A RIDGE AND CORNER PRESERVING MODEL FOR SURFACE RESTORATION

RONGJIE LAI *, XUE-CHENG TAI † , AND TONY F. CHAN ‡

Abstract. One challenge in surface restoration is to design surface diffusion preserving ridges and sharp corners. In this paper, we propose a new surface restoration model based on the observation that surfaces' implicit representations are continuous functions whose first order derivatives have discontinuities at ridges and sharp corners. Regularized by vectorial total variation on the derivatives of surfaces' implicit representation functions, the proposed model has ridges and corners preserving properties validated by numerical experiments. To solve the proposed fourth order and convex problem efficiently, we further design a numerical algorithm based on the augmented Lagrangian method. Moreover, the theoretical convergence analysis of the proposed algorithm is also provided. To demonstrate the efficiency and robustness of the proposed method, we show restoration results on several different surfaces and also conduct comparisons with mean curvature flow method and nonlocal mean method.

Keywords: Surface restoration, vectorial total variation, Hessian, augmented Lagrangian.

1. Introduction. With the rapid development of data acquisition technology, more and more 3D surfaces can be easily collected and widely used in many fields, such as computer graphics, computer vision as well as medical image analysis. However, 3D surfaces obtained from laser scanner, computed tomography (CT), magnetic resonance (MR) or 3D ultrasound devices are usually contaminated by some noise due to local measurement errors. Typically, these surfaces have crucial features represented as ridges and sharp corners. Thus, it is critical to process these noisy surfaces using ridges and sharp corners preserving methods.

In the last decade, several feature-preserving methods have been developed for 3D surface fairing in the computer graphics society [32, 25, 20, 36, 2]. Among all other techniques, surface diffusion methods via different curvature flows have beeb successfully used in surface processing. The most natural way for surface smoothing is motion by mean curvature flow (MCF) or Laplace smoothing [12, 48, 40], which is essentially the process of reducing the surface area. As a second order model, MCF is effective to remove oscillations on noisy surfaces. However, it has limitations for preserving ridges and sharp corners on surfaces. To tackle the challenge of ridges and sharp corners preserving problems, several other second order models of the anisotropic surface diffusions are considered in [13, 10, 14, 3, 37, 11], which can be viewed as processes of reducing weighted surface area. Besides the above second order models, the fourth order isotropic or anisotropic flows based on minimizing the surface total curvature or weighted total curvature are discussed in [46, 47].

More recently, there has been an increasing interest to study surface processing by adapting ideas from image processing. Based on the fact that surfaces usually have similar repeated patterns, nonlocal approaches in 2D imaging [7, 21] are adapted to study 3D surface smoothing in [15, 33], where numerical experiments provide ridges and sharp corners preserving results. A critical limitation of the nonlocal mean method is that the computation of the nonlocal weight function is very time consuming, thus a semi-nonlocal process with a small patch size is usually considered in practice. Inspired by the success of the total variation (TV) in 2D imaging, Elsey and Esedoglu [18] introduce an analogue of the total variation denoising model for surface processing by minimizing the absolute value of the Gauss curvature as regularization. Their model can guarantee preservation of ridges and sharp corners for *convex* surfaces, or surfaces with local constant sign of Gauss curvature on a fine scale. For surfaces

^{*}Department of Mathematics, University of Southern California, U.S.A. (rongjiel@usc.edu).

[†]Department of Mathematics, University of Bergen, Norway and Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 637371, Singapore, (tai@math.uib.no).

[‡]Department of Mathematics and Department of Computer Science and Engineering, Hong Kong University of Science and Technology, Hong Kong. (tonyfchan@ust.hk).

not satisfying either of these two conditions, which are also commonly seen in practice, their ridges and sharp corners might be lost using the model proposed in [18].

In this work, we propose a new model for 3D surface processing with ridges and sharp corner preserving properties based on a simple observation as follows. In image processing, the representation functions are piecewise smooth functions; thus the total variation provides a powerful tool to study images with jumps preserving properties [38, 42]. Comparing with representation functions for images, the implicit representation functions of curves or surfaces have their own essential differences. "Image functions" are normally discontinuous, and the salient features are contained in the discontinuities. "Implicit surface functions" are normally continuous [35], while their derivatives have discontinuities due to the ridges and sharp corners of the represented surfaces. If the jumps of the derivatives of the implicit functions can be well preserved, then the corresponding ridges and sharp corners of surfaces can be also well preserved. Therefore, a natural choice for surface processing would be to choose the total variation to process the derivatives of the implicit functions of surfaces. More precisely, given a surface with implicit function representation $\phi: D \subset \mathbb{R}^3 \to \mathbb{R}$ and its three directional derivatives $\partial_x \phi$, $\partial_y \phi$, $\partial_z \phi$, we need to consider the total variation for the vector $(\partial_x \phi, \partial_y \phi, \partial_z \phi)$. As a generalization of the total variation for vector-valued images, the vectorial total variation is studied in [1, 6] for color image processing. Thus, we can utilize the vectorial total variation to have the jump preservation of the derivative of the implicit function ϕ , which leads to the ridges and sharp corners preserving property as we desired for surface processing. This nice property of the proposed model will be experimentally illustrated in section 5.

As an important distinction of the proposed model, the optimization problem we proposed is convex. Thus, the global minimizer of our model will provide the desired solution, which avoids terminating the processing in artificial given iteration steps as geometric diffusion methods in [46, 47, 40, 11]. Overall, the gradient flow of the proposed model leads to a fourth order problem, which is usually extremely time consuming to approach the optimizer. To solve the proposed minimization problem efficiently, we design a fast algorithm based on the augmented Lagrangian method. With the help of auxiliary variables, the solution of the original optimization problem can be iteratively approached by computation of several easy-solving subproblems. This algorithm is inspired by a series work of operator splitting and split Bregman iterations [34, 49, 24, 23, 45, 50, 44], which popularize the idea of using operator splitting to solve optimization problems. The equivalence of the split Bregman iterations to the alternating direction method of multipliers (ADMM), Douglas-Rachford splitting and augmented Lagrangian method can be found in [19, 39, 45, 50]. Moreover, we also show the theoretical convergence analysis of the proposed algorithm inspired by the general discussion of the augmented Lagrangian method in [22].

To summarize, this paper proposes a fourth order PDE based surface restoration model which has numerical validation of ridges and sharp corners preserving properties and also has a corresponding fast algorithm. The rest of this paper is organized as follows. In section 2, motivated by considering vectorial total variation for the derivatives of the surface implicit function, we first introduce a variational model for 3D surface restoration, and then explain the proposed model in a toy example. In section 3, we design fast algorithms to solve the proposed model based on the augmented Lagrangian method. Meanwhile, the theoretical convergence analysis is also provided. After that, the numerical implementations are discussed in section 4. To demonstrate the robustness and efficiency of the proposed model and algorithms, we report experimental results and comparisons with previous methods in section 5. In addition, we also illustrate the potential applications of the proposed model to surface processing in medical imaging. Finally, conclusions are made in section 6.

2. A Fourth Order Variational Model for Surface Restoration. The total variation model plays an important role in image processing due to its edge preserving property. By observing that the derivatives of the surface implicit representation functions are discontinuous on ridges and sharp corners, we propose a noisy surface restoration model with the regularization as the vectorial total variation of the first order derivatives of the surface implicit functions.

2.1. Motivation and model. In the pioneering work [38], Rudin, Osher and Fatemi introduce the ROF denoising model for gray-scale/scalar image $u_0 : \Omega \to \mathbb{R}$ based on the total variation (TV) as follows:

$$\min_{u} \int_{\Omega} |\nabla u| + \frac{\eta}{2} \int_{\Omega} (u - u_0)^2 \tag{2.1}$$

Due to the edge preserving properties of the total variation (TV), the TV related models have had remarkable success on image processing. In the case of color image $\vec{u} = (u^1, \dots, u^m)$: $\Omega \to \mathbb{R}^m$, the above ROF model can be naturally generalized as the following color TV-L2 model [1, 6] (also related to [5]):

$$\min_{\vec{u}} \int_{\Omega} \sqrt{\|\nabla u^1\|^2 + \dots + \|\nabla u^m\|^2} + \frac{\eta}{2} \int_{\Omega} (\vec{u} - \vec{u}_0)^2$$
(2.2)

The regularization referred as vectorial total variation has also been studied in [26, 16] for color image processing.

In the problem of surface processing, one of the major challenges is diffusing surfaces with ridges and sharp corner preserving properties. Compared with image functions, an essential distinction of the implicit function representations for surfaces is that surface representation functions are not piecewise continuous but continuous. However, derivatives of the implicit functions may have discontinuities due to the ridges and sharp corners of surfaces. If the jumps of the derivatives of the implicit functions can be well preserved, then the corresponding ridges and sharp corners of surfaces can be also well preserved. Therefore, it is natural to consider the above vectorial total variation for the derivatives of implicit functions.

More precisely, let \mathcal{M} be a closed surface in \mathbb{R}^3 with implicit signed distance function representation ϕ [35]. Namely, $\mathcal{M} = \phi^{-1}(0) = \{(x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3 \mid \phi(x_1, x_2, x_3) = 0\}$. Then ϕ is a continuous function with possible discontinuity of its first order derivatives (ϕ_x, ϕ_y, ϕ_z) if \mathcal{M} has ridges or sharp corners. We consider a surface restoration model regularized by the vectorial total variation for the vector field (ϕ_x, ϕ_y, ϕ_z) as follows:

$$\min_{\phi} \int_{\Omega} \sqrt{\|\nabla \phi_x\|^2 + \|\nabla \phi_y\|^2 + \|\nabla \phi_z\|^2} + \frac{\eta}{2} \int_{\Omega} (\phi - \phi_0)^2$$

$$= \min_{\phi} \int_{\Omega} \sqrt{\sum_{\alpha,\beta=1}^3 \left(\frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta}\right)^2} + \frac{\eta}{2} \int_{\Omega} (\phi - \phi_0)^2 \tag{2.3}$$

If we denote the Hessian matrix of ϕ by $H(\phi) = \left(\frac{\partial^2 \phi}{\partial x_{\alpha} \partial x_{\beta}}\right)_{3 \times 3}$ and its Frobenius norm

by $|H(\phi)| = \sqrt{\sum_{\alpha,\beta=1}^{3} \left(\frac{\partial^2 \phi}{\partial x_{\alpha} \partial x_{\beta}}\right)^2}$. Then we can write down the above model as the following simple expression:

$$\min_{\phi} \int_{\Omega} |H(\phi)| + \frac{\eta}{2} \int_{\Omega} (\phi - \phi_0)^2 \tag{2.4}$$

Moreover, this vectorial total variation model for the first order derivatives only need onestep processing instead of two-step methods as in [31, 47] with the first step of processing the derivatives of the surface and the second step of recovering the surface from the derivatives.

2.2. Discretization. In practice, we assume the implicit representation ϕ is defined on regular $N_1 \times N_2 \times N_3$ grids with grid width h. Let's denote V as the Euclidean space $\mathbb{R}^{N_1 \times N_2 \times N_3}$. Then, the Hessian matrix of ϕ is defined on $\mathcal{W} = \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{9}$. For convenience, we use the following notations for any $\phi \in \mathcal{V}, P = (p_{\alpha\beta})_{3\times 3}, Q = (q_{\alpha\beta})_{3\times 3} \in \mathcal{W}$

in the rest of this paper:

$$\begin{aligned} |Q|_{\mathcal{W}} &= \sum_{i,j,k} \left(\sum_{\alpha,\beta=1}^{3} q_{\alpha\beta}^{2}(i,j,k) \right)^{1/2}, \qquad ||\phi||_{\mathcal{V}}^{2} = \sum_{i,j,k} \phi^{2}(i,j,k), \\ \langle Q, P \rangle_{\mathcal{W}} &= \sum_{i,j,k} \left(\sum_{\alpha,\beta=1}^{3} q_{\alpha\beta}(i,j,k) p_{\alpha\beta}(i,j,k) \right), \qquad ||Q||_{\mathcal{W}}^{2} = \langle Q, Q \rangle_{\mathcal{W}}, \end{aligned}$$

We denote the backward and forward discretization of derivative with periodic boundary condition as:

$$\begin{split} \partial_{x_1}^{-}\phi(i,j,k) &= \begin{cases} (\phi(i,j,k) - \phi(i-1,j,k)), &1 < i \le N_1; \\ (\phi(1,j,k) - \phi(N_1,j,k)), &i = 1. \end{cases} \\ \partial_{x_1}^{+}\phi(i,j,k) &= \begin{cases} (\phi(i+1,j,k) - \phi(i,j,k)), &1 \le i < N_1; \\ (\phi(1,j,k) - \phi(N_1,j,k)), &i = N_1. \end{cases} \\ \partial_{x_2}^{-}\phi(i,j,k) &= \begin{cases} (\phi(i,j,k) - \phi(i,j-1,k)), &1 < j \le N_2; \\ (\phi(i,1,k) - \phi(i,N_2,k)), &j = 1. \end{cases} \\ \partial_{x_2}^{+}\phi(i,j,k) &= \begin{cases} (\phi(i,j+1,k) - \phi(i,j,k)), &1 \le j < N_2; \\ (\phi(i,1,k) - \phi(i,N_2,k)), &j = N_2. \end{cases} \\ \partial_{x_3}^{-}\phi(i,j,k) &= \begin{cases} (\phi(i,j,k) - \phi(i,j,k-1)), &1 < k \le N_3; \\ (\phi(i,j,1) - \phi(i,j,N_3)), &k = 1. \end{cases} \\ \partial_{x_3}^{+}\phi(i,j,k) &= \begin{cases} (\phi(i,j,k+1) - \phi(i,j,N_3)), &1 \le k < N_3; \\ (\phi(i,j,1) - \phi(i,j,N_3)), &k = N_3. \end{cases} \end{split}$$

The corresponding discretization of second order derivatives are given by $(\alpha, \beta = 1, 2, 3)$:

$$\begin{split} \partial^{-+}_{x_{\alpha}x_{\beta}}\phi(i,j,k) &= (\partial^{-}_{x_{\alpha}}(\partial^{+}_{x_{\beta}}\phi))(i,j,k), \quad \partial^{+-}_{x_{\alpha}x_{\beta}}\phi(i,j,k) = (\partial^{+}_{x_{\alpha}}(\partial^{-}_{x_{\beta}}\phi))(i,j,k) \\ \partial^{--}_{x_{\alpha}x_{\beta}}\phi(i,j,k) &= (\partial^{-}_{x_{\alpha}}(\partial^{-}_{x_{\beta}}\phi))(i,j,k), \quad \partial^{++}_{x_{\alpha}x_{\beta}}\phi(i,j,k) = (\partial^{+}_{x_{\alpha}}(\partial^{+}_{x_{\beta}}\phi))(i,j,k), \end{split}$$

We denote the discretization of Hessian $H : \mathcal{V} \to \mathcal{W}$ as follows:

$$H(\phi) = \begin{pmatrix} \partial_{x_1x_1}^{-+} \phi & \partial_{x_1x_2}^{++} \phi & \partial_{x_1x_3}^{++} \phi \\ \partial_{x_2x_1}^{++} \phi & \partial_{x_2x_2}^{-+} \phi & \partial_{x_2x_3}^{++} \phi \\ \partial_{x_3x_1}^{++} \phi & \partial_{x_3x_2}^{++} \phi & \partial_{x_3x_3}^{-+} \phi \end{pmatrix}$$
(2.5)

Then, the discretization of the surface restoration model (2.4) can be written as:

$$\min_{\phi \in \mathcal{V}} \mathcal{E}(\phi) = |H(\phi)|_{\mathcal{W}} + \frac{\eta}{2} \|\phi - \phi_0\|_{\mathcal{V}}^2$$
(2.6)

We write $\mathcal{R}_{L^1}: \mathcal{W} \to \mathbb{R}, \mathcal{R}_{L^1}(Q) = |Q|_{\mathcal{W}}$, then R is a convex function defined on \mathcal{W} . Thus, $\mathcal{E}(\phi) = \mathcal{R}_{L^1}(H\phi) + \frac{\eta}{2} \|\phi - \phi_0\|_{\mathcal{V}}^2$ is also a convex function. In addition, it is easy to see that \mathcal{E} is continuous and coercive. Therefore, the following results can be obtained from a standard result in convex optimization [17]:

THEOREM 2.1. The problem (2.6) has a unique solution ϕ^* satisfying:

$$0 \in \eta(\phi^* - \phi_0) - H^*(\partial \mathcal{R}_{L^1}(H\phi^*))$$
(2.7)

where $\partial \mathcal{R}_{L^1}(H\phi^*)$ is the sub-differential of \mathcal{R}_{L^1} at $H\phi^*$ and H^* is the dual operator of H.

2.3. Discussion about the proposed model. To clearly explain in what sense the proposed model preserves ridges and sharp corners, we first check the model for a 1D curve example, the graph of h(x) = 1 - |x|. Remember that the regularization of our model is the total variation(TV) of the derivative. A well-known result for TV-L2 model is that the output will preserve jumps, but lose ontrast [43, 4, 42]. In our case, jump is the sharp corner at x = 0 and the contrast is the slope of h(x). We can expect that our model in 1D case will keep the corner at x = 0 and shrink the angle of the corner. However, the mean curvature flow in this case will smooth out the sharp corner. Thus, we expect the proposed model preserves ridges and corners in the sense that the sharpness of ridges and corners will be preserved, but their angles might have small shrinkage. However, theoretical analysis in high dimension case is not straightforward, which will be investigated in our future work. Here, we would like to take a toy surface example to illustrate this point numerically.

Let \mathcal{M}_c be a surface defined as the graph of a function $f_c(x, y) = 1 - |x| - |y|, (x, y) \in [-1, 1]^2$. Namely, $\mathcal{M}_c = \{(x, y) \in [-1, 1]^2 \mid (x, y, f_c(x, y))\}$, which can also be represented as the zero level set of the function $\phi_c(x, y, z) = f_c(x, y) - z : [-1, 1]^3 \to \mathbb{R}$. This surface and the corresponding level curves of f_c in $[-1, 1]^2$ are with ridges and sharp corners. In this special example, we consider all surfaces given as graphs of function f(x, y) on $[-1, 1]^2$ with implicit representation $\phi(x, y, z) = f(x, y) - z$. Then the proposed model (2.4) can be written as:

$$\phi^* = \arg\min_{\phi} \int \sqrt{\|\nabla \partial_x \phi\|^2 + \|\nabla \partial_y \phi\|^2 + \|\nabla \partial_z \phi\|^2} + \frac{\eta}{2} \int (\phi - \phi_0)^2$$
$$\implies f^* = \arg\min_{f} \int_{[-1,1]^2} \sqrt{f_{xx}^2 + f_{xy}^2 + f_{yx}^2 + f_{yy}^2} + \frac{\eta}{2} \int_{[-1,1]^2} (f - f_0)^2 \qquad (2.8)$$

This is exactly the LLT model proposed in [30] for image processing. In fact, the second order norm in the above minimization has also been used in Chambolle-Lions [8] together with the TV-norm in a setting for image decomposition. The LLT model has been analyzed in a number of papers [27, 28, 29].

Here, we would like to first analyze the minimizer f^* in the case that $f_0 = f_c$. Since the closed-form solution is not straightforward to obtain even for this simple case, we numerically illustrate the relation of f^* and η to demonstrate its ridge and shape corners preserving properties.



FIG. 2.1. Energy curves of $E_{\eta}^{c}, E_{\eta}^{\epsilon}, E_{\eta}$ via different value of η are color-coded as red, black and blue curves, respectively.

By choosing the grid size h = 0.01, we write the discretized energy $\mathcal{E}_{\eta}(f) = |H(f)|_{\mathcal{W}} + \frac{\eta}{2} ||f - f_c||_{\mathcal{V}}^2$ using the similar notation introduced in section 2.2. A straightforward calculation indicates that a smoothed version of \mathcal{M}_c given by $f_{\epsilon}(x, y, z) = 1 - \sqrt{x^2 + \epsilon^2} - \sqrt{y^2 + \epsilon^2}$



FIG. 2.2. Numerical test of (2.8) with $f_0 = 1 - |x| - |y|$ and $\eta = 0.5, 10, 20$. The first row: output surfaces f^* and corresponding red color-coded level contours using $\eta = 0.5, 10, 20$ respectively. The second row: level contours comparison of the resulting surfaces f^* with the input clean surface and resulting surface f_{ϵ}^* . The third row: zoom-in images of the second row.

may have a smaller energy $\mathcal{E}_{\eta}(f_{\epsilon})$ than $\mathcal{E}_{\eta}(f_{c})$. This might give an intuition that the proposed model is energetically favorable to choose f_{ϵ} , which will lead to the ridges and corners smoothing instead of preserving. However, a more careful numerical calculation shows that f_{ϵ} never attains the global minimal value for the convex energy \mathcal{E}_{η} . If we write $E_{\eta}^{c} =$ $\mathcal{E}_{\eta}(f_c), E_{\eta}^{\epsilon} = \min_{\epsilon} \mathcal{E}_{\eta}(f_{\epsilon}), E_{\eta} = \min_{f} \mathcal{E}_{\eta}(f)$, a comparison among these three values is reported in Figure. 2.1 with different values of η . It is clear to see that $E_{\eta}^{c} \ge E_{\eta} \ge E_{\eta}$. In other words, the ridges and corners smoothed version f_{ϵ} is not the minimizer of the proposed model for all test value η . In Figure. 2.2, we report resulting surfaces using $\eta = 0.5, 10, 20$. One can observe that the larger η is chosen, then the smaller difference we can have between the output f^* and f_c , while $f^*_{\epsilon} = \arg \min_{\epsilon} \mathcal{E}_{\eta}(f_{\epsilon})$ always smears out ridges and corners for different values of $\eta = 0.5, 10, 20$. To clearly show the ridges preserving property, we project the level contours of resulting surfaces from f^* and f^*_{ϵ} to the 2D plane and illustrate comparisons with clean surface f_c in the second and third rows of Figure. 2.2. One can observe that a small value of η will lead to ridges and corners smoothing although the input surface is completely noisy free. However, a suitable scale of η in the proposed model can successfully preserve the ridges and corners of the input surface. A careful check of the last four zoom-in images in Figure. 2.2 will find that, although resulting f^* are not exact the same as the input clean surface f_c , differences of ridges and corners of f^* and f_c are close enough in the way that the sharpness of ridges and corners are preserved while the angles of f^* may have tiny shrinkage. However, f_{ϵ}^* can not preserve ridges and corners for all test values $\eta = 0.5, 10, 20$. In this sense, we call f^* preserve ridges and corners of f_c .

To have a further test to the proposed model for noisy surface restoration, we consider

a noise contaminated surface $\mathcal{M}_{\sigma} = \{(x, y) \in [-1, 1]^2 \mid (x, y, f_{\sigma}(x, y))\}$, where $f_{\sigma} = f_c + \sigma * \xi$ and ξ satisfies the standard normal distribution $\mathcal{N}(0, 1)$. In Figure 2.3, we illustrate a preliminary result of the above model using $\eta = 20$ to a noisy input surface f_{σ} with $\sigma = 0.01$. Figure 2.3 (b) and (c) illustrate a promising surface restoration result with well-preserved ridges and sharp corners.



FIG. 2.3. (a)The input surface with Gaussian noise and the corresponding contours; (b) The output surface restoration result and the corresponding contours. (c) A Comparison of contours from the input noisy surface, the output restoration result and the clean surface with blue, red and black color respectively.

3. Fast algorithms using augmented Lagrangian methods. Because of the second order derivates and the non-differentiability of absolute value in model (2.4), the gradient descent approach needs extreme carefulness about the way of choosing the time step, and it is highly time consuming. Instead of using gradient descent approach, dual methods have been proposed in [9, 41]. More recently, Wu et al. [50] propose another fast algorithm to solve 2D LLT [30] for image processing based on augmented Lagrangian methods. Here, we adapt the augmented Lagrangian method in [50] to solve the proposed surface restoration model (2.4) in 3D. More importantly, we theoretically prove the convergence of the proposed algorithm based on the augmented Lagrangian method.

If we introduce an auxiliary variable Q for $H(\phi)$, then the discretized variational problem (2.6) is equivalent to the following constrained problem:

$$\min_{\phi \in \mathcal{V}, Q \in \mathcal{W}} \mathcal{E}_2(\phi, Q) = |Q|_{\mathcal{W}} + \frac{\eta}{2} \|\phi - \phi_0\|_{\mathcal{V}}^2 \quad \text{s.t.} \quad Q = H(\phi)$$
(3.1)

To solve the above constrained problem, we define the augmented Lagrangian:

$$\mathcal{L}(\phi, Q; \Lambda) = |Q|_{\mathcal{W}} + \frac{\eta}{2} \|\phi - \phi_0\|_{\mathcal{V}}^2 + \frac{r}{2} \|Q - H(\phi)\|_{\mathcal{W}}^2 + \langle \Lambda, Q - H(\phi) \rangle_{\mathcal{W}}$$
(3.2)

whose corresponding saddle-point problem can be described as follows:

Find
$$(\phi^*, Q^*; \Lambda^*) \in \mathcal{V} \times \mathcal{W} \times \mathcal{W}$$

s.t. $\mathcal{L}(\phi^*, Q^*; \Lambda) \leq \mathcal{L}(\phi^*, Q^*; \Lambda^*) \leq \mathcal{L}(\phi, Q; \Lambda^*), \quad \forall (\phi, Q; \Lambda)$ (3.3)

Theoretically, the following statement about the above saddle-point problem holds:

THEOREM 3.1. $\phi^* \in \mathcal{V}$ is a solution of the problem (2.6) if and only if there exist $Q^*, \Lambda^* \in \mathcal{W}$ such that $(\phi^*, Q^*; \Lambda^*)$ is a solution of the saddle point problem (3.3).

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[Proof]: Suppose $(\phi^*, Q^*; \Lambda^*)$ is a solution of the saddle point problem (3.3). Then $Q^* = H(\phi^*)$ from the first inequality of (3.3). By using the second inequality of (3.3), we can have:

$$|H(\phi^*)|_{\mathcal{W}} + \frac{\eta}{2} ||\phi^* - \phi_0||_{\mathcal{V}}^2 \le |Q|_{\mathcal{V}} + \frac{\eta}{2} ||\phi - \phi_0||_{\mathcal{V}}^2 + \frac{r}{2} ||Q - H(\phi)||_{\mathcal{W}}^2 + \langle \Lambda^*, Q - H(\phi) \rangle_{\mathcal{W}}$$

By taking $Q = H(\phi)$, we have:

$$|H(\phi^*)|_{\mathcal{W}} + \frac{\eta}{2} ||\phi^* - \phi_0||_{\mathcal{V}}^2 \le |H(\phi)|_{\mathcal{W}} + \frac{\eta}{2} ||\phi - \phi_0||_{\mathcal{V}}^2, \quad \forall \phi \in \mathcal{V}$$
(3.4)

On the other hand, if ϕ^* is a solution of the problem (2.6), we choose $Q^* = H(\phi^*)$. According to the theorem 2.1, there exit a $\Lambda^* \in \partial \mathcal{R}_{L^1}(H\phi^*)$, such that $H^*(\Lambda^*) = \eta(\phi^* - \phi_0)$. To verify $(\phi^*, Q^*; \Lambda^*)$ is a saddle point of the problem (3.3), we need to prove

$$\mathcal{L}(\phi^*, Q^*; \Lambda) \le \mathcal{L}(\phi^*, Q^*; \Lambda^*) \le \mathcal{L}(\phi, Q; \Lambda^*), \quad \forall (\phi, Q; \Lambda) \in \mathcal{V} \times \mathcal{W} \times \mathcal{W} \quad (3.5)$$

This first inequality is easy to check since $Q^* = H(\phi^*)$. To verify the second inequality, we only need to illustrate:

$$|Q^*|_{\mathcal{W}} + \frac{\eta}{2} ||\phi^* - \phi_0||_{\mathcal{V}}^2 \le |Q|_{\mathcal{W}} + \frac{\eta}{2} ||\phi - \phi_0||_{\mathcal{V}}^2 + \frac{r}{2} ||Q - H(\phi)||_{\mathcal{W}}^2 + \langle \Lambda^*, Q - H(\phi) \rangle_{\mathcal{W}}$$
(3.6)

Due to the construction of Q^* and Λ^* , we can have the following inequalities:

$$\frac{\eta}{2}||\phi - \phi_0||_{\mathcal{V}}^2 - \frac{\eta}{2}||\phi^* - \phi_0||_{\mathcal{V}}^2 + \langle \Lambda^*, H(\phi^* - \phi) \rangle_{\mathcal{W}} \ge 0, \quad \forall \phi \in \mathcal{V}$$
(3.7)

$$|Q|_{\mathcal{W}} - |Q^*|_{\mathcal{W}} + \langle \Lambda^*, Q - Q^* \rangle_{\mathcal{W}} \ge 0, \qquad \forall Q \in \mathcal{W}$$
(3.8)

Then the inequality (3.6) can be obtained directly for the summation of (3.7) and (3.8). This completes the proof. $\hfill \Box$

According to the theorem 3.1, a solution of the variational problem (2.6) can be obtained by solving the saddle point problem (3.3). We can iteratively approach a solution of (3.3) by solving a series of minimization problems which lead to the first algorithm.

Algorithm 1: Augmented Lagrangian method for the variational problem (2.6)

- 1. Initialization: $Q^0 = H(\phi_0), \Lambda^1 = 0,$
- 2. Update ϕ , Q;

$$(\phi^n, Q^n) = \arg\min_{\phi, Q} \mathcal{L}(\phi, Q; \Lambda^n)$$
(3.9)

3. Update Lagrange multipliers:

$$\Lambda^{n+1} = \Lambda^n + r(Q^n - H(\phi^n)) \tag{3.10}$$

The solution of the minimization problem (3.9) can be computed by the alternating minimization method. We here particularly choose one alternating step and obtain the second algorithm. Moreover, similar as the algorithms of augmented Lagrangian provided by Glowinski in [22], we have the following theorem about the convergence analysis of the proposed Algorithm 2.

Algorithm 2: Augmented Lagrangian method for the variational problem (2.6)

1. Initialization: $Q^0 = H(\phi_0), \Lambda^1 = 0,$

2. Update ϕ ;

$$\phi^n = \arg\min_{\phi} \mathcal{L}(\phi, Q^{n-1}; \Lambda^n)$$
(3.11)

3. Update Q;

$$Q^{n} = \arg\min_{Q} \mathcal{L}(\phi^{n}, Q; \Lambda^{n})$$
(3.12)

4. Update Lagrange multipliers:

$$\Lambda^{n+1} = \Lambda^n + r(Q^n - H(\phi^n))$$

THEOREM 3.2 (Convergence analysis). Assume $(\phi^*, Q^*; \Lambda^*)$ is a saddle-point of (3.3), then the asymptotical behaviors of ϕ_n , Q_n and Λ_n satisfy:

$$\lim_{n \to \infty} \phi^n = \phi^*, \quad \lim_{n \to \infty} Q^n = Q^*, \quad \lim_{n \to \infty} \Lambda^n = \Lambda^*$$
(3.13)

Therefore, $\lim_{n\to\infty} \mathcal{L}(\phi^n, Q^n; \Lambda^n) = \mathcal{L}(\phi^*, Q^*; \Lambda^*) = \mathcal{E}_1(\phi^*)$ and ϕ^n will converge to the solution of (2.6).

[Proof]: Define all errors as follows:

$$e_{\phi}^{n} = \phi^{n} - \phi^{*}, \quad e_{Q}^{n} = Q^{n} - Q^{*}, \quad e_{\Lambda}^{n} = \Lambda^{n} - \Lambda^{*}.$$
 (3.14)

According to the theorem 3.1, we have $Q^* = H(\phi^*)$. Therefore, we have:

$$e_{\Lambda}^{n+1} = e_{\Lambda}^{n} + r(e_{Q}^{n} - He_{\phi}^{n})$$

$$\implies ||e_{\Lambda}^{n}||_{\mathcal{W}}^{2} - ||e_{\Lambda}^{n+1}||_{\mathcal{W}}^{2} = -2r\langle e_{\Lambda}^{n}, e_{Q}^{n} - He_{\phi}^{n}\rangle_{\mathcal{W}} - r^{2}||e_{Q}^{n} - He_{\phi}^{n}||_{\mathcal{W}}^{2} \quad (3.15)$$

Since $(\phi^*,Q^*;\Lambda^*)$ is a saddle-point of (3.3), we have:

$$\phi^* = \arg\min_{\phi \in \mathcal{V}} \mathcal{L}(\phi, Q^*; \Lambda^*) \quad \text{and} \quad Q^* = \arg\min_{Q \in \mathcal{W}} \mathcal{L}(\phi^*, Q; \Lambda^*)$$
(3.16)

Therefore, for any $\phi \in \mathcal{V}, Q \in \mathcal{W}$,

$$\frac{\eta}{2} ||\phi - \phi_0||_{\mathcal{V}}^2 - \frac{\eta}{2} ||\phi^* - \phi_0||_{\mathcal{V}}^2 + \langle \Lambda^*, H(\phi^* - \phi) \rangle_{\mathcal{W}} + r \langle Q^* - H\phi^*, H(\phi^* - \phi) \rangle_{\mathcal{W}} \ge 0,$$
(3.17)
$$|Q|_{\mathcal{W}} - |Q^*|_{\mathcal{W}} + \langle \Lambda^*, Q - Q^* \rangle_{\mathcal{W}} + r \langle Q^* - H\phi^*, Q - Q^* \rangle_{\mathcal{W}} \ge 0,$$
(3.18)

Similarly, from the construction of ϕ^n and Q^n , we have:

$$\phi^n = \arg\min_{\phi \in \mathcal{V}} \mathcal{L}(\phi, Q^{n-1}; \Lambda^n) \quad \text{and} \quad Q^n = \arg\min_{Q \in \mathcal{W}} \mathcal{L}(\phi^n, Q; \Lambda^n)$$
(3.19)

Therefore, for any $\phi \in \mathcal{V}, Q \in \mathcal{W}$,

$$\frac{\eta}{2} ||\phi - \phi_0||_{\mathcal{V}}^2 - \frac{\eta}{2} ||\phi^n - \phi_0||_{\mathcal{V}}^2 + \langle \Lambda^n, H(\phi^n - \phi) \rangle_{\mathcal{W}} + r \langle Q^{n-1} - H\phi^n, H(\phi^n - \phi) \rangle_{\mathcal{W}} \ge 0$$
(3.20)
$$|Q|_{\mathcal{W}} - |Q^n|_{\mathcal{W}} + \langle \Lambda^n, Q - Q^n \rangle_{\mathcal{W}} + r \langle Q^n - H\phi^n, Q - Q^n \rangle_{\mathcal{W}} \ge 0$$
(3.21)

Let $\phi = \phi^n$ in (3.17), $\phi = \phi^*$ in (3.20), respectively, and then add them together:

$$\langle e_{\Lambda}^{n}, He_{\phi}^{n} \rangle_{\mathcal{W}} + r \langle e_{Q}^{n-1} - He_{\phi}^{n}, He_{\phi}^{n} \rangle_{\mathcal{W}} \ge 0$$
(3.22)

Let $Q = Q^n$ in (3.18), $Q = Q^*$ in (3.21), respectively, and then add them together:

$$\langle e^n_{\Lambda}, -e^n_Q \rangle_{\mathcal{W}} + r \langle -e^n_Q + He^n_{\phi}, e^n_Q \rangle_{\mathcal{W}} \ge 0$$
(3.23)

We add (3.22) and (3.23) together:

$$\langle e_{\Lambda}^{n}, He_{\phi}^{n} - e_{Q}^{n} \rangle_{\mathcal{W}} + r \langle e_{Q}^{n-1} - He_{\phi}^{n}, He_{\phi}^{n} \rangle_{\mathcal{W}} + r \langle -e_{Q}^{n} + He_{\phi}^{n}, e_{Q}^{n} \rangle_{\mathcal{W}} \ge 0$$

$$\Longrightarrow \quad - \langle e_{\Lambda}^{n}, e_{Q}^{n} - He_{\phi}^{n} \rangle_{\mathcal{W}} - r ||e_{Q}^{n} - He_{\phi}^{n}||_{\mathcal{W}}^{2} + r \langle He_{\phi}^{n}, e_{Q}^{n} - e_{Q}^{n-1} \rangle_{\mathcal{W}} \ge 0$$

$$(3.24)$$

By combining (3.15) and (3.24), we can have:

$$|e_{\Lambda}^{n}||_{\mathcal{W}}^{2} - ||e_{\Lambda}^{n+1}||_{\mathcal{W}}^{2} \ge r^{2}||e_{Q}^{n} - He_{\phi}^{n}||_{\mathcal{W}}^{2} + 2r^{2}\langle He_{\phi}^{n}, e_{Q}^{n} - e_{Q}^{n-1}\rangle_{\mathcal{W}}$$
(3.25)

We further analyze $\langle He_{\phi}^n, e_Q^n - e_Q^{n-1} \rangle_{\mathcal{W}}$ by expanding it as:

$$\langle He_{\phi}^{n}, e_{Q}^{n} - e_{Q}^{n-1} \rangle_{\mathcal{W}} = \langle He_{\phi}^{n} - He_{\phi}^{n-1}, e_{Q}^{n} - e_{Q}^{n-1} \rangle_{\mathcal{W}} + \langle He_{\phi}^{n-1} - e_{Q}^{n-1}, e_{Q}^{n} - e_{Q}^{n-1} \rangle_{\mathcal{W}} + \langle e_{Q}^{n-1}, e_{Q}^{n} - e_{Q}^{n-1} \rangle_{\mathcal{W}} (3.26)$$

Notice that $Q^{n-1} = \arg\min_{\phi\in\mathcal{V}}\mathcal{L}(\phi^{n-1},\phi;\Lambda^{n-1})$, we can have similar inequality as (3.21): $|Q|_{\mathcal{W}} - |Q^{n-1}|_{\mathcal{W}} + \langle\Lambda^{n-1},Q-Q^{n-1}\rangle_{\mathcal{W}} + r\langle Q^{n-1}-H\phi^{n-1},Q-Q^{n-1}\rangle_{\mathcal{W}} \ge 0, \ \forall Q \in \mathcal{W}$ (3.27)

Let $Q = Q^{n}$ in (3.27), $Q = Q^{n-1}$ in (3.21), respectively, and then add them together: $\langle \Lambda^{n-1} - \Lambda^{n}, Q^{n} - Q^{n-1} \rangle_{\mathcal{W}} + r \langle Q^{n-1} - Q^{n} + H(\phi^{n} - \phi^{n-1}), Q^{n} - Q^{n-1} \rangle_{\mathcal{W}} \ge 0,$ $\Rightarrow \langle \Lambda^{n} - \Lambda^{n-1}, Q^{n} - Q^{n-1} \rangle_{\mathcal{W}} + r ||Q^{n} - Q^{n-1}||_{\mathcal{W}}^{2} - r \langle H(\phi^{n} - \phi^{n-1}), Q^{n} - Q^{n-1} \rangle_{\mathcal{W}} \le 0$ Since $\Lambda^{n} - \Lambda^{n-1} = r(Q^{n-1} - H\phi^{n-1}), Q^{n} - Q^{n-1} = e_{Q}^{n} - e_{Q}^{n-1}, \phi^{n} - \phi^{n-1} = e_{\phi}^{n} - e_{\phi}^{n-1},$ $\langle He_{\phi}^{n-1} - e_{Q}^{n-1}, e_{Q}^{n} - e_{Q}^{n-1} \rangle_{\mathcal{W}} + \langle He_{\phi}^{n} - He_{\phi}^{n-1}, e_{Q}^{n} - e_{Q}^{n-1} \rangle_{\mathcal{W}} \ge ||e_{Q}^{n} - e_{Q}^{n-1}||_{\mathcal{W}}^{2}$ (3.28)

By combining (3.28), (3.25) and (3.26), we can have:

$$\begin{split} ||e_{\Lambda}^{n}||_{\mathcal{W}}^{2} - ||e_{\Lambda}^{n+1}||_{\mathcal{W}}^{2} \geq r^{2}||e_{Q}^{n} - He_{\phi}^{n}||_{\mathcal{W}}^{2} + 2r^{2}(||e_{Q}^{n} - e_{Q}^{n-1}||_{\mathcal{W}}^{2} + \langle e_{Q}^{n-1}, e_{Q}^{n} - e_{Q}^{n-1}\rangle_{\mathcal{W}}) \\ &= r^{2}||e_{Q}^{n} - He_{\phi}^{n}||_{\mathcal{W}}^{2} + r^{2}(||e_{Q}^{n}||_{\mathcal{W}}^{2} - ||e_{Q}^{n-1}||_{\mathcal{W}}^{2} + ||e_{Q}^{n} - e_{Q}^{n-1}||_{\mathcal{W}}^{2}) \\ &= r^{2}||Q^{n} - H\phi^{n}||_{\mathcal{W}}^{2} + r^{2}(||e_{Q}^{n}||_{\mathcal{W}}^{2} - ||e_{Q}^{n-1}||_{\mathcal{W}}^{2} + ||Q^{n} - Q^{n-1}||_{\mathcal{W}}^{2}) \end{split}$$

This implies:

$$(||e_{\Lambda}^{n}||_{\mathcal{W}}^{2} + r^{2}||e_{Q}^{n-1}||_{\mathcal{W}}^{2}) - (||e_{\Lambda}^{n+1}||_{\mathcal{W}}^{2} + r^{2}||e_{Q}^{n}||_{\mathcal{W}}^{2}) \ge r^{2}||Q^{n} - H\phi^{n}||_{\mathcal{W}}^{2} + r^{2}||Q^{n} - Q^{n-1}||_{\mathcal{W}}^{2}$$

$$(3.29)$$

Therefore, the sequence $\{||e_{\Lambda}^{n}||_{W}^{2} + r^{2}||e_{Q}^{n-1}||_{W}^{2}\}_{n}$ is a decreasing sequence, thus it is convergent. This indicates that the left hand side of (3.29) will converge to 0, which further implies:

$$\begin{cases} \{\Lambda^n\}_n, \{Q^n\}_n, \{H\phi^n\}_n \text{ are bounded sequences} \\ \lim_{n \to \infty} ||Q^n - H\phi^n||_{\mathcal{W}}^2 = 0, \text{ and } \lim_{n \to \infty} ||Q^n - Q^{n-1}||_{\mathcal{W}}^2 = 0 \end{cases} (3.30)$$

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Furthermore, since (ϕ^*, Q^*, Λ^*) is a solution of the saddle-point problem (3.3), we have:

$$\mathcal{E}_2(\phi^*, Q^*) \le \mathcal{E}_2(\phi^n, Q^n) + \langle \Lambda^*, Q^n - H\phi^n \rangle_{\mathcal{W}} + \frac{r}{2} ||Q^n - H\phi^n||_{\mathcal{W}}^2$$
(3.31)

In addition, we take $\phi = \phi^*$ in (3.20) and $Q = Q^*$ in (3.21), the summation of them would be:

$$\mathcal{E}_{2}(\phi^{*},Q^{*}) \geq E_{2}(\phi^{n},Q^{n}) + \langle \Lambda^{n},Q^{n} - H\phi^{n} \rangle_{\mathcal{W}} + r||Q^{n} - H\phi^{n}||_{\mathcal{W}} + r\langle He_{\phi}^{n},Q^{n} - Q^{n-1} \rangle$$
(3.32)

By combining (3.30), (3.31), and (3.32), we have:

$$\lim \inf \mathcal{E}_{2}(\phi^{n}, Q^{n}) \geq \mathcal{E}_{2}(\phi^{*}, Q^{*}) \geq \limsup \mathcal{E}_{2}(\phi^{n}, Q^{n})$$
$$\implies \qquad \lim_{n \to \infty} \mathcal{E}_{2}(\phi^{n}, Q^{n}) = \mathcal{E}_{2}(\phi^{*}, Q^{*})$$
(3.33)

Moreover, since $(\phi^*, Q^*; \Lambda^*)$ is a saddle point of $\mathcal{L}(\phi, Q; \Lambda)$, we have:

$$\Lambda^* \in \partial \mathcal{R}_{L^1}(Q^*) \Longrightarrow H^*(\Lambda^*) = \eta(\phi^* - \phi_0),$$

$$|Q^n|_{\mathcal{W}} - |Q^*|_{\mathcal{W}} + \langle \Lambda^*, Q^n - Q^* \rangle_{\mathcal{W}} \ge 0,$$
(3.34)

Then we have:

$$\mathcal{E}_{2}(\phi^{n}, Q^{n}) + \langle \Lambda^{*}, Q^{n} - H\phi^{n} \rangle_{\mathcal{W}} \\
= |Q^{n}|_{\mathcal{W}} + \frac{\eta}{2} ||\phi^{n} - \phi_{0}||_{\mathcal{V}}^{2} + \langle \Lambda^{*}, Q^{n} - H\phi^{n} \rangle_{\mathcal{W}} \\
\geq |Q^{*}|_{\mathcal{W}} - \langle \Lambda^{*}, Q^{n} - Q^{*} \rangle_{\mathcal{W}} + \frac{\eta}{2} ||\phi^{n} - \phi_{0}||_{\mathcal{V}}^{2} + \langle \Lambda^{*}, Q^{n} - H\phi^{n} \rangle_{\mathcal{W}} \\
\geq |Q^{*}|_{\mathcal{W}} + \frac{\eta}{2} ||\phi^{n} - \phi_{0}||_{\mathcal{V}}^{2} + \langle \Lambda^{*}, H\phi^{*} - H\phi^{n} \rangle_{\mathcal{W}} \\
= \mathcal{E}_{2}(\phi^{*}, Q^{*}) - \frac{\eta}{2} ||\phi^{*} - \phi_{0}||_{\mathcal{V}}^{2} + \frac{\eta}{2} ||\phi^{n} - \phi_{0}||_{\mathcal{V}}^{2} + \langle \Lambda^{*}, H\phi^{*} - H\phi^{n} \rangle_{\mathcal{W}} \\
= \mathcal{E}_{2}(\phi^{*}, Q^{*}) + \frac{\eta}{2} \langle \phi^{*} + \phi^{n} - 2\phi_{0}, \phi^{n} - \phi^{*} \rangle_{\mathcal{V}} + \eta \langle \phi^{*} - \phi_{0}, \phi^{*} - \phi^{n} \rangle_{\mathcal{V}} \\
= \mathcal{E}_{2}(\phi^{*}, Q^{*}) + \frac{\eta}{2} ||\phi^{*} - \phi^{n}||_{\mathcal{V}}^{2}$$
(3.35)

By combining (3.35) with (3.30) and (3.33), we can have:

$$\lim_{n \to \infty} \phi^n = \phi^*, \quad \lim_{n \to \infty} Q^n = Q^*$$
(3.36)

In addition, by the construction of Λ^n , it is easy to have $\lim_{n\to\infty} \Lambda^n = \Lambda^*$. This completes the proof.

Theorem 3.2 clearly indicates the convergence of the Algorithm 2. Based on this theoretical guarantee, we can solve the variational model (2.4) by approaching the solution of its discrete model (2.6). Numerically, we need to solve two minimization problems in Algorithm 2, which will be discussed in the next section for more details.

4. Numerical Implementation. According to the Algorithm 2, we need to solve two minimization problems (3.11) and (3.12). Let's define an operator $H^* : W \to V$ by:

$$H^{*}((q_{\alpha\beta})_{3\times3}) = \partial_{x_{1}x_{1}}^{+} q_{11} + \partial_{x_{1}x_{2}}^{-} q_{12} + \partial_{x_{1}x_{3}}^{-} q_{13} + \partial_{x_{2}x_{1}}^{-} q_{21} + \partial_{x_{2}x_{2}}^{+} q_{22} + \partial_{x_{2}x_{3}}^{-} q_{23} + \partial_{x_{3}x_{1}}^{-} q_{31} + \partial_{x_{3}x_{2}}^{-} q_{32} + \partial_{x_{3}x_{3}}^{+} q_{33}$$
(4.1)

Then it is easy to verify that $\langle H(\phi), Q \rangle_{\mathcal{W}} = \langle \phi, H^*(Q) \rangle_{\mathcal{V}}$. Namely, H^* is the adjoint operator of H. The minimizer of the problems (3.11) is the same as the minimizer of the following problem:

$$\phi^{n} = \arg\min_{\phi} \mathcal{L}(\phi, Q^{n-1}; \Lambda^{n})$$

= $\arg\min_{\phi} \frac{\eta}{2} \|\phi - \phi_{0}\|_{\mathcal{V}}^{2} + \frac{r}{2} \|Q^{n-1} - H(\phi)\|_{\mathcal{W}}^{2} - \langle \Lambda^{n}, H(\phi) \rangle_{\mathcal{W}}$ (4.2)

Whose minimizer ϕ^n satisfies the Euler-Lagrange equation:

$$\eta(\phi^{n} - \phi_{0}) + rH^{*}H(\phi^{n}) - rH^{*}(Q^{n-1}) - H^{*}(\Lambda^{n}) = 0$$

(\eta + rH^{*}H)\phi^{n} = rH^{*}(Q^{n-1}) + H^{*}(\Lambda^{n}) + \eta \phi_{0} (4.3)

Since the periodic boundary condition is imposed, the above equation (4.3) can be efficiently solved using the fast Fourier transform (FFT). If we write \mathcal{F} as the discrete Fourier transform, then

$$\begin{split} \mathcal{F}\partial_{x_{1}}^{+}\phi(i,j,k) &= (e^{\sqrt{-1}\theta_{i}^{1}} - 1)\mathcal{F}\phi(i,j,k), \quad \mathcal{F}\partial_{x_{1}}^{-}\phi(i,j,k) = (1 - e^{-\sqrt{-1}\theta_{i}^{1}})\mathcal{F}\phi(i,j,k) \\ \mathcal{F}\partial_{x_{2}}^{+}\phi(i,j,k) &= (e^{\sqrt{-1}\theta_{j}^{2}} - 1)\mathcal{F}\phi(i,j,k), \quad \mathcal{F}\partial_{x_{2}}^{-}\phi(i,j,k) = (1 - e^{-\sqrt{-1}\theta_{j}^{2}})\mathcal{F}\phi(i,j,k) \\ \mathcal{F}\partial_{x_{3}}^{+}\phi(i,j,k) &= (e^{\sqrt{-1}\theta_{k}^{3}} - 1)\mathcal{F}\phi(i,j,k), \quad \mathcal{F}\partial_{x_{3}}^{-}\phi(i,j,k) = (1 - e^{-\sqrt{-1}\theta_{k}^{3}})\mathcal{F}\phi(i,j,k) \end{split}$$

where

$$\theta_i^1 = \frac{2\pi}{N_1}i, \ i = 1, \cdots, N_1, \quad \theta_j^2 = \frac{2\pi}{N_2}j, \ j = 1, \cdots, N_2, \quad \theta_k^3 = \frac{2\pi}{N_3}k, \ k = 1, \cdots, N_3$$

By applying FFT on the both sides of (4.3), we can have:

$$(\eta + 4r(\cos\theta_i^1 + \cos\theta_j^2 + \cos\theta_k^3 - 3)^2)\mathcal{F}\phi(i, j, k) = \mathcal{F}g(i, j, k)$$
(4.4)

where $g = rH^{*}(Q^{n-1}) + H^{*}(\Lambda^{n}) + \eta\phi_{0}$.

On the other hand, the minimizer of the problems (3.12) is the same as the minimizer of the following problem:

$$Q^{n} = \arg \min_{Q} \mathcal{L}(\phi^{n}, Q; \Lambda^{n})$$

= $\arg \min_{Q} |Q|_{\mathcal{W}} + \frac{r}{2} ||Q - H(\phi^{n})||_{\mathcal{W}}^{2} + \langle \Lambda^{n}, Q \rangle_{W}$
= $\arg \min_{Q} |Q|_{\mathcal{W}} + \frac{r}{2} ||Q - (H(\phi^{n}) - \frac{\Lambda^{n}}{r})||_{\mathcal{W}}^{2}$ (4.5)

Whose solution is analytically given by [49]:

$$Q^{n}(i,j,k) = \max\{0, 1 - \frac{1}{r|B^{n}(i,j,k)|}\}B^{n}(i,j,k).$$
(4.6)

Where $B^n = H(\phi^n) - \frac{\Lambda^n}{r}$.

In summary, by utilizing formula (4.4) and (4.6) in the Algorithm 2, we can easily compute the solution to the variational model (2.4). As the common properties of augmented Lagrangian and operator splitting method, the original nonlinear and high order variational model (2.4) can be split to several subproblems, which can be either solved with analytical forms or by fast algorithms. Therefore, we can expect that the proposed augmented Lagrangian algorithm for our model will be solved efficiently. In the next section, we will show several numerical results to demonstrate the proposed model and Algorithm 2 for 3D surface restoration. **5.** Numerical Results. In this section, we illustrate experimental results on several synthetic surfaces using the proposed model and algorithm. We first design a numerical experiment to show the parameter dependence of the proposed method. In order to demonstrate the efficiency and robustness of the proposed method, we also conduct numerical comparisons with the mean curvature flow method [40] and nonlocal mean method [15]. In addition, we indicate the possible applications of our method to surface processing in 3D medical imaging. All experiments are implemented by C++ in a PC with a 4G RAM and a 2.66 GHz CPU.

Given a clean surface \mathcal{M}_c represented by its signed distance function $\phi_{\mathcal{M}_c} : \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3} \to \mathbb{R}$, we consider a synthetic noise contaminated surface \mathcal{M}_n implicitly represented by $\phi^0 = \phi_{\mathcal{M}_c} + \sigma \xi$, where ξ satisfies the standard normal distribution $\mathcal{N}(0, 1)$. Using the proposed Algorithm 2, a sequence $\{\phi^k \mid k = 1, 2, \cdots\}$ can be obtained to approximate the minimizer of the model (2.4). We fix a small number ϵ as the tolerance and stop the iteration at the k-th step if ϕ^k satisfies:

$$\frac{1}{N_1 \cdot N_2 \cdot N_3} \|\phi^k - \phi^{k-1}\| < \epsilon$$
(5.1)

In the first numerical experiment, we test the parameter dependence of the proposed model (2.4). The parameter r in Algorithm 2 comes from the augmented Lagrangian method. The scale of r controls the difference between the auxiliary variable Q and $H(\phi)$. According to the theorem 3.2, the proposed Algorithm 2 will converge for any positive number r. However, the rate of convergence will depend on the choice of r. Another parameter η controls the weight of the fidelity term in the energy functional. The smaller η is chosen, the smoother output surface will be obtained. As all variational methods for denoising problems, the choice of η is dependent on the scale of noise of the input surface.

Figure. 5.1 reports experiments for testing the parameter dependence of the proposed surface restoration model. In our experiment, we fix the stopping criteria $\epsilon = 10^{-8}$ and use a noisy contaminated surface, an octa-flower surface ¹ with size $124 \times 123 \times 90$, showed in the first row of Figure. 5.1, then we test the proposed Algorithm 2 in two sets of parameters. First, we fix $\eta = 10$ and set r = 1, 10, 50, 100 to test the effect of the parameter r. The error and energy evolution curves via iteration numbers are showed in the first two pictures on the second row of Figure. 5.1, and the corresponding surface restoration results are showed in the third row of Figure. 5.1. From our results, it is clear to see that different choices of r will provide similar surface restoration results, while the algorithm may have different rate of convergence in term of different values of r. Second, we fix r = 10 and set $\eta = 1, 10, 50, 100$ to test the effect of the parameter η . According to the error, energy evolution curves and restoration results showed in Figure. 5.1, we can observe that the scale of η will control the smoothness of the restoration results. The smaller η will provide the smoother output surface. A suitable choose of η will provide a ridge and corner preserving restoration result, which is compatible with the analysis discussed in Section 2.3. According the energy evolution curves showed in Figure. 5.1, it can also be observed that the proposed algorithm does provide an efficient approach to solve the proposed fourth order model. Although different combinations of the parameters are chosen, only small numbers of iterations are needed such that the auxiliary variable Q converges to H(Q), the energy converges to steady states and satisfactory results are produced in all experiments.

In the second experiment, we compare our method with the mean curvature flow method [40] and nonlocal mean method [15]. To test the restoration results for all three methods, we synthesize noise contaminated surfaces. Given a clean surface \mathcal{M}_c with signed distance function

¹The octa-flower surface and other three synthetic surfaces are obtained from the public available database SHARP3D



FIG. 5.1. Parameters dependence of the proposed model. The first row: the clean surface and the input surface contaminated by Gaussian white noise ($\sigma = 0.1$); The second row: numerical error of Lagrangian multipliers $\log(||Q - H(\phi)||_W^2)$ and energy evolution curves via iteration numbers; The third row: surface restoration results by fixing $\eta = 10$ and choosing r = 1, 10, 50, 100 respectively; The fourth row: surface restoration results by fixing r = 10 and choosing $\eta = 1, 10, 50, 100$ respectively.

representation $\phi_{\mathcal{M}_c}$, Gaussian white noise is added to $\phi_{\mathcal{M}_c}$ to have the input noisy surface ϕ^0 . The output ϕ^k from all three methods will not only smooth out \mathcal{M}_n as the zero level set of ϕ^0 , but also smooth out each layer of ϕ^0 near \mathcal{M}_n . Thus, we measure the difference between ϕ^k and $\phi_{\mathcal{M}_c}$ in a narrow band of \mathcal{M}_c to have a quantitative description of the restoration result. In other words, we propose the following signal-to-noise ratio (SNR):

$$SNR_{in} = 10\log_{10}\frac{\int_{\mathcal{D}_{\epsilon}}\phi_{\mathcal{M}_{c}}^{2}\mathrm{d}x}{\int_{\mathcal{D}_{\epsilon}}(\phi_{\mathcal{M}_{c}}-\phi^{0})^{2}\mathrm{d}x}, \quad SNR_{out} = 10\log_{10}\frac{\int_{\mathcal{D}_{\epsilon}}\phi_{\mathcal{M}_{c}}^{2}\mathrm{d}x}{\int_{\mathcal{D}_{\epsilon}}(\phi_{\mathcal{M}_{c}}-\phi^{k})^{2}\mathrm{d}x}$$
(5.2)

where \mathcal{D}_{ϵ} is the narrow band of \mathcal{M}_{c} within 2 × grid width.

Four clean surfaces used in our experiment are plotted in Figure 5.2. We contaminate these four surfaces with two scales, $\sigma = 0.1$ and $\sigma = 0.15$, of Gaussian noise as the input surfaces illustrated in the first column of Figure 5.3 and Figure 5.4. For all three methods,



FIG. 5.2. The synthetic clean surface data set.

parameters are tuned to produce the best results we can have. We choose $\eta = 10, r = 5, \epsilon =$ 10^{-8} in our method and set time step dt = 0.0001 in mean curvature flow method. The code of a nonlocal mean method is obtained from the authors in [15], where computation is conducted in a surface narrow band to save computation time and the patch size is chosen to be $11 \times 11 \times 11$. We would like to point out that a larger patch size in nonlocal mean method will slightly improve the restoration results, but the computation will be extremely time consuming. Thus we only report results using patch size as $11 \times 11 \times 11$. We choose the weight for similarity function as c = 0.85 in the case of $\sigma = 0.1$ and c = 1 in the case of $\sigma =$ 0.15. Figure 5.3 and Figure 5.4 report the surface restoration results obtained from these three methods. The quantitative comparison is listed in the Table 5.1. As predicted by the theory, mean curvature flow will smear ridges and sharp corners, which can not provide satisfactory surface restoration results. The nonlocal mean method will provide much better behavior for preserving ridges and sharp corners. However, the processing to compute nonlocal weight is time consuming even though it is only conducted in the narrow band. Our proposed method can simultaneously have computation efficiency and properties of preserving ridges and sharp corners. In addition, we would like to point out that the efficiency of proposed algorithm can be further improved by processing on the surface narrow band.

$\sigma = 0.1$	SNR _{in}	The Proposed Method ($\eta = 10, r = 5$)			MCF ($dt = 0.0001$)			NL (c = 0.85)		
		iterations	time(s)	SNR _{out}	iterations	time(s)	SNR _{out}	iterations	time(s)	SNR _{out}
sharp sphere	7.23	32	29.43	21.49	150	11.37	15.60	120	233.66	19.31
twirl	7.33	34	44.18	18.77	150	17.70	14.73	120	174.28	19.18
block	7.45	36	56.40	22.54	150	18.66	16.84	120	218.17	21.16
octa flower	10.78	29	21.74	21.01	150	11.69	16.58	120	105.43	19.32
$\sigma = 0.15$	SNR.	The Proposed Method ($\eta = 10, r = 5$)			MCF ($dt = 0.0001$)			NL (c = 1)		
	SAR.									
$\sigma = 0.15$	SNR _{in}	iterations	time(s)	SNRout	iterations	time(s)	SNR _{out}	iterations	time(s)	SNR _{out}
$\sigma = 0.15$ sharp sphere	3.74	iterations 30	time(s) 26.89	SNR _{out} 20.57	iterations 150	time(s) 12.03	<i>SNR_{out}</i> 12.49	iterations 200	time(s) 244.08	SNR _{out} 16.35
$\sigma = 0.15$ sharp sphere twirl	3.74 3.82	iterations 30 33	time(s) 26.89 46.08	SNR _{out} 20.57 18.13	iterations 150 150	time(s) 12.03 18.08	SNR _{out} 12.49 10.22	iterations 200 200	time(s) 244.08 190.12	SNR _{out} 16.35 15.72
$\sigma = 0.13$ sharp sphere twirl block	SNRin 3.74 3.82 3.94	iterations 30 33 34	time(s) 26.89 46.08 50.60	SNR _{out} 20.57 18.13 21.74	iterations 150 150 150	time(s) 12.03 18.08 20.14	SNR _{out} 12.49 10.22 12.95	iterations 200 200 200	time(s) 244.08 190.12 241.22	SNR _{out} 16.35 15.72 18.11
$\sigma = 0.13$ sharp sphere twirl block octa flower	SNR _{in} 3.74 3.82 3.94 7.25	iterations 30 33 34 29	time(s) 26.89 46.08 50.60 21.45	SNR _{out} 20.57 18.13 21.74 20.56	iterations 150 150 150 150	time(s) 12.03 18.08 20.14 11.71	SNR _{out} 12.49 10.22 12.95 15.05	iterations 200 200 200 200	time(s) 244.08 190.12 241.22 112.83	SNR _{out} 16.35 15.72 18.11 16.27

Comparison of proposed method with the mean curvature flow (MCF) method and the nonlocal mean method (NL)

Finally, we illustrate a potential application of our method to medical image processing.



FIG. 5.3. Surface restoration comparison. The first column: input surfaces contaminated by Gaussian white noise ($\sigma = 0.1$). The second column: results obtained by the proposed method. The third column: results obtained by nonlocal mean method. The fouth column: results obtained by the mean curvature flow method.

3D surfaces collected from CT, MR, or 3D ultrasound devices are usually contaminated by certain noise due to local measurement error, which will further affect the surface analysis afterwards. In addition, these medical data have crucial features represented as ridges and sharp corners which need to be preserved. In Figure 5.5, we demonstrate our method on surface restorations in medical imaging. From the second row of Figure 5.5, it is clear to see that our method provide promising results with ridges and sharp corners preserving properties.

6. Conclusion. In this work, we propose a ridges and sharp corner preserving model for 3D surface restoration based on vectorial total variation for the derivatives of the surface implicit representation functions. Moreover, an efficient numerical algorithm is proposed to solve the minimization problem based on augmented Lagrangian method. Meanwhile, we theoretically prove the convergence of the proposed algorithm. To demonstrate the robustness and efficiency of the proposed method, we compare our method with a mean curvature flow



FIG. 5.4. Surface restoration comparison. The first column: surfaces contaminated by Gaussian white noise ($\sigma = 0.15$). The second column: results obtained by the proposed method. The third column: results obtained by nonlocal mean method. The fouth column: results obtained by the mean curvature flow method.

method and a nonlocal mean method. In addition, we also illustrate possible applications of the proposed method to surface restoration in medical imaging.

Acknowledgement. Rongjie Lai's work is supported by Zumberge Individual Award from USC's James H. Zumberge Faculty Research and Innovation Fund and NSF IIS-0914580 and ONR N00014-09-1-0105. Xue-cheng Tai has been supported by Singapore MOE Grant T207B2202 and Singapore NRF2007IDM-IDM002-010. In addition, we would like to thank the valuable discussion and the nonlocal mean code provided by Professor S. Osher and Professor B. Dong. Last but not least, we also thank the valuable comments and suggestions from the anonymous referees.



FIG. 5.5. The first row: noisy surfaces. The second row: results obtained by the proposed method.

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