

Active Contour Without Edges for Multiphase Segmentations with the Presence of Poisson Noise

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Abstract

We propose a multi-phase (piecewise-constant) model to segment images corrupted by Poisson noise. Our motivation comes from the Chan-Vese model and the Mumford-Shah model, where now we consider the observed data being corrupted by Poisson noise instead of the usual additive Gaussian noise. We reformulate the model so that the Split Bregman method can be applied and present a fast algorithm for the reformulated model. Comparisons of the proposed model with multi-phase Chan-Vese model through numerical experiments are also presented.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and $f : \Omega \rightarrow \mathbb{R}$ be the given image. A classical image segmentation model is the Mumford-Shah model [20] where f is assumed to be in $L^2(\Omega)$ and consists of $u + v$, where u is piecewise smooth and v is additive Gaussian noise. Denote by Γ the edge set of u , then the variational Mumford-Shah model is given by

$$\inf_{\Gamma, u} \left\{ \mathcal{E}_{MS} = \mathcal{H}^1(\Gamma) + \alpha \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \lambda \int_{\Omega} |f - u|^2 dx : u \in W^{1,2}(\Omega \setminus \Gamma) \right\}. \quad (1.1)$$

Here \mathcal{H}^1 denotes the 1-dimensional Hausdorff measure.

In the Chan-Vese model [8], the segmented image u is assumed to be piecewise constant. In the two-phase segmentation model, we have $u = c_1\chi_{\Omega_1} + c_2\chi_{\Omega_2}$, where $\Omega_2 = \Omega \setminus \bar{\Omega}_1$. Let $\Gamma = \partial\Omega \setminus \partial\Omega_1$. Then the two-phase Chan-Vese model [8] is given by

$$\inf_{\Gamma, c_i} \left\{ \mathcal{E}_{CV} = \mathcal{H}^1(\Gamma) + \lambda \sum_{i=1}^2 \int_{\Omega_i} |f - c_i|^2 dx \right\}. \quad (1.2)$$

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In [8], the authors successfully implemented this model using the level set method. Let $\phi : \Omega \rightarrow \mathbb{R}$ be the level set function, representing $\Gamma = \{\phi(x) = 0\}$, $\Omega_1 = \{\phi(x) > 0\}$ and $\Omega_2 = \{\phi(x) < 0\}$. Let H be the Heaviside function. Then (1.2) can be reformulated as

$$\inf_{\phi, c_i} \left\{ \mathcal{E}_{CV} = \int_{\Omega} |DH(\phi)| + \lambda \left[\int_{\Omega} |f - c_1|^2 H(\phi) + |f - c_2|^2 (1 - H(\phi)) dx \right] \right\}. \quad (1.3)$$

Here Dg denotes the distributional derivative of the function g . See for instance [1] for the precise definition.

Extensions to piecewise constant multi-phase segmentation models using multiple level set functions are proposed by Vese-Chan in [26]. For instance, using two level set functions ϕ_1 and ϕ_2 , one can represent four regions $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ as follows.

$$\begin{aligned} \Omega_1 &= \{\phi_1(x) > 0\} \cap \{\phi_2(x) > 0\}, & \Omega_2 &= \{\phi_1(x) < 0\} \cap \{\phi_2(x) > 0\}, \\ \Omega_3 &= \{\phi_1(x) > 0\} \cap \{\phi_2(x) < 0\}, & \Omega_4 &= \{\phi_1(x) < 0\} \cap \{\phi_2(x) < 0\}. \end{aligned}$$

The Vese-Chan multiphase piecewise constant model becomes

$$\begin{aligned} \inf_{\phi_i, c_i} \left\{ \mathcal{E}_{VC}(\phi_i, c_i) &= \sum_i^2 \int_{\Omega} |DH(\phi_i)| \right. \\ &+ \lambda \left[\int_{\Omega} |f - c_1|^2 H(\phi_1) H(\phi_2) dx \right. \\ &+ \int_{\Omega} |f - c_2|^2 (1 - H(\phi_1)) H(\phi_2) dx \\ &+ \int_{\Omega} |f - c_3|^2 H(\phi_1) (1 - H(\phi_2)) dx \\ &\left. \left. + \int_{\Omega} |f - c_4|^2 (1 - H(\phi_1)) (1 - H(\phi_2)) dx \right] \right\}. \end{aligned} \quad (1.4)$$

Following these variational models, various extensions and analysis have been studied [6], [2], [9], [10], [13], [17], [18], [23], [24], [25], [5], [4], [16], among others. A segmentation model with a weaker constraint on Γ is studied in [3].

In this paper, we consider the problem of segmenting images corrupted by Poisson noise, which is neither additive nor multiplicative. In [14], the authors proposed the following variational model for denoising images corrupted by Poisson noise:

$$\inf_{u \in BV(\Omega)} \left\{ |u|_{BV} + \lambda \int_{\Omega} (u - f \log(u)) dx \right\}. \quad (1.5)$$

Here $BV(\Omega)$ is the space of functions of bounded variation defined on Ω , and $|u|_{BV} = \int_{\Omega} |Du|$, having Du as the distributional derivative of u [1]. The model (1.5) is an extension of the Rudin-Osher-Fatemi model [22] replacing $\|f - u\|_{L^2}^2$ with $\int_{\Omega} (u - f \log(u)) dx$, which is more suitable for Poisson noise.

In this paper, we study the multiphase segmentation problem, where we assume that the given image f is given by a piecewise-constant image u corrupted by Poisson noise. Motivated from [14], we replace the fidelity term $\|f - u\|_{L^2}^2$ in the Chan-Vese model (1.2), where $u = c_1\chi_{\Omega_1} + c_2\chi_{\Omega_2}$, with $\int_{\Omega}(u - f \log(u)) dx$. The organization of the paper is as follows. In section 2, we give the description of the proposed multi-phase piecewise constant segmentation model for image corrupted by Poisson noise using level sets. In section 3, we reformulate this model into a new one so that the fidelity term is smooth. The idea comes from [21], which reformulated the 2-phase piecewise constant Chan-Vese segmentation model into a convex model. In section 4, we use Split Bregman method to decouple the smooth fidelity term and the non-differentiable BV term. We also propose an algorithm that doesn't require regularizing the BV terms. In section 5, we present some numerical results which compare our model with Chan-Vese model applying to data corrupted by Poisson noise. An example of salt and pepper noise is also presented, where again we reformulated the fidelity term to fit the model noise.

Remark 1. We note that the proposed model can be extended to the piecewise-smooth case where f is assumed to be given by uw corrupted by Poisson noise. Thus the fidelity term becomes $\int_{\Omega}(uw - f \log(uw)) dx$, where w is smooth. In medical imaging, the factor w is called a bias field or intensity inhomogeneity. A related piecewise-smooth model has been proposed in [19], where the bias field is smooth, and f is assumed to be affected by additive Gaussian noise of zero mean. In other words, $f = uw + v$, where v is additive Gaussian noise. Another related work is [15] where the authors studied segmentation models where f is effected by multiplicative noise, that is $f = u w v$, where v is multiplicative noise of mean one.

2 Description of the Proposed Model

Let f be the observed image which is the true image u corrupted by Poisson noise. To reconstruct u , the authors in [14] proposed the following variational problem,

$$\inf_u \left\{ |u|_{BV} + \lambda \int_{\Omega} (u - f \log(u)) dx \right\}, \quad (2.1)$$

where the fidelity term $F(u) = \int_{\Omega}(u - f \log(u)) dx$ replaces $\int_{\Omega} |f - u|^2 dx$ in the Rudin-Osher-Fatemi model [22]. We refer the reader to [14] for the derivation of this fidelity term using Bayesian statistics. From the variational approach for image reconstruction, $F(u)$ has shown to be more suitable for Poisson noise. To incorporate $F(u)$ into a piecewise constant segmentation model, we assume Ω is partitioned into $\Omega_1, \dots, \Omega_n$, and $u = \sum_{i=1}^n c_i \chi_{\Omega_i}$, and study the following variational problem

$$\inf_u \left\{ \mathcal{E}_n(u) = \mathcal{H}^1(\Gamma) + \lambda \int_{\Omega} (u - f \log(u)) dx \right\}, \quad (2.2)$$

where $u = \sum_{i=1}^n c_i \chi_{\Omega_i}$, and Γ is the discontinuity set of u , $\mathcal{H}^1(\Gamma)$ is the one dimensional Hausdorff measure (length) of Γ . In particular, when $n = 2$, we have $u = c_1 \chi_{\Omega_1} + c_2 \chi_{\Omega_2}$, and

$$\mathcal{E}_2(u) = \mathcal{H}^1(\Gamma) + \lambda \sum_{i=1}^2 \int_{\Omega_i} (c_i - f \log(c_i)) dx, \quad (2.3)$$

which is similar to the Chan-Vese model (1.2) except the fidelity term. The level set formulations of (2.3) is

$$\inf_{\phi, c_1, c_2} \left\{ \mathcal{E}_2(\phi, c_1, c_2) = \int_{\Omega} |DH(\phi)| + \lambda \int_{\Omega} \left[(c_1 - f \log(c_1)) H(\phi) + (c_2 - f \log(c_2)) (1 - H(\phi)) \right] \right\}. \quad (2.4)$$

The differential of \mathcal{E}_2 at ϕ (assuming to be in $W^{1,1}$) is given by

$$\begin{aligned} \frac{\partial \mathcal{E}_{2,\epsilon}}{\partial \phi} &= \delta(H(\phi)) \left[\operatorname{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) + \lambda [(c_1 - f \log(c_1)) - (c_2 - f \log(c_2))] \right] \\ &= \delta(H(\phi)) \left[\operatorname{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) + \lambda \left[(c_1 - c_2) + f \left(\log \left(\frac{c_2}{c_1} \right) \right) \right] \right]. \end{aligned} \quad (2.5)$$

Similarly, the differential of $\mathcal{E}_{CV}(\phi)$ at $\phi \in W^{1,1}$ is

$$\begin{aligned} \frac{\partial \mathcal{E}_{CV,\epsilon}}{\partial \phi}(\phi) &= \delta(H(\phi)) \left[\operatorname{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) + \lambda [(f - c_1)^2 - (f - c_2)^2] \right] \\ &= \delta(H(\phi)) \left[\operatorname{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) + \lambda [(c_1^2 - c_2^2) + 2f(c_2 - c_1)] \right]. \end{aligned} \quad (2.6)$$

Equations (2.5) and (2.6) are similar, except in the fidelity term. The fidelity terms related to f in both equations are $\lambda \log(\frac{c_2}{c_1})f$ and $\lambda(c_2 - c_1)f$. So, in the Poisson model, the fidelity force acting on the level set function depends on the ratio between c_1 and c_2 which is more suitable for Poisson noise because the magnitude of noise depends linearly with respect to the image intensity. For the two-phase segmentation, λ can be chosen according to $\frac{c_2}{c_1}$ to compensate the weakness of the Chan-Vese model. So, the advantage of this model appears only in multi-phase segmentations.

3 Reformulation

In [21], the authors observed that if the term $\delta(H(\phi))$ in (2.6) is removed, then the minimizing energy becomes

$$\mathcal{E}_{NEC}(\phi, u) = \int_{\Omega} |D\phi| + \lambda \int_{\Omega} [(f - c_1)^2 - (f - c_2)^2] \phi dx.$$

Here, ϕ is not viewed as a level set function but a function in BV with values in $[0, 1]$. Fixing u , the functional $\mathcal{E}_{NEC}(\phi, u)$ is convex in ϕ . They showed that

\mathcal{E}_{NEC} is the average of \mathcal{E}_{CV} , that is, $\mathcal{E}_{NEC}(\phi) = \int_0^1 \mathcal{E}_{CV}(1_{\phi > \mu}) d\mu$, which implies that a minimizer of $\mathcal{E}_{NEC}(\phi)$ is also a minimizer of $\mathcal{E}_{CV}(\phi)$.

For multi-phase segmentation, we use Vese-Chan model (1.4) and use the same approach as above to reformulate our functional from (2.2). For simplicity, we only discuss the case $n = 4$. Given $f > 0$, our proposed new functional is

$$\begin{aligned} \tilde{\mathcal{E}}_4(c_i, u_1, u_2) &= \int_{\Omega} |Du_1| + \int_{\Omega} |Du_2| + \int_{\Omega} f_1 u_1 u_2 dx + \int_{\Omega} f_2 (1 - u_1) u_2 dx \\ &+ \int_{\Omega} f_3 u_1 (1 - u_2) dx + \int_{\Omega} f_4 (1 - u_1) (1 - u_2) dx, \end{aligned}$$

where $f_i = \lambda(c_i - f \log(c_i))$, and $u = \sum_{i=1}^4 c_i \chi_{\Omega_i}$, with the constraint that $0 \leq u_i \leq 1$ and $c_i > 0$. We note that the functional $\tilde{\mathcal{E}}_4(c_i, u_1, u_2)$ is convex in each variable. For instance, fixing u_j and c_2, c_3, c_4 , we have

$$\frac{\partial^2 \tilde{\mathcal{E}}_4}{\partial c_1^2} = \int_{\Omega} \frac{f}{c_1^2} u_1 u_2 dx \geq 0.$$

Similarly, one can show that $\frac{\partial^2 \tilde{\mathcal{E}}_4}{\partial c_i^2} > 0$ for all $i = 2, 3, 4$. The convexity in u_i is also clear.

Using the arguments from [14], one can show that a minimizer for \mathcal{E}_4 exists. Next, we would like to show that the result from [21] relating minimizers of \mathcal{E}_{CV} and \mathcal{E}_{NEC} also holds for \mathcal{E}_4 and $\tilde{\mathcal{E}}_4$.

Theorem 1. *Assume $\lambda > 0$, $1 \geq c_i > 0$. Let Ω be a bounded open subset of \mathbb{R}^2 with Lipschitz boundary. Let f be a positive bounded function. Let the domain of \mathcal{E}_4 be $\{(u_1, u_2) : u_i \in BV(\Omega)\}$ and the domain of $\tilde{\mathcal{E}}_4$ be $\{(u_1, u_2) : 0 \leq u_i \leq 1 \text{ and } u_i \in BV(\Omega)\}$. Then $\tilde{\mathcal{E}}_4$ has a minimizer. Furthermore, any minimizer of $\tilde{\mathcal{E}}_4$ is a minimizer of \mathcal{E}_4 .*

Proof. Since $\tilde{\mathcal{E}}_4$ is positive, we can pick a minimizing sequence $\{c_{i,n}, u_{j,n} : i = 1, \dots, 4, j = 1, 2\}$ such that $0 \leq u_{i,n} \leq 1$. Since $\|u_{j,n}\|_{BV}$ is uniformly bounded, then up to a subsequence, which we still denote $u_{j,n}$, we have $u_{j,n} \rightarrow u_j$ strongly in L^1 . By taking subsequences if necessary, we may assume $u_{j,n}$ converges to u_j pointwise. Since $\{c_{i,n}\}$ is uniformly bounded, then up to a subsequence, we obtain $c_{i,n} \rightarrow c_i$. Using the semicontinuity of BV norm, we deduce that $\tilde{\mathcal{E}}_4(c_i, u_j) \leq \liminf \tilde{\mathcal{E}}_4(c_{i,n}, u_{j,n})$. So, $\{c_i, u_j\}$ is a minimizer.

For the relation between \mathcal{E}_4 and $\tilde{\mathcal{E}}_4$, pick any (u_1, u_2) in the domain of $\tilde{\mathcal{E}}_4$. Using coarea formula and $0 \leq u_i \leq 1$, we have

$$\int_{\Omega} |Du_i| = \int_0^1 \mathcal{H}^1(\{\partial(u_i > \mu)\}) d\mu = \int_0^1 \int_{\Omega} |D(H(u_i - \mu))| d\mu$$

Also,

$$\begin{aligned} \int_{\Omega} f_1 u_1 u_2 dx &= \int_{\Omega} f_1 \int_0^1 \int_0^1 1_{u_1 > \mu} 1_{u_2 > \nu} d\mu d\nu dx \\ &= \int_0^1 \int_0^1 \int_{\Omega} f_1 H(u_1 - \mu) H(u_2 - \nu) dx d\mu d\nu \end{aligned}$$

It is similar for the other three terms. Thus,

$$\begin{aligned}
\tilde{\mathcal{E}}_4(u) &= \int_0^1 \int_0^1 \int_{\Omega} |D(H(u_1 - \mu))| d\mu d\nu + \\
&\quad \int_0^1 \int_0^1 \int_{\Omega} |D(H(u_2 - \nu))| d\mu d\nu + \\
&\quad \int_0^1 \int_0^1 \int_{\Omega} f_1 H(u_1 - \mu) H(u_2 - \nu) dx d\mu d\nu + \dots \\
&= \int_0^1 \int_0^1 \mathcal{E}_4(u_1 - \mu, u_2 - \nu) d\mu d\nu.
\end{aligned}$$

Thus, $\min \mathcal{E}_4 \leq \min \tilde{\mathcal{E}}_4$. For opposite direction, since the value \mathcal{E}_4 only depends on the sign of u_i , we have

$$\begin{aligned}
\tilde{\mathcal{E}}_4(1_{u_1>0}, 1_{u_2>0}) &= \int_0^1 \int_0^1 \mathcal{E}_4(1_{u_1>0} - \mu, 1_{u_2>0} - \nu) d\mu d\nu \\
&= \mathcal{E}_4(u_1, u_2).
\end{aligned}$$

Thus, $\min \tilde{\mathcal{E}}_4 \leq \min \mathcal{E}_4$. □

4 Algorithm

The BV semi-norm $|u|_{BV} = \int |Du|$ is non-differentiable. This can be viewed as minimizing the L^1 -norm of $\Phi(u) = Du$, which is difficult to solve by conventional method. In [12], authors propose a split Bregman method to solve L1-Regularized problem:

$$\min_u \int_{\Omega} |\Phi(u)| + H(u), \tag{4.1}$$

where $H(u)$ is a convex functional. They decoupled the L1-problem with the H-problem by introducing the auxiliary variable $d = \Phi(u)$:

$$\min_{u,d} \int_{\Omega} |d| + H(u) \quad \text{given } d = \Phi(u).$$

This problem is then converted into an unconstrained minimization problem by adding a quadratic penalty term imposing d to be close to $\Phi(u)$ in L^2 -sense, that is

$$\min_{u,d} \int_{\Omega} |d| dx + H(u) + \frac{\mu}{2} \int_{\Omega} |d - \Phi(u)|^2 dx$$

for some large positive constant μ . For a fixed u , d can be solved explicitly. Also, the original problem of minimizing $\int_{\Omega} |\Phi(u)| + H(u)$ over u becomes a minimization of $\frac{\mu}{2} \int_{\Omega} |d - \Phi(u)|^2 + H(u)$ which is easier to solve. In order to

enforce $d = \Phi(u)$, they add a variable into the penalty, that acts like adding back the difference between d and $\Phi(u)$. The iterative algorithm involves solving:

$$\begin{aligned} (u^{k+1}, d^{k+1}) &= \arg \min_{u, d} \|d\|_{L^1} + H(u) + \frac{\mu}{2} \|d - \Phi(u) - b^k\|_{L^2}^2 \quad (4.2) \\ b^k &= b^k + \Phi(u^k) - d^k, \end{aligned}$$

where the superscript k means the k^{th} step in the iteration.

The authors also proved that this algorithm converges to some solution of (4.1). In particular, they demonstrated this method is more efficient than using dual formulation [7] for TV-denoising [12] and 2-phase segmentation [11]. Our algorithm is quite analogous to the 2-phase segmentation [11]. That is, we iteratively compute

$$\begin{aligned} (u_j^{k+1}, d_j^{k+1}) &= \arg \min_{0 \leq u_j \leq 1, d_j} \left\{ \sum_{j=1}^2 \|d_j\|_{L^1} + \lambda \sum_{i=1}^4 \int_{\Omega} [f_i u_1 u_2 + \dots] dx \right. \\ &\quad \left. + \frac{\mu}{2} \sum_{j=1}^2 \|d_j - \Phi(u_j) - b_j^k\|_{L^2}^2 \right\}, \\ b_j^{k+1} &= b_j^k + \Phi(u_j^k) - d_j^k, \text{ for } j = 1, 2. \end{aligned}$$

Let $U = (u_1, u_2)$, $D = (d_1, d_2)$, $B = (b_1, b_2)$ and $\Phi(U) = (\Phi(u_1), \Phi(u_2))$. Define

$$\begin{aligned} Reg(D) &:= \sum_{j=1}^2 \|d_j\|_{L^1}, \\ Fid(U) &:= \sum_{i=1}^4 \int_{\Omega} [f_i u_1 u_2 + \dots] dx, \text{ and} \\ \|D - \Phi(U) - B^k\|_{L^2}^2 &:= \sum_{j=1}^2 \|d_j - \Phi(u_j) - b_j^k\|_{L^2}^2. \end{aligned}$$

Then in vector form, the above algorithm becomes

$$\begin{aligned} (U^{k+1}, D^{k+1}) &= \arg \min_{U \in [0,1]^2, D} \left\{ Reg(D) + \lambda Fid(U) + \frac{\mu}{2} \|D - \Phi(U) - B^k\|_{L^2}^2 \right\}, \\ B^{k+1} &= B^k + \Phi(U^k) - D^k. \end{aligned} \quad (4.3)$$

The problem (4.3) is solved by iteratively minimizing over u_1 , u_2 , d_1 , and d_2 .

For the u_1 -problem, we need to find

$$\arg \min_{0 \leq u_1 \leq 1} \int_{\Omega} f_1 u_1 u_2 + \dots + f_4 (1 - u_1)(1 - u_2) dx + \frac{\mu}{2} \|d_1 - \Phi(u_1) - b_1^k\|_{L^2}^2.$$

If we ignore the constraint, it is a Poission equation. So, we can solve it by Gauss-Seidel method, and then project it into $[0, 1]$ (as in [11]). In other words,

denote

$$\begin{aligned} [x] &= \min(1, \max(0, x)), \text{ and} \\ \operatorname{div}(f)(i, j) &= \partial_x f(i, j) - \partial_x f(i-1, j) + \partial_y f(i, j) - \partial_y f(i, j-1). \end{aligned}$$

Then explicitly, we solve

$$u_1^{k+1}(i, j) = \left[\frac{1}{4} (u_1^{k+1}(i-1, j) + u_1^{k+1}(i, j-1) + u_1^k(i, j+1) + u_1^k(i+1, j)) - h_1^k(i, j) \right], \quad (4.4)$$

with $h_1^k = \operatorname{div}(d_1^k - b_1^k) + (f_1 - f_2)u_2^k + (f_3 - f_4)(1 - u_2^k)$.

For d_1 -problem, it can be solved by a shrinkage operator

$$d_1^k(i, j) = \max \left(0, 1 - \frac{1}{\mu |l_1^k(i, j)|} \right) l_1^k(i, j), \quad (4.5)$$

where

$$l_1^k(i, j) = (u_1^k(i+1, j) - u_1^k(i, j) + b_{1,x}^k(i, j), u_1^k(i, j+1) - u_1^k(i, j) + b_{1,y}^k(i, j)).$$

The u_2 -problem and d_2 -problem are solved similarly.

Combining (4.2), (4.4) and (4.5), we have a fast algorithm to solve the piecewise-constant multiphase segmentation model. Notice that this algorithm is suitable for either Poisson noise or Gaussian noise. In the numerical result, we will show that it is also applicable for salt & pepper noise by changing the fidelity term.

Algorithm 1: Split Bregman for piecewise-constant multiphase model: Fix $\varepsilon > 0$

1. $U^1 = 0, D^1 = 0, B^1 = 0$
 2. Do
 3. Find $U^{k+1} = \operatorname{argmin}_{U \in [0,1]^2} \left\{ \operatorname{Fid}(U) + \frac{\mu}{2} \|D - \Phi(U) - B^k\|_{L^2}^2 \right\}$ by using (4.4)
 4. Find $D^{k+1} = \operatorname{argmin}_D \left\{ \operatorname{Reg}(D) + \frac{\mu}{2} \|D - \Phi(U) - B^k\|_{L^2}^2 \right\}$ by using (4.5)
 5. $B^{k+1} = B^k + \Phi(U^k) - D^k$
 6. $c_1^{k+1} = \frac{\int_{\Omega} f u_1^k u_2^k dx}{\int_{\Omega} u_1^k u_2^k dx}, c_2^{k+1} = \frac{\int_{\Omega} f (1-u_1^k) u_2^k dx}{\int_{\Omega} (1-u_1^k) u_2^k dx}, \dots$
 7. While $\|U^{k+1} - U^k\|_{L^2} > \varepsilon$
- $$u = c_1 1_{u_1 > \frac{1}{2}} 1_{u_2 > \frac{1}{2}} + c_2 1_{u_1 > \frac{1}{2}} 1_{u_2 \leq \frac{1}{2}} + c_3 1_{u_1 \leq \frac{1}{2}} 1_{u_2 > \frac{1}{2}} + c_4 1_{u_1 \leq \frac{1}{2}} 1_{u_2 \leq \frac{1}{2}}.$$

5 Numerical result

In all numerical result, we used Algorithm 1 to find minimizer of $\tilde{\mathcal{E}}_4(u)$. It is implemented in C++ and didn't use GPU. All data is processed in single precision because only 1 bit precision will retain after the thresholding at the end. It is run in a laptop with Pentium Dual Core T440. All of the image is of the size 256×256 pixels and renormalized the grey value to 0 to 1. We chose μ

and λ by hand. We didn't handle the c_i seriously because we focus on the effect on different noise model.

In Figure 5.1, our initial $c_i = \{0.1, 0.4, 0.2, 0.6\}$ while the true value is $\{0.0627, 0.4353, 0.1961, 0.6275\}$. Compare with our model, using Chan-Vese model either lose some thin strip (c) or fail to remove some noise (d).

In Figure 5.2, we given the true value $c_i = \{0.2, 0.4, 0.102, 0.051\}$ and skip step 6 in the algorithm. It is because the Chan-Vese model have difficulty on finding c_i . In Chan-Vese model and small λ , (c) '1' disappears because the intensity of '1' is too close to background in the sense of Gaussian noise. For large λ , there is more noise point in the square of '4' and the '1' become too regularized because the model assume it is uniform variance regardless of image intensity.

In Figure 5.3, we given the true value $c_i = \{0.2, 0.4, 0.102, 0.051\}$ and skip step 6 because step 6 doesn't make sense for this noise. Also, we use Salt & Pepper model, that is, $f_i = |f - c_i|1_{f \neq 1}1_{f \neq 0}$. This explains that the regularity of f_i respect to c_i is not important for 4-phase segmentation.

Although we haven't take advantage of parallel technique available, the performance is still quite fast, it takes less than 0.4 sec in all examples.

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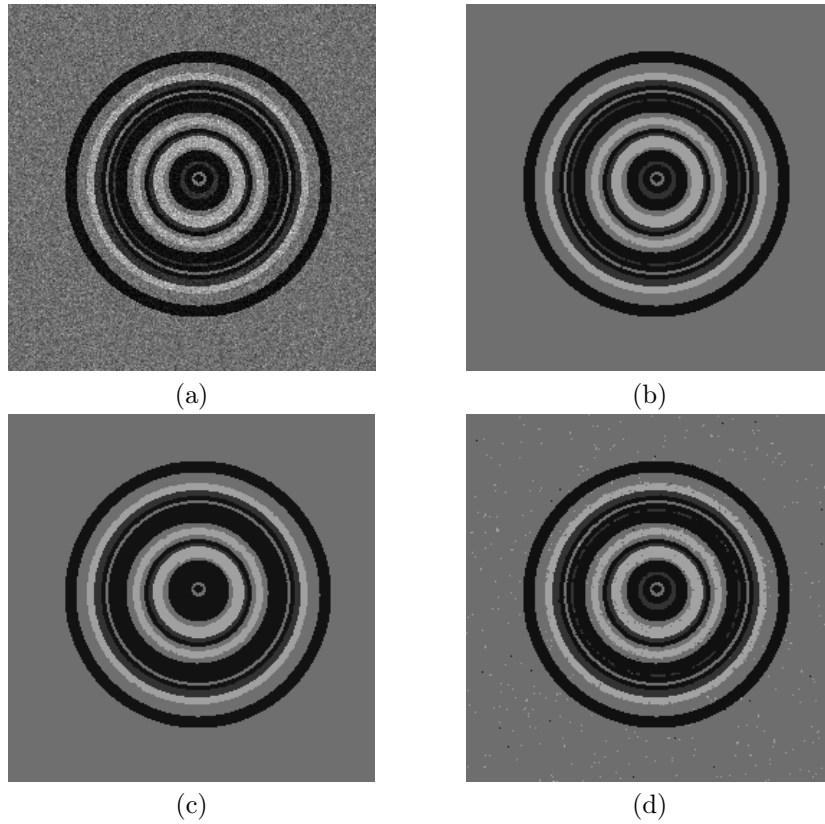


Figure 5.1: (a) Image is corrupted by Poisson noise (b) Recovered by Poisson Model with $\lambda = 30, \mu = 0.5$. Iteration = 18, Time = 0.097sec. (c) Recovered by Gaussian Model with $\lambda = 30, \mu = 0.5$. Iteration = 26, Time = 0.123 sec. (d) Recovered by Gaussian Model with $\lambda = 90, \mu = 2$. Iteration = 33, Time = 0.161sec.

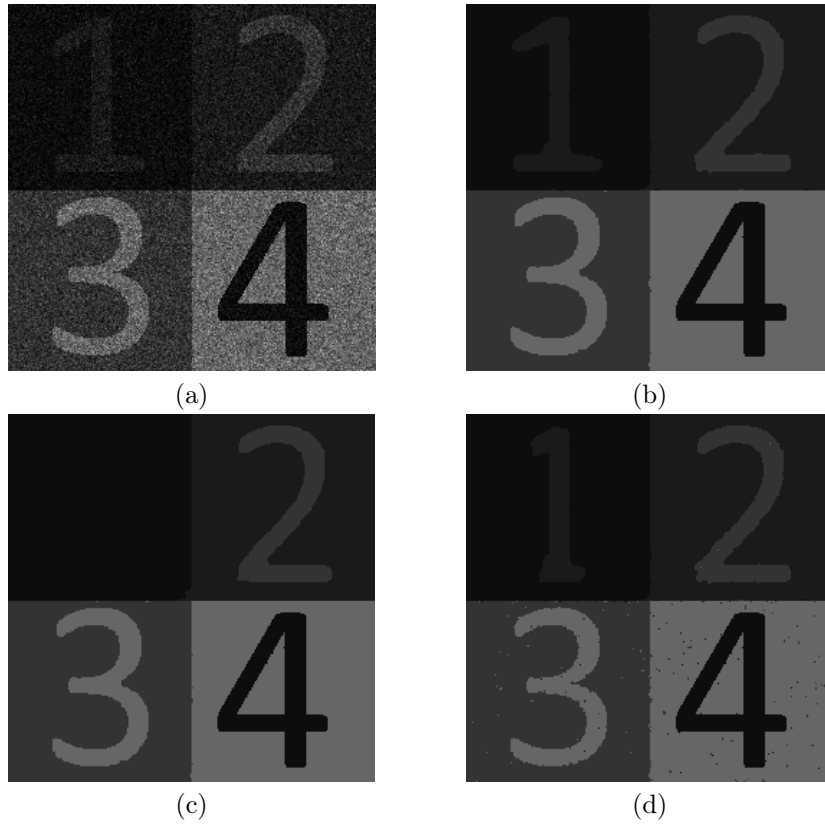


Figure 5.2: (a) Image is corrupted by Poisson noise (b) Recovered by Poisson Model with $\lambda = 13, \mu = 0.5$. Iteration = 33, Time = 0.140sec. (c) Recovered by Gaussian Model with $\lambda = 26, \mu = 0.5$. Iteration = 104, Time = 0.392 sec. (d) Recovered by Gaussian Model with $\lambda = 63, \mu = 0.5$. Iteration = 59, Time = 0.227sec.

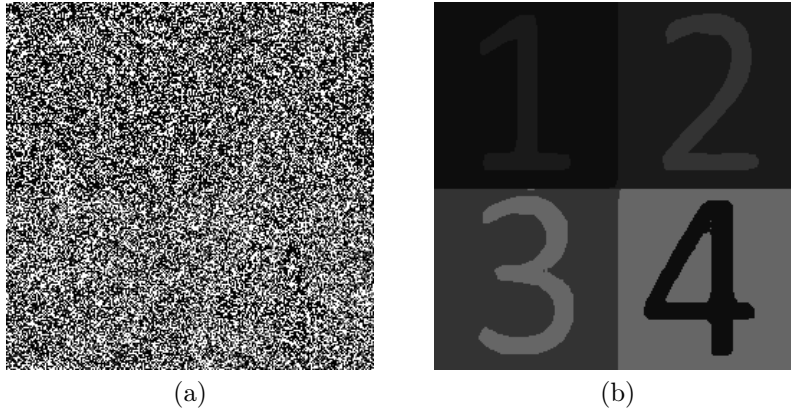


Figure 5.3: (a) Image is corrupted by 80% Salt & Pepper noise (b) Recovered by Salt & Pepper Model with $\lambda = 10^8$, $\mu = 0.5$. Iteration = 34, Time = 0.197sec.

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