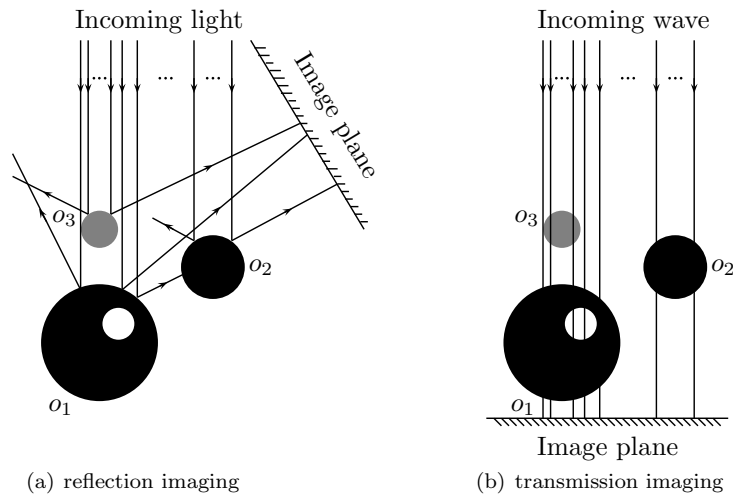


FIG. 2.1. A simple illustration of reflection and transmission imaging

The principle and implementation of an imaging system depends on the physical properties of both the objects to be imaged and the source (e.g., visible light, X-ray, and electron beam) used for imaging. Usually the source produces a composition of waves with frequencies in a certain range (including the case of just one single frequency). Waves with different frequencies have different reflection and transmission abilities. The visible light in our everyday life has lower frequencies and thus is difficult to transmit most objects. When it hits most objects, a portion of it reflects while the remainder is absorbed by the objects. Different objects can absorb waves with different frequencies. Consequently waves with different frequencies are reflected. We do recognize different objects by these reflected waves, which exhibit different colors. Digital cameras rely on this reflection phenomenon of the visible light. See Fig. 2.1 (a), where  $o_1, o_2, o_3$  are three objects and the image plane records the reflection light from  $o_1, o_2, o_3$ . Note that not all the reflection light of an object can be observed by the image plane, since some portions of the reflection light may be blocked by other objects.

The situation is totally different in transmission imaging case. Transmission imaging adopts the strong transmission ability of the source, e.g.,  $\gamma$  radial, X-ray and high speed electron beams, whose frequencies are much higher. This kind of imaging techniques are mostly applied to medical diagnosis and biological structure analysis [15, 10]. Ideally, the source produces a wave with a single frequency during one single measurement. When hitting a medical or biological scene (e.g., some biological tissue or specimen), the wave transmits the objects and then arrives at the image plane. During the transmission, the wave interacts with the objects. Usually this interaction is very complicated and very hard to precisely describe, especially in micro structure analysis at molecular or atomic level. Fig. 2.1 (b) is an illustration, where  $o_1, o_2, o_3$  are three (biological) objects and the image plane records the electrons successfully transmitting the objects. As mentioned before, here we consider transmission imaging with parallel line geometry of the wave beam [15, 10, 19]. Some other types of geometry such as fan beam are introduced in [15, 19].

In the following, we explain these two imaging procedures in mathematical point

of view. As shown in Fig. 2.1, we assume three objects  $o_1, o_2, o_3$  to be imaged. We represent them via their density functions  $\rho_1(\mathbf{r}), \rho_2(\mathbf{r}), \rho_3(\mathbf{r})$  ( $\mathbf{r} \in \mathbf{R}^3$ ). These functions can be mass density, electron density, or others of the objects, depending on by which density the objects will interact with the source wave.

In the reflection imaging case, the light wave hits the objects and reflects. The reflection depends on the material and the position of the outer boundary of the object. (Note that we can always assume that the material of a single object is nearly homogeneous; otherwise we can further decompose it.) Each object will reflect waves with some particular small range of frequencies (even particular single frequency in many cases), which indicate the colors of the object. The direction of the reflected wave and within which domain of the image plane the wave will arrive at depend on the position of the outer boundary of the object. The inner boundary of an object (see  $o_1$ ), and actually its whole interior body, have no effect on the reflection. Therefore, if we consider only  $o_1$  and remove  $o_2, o_3$  from the scene, an intensity function, which is nearly constant within its domain and usually discontinuous at the boundary, will be recorded on the image plane. Suppose that the intensity function generated by  $o_1$  is  $I^r(o_1; r)$ . Similarly, we have  $I^r(o_2; r)$  and  $I^r(o_3; r)$ . Here the super script  $r$  means ‘‘reflection imaging’’ and the argument  $r$  is the coordinate in the image plane. Now, we consider the whole scene. If no object is warded off by others, then the total intensity in the image plane is simply a linear combination:

$$I^r(o_1; r) + I^r(o_2; r) + I^r(o_3; r).$$

However, in most scenes some objects are warded off by others. A portion of the reflection light of the warded object is blocked. The image plane records the sum of the blocked reflection light, which reads

$$I^r(o_1 + o_2 + o_3; r) = b_1(r)I^r(o_1; r) + b_2(r)I^r(o_2; r) + b_3(r)I^r(o_3; r), \quad (2.1)$$

where  $b_i(r), i = 1, 2, 3$  are some truncation functions indicating the block effect.  $b_i, i = 1, 2, 3$  depend on the positions of the outer boundaries of the objects and are actually indicator functions. Therefore, we have the following conclusion:

- As an operator, reflection imaging is in general nonlinear with respect to the objects in the scene.

Since the functions  $I^r(o_i; r), \forall i$  are piecewise smooth and usually discontinuous at the boundaries of their supports, and the functions  $b_i(r), \forall i$  are piecewise constant (thus discontinuous),  $I^r(o_1 + o_2 + o_3; r)$  is piecewise smooth and discontinuous at the common boundaries between objects. In most cases, the image is piecewise constant. This is exactly the classic mathematical modeling of reflection images in [18]. Nevertheless, we conclude that:

- In reflection imaging, the images are discontinuous functions. In most cases, they are piecewise constant.

This is a very important property, based on which many image processing and segmentation techniques, models, and algorithms have been proposed in the literature.

Now let us turn to the transmission imaging. See Fig. 2.1 (b). In transmission imaging, the source wave transmits the scene (such as some biological specimen) and a portion of the wave arrives at the image plane. The information recorded in the image plane is then used to infer the structure of the scene. The interaction between the wave and the objects in the scene depends on some certain density functions of the objects. Mathematically this procedure can be modeled by the Radon transform [22]. For convenience of description, we assume the direction of the wave beam in

Fig. 2.1 (b) is  $z$ . Let us first consider  $o_1$  and remove  $o_2, o_3$  from the scene. The image generated by  $o_1$  is as follows

$$I^t(o_1; r) = \int \rho_1(\mathbf{r}) dz, \quad (2.2)$$

where we use the super script  $t$  to denote “transmission imaging”. (Physically speaking, the intensity sensed in the image plane is actually  $C - I^t(o_1; r)$  with a constant  $C$  indicating the strongness of the source wave. In mathematical modeling and analysis, there is no difference between them.) Putting  $o_1, o_2, o_3$  all together in the scene, the total density function is  $\rho_1(\mathbf{r}) + \rho_2(\mathbf{r}) + \rho_3(\mathbf{r})$ , since in many cases we can omit the interaction between these objects. Therefore, the total intensity sensed in the image plane is

$$I^t(o_1 + o_2 + o_3; r) = \int \rho_1(\mathbf{r}) + \rho_2(\mathbf{r}) + \rho_3(\mathbf{r}) dz = I^t(o_1; r) + I^t(o_2; r) + I^t(o_3; r). \quad (2.3)$$

Hence we conclude:

- As an operator, transmission imaging is linear with respect to the objects in the scene for given projection direction.

In the following section we focus on analyze the features of 2D images (projections) recorded by the image plane. As mentioned previously, in transmission imaging with parallel line geometry, people take images from many projection directions. Each projection direction corresponds to an image. We will show that most images generated in transmission imaging can be modeled as continuous functions, even if the density functions of objects are discontinuous. This is totally different from reflection imaging.

**3. The continuity property of images generated by transmission imaging.** In this section, we study the continuity property of images taken by transmission imaging with parallel line geometry of wave beam. We will discuss the topic in two cases, which are very common in real applications. In the first case compactly supported density functions are considered. Real applications with density functions such as mass density fall into this case. We will prove that for almost every projection direction, the image taken in this direction is a continuous function. The set of discontinuous images corresponds to a set of projection directions which has measure zero. In the second case we consider radial density functions (not restricted to be compactly supported). We will show that in this case, all images from all the projection directions are continuous functions. All of these results will be presented in general Euclidean space  $\mathbf{R}^n$  (typically  $n = 2, 3$ ).

We first give some notation. We use  $B_{\mathbf{r}, \delta}$  to denote the open ball centered at  $\mathbf{r}$  with radii  $\delta$ .  $S^{n-1}$  is denoted for the  $n - 1$  dimensional unit sphere. Its upper part is denoted by

$$S_+^{n-1} = \{\mathbf{r} = (r_1, r_2, \dots, r_n) \in \mathbf{R}^n : |\mathbf{r}| = 1, r_n > 0\} . \quad (3.1)$$

A straight line  $l \subset \mathbf{R}^n$  is expressed as

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t, \quad -\infty < t < +\infty, \quad (3.2)$$

where  $\mathbf{r}_0 \in \mathbf{R}^n$ ; and  $\mathbf{v} \in \mathbf{R}^n$  is a unit direction vector, i.e.,  $|\mathbf{v}| = 1$ . For simplicity, we denote a straight line determined by  $\mathbf{r}_0$  and  $\mathbf{v}$  as  $l_{\mathbf{r}_0, \mathbf{v}}$  in the following.

In transmission imaging, people take images of objects from different projection directions by computing the integrals of the density of the objects along straight lines with those directions. Regardless of the symmetry of direction, we need only to consider those images generated in the projection directions, which are elements of one half of the  $n - 1$  dimensional sphere denoted by

$$S_{\frac{1}{2}}^{n-1} = \{\mathbf{v} \in \mathbf{R}^n : |\mathbf{v}| = 1, (\theta_1, \theta_2, \dots, \theta_{n-1}) \in [0, \pi]^{n-2} \times [0, \pi),$$

$$(\theta_1, \theta_2, \dots, \theta_{n-1}) \text{ are the Euler angles of } \mathbf{v}\} \quad (3.3)$$

For each  $\mathbf{v}$  in  $S_{\frac{1}{2}}^{n-1}$ , people take one image from that projection direction. We denote the image generated by  $\mathbf{v} \in S_{\frac{1}{2}}^{n-1}$  as  $I_{\mathbf{v}}^t$  (we remind that the super script  $t$  means “transmission imaging”). If there are no repeated images, the images are as many as the points in  $S_{\frac{1}{2}}^{n-1}$ .

**3.1. Compactly supported density functions.** Before presenting our results, we give an example in  $\mathbf{R}^2$ . See Fig. 3.1. Here the object to be imaged is a rectangle  $o$ , whose density function is supported in the rectangle and is constant  $\rho(\mathbf{r}) \equiv 0.5$ . In transmission imaging, the object is projected from different directions to the corresponding image planes. The projection from each direction generates one image. In  $\mathbf{R}^2$ , the projection direction is  $\mathbf{v} \in S_{\frac{1}{2}}^1$ , or, equivalently, can be described by an azimuthal  $\theta \in [0, \pi)$ . That is, the unit vectors indicating the projection directions are on a half circle. In Fig. 3.1, we only show two projection directions and the corresponding images of the object. As one can see, the image in Fig. 3.1 (a) is continuous over the whole image plane ( $\mathbf{R}^1$  here), whereas the image in Fig. 3.1 (b) is discontinuous. This example shows that different projection directions may generate images with different smoothness, even for the object of constant density. In this example, there are in total 2 projection directions (parallel to the edges of the rectangle respectively) which generate discontinuous images. In the following, we will prove that the set of projection directions generating discontinuous images has measure zero within the set of all the projection directions. We will present our results in general Euclidean space  $\mathbf{R}^n$ .

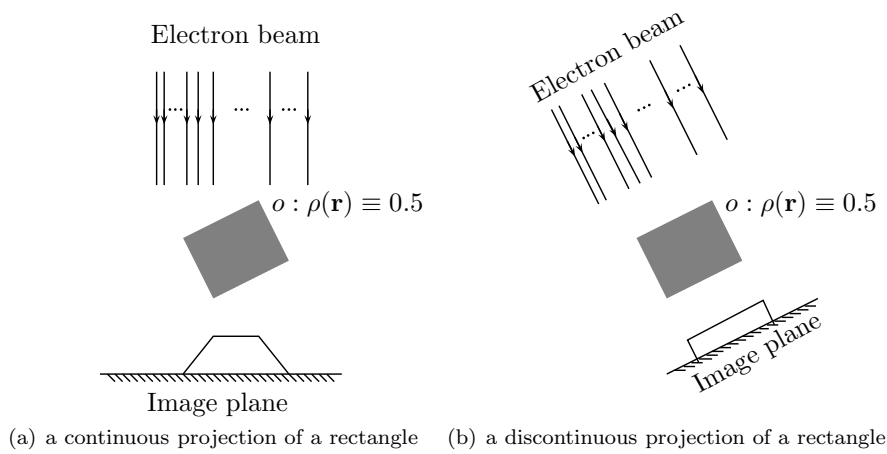


FIG. 3.1. An example of transmission imaging in  $\mathbf{R}^2$ .

By the linearity of the transmission imaging operator, we only need to consider just one single object. We denote it by  $o$ , whose density function is  $\rho(\mathbf{r})$ . The support of  $\rho(\mathbf{r})$  is  $D$  with boundary  $B$ . In almost all real applications, we can assume:

Assump. 1  $\rho(\mathbf{r})$  is compactly supported, i.e.,  $D$  is bounded;

Assump. 2  $\rho(\mathbf{r})$  is continuous over  $D$ ;

Assump. 3  $B$  is a continuous and piecewise  $C^1$  hypersurface;

Assump. 4 For any straight line  $l$  in  $\mathbf{R}^n$ ,  $l$  intersects  $B$  at either empty set, or finite number of points, or finite number of line segments, or the union of finite number of points and finite number of line segments.

A common example is that  $D$  is a polyhedron and  $\rho(\mathbf{r})$  is constant over  $D$ .

LEMMA 3.1. Assume that  $g$  is the metric tensor of the mapping  $\varphi$  from the  $n-1$  dimensional open ball  $B_{\mathbf{0}_{n-1},1}$  ( $\mathbf{0}_{n-1}$  is the origin of  $\mathbf{R}^{n-1}$ ) to  $S_+^{n-1}$  defined by

$$(v_1, v_2, \dots, v_{n-1}) \rightarrow (v_1, v_2, \dots, v_{n-1}, v_n) = (v_1, v_2, \dots, v_{n-1}, \sqrt{1 - \sum_{1 \leq i \leq n-1} v_i^2}).$$

Then  $\det g = \frac{1}{1 - \sum_{1 \leq i \leq n-1} v_i^2} = \frac{1}{v_n^2}$ .

**Proof** By basic calculation we have

$$g = I + V,$$

where  $I$  is the identity matrix and  $V = (v_{ij})$  with elements  $v_{ij} = \frac{v_i v_j}{v_n^2}$ . From linear algebra, we know

$$\det g = 1 + \sum_{1 \leq k \leq n-1} \sum_{i_1 < i_2 < \dots < i_k} V \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ i_1 & i_2 & \dots & i_k \end{pmatrix},$$

where  $V \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ i_1 & i_2 & \dots & i_k \end{pmatrix}$  is a principle minor of  $V$ . It is easy to see that all the principle minors with orders greater than 1 have zero determinant. Therefore,

$$\det g = 1 + \sum_{i_1} V \begin{pmatrix} i_1 \\ i_1 \end{pmatrix} = \frac{1}{v_n^2},$$

which completes the proof. ■

Based on Lemma 3.1, we can prove the following result.

THEOREM 3.2. Assume  $Z = \{\mathbf{v} \in S_{\frac{1}{2}}^{n-1} : \exists \mathbf{r}_0, \text{ s.t. } m(l_{\mathbf{r}_0, \mathbf{v}} \cap B) > 0\}$  where  $m(l_{\mathbf{r}_0, \mathbf{v}} \cap B)$  is the Lebesgue measure in  $\mathbf{R}^1$  (along the straight line). Then  $Z$  has zero spherical measure.

**Proof** We have

$$Z = Z_+ \cup Z_0,$$

where

$$\begin{aligned} Z_+ &= \{\mathbf{v} = (v_1, v_2, \dots, v_n) \in S_{\frac{1}{2}}^{n-1} : v_n > 0, \exists \mathbf{r}_0, \text{ s.t. } m(l_{\mathbf{r}_0, \mathbf{v}} \cap B) > 0\} \\ &= \{\mathbf{v} = (v_1, v_2, \dots, v_n) \in S_+^{n-1} : \exists \mathbf{r}_0, \text{ s.t. } m(l_{\mathbf{r}_0, \mathbf{v}} \cap B) > 0\}, \end{aligned}$$

and

$$Z_0 = \{\mathbf{v} = (v_1, v_2, \dots, v_n) \in S_{\frac{1}{2}}^{n-1} : v_n = 0, \exists \mathbf{r}_0, \text{ s.t. } m(l_{\mathbf{r}_0, \mathbf{v}} \cap B) > 0\}.$$

As a subset of  $\{\mathbf{v} = (v_1, v_2, \dots, v_n) \in S^{n-1} : v_n = 0\}$ ,  $Z_0$  has zero spherical measure.

To prove that  $Z_+$  also has zero spherical measure, we introduce the following set

$$P_{\mathbf{r}_0, \mathbf{v}} = \{r_n \in \mathbf{R}^1 : r_n = \mathbf{r} \cdot (0, 0, \dots, 0, 1), \forall \mathbf{r} \in (l_{\mathbf{r}_0, \mathbf{v}} \cap B)\},$$

which is actually the projection of  $l_{\mathbf{r}_0, \mathbf{v}} \cap B$  to the  $(0, 0, \dots, 0, 1)$  direction. We then have

$$Z_+ = \{\mathbf{v} = (v_1, v_2, \dots, v_n) \in S_+^{n-1} : \exists \mathbf{r}_0, \text{ s.t. } m(P_{\mathbf{r}_0, \mathbf{v}}) > 0\},$$

which can be written as

$$\begin{aligned} Z_+ &= \bigcup_{k=1}^{\infty} \{\mathbf{v} = (v_1, v_2, \dots, v_n) \in S_+^{n-1} : \exists \mathbf{r}_0, \text{ s.t. } m(P_{\mathbf{r}_0, \mathbf{v}}) > \frac{1}{k}\} \\ &= \bigcup_{k=1}^{\infty} Z_+^k. \end{aligned}$$

Now let us consider the set

$$\begin{aligned} A_+ &= \{\mathbf{v}_{n-1} = (v_1, v_2, \dots, v_{n-1}) \in \mathbf{R}^{n-1} : |\mathbf{v}_{n-1}| < 1, \\ &\quad \mathbf{v} = (v_1, v_2, \dots, v_{n-1}, \sqrt{1 - \sum_{1 \leq i \leq n-1} v_i^2}) \in S_+^{n-1}, \exists \mathbf{r}_0, \text{ s.t. } m(P_{\mathbf{r}_0, \mathbf{v}}) > 0\}, \end{aligned}$$

which can be further written as

$$\begin{aligned} A_+ &= \bigcup_{k=1}^{\infty} \{\mathbf{v}_{n-1} = (v_1, v_2, \dots, v_{n-1}) \in \mathbf{R}^{n-1} : |\mathbf{v}_{n-1}| < 1, \\ &\quad \mathbf{v} = (v_1, v_2, \dots, v_{n-1}, \sqrt{1 - \sum_{1 \leq i \leq n-1} v_i^2}) \in S_+^{n-1}, \exists \mathbf{r}_0, \text{ s.t. } m(P_{\mathbf{r}_0, \mathbf{v}}) > \frac{1}{k}\} \\ &= \bigcup_{k=1}^{\infty} A_+^k. \end{aligned}$$

As one can see,  $A_+$  and  $A_+^k$  are the restrictions of  $Z_+$  and  $Z_+^k$  in the  $\mathbf{R}^{n-1}$ .

We claim that, for any  $k$ ,  $A_+^k$  has zero outer Lebesgue measure in  $\mathbf{R}^{n-1}$ , i.e.,  $m^*(A_+^k) = 0$ . Otherwise, let us suppose  $m^*(A_+^k) > 0$ . Then the following set

$$\begin{aligned} E_{\mathbf{r}_0, \mathbf{v}} &= \{\mathbf{r} = (v_1, v_2, \dots, v_{n-1}, r_n) \in \mathbf{R}^n : (v_1, v_2, \dots, v_{n-1}) \in A_+^k, r_n \in P_{\mathbf{r}_0, \mathbf{v}}, \\ &\quad \mathbf{v} = (v_1, v_2, \dots, v_{n-1}, \sqrt{1 - \sum_{1 \leq i \leq n-1} v_i^2})\} \end{aligned}$$

has positive outer Lebesgue measure in  $\mathbf{R}^n$ . Since any open cover of  $l_{\mathbf{r}_0, \mathbf{v}} \cap B$  can induce an open cover of  $E_{\mathbf{r}_0, \mathbf{v}}$  by some translations, we conclude that the following set

$$\bigcup_{\mathbf{v} \in Z_+^k} (l_{\mathbf{r}_0, \mathbf{v}} \cap B),$$

as a subset of  $B$ , has positive outer Lebesgue measure in  $\mathbf{R}^n$ . This contradicts that  $B$  has zero Lebesgue measure in  $\mathbf{R}^n$ .



Therefore, we obtain  $m(A_+) = 0$  in  $\mathbf{R}^{n-1}$ . On the other hand,  $A_+$  can be reformulated as

$$A_+ = \bigcup_{k=1}^{\infty} \{ \mathbf{v}_{n-1} \in A_+ : v_n = \sqrt{1 - \sum_{1 \leq i \leq n-1} v_i^2} > \frac{1}{k} \}.$$

By Lemma 3.1, one can show that  $Z_+$  has zero spherical measure. The theorem is proven.  $\blacksquare$

**THEOREM 3.3.** *In transmission imaging with parallel line geometry, for almost every projection direction  $\mathbf{v} \in S_{\frac{1}{2}}^{n-1}$ , the image taken in direction  $\mathbf{v}$  is a continuous function. The subset of projection directions generating discontinuous images has zero measure on  $S_{\frac{1}{2}}^{n-1}$ .*

**Proof** We will prove that  $\forall \mathbf{v} \in S_{\frac{1}{2}}^{n-1} \setminus Z$ ,  $I_{\mathbf{v}}^t$  is a continuous function. Here  $Z$  is as in Theorem 3.2.

According to our assumptions,  $\forall \mathbf{v} \in S_{\frac{1}{2}}^{n-1} \setminus Z$ , any straight line with direction  $\mathbf{v}$  intersects with  $B$  at empty set, or finite number of points.

Without loss of generality, we can assume  $\mathbf{v} = (0, 0, \dots, 0, 1)$  (if  $(0, 0, \dots, 0, 1) \in Z$ , we can rotate the coordinate system to make sure in the new system the  $n$ 'th axis is not in  $Z$ ). In this case, we will generate an image whose pixels are in the  $\mathbf{R}^{n-1}$  space with coordinates  $\mathbf{r}_{n-1} = (r_1, r_2, \dots, r_{n-1})$ . We denote this image as  $I_{\mathbf{v}}^t(o; r_1, r_2, \dots, r_{n-1})$ . We now show that it is a continuous function with respect to the image coordinates  $(r_1, r_2, \dots, r_{n-1})$ . At any  $\mathbf{r}_{n-1}^* = (r_1^*, r_2^*, \dots, r_{n-1}^*) \in \mathbf{R}^{n-1}$ , the straight line for integration can be written as

$$\mathbf{r} = (\mathbf{r}_{n-1}^*, 0) + \mathbf{v}t, \quad -\infty < t < +\infty,$$

which is, in short,  $l_{(\mathbf{r}_{n-1}^*, 0), \mathbf{v}}$ . We discuss the continuity in three cases, respectively.

- a.  $l_{(\mathbf{r}_{n-1}^*, 0), \mathbf{v}} \cap B = \emptyset$ . In this case, there exists a neighborhood  $B_{\mathbf{r}_{n-1}^*, \delta} \subset \mathbf{R}^{n-1}$ , such that  $\forall \mathbf{r}_{n-1} \in B_{\mathbf{r}_{n-1}^*, \delta}$ ,  $l_{(\mathbf{r}_{n-1}, 0), \mathbf{v}} \cap B = \emptyset$ . Therefore, the image intensity  $I_{\mathbf{v}}^t(o; \mathbf{r}_{n-1}) = 0$  for  $\forall \mathbf{r}_{n-1} \in B_{\mathbf{r}_{n-1}^*, \delta}$ . It is thus continuous at  $\mathbf{r}_{n-1}^*$ .
- b.  $l_{(\mathbf{r}_{n-1}^*, 0), \mathbf{v}} \cap B = \{(\mathbf{r}_{n-1}^*, r_n^{*,i}) : i = 1, \dots, 2M, r_n^{*,1} < r_n^{*,2} < \dots < r_n^{*,2M}\}$  and all the intersecting points are secant points. By the continuity of the boundary  $B$  and that any intersecting point is a secant point,  $\forall 1 \leq i \leq 2M$ , there exists a local continuous mapping  $\varphi^i$  from a small neighborhood  $B_{\mathbf{r}_{n-1}^*, \delta^i} \subset \mathbf{R}^{n-1}$  to a small neighborhood of  $(\mathbf{r}_{n-1}^*, r_n^{*,i})$  on  $B$ , which is

$$\varphi^i(\mathbf{r}_{n-1}) = (\mathbf{r}_{n-1}, h^i(\mathbf{r}_{n-1})), \quad \forall \mathbf{r}_{n-1} \in B_{\mathbf{r}_{n-1}^*, \delta^i},$$

for some continuous function  $h^i(\mathbf{r}_{n-1})$  (otherwise there will be more than one intersections between  $l_{(\mathbf{r}_{n-1}^*, 0), \mathbf{v}}$  and  $B$  around this secant point). This also means that,  $\forall 1 \leq i \leq 2M$ , the secant point  $(\mathbf{r}_{n-1}^*, r_n^{*,i})$  is in the interior of the set  $\varphi^i(B_{\mathbf{r}_{n-1}^*, \delta^i})$ . It follows that,  $\forall \mathbf{r}_{n-1} \in B_{\mathbf{r}_{n-1}^*, \delta^i}$ ,  $l_{(\mathbf{r}_{n-1}, 0), \mathbf{v}}$  intersects with  $\varphi^i(B_{\mathbf{r}_{n-1}^*, \delta^i})$  at a secant point. Note that we can choose  $\delta^i$  small enough so that  $\varphi^i(B_{\mathbf{r}_{n-1}^*, \delta^i}) \cap \varphi^j(B_{\mathbf{r}_{n-1}^*, \delta^j}) = \emptyset$ ,  $\forall i \neq j$ , since  $r_n^{*,1} < r_n^{*,2} < \dots < r_n^{*,2M}$ . Let  $\delta = \min(\delta_1, \delta_2, \dots, \delta_{2M})$ . We conclude that,  $\forall \mathbf{r}_{n-1} \in B_{\mathbf{r}_{n-1}^*, \delta}$ ,  $l_{(\mathbf{r}_{n-1}, 0), \mathbf{v}}$  intersects with  $B$  at a series of secant points  $\{(\mathbf{r}_{n-1}, r_n^i) : 1 \leq i \leq 2M, r_n^1 < r_n^2 < \dots < r_n^{2M}\}$ . Again by the continuity,  $\forall i = 1, 2, \dots, 2M$ ,  $|r_n^i - r_n^{*,i}| < \hat{\delta}$ ;

and  $\hat{\delta} \rightarrow 0$  as  $\delta \rightarrow 0$ . This, together with the continuity of the density function  $\rho(\mathbf{r})$ , indicates that

$$\begin{aligned}
& I_{\mathbf{v}}^t(o; \mathbf{r}_{n-1}) - I_{\mathbf{v}}^t(o; \mathbf{r}_{n-1}^*) \\
&= \int_{l_{(\mathbf{r}_{n-1}, 0), \mathbf{v}}} \rho(\mathbf{r}) dl - \int_{l_{(\mathbf{r}_{n-1}^*, 0), \mathbf{v}}} \rho(\mathbf{r}) dl \\
&= \sum_{1 \leq i \leq M} \int_{r_n^{2i-1}}^{r_n^{2i}} \rho((\mathbf{r}_{n-1}, t)) dt - \sum_{1 \leq i \leq M} \int_{r_n^{*, 2i-1}}^{r_n^{*, 2i}} \rho((\mathbf{r}_{n-1}^*, t)) dt \\
&= \sum_{1 \leq i \leq M} \left( \int_{r_n^{2i-1}}^{r_n^{2i}} \rho((\mathbf{r}_{n-1}, t)) dt - \int_{r_n^{*, 2i-1}}^{r_n^{*, 2i}} \rho((\mathbf{r}_{n-1}^*, t)) dt \right) \\
&\rightarrow 0,
\end{aligned}$$

as  $\delta \rightarrow 0$ . Hence the image is continuous at  $\mathbf{r}_{n-1}^*$ .

- c.  $l_{(\mathbf{r}_{n-1}^*, 0), \mathbf{v}} \cap B = \{(\mathbf{r}_{n-1}^*, r_n^{*, i}) : i = 1, \dots, 2M + K, r_n^{*, 1} < r_n^{*, 2} < \dots < r_n^{*, 2M+K}\}$  within which  $2M$  points are secant points and the remainder are touching points. Assume that the index set of the secant points is  $A_1$  whereas the index set of the touching points is  $A_2$ . First, let us consider the secant points. By the argument in the above case, there always exists a neighborhood  $B_{\mathbf{r}_{n-1}^*, \delta_1}$  (with  $\delta_1$  small enough) such that  $\forall \mathbf{r}_{n-1} \in B_{\mathbf{r}_{n-1}^*, \delta_1}$  and  $\forall i \in A_1$ ,  $l_{(\mathbf{r}_{n-1}, 0), \mathbf{v}}$  intersects with  $B$  around the secant point  $(\mathbf{r}_{n-1}^*, r_n^{*, i})$  at another secant point  $(\mathbf{r}_{n-1}, r_n^i)$ . We have  $r_n^{i_1} < r_n^{i_2}, \forall i_1, i_2 \in A_1, i_1 < i_2$ . In addition, by the continuity,  $\forall i \in A_1, |r_n^i - r_n^{*, i}| < \hat{\delta}_1$ ; and  $\hat{\delta}_1 \rightarrow 0$  as  $\delta_1 \rightarrow 0$ .

Now let us consider the touching points.  $\forall i \in A_2$ , we decompose a small enough neighborhood of  $(\mathbf{r}_{n-1}^*, r_n^{*, i})$  on  $B$  to the union of the upper half  $U$  with the  $n$ 'th coordinate  $r_n \geq r_n^{*, i}$  and the lower half  $L$  with the  $n$ 'th coordinate  $r_n \leq r_n^{*, i}$ . There exist a continuous mapping from a subset of a neighborhood  $B_{\mathbf{r}_{n-1}^*, \delta_2^{i,+}}$  to  $U$ , and a continuous mapping from a subset of a neighborhood  $B_{\mathbf{r}_{n-1}^*, \delta_2^{i,-}}$  to  $L$ , respectively. Let  $\delta_2^i = \min(\delta_2^{i,+}, \delta_2^{i,-})$ . Then,  $\forall \mathbf{r}_{n-1} \in B_{\mathbf{r}_{n-1}^*, \delta_2^i}$ , the straight line  $l_{(\mathbf{r}_{n-1}, 0), \mathbf{v}}$  intersects  $U \cup L$  at an empty set, or a single touching point  $r_n^i$ , or two secant points  $r_n^{i,+}, r_n^{i,-}$ . Again, by the continuity of the mappings,  $|r_n^{i,+} - r_n^{*, i}| < \hat{\delta}_2^i, |r_n^{i,-} - r_n^{*, i}| < \hat{\delta}_2^i$ ; and  $\hat{\delta}_2^i \rightarrow 0$  as  $\delta_2^i \rightarrow 0$ . Now we let  $\delta_2 = \min\{\delta_2^i : i \in A_2\}$ . Therefore,  $\forall \mathbf{r}_{n-1} \in B_{\mathbf{r}_{n-1}^*, \delta_2}$  and  $\forall i \in A_2$ , the straight line  $l_{(\mathbf{r}_{n-1}, 0), \mathbf{v}}$  intersects  $B$  around the touching point  $(\mathbf{r}_{n-1}^*, r_n^{*, i})$  at an empty set, or a single point, or two secant points  $r_n^{i,+}, r_n^{i,-}$ . In addition,  $|r_n^{i,+} - r_n^{*, i}| < \hat{\delta}_2, |r_n^{i,-} - r_n^{*, i}| < \hat{\delta}_2$ ; and  $\hat{\delta}_2 \rightarrow 0$  as  $\delta_2 \rightarrow 0$ .

Let  $\delta = \min(\delta_1, \delta_2)$ . We conclude that,  $\forall \mathbf{r}_{n-1} \in B_{\mathbf{r}_{n-1}^*, \delta}$ ,

$$I_{\mathbf{v}}^t(o; \mathbf{r}_{n-1}) - I_{\mathbf{v}}^t(o; \mathbf{r}_{n-1}^*) \rightarrow 0, \text{ as } \delta \rightarrow 0,$$

by a similar argument as in the previous case, with a trick to delete some of the touching points  $\{(\mathbf{r}_{n-1}^*, r_n^{*, i}) : i \in A_2\}$  and make the remainder of multiplicity 2 according to how many intersections there are between the line and  $B$  around the touching points. The image is also continuous at  $\mathbf{r}_{n-1}^*$  in this case.  $\blacksquare$

Theorem 3.2 and 3.3 show that in transmission imaging with parallel line geometry, images taken in almost all the projection directions are continuous functions.

The discontinuous images are so sparse that in real applications they seldom appear. This will provide some information for processing images generated by transmission imaging, which is totally different from reflection imaging.

The following corollary is on our results applied to a combination of finite objects.

**COROLLARY 3.4.** *Suppose that a physical scene consists of finite objects  $\{o_i : 1 \leq i \leq M\}$  with compactly supported density functions  $\{\rho_i(\mathbf{r}) : 1 \leq i \leq M\}$  which are continuous over their supports  $\{D_i : 1 \leq i \leq M\}$ , respectively. The set of projection directions generating discontinuous images has measure zero on  $S_{\frac{1}{2}}^{n-1}$ . In particular, even if  $\{\rho_i(\mathbf{r}) \equiv c_i, \mathbf{r} \in D_i : c_i \in \mathbf{R} \text{ is a constant}, 1 \leq i \leq M\}$ , i.e., the whole density is piecewise constant, the result still holds.*

Since in real applications the case of  $\mathbf{R}^3$  is very important, we give the following corollary.

**COROLLARY 3.5.** *Assume that a physical scene in  $\mathbf{R}^3$  consists of finite objects with compactly supported density functions which are continuous over their domains. The set of projection directions generating discontinuous images has measure zero on  $S_{\frac{1}{2}}^2$ .*

**3.2. Radial density functions.** In some applications, the density functions are not compactly supported. However, they are radial functions or combinations of some translations of radial functions. This subsection contributes to this case.

Again we need only consider one single object. We denote it by  $o$  and assume the density function is  $\rho(|\mathbf{r}|)$ ,  $\mathbf{r} \in \mathbf{R}^n$ . Since  $\rho$  depends only on  $|\mathbf{r}|$ , the images are independent of projection directions. All the projection directions will generate the same image. Therefore, without loss of generality, we can assume the projection direction is  $\mathbf{v} = (0, 0, \dots, 0, 1)$ , for convenience of description. The coordinates in the image plane is thus  $\mathbf{r}_{n-1}$ . We then have

$$I_{\mathbf{v}}^t(o; \mathbf{r}_{n-1}) = \int_{l_{(\mathbf{r}_{n-1}, 0), \mathbf{v}}} \rho(|\mathbf{r}|) dl = \int_{-\infty}^{+\infty} \rho(\sqrt{|\mathbf{r}_{n-1}|^2 + r_n^2}) dr_n. \quad (3.4)$$

It is straightforward to verify that the image  $I_{\mathbf{v}}^t(o; \mathbf{r}_{n-1})$  is a radial function of  $\mathbf{r}_{n-1}$ . Under some mild assumptions, it is continuous.

**THEOREM 3.6.** *The image  $I_{\mathbf{v}}^t(o; \mathbf{r}_{n-1})$  is also radial, when the density of the object is a radial function.*

**THEOREM 3.7.** *Assume that  $\forall \mathbf{r}_{n-1} \in \mathbf{R}^{n-1}$ ,  $\int_{-\infty}^{+\infty} \rho^2(\sqrt{|\mathbf{r}_{n-1}|^2 + r_n^2}) dr_n < +\infty$ . Then the image  $I_{\mathbf{v}}^t(o; \mathbf{r}_{n-1})$  is continuous with respect to  $\mathbf{r}_{n-1}$ .*

**Proof** Assume that  $\mathbf{r}_{n-1}^* \in \mathbf{R}^{n-1}$ ,  $\mathbf{r}_{n-1} \in \mathbf{R}^{n-1}$  and  $\mathbf{r}_{n-1} \rightarrow \mathbf{r}_{n-1}^*$ . Without loss of generality, we consider the case of  $|\mathbf{r}_{n-1}| > |\mathbf{r}_{n-1}^*|$  (the case of  $|\mathbf{r}_{n-1}| < |\mathbf{r}_{n-1}^*|$  can be treated similarly). Let  $z = \sqrt{|\mathbf{r}_{n-1}|^2 - |\mathbf{r}_{n-1}^*|^2}$ . We have

$$\begin{aligned} I_{\mathbf{v}}^t(o; \mathbf{r}_{n-1}) - I_{\mathbf{v}}^t(o; \mathbf{r}_{n-1}^*) &= \int_{-\infty}^{+\infty} \rho(\sqrt{|\mathbf{r}_{n-1}|^2 + r_n^2}) dr_n - \int_{-\infty}^{+\infty} \rho(\sqrt{|\mathbf{r}_{n-1}^*|^2 + r_n^2}) dr_n \\ &= \int_{-\infty}^{+\infty} \rho(\sqrt{|\mathbf{r}_{n-1}|^2 + r_n^2}) dr_n \\ &\quad - \left( \int_{-\infty}^{-z} + \int_{-z}^z + \int_z^{+\infty} \right) \rho(\sqrt{|\mathbf{r}_{n-1}^*|^2 + r_n^2}) dr_n \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^{+\infty} \rho(\sqrt{|\mathbf{r}_{n-1}|^2 + r_n^2}) dr_n - 2 \int_z^{+\infty} \rho(\sqrt{|\mathbf{r}_{n-1}^*|^2 + r_n^2}) dr_n \\
&\quad - \int_{-z}^z \rho(\sqrt{|\mathbf{r}_{n-1}^*|^2 + r_n^2}) dr_n \\
&= 2 \int_0^{+\infty} \rho(\sqrt{|\mathbf{r}_{n-1}|^2 + r_n^2}) \left(1 - \frac{r_n}{\sqrt{r_n^2 + z^2}}\right) dr_n \\
&\quad - \int_{-z}^z \rho(\sqrt{|\mathbf{r}_{n-1}^*|^2 + r_n^2}) dr_n \\
&= \mathbf{I}_1 + \mathbf{I}_2.
\end{aligned}$$

We now estimate the integral  $\mathbf{I}_1$ .

$$\begin{aligned}
|\mathbf{I}_1|^2 &\leq 4 \int_0^{+\infty} \rho^2(\sqrt{|\mathbf{r}_{n-1}|^2 + r_n^2}) dr_n \int_0^{+\infty} \left(1 - \frac{r_n}{\sqrt{r_n^2 + z^2}}\right)^2 dr_n \\
&= 4 \int_0^{+\infty} \rho^2(\sqrt{|\mathbf{r}_{n-1}|^2 + r_n^2}) dr_n \int_0^{+\infty} \left(\frac{\sqrt{r_n^2 + z^2} - r_n}{\sqrt{r_n^2 + z^2}}\right)^2 dr_n \\
&< 4 \int_0^{+\infty} \rho^2(\sqrt{|\mathbf{r}_{n-1}|^2 + r_n^2}) dr_n \int_0^{+\infty} \frac{z^2}{r_n^2 + z^2} dr_n \\
&= \pi z \int_{-\infty}^{+\infty} \rho^2(\sqrt{|\mathbf{r}_{n-1}|^2 + r_n^2}) dr_n.
\end{aligned}$$

As for the integral  $\mathbf{I}_2$ , the absolute continuity of integral gives that  $|\mathbf{I}_2| \rightarrow 0$ , as  $z \rightarrow 0$ .

Therefore, as  $\mathbf{r}_{n-1} \rightarrow \mathbf{r}_{n-1}^*$  and thus  $z \rightarrow 0$ , we have  $|\mathbf{I}_1| + |\mathbf{I}_2| \rightarrow 0$ , which implies  $I_{\mathbf{v}}^t(o; \mathbf{r}_{n-1})$  is continuous at  $\mathbf{r}_{n-1}^*$ .  $\blacksquare$

**COROLLARY 3.8.** *Suppose that a physical scene consisting of finite objects  $\{o_i : 1 \leq i \leq M\}$  with translations of radial density functions  $\{\rho_i(|\mathbf{r} - \mathbf{r}_i|), \mathbf{r}_i \in \mathbf{R}^n : 1 \leq i \leq M\}$  satisfying the condition in Theorem 3.7. Then  $\forall \mathbf{v} \in S_{\frac{1}{2}}^{n-1}$ ,  $I_{\mathbf{v}}^t(o_1 + o_2 + \dots + o_M; \cdot)$  is continuous.*

**4. Improving TV regularization for Poisson noise removal.** In this section we present a straightforward application of our analysis, i.e., a simple yet effective improvement of the popular TV regularization for Poisson noise reduction of images generated by transmission imaging. We should mention that there are also many other very successful image denoising methods which are not based on energy minimization and variational principle.

TV regularization was first introduced in [23] and has been extensively applied in various digital image processing problems; see [7] and references therein. Very recently, many scholars studied the combination of TV and the Kullback-Leibler (KL) fidelity [21, 16] for Poisson noise removal [2, 8, 16, 11, 21, 25, 29, 32] and many numerical tests therein used piecewise constant images. The minimization model reads:

$$\min_u \int_{\Omega} |\nabla u| d\Omega + \alpha \int_{\Omega} (u - f \log u) d\Omega, \quad (4.1)$$

where  $f$  is an observed image defined on  $\Omega \subset \mathbf{R}^2$ ;  $\alpha$  is a model parameter. As in [29], we call this model as TV-KL model.

However, according to our results in the previous section, images generated by transmission imaging are usually continuous. Since TV often suffers from staircase

effect and generates piecewise constant solutions, it destroys the continuity structure of the true images by transmission imaging. Hence TV-KL model need to be improved.

In this section, we propose to solve the following one:

$$\min_u \int_{\Omega} \sqrt{u_x^2 + u_y^2 + u_{xx}^2 + u_{yy}^2 + u_{xy}^2 + u_{yx}^2} d\Omega + \alpha \int_{\Omega} (u - f \log u) d\Omega, \quad (4.2)$$

which combines the first order and the second order derivatives  $\nabla u$  and  $\text{Hessian}(u)$ . We call (4.2) as MoTV(Multi-order TV)-KL model, since it uses a combination of  $\nabla u$  and  $\text{Hessian}(u)$ . Here we give the following remarks to this model.

- *Remark 1.* The regularizer in (4.2) is an extension of TV and not new. Here we just combine it with KL fidelity to remove Poisson noise.
- *Remark 2.* Since the underlying density functions of imaged objects are diverse, the corresponding most suitable regularizers should be also diverse. To the best of our understanding, the proposed MoTV-KL model (4.2) may be the simplest improvement of TV-KL model. Although it seems to favor piecewise linear images, which are the simplest non-constant continuous images, it also at least improves the effectiveness of TV-KL model for more complicated continuous images (our test images are not restricted to be piecewise linear images). The aim of this section is not a perfect denoising model for all continuous images, but a simple yet effective improvement of TV-KL model, showing a straightforward application of our theoretical analysis.
- *Remark 3.* According to our test, the model involving only the second order derivatives  $\text{Hessian}(u)$  does not work well. Therefore, the  $\nabla u$  in (4.2) cannot be deleted. This indicates that, in denoising problems,  $\nabla u$  in regularization is even quite important for images with no discontinuity. This also verifies *Remark 2* in some sense that the involvement of  $\text{Hessian}(u)$  in MoTV-KL model is helpful for continuous images with continuity orders higher than piecewise linearity.

The problem (4.2) is an  $L_1$  minimization problem. Recently many efficient methods have been developed to solve this kind of problems; see, e.g., [5, 6, 4, 3, 9, 13, 17, 20, 26, 24, 27, 28, 29, 30, 31, 33] and references therein. In our implementation we applied operator splitting and augmented Lagrangian method with single inner iteration [12] to solve it based on the code of our previous work [26, 28, 29].

Two numerical examples are shown in Fig. 4.1 and 4.2. In Fig. 4.1, the image is synthesized by projecting 4 objects (1 sphere + 2 polyhedrons + 1 ellipse, with constant density functions within their domains, respectively) to an image plane. The molecular image in Fig. 4.2 was downloaded from <http://people.csail.mit.edu/gdp/cryoem.html>. The experiments were performed under Windows Vista and Matlab R14 (Version 7.0.4) on a laptop with Intel CPU (Core 2) at 2.53GHz and 4GB memory. We used  $\frac{\|u^k - u^{k-1}\|_F}{\|u^{k-1}\|_F} \leq 10^{-2}$  as the stopping condition of the iteration for all the examples, where  $\|\cdot\|_F$  denotes the Frobenius norm of the data expressed as a matrix. From Fig. 4.1 and 4.2, one can clearly see the continuity of the clean images. The MoTV-KL model (4.2) generates much smoother results with much higher SNRs than the TV-KL model (4.1). In the tests, we adjusted the model parameter  $\alpha$  to figure out how high SNRs the two models can achieve respectively. For the parameter value with which the TV-KL model generates nearly highest SNR, we computed the result by MoTV-KL model, and vice versa. We found that MoTV-KL model gives results with much higher SNRs, even with the parameter value which is best for TV-KL model. Besides, TV-KL model suffers from staircase effect, whereas MoTV-KL model

does not. The fact that the smoother solutions have higher SNRs also demonstrates our analysis on the continuity of the images generated by transmission imaging with parallel line geometry.

**5. Conclusions and future work.** In this paper, we analyzed the continuity property of images generated by transmission imaging with parallel line geometry of wave beam (which is a fundamental and widely used geometry) and showed a simple yet natural application of our theoretical analysis. Transmission imaging is widely applied in astronomy and biomedical sciences for macro and micro scale objects, which is quite different from reflection imaging frequently used in our everyday life for common scale objects. The physical mechanisms and thus mathematical models of these two imaging principles are totally different. In contrast to the well known modeling images by reflection imaging as discontinuous functions, we showed that almost all images generated by transmission imaging are continuous functions, even if the density functions of the imaged objects are discontinuous. Discontinuous images scarcely appear in real applications. This analysis has not appeared yet in the literature. Although the central topic in transmission imaging is the reconstruction of density functions of objects, processing of the image (projection) data before reconstruction is also sometimes important due to the fact that these data usually involve degradations such as Poisson noise. Our theoretical analysis may help us to understand the structures of images generated by transmission imaging and provide some information for designing and testing image processing techniques for transmission imaging. As a straightforward application, we proposed a simple yet effective improvement of the most popular TV regularization applied to Poisson noise removal. In addition, more reasonable test images for models and algorithms should be continuous, instead of the currently widely used piecewise constant phantom image and other discontinuous images. Numerical tests and comparisons demonstrated our analysis and the effectiveness of the improved variational model.

There are some future works. In this paper we only considered the transmission imaging with parallel line geometry. The corresponding results for fan beam geometry will be reported in our near future. In addition, the variational model presented here is just our first try to handle this kind of images after our realizing the continuity of the images. We believe that better restoration models exist and need to be found.

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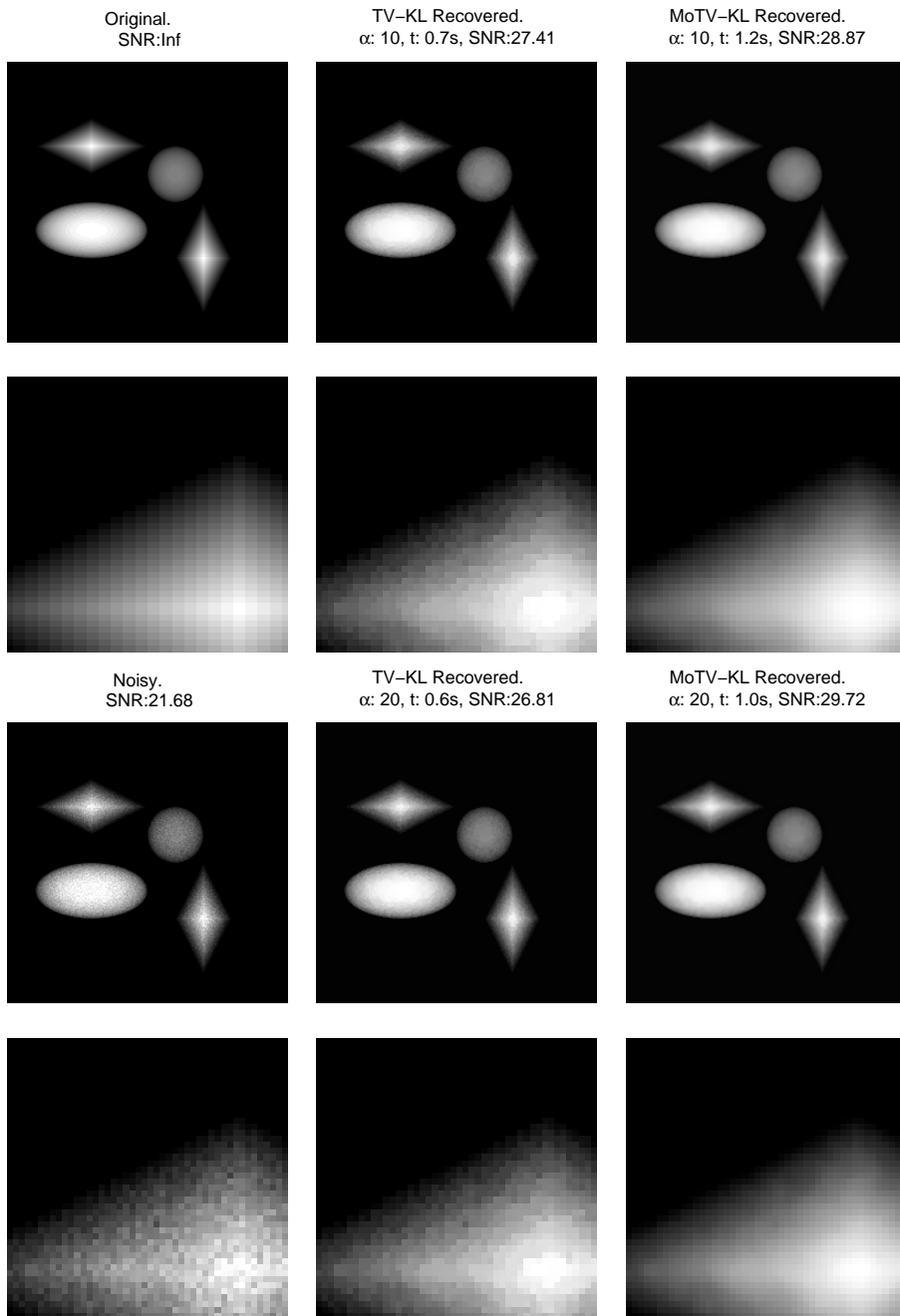


FIG. 4.1. Comparisons between TV-KL and MoTV-KL models for a  $256 \times 256$  synthetic image. The second and fourth rows are zoom-in images of the first and third rows, respectively. The first column: original and noisy (with Poisson noise) images. The second column: TV-KL recovered images. The third column: MoTV-KL recovered images. According to our test, TV-KL model gives a result with nearly highest SNR with the model parameter  $\alpha = 10$ , whereas MoTV-KL model generates a result with nearly highest SNR with  $\alpha = 20$ . For both parameter values, MoTV-KL model gives much higher SNR than TV-KL model. In addition, TV-KL model suffers from staircase effect, whereas MoTV-KL model does not.



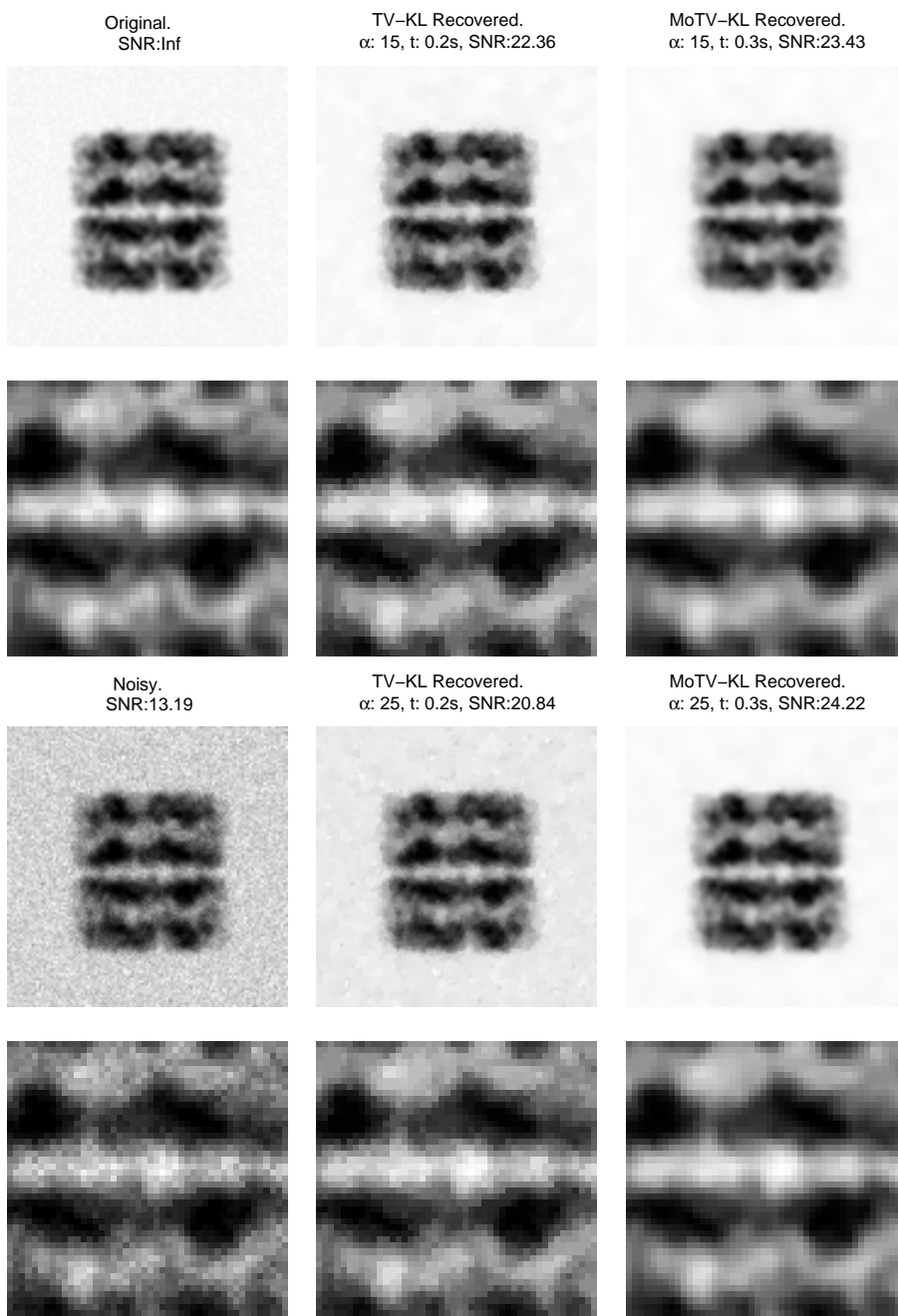


FIG. 4.2. Comparisons between TV-KL and MoTV-KL models for a  $115 \times 115$  molecular image. The second and fourth rows are zoom-in images of the first and third rows, respectively. The first column: original and noisy (with Poisson noise) images. The second column: TV-KL recovered images. The third column: MoTV-KL recovered images. According to our test, TV-KL model gives a result with nearly highest SNR with the parameter  $\alpha = 15$ , whereas MoTV-KL model gives a result with nearly highest SNR with  $\alpha = 25$ . For both parameters, MoTV-KL model gives much higher SNR than TV-KL model. In addition, TV-KL model suffers from staircase effect, whereas MoTV-KL model does not.