Anomalous exponents of self-similar blow-up solutions to an aggregation equation in odd dimensions

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Abstract

We calculate the scaling behavior of the second-kind self-similar blow-up solution of an aggregation equation in odd dimensions. This solution describes the radially symmetric finite-time blowup phenomena and has been observed in numerical simulations of the dynamic problem. The nonlocal equation for the self-similar profile is transformed into a system of ODEs that is solved using a shooting method. The anomalous exponents are then retrieved from this transformed system.

Keywords: aggregation equation, self-similarity solution of the second kind, finite-time blow-up, shooting methods

1. Introduction

We consider the radially symmetric, self-similar blowup solution of the equation

$$u_t = \nabla \cdot (u \nabla K * u) \quad \text{on } \mathbb{R}^n \times [0, T), \tag{1}$$

where u represents a mass density and $K * u$ is the convolution with kernel $K(x) = |x|$. Equations of this type are used to describe the aggregation of biological species [1, 2, 3]. It is proved in [4] that the solution blows up in finite time for more general kernels. Radially symmetric self-similar blow-up solutions of the form form [5, 6],

$$u(x,t) = (T-t)^{-\alpha} U(r), \quad r = |x|(T-t)^{-\beta} \tag{2}$$

are observed in numerical simulations. These solutions are of second-kind [7] since the exponents α, β cannot be determined from scaling arguments, symmetries, or conservation laws alone. First kind similarity solutions typically take the form of delta rings [4, 8, 9].

In this letter we consider the special case of second kind similarity solutions in odd dimensions $n = 2N + 1$ with $N = 1, 2, \cdots$. Using the fact that the successive Laplacians of kernel $K(x) = |x|$ are proportional to the fundamental solution of the Laplace equation, we can transform the equation (3) for the blowup profile U into an equivalent system of ordinary differential equations. The anomalous exponents α, β can be determined after solving the boundary value problem for this system using a shooting method.

2. Equivalent system and shooting methods

Substituting ansatz (2) into equation (1) yields

$$\alpha U + \beta r \frac{dU}{dr} = \nabla \cdot (U \nabla K * U), \tag{3}$$

where the relation $\alpha = 2N\beta + 1$ is determined by the scaling symmetry of (1) with respect to $(T-t)$.

2.1. Equivalent system of ODEs

We introduce the variables $U_0(=U), U_1, \cdots, U_{N+1}$ such that

$$-\Delta U_1 = U_0, \qquad \cdots, \qquad -\Delta U_N = U_{N-1}, \qquad \Delta U_{N+1} = c_N U_N, \tag{4}$$

where $\Delta(\cdot) = \frac{1}{r^{2N}}\frac{d}{dr}(r^{2N}\frac{d}{dr}(\cdot))$ is the axisymmetric Laplacian on $\mathbb{R}^{2N+1}$ in the radial coordinate r and $c_N = (2\pi)^N\sqrt{\pi}(2N+1)N!/\Gamma(N+1+\frac{1}{2})$. Using the relation $\Delta K * U_0 = c_N U_N$, we can write (3) in the form

$$\frac{dU_0}{dr} = -\frac{2N\beta + 1 - c_N U_N}{\beta r - dU_{N+1}/dr} U_0, \tag{5}$$

where dU_{N+1}/dr can be expressed as an integral involving U_N:

$$\frac{dU_{N+1}}{dr} = \frac{c_N}{r^{2N}}\int_0^r s^{2N} U_N(s)ds. \tag{6}$$

The system (4), together with (5), forms a closed system of second order ODEs for $(U_0, U_1, \cdots, U_N, U_{N+1})$ on $[0,\infty)$. The localized nature of the blow-up observed in numerical simulations [5] of (1) predicts the far-field algebraic decay of the solution $U_0(r) = O(r^{-\alpha/\beta}) = O(r^{-2N-1/\beta})$, implying the asymptotic boundary conditions $U_i(r) = O(r^{-2(N-i)-1/\beta})$, $i = 1, \cdots, N$ and

$$U_0(\infty) = U_1(\infty) = \cdots = U_N(\infty) = 0. \tag{7}$$

Equation (3) has a scaling symmetry: if $U(r)$ is a solution, then so is $\lambda^{2N}U(\lambda r)$. This allows us to normalize the solution using the condition $U_0(0) = 1$. Smoothness at the origin also forces $U_i'(0) = 0$ for $i = 0, 1, \cdots, N$. Additionally at the origin, equation (3) reduces to $\alpha = \Delta K * U|_{r=0} = c_N U_N(0)$, giving $U_N(0) = \alpha/c_N = (2N\beta+1)/c_N$.

A key observation to simplify the calculation of the exponent β is that we can get a β-independent first order system of ODEs from (4)-(5) by two additional transformations. The first is the scaling transformation

$$\begin{cases} y_{2i}(r) &= (\beta-1)^{-\frac{i}{N}} U_i((\beta-1)^{\frac{1}{2N}} r), \qquad i = 0, 1, \cdots, N+1 \\ y_{2i-1}(r) &= y_{2i}'(r), \qquad i = 1, \cdots, N \end{cases} \tag{8}$$

motivated by the $\beta - 1$ factor stemming from the denominator in (5) for $r \to 0$. This is followed by the second transformation

$$z_i(r) = y_i(r), \qquad i = 0, 1, \cdots, 2N-1, \tag{9}$$

$$z_{2N}(r) = y_{2N}(r) - \frac{2N+1}{c_N(\beta-1)}, \qquad z_{2N+1}(r) = y_{2N+1}(r) - \frac{1}{\beta-1}r, \tag{10}$$

which factors out the dependence on β from equation (5).

All these give the final β-independent system for z_i:

$$\begin{aligned} \frac{dz_0}{dr} &= -\frac{2N - c_N z_{2N}}{r - z_{2N+1}} z_0, \\ &\begin{cases} \frac{dz_{2i-1}}{dr} = -z_{2i-2} - \frac{2N}{r} z_{2i-1}, \\ \frac{dz_{2i}}{dr} = z_{2i-1}, \end{cases} \qquad i = 1, 2, \cdots, N \\ \frac{dz_{2N+1}}{dr} &= c_N z_{2N} - \frac{2N}{r} z_{2N+1}, \end{aligned} \tag{11}$$

with the β-independent initial conditions

$$z_0(0) = 1, \ z_{2N}(0) = 2N, \quad \text{and} \quad z_{2i+1}(0) = 0 \quad \text{for } i = 0, 1, \cdots, N. \tag{12}$$

The system must also satisfy boundary conditions at infinity stemming from (6),

$$z_i(\infty) = 0 \quad \text{for } i = 0, 1, \cdots, 2N-1, \tag{13}$$

with $z_{2N}(\infty)$ finite. The value $z_{2N}(\infty)$ recovers the anomalous exponent β from the first equation in (10),

$$z_{2N}(\infty) = y_{2N}(\infty) - \frac{2N+1}{c_N(\beta-1)} = -\frac{2N+1}{c_N(\beta-1)}. \tag{14}$$

In actual computation, system (11) will be solved using a shooting method with the rest of the initial conditions as shooting parameters:

$$z_{2i}(0) = s_i \quad \text{for } i = 1, 2, \cdots, N-1. \tag{15}$$

The corresponding solution $\mathbf{z} = (z_0, z_1, \cdots, z_{2N+1})$ then will be computed on a finite interval $[0, L]$ for sufficiently large L, with $z_{2N}(\infty)$ approximated by $z_{2N}(L)$. Because of the sensitive dependence of $u_{2N}(L)$ on the initial conditions, the anomalous exponent β is recovered not from (14) but the following equivalent and much more stable relation

$$\frac{2N\beta+1}{\beta-1} = z_{2N}(0) - z_{2N}(\infty) = \frac{1}{(N-1)!}\int_0^\infty r^{2N} z_0(r)dr, \tag{16}$$

from successive integrations of the system (11).

2.2. Local expansion at the origin

The system (11) has a removable singularity at the origin and may not be integrated directly. However, the solution $z_i(r)$ near $r = 0$ can be expanded as a convergent power series. We assume $z_0(r) = \sum_{k=0}^{\infty} u_{2k} r^{2k}$ containing only even order terms due to radial symmetry. Integrating the system (11) with the initial conditions (12) and (15), we obtain

$$\begin{aligned} z_{2i}(r) = {} & (2N-1)!! \sum_{j=0}^{i-1} (-1)^j \frac{1}{2^j j!(2N+2j-1)!!} s_{i-j} r^{2j} \\ & + \frac{(-1)^i}{2^i} \sum_{k=0}^{\infty} \frac{k!(2k+2N-1)!!}{(k+i)!(2k+2N+2i-1)!!} u_{2k} r^{2k+2i}, i = 1, 2, \cdots, N, \end{aligned} \tag{17}$$

$$\begin{aligned} z_{2N+1}(r) = {} & c_N(2N-1)!! \sum_{j=0}^{N-1} (-1)^j \frac{1}{2^j j!(2N+2j+1)!!} s_{N-j} r^{2j+1} \\ & + \frac{(-1)^N c_N}{2^N} \sum_{k=0}^{\infty} \frac{k!(2k+2N-1)!! u_{2k}}{(k+N)!(2k+4N+1)!!} r^{2k+2N+1}. \end{aligned} \tag{18}$$

Substituting the expression z_0, z_{2N}, z_{2N+1} into the first equation in (11), we have

$$c_N \left[(2N-1)!! \sum_{j=0}^{N-1} (-1)^j \frac{1}{2^j j!(2N+2j-1)!!} s_{N-j} r^{2j} \right.$$

$$
\begin{aligned}
&+\frac{(-1)^N}{2^N}\sum_{k=0}^{\infty}\frac{k!(2k+2N-1)!!u_{2k}}{(k+N)!(2k+4N-1)!!}r^{2k+2N}\Bigg]\sum_{k=0}^{\infty}u_{2k}r^{2k}\\
&+c_N\Bigg[(2N-1)!!\sum_{j=0}^{N-1}(-1)^j\frac{1}{2^j j!(2N+2j+1)!!}s_{N-j}r^{2j}\\
&+\frac{(-1)^N}{2^N}\sum_{k=0}^{\infty}\frac{k!(2k+2N-1)!!u_{2k}}{(k+N)!(2k+4N+1)!!}r^{2k+2N}\Bigg]\sum_{k=0}^{\infty}2ku_{2k}r^{2k}\\
=\ &\sum_{k=0}^{\infty}(2N+2k)u_{2k}r^{2k}. \qquad (19)
\end{aligned}
$$

The matching condition for the coefficients of r^{2k} gives the following recursive relations for u_{2k},

$$
u_{2j}=\frac{c_N(2N+1)!!(2N+2j+1)}{2j}\sum_{l=1}^{j}\frac{(-1)^l}{2^l l!(2N+2l+1)!!}u_{2j-2l}s_{N-l}, \quad j=1,\cdots,N-1,
$$

$$
\begin{aligned}
u_{2N+2k}=&\frac{(-1)^N c_N(4N+2k+1)(2N+1)}{2^{N+1}(N+k)}\sum_{l=0}^{k}\frac{l!(2N+2l-1)!!}{(N+l)!(4N+2l+1)!!}u_{2l}u_{2k-2l}\\
&+\frac{c_N(2N+1)!!(4N+2k+1)}{2(N+k)}\sum_{j=1}^{N-1}\frac{(-1)^j s_{N-j}}{2^j j!(2N+2j+1)!!}u_{2N+2k-2j}, \qquad k=0,1,\cdots.
\end{aligned}
$$

Numerical simulations indicates that the coefficients u_{2k} converge geometrically, and the corresponding power series has a finite radius of convergence (approximately 0.87 in dimension three for z_0). When r is small, with fixed shooting parameters $\mathbf{s}$, $z_i(r)$ is obtained accurately with just a few leading terms in the expansion and is continued by numerical integration up to $r=L$.

3. The shooting method and numerical results

Starting from any set of shooting parameters $\mathbf{s}=(s_1,\cdots,s_{N-1})$ we can find a perturbation, $\delta\mathbf{s}$ to make the solution $\mathbf{z}(L;\mathbf{s}+\delta\mathbf{s})$ to (11) approach the boundary conditions (13). Based on the observation that the series (17), when it converges, is dominated by the first summation, approximate $\delta\mathbf{z}(L)$ by

$$
\delta z_{2i}(L)=(2N-1)!!\sum_{j=0}^{i-1}\frac{(-1)^j}{2^j j!(2N+2j-1)!!}L^{2j}\delta s_{i-j}, \quad i=1,2,\cdots,N-1. \qquad (20)
$$

This suggests the following iterative scheme

$$
\mathbf{s}^{m+1}=\mathbf{s}^m+\omega\delta\mathbf{s}^m, \qquad (21)
$$

where $\mathbf{s}^m=(s_1^m,s_2^m,\cdots,s_{N-1}^m)$ is the set of shooting parameters at m-th iteration, $\delta\mathbf{s}^m$ is solved from (20) with $\delta z_{2i}(L)=-z_{2i}(L)$ and $\omega(<1)$ is a positive relaxation parameter. The triangular system (20) can be solved easily. For instance, in dimension five

$$
\delta s_1=-z_2(L)
$$

and in dimension seven

$$\delta s_1 = -z_2(L), \quad \delta s_2 = -\left(z_4(L) + \frac{L^2}{14} z_2(L)\right). \tag{22}$$

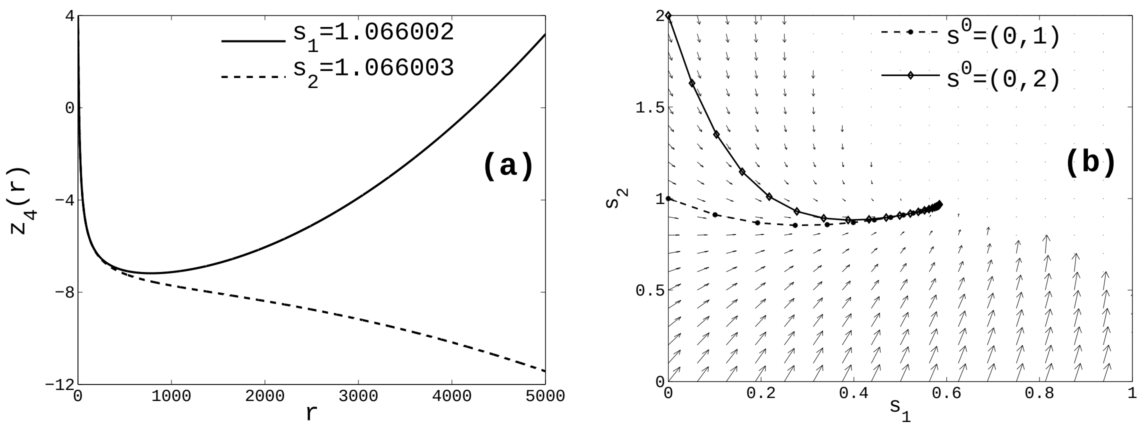

Figure 1: **(a)** In dimension five, the solution $z_4(r)$ depends very sensitively on the only shooting parameter s_1 when r is large. **(b)** The gradient field $(\delta s_1, \delta s_2)$ defined by (22) and two sample trajectories of the scheme (21) in dimension seven with $L = 100$ and $\omega = 0.2$. The gradient field is normalized by the factor $(1 + \delta s_1^2 + \delta s_2^2)^{-1/2}$ for better visualization.

3.1. Dimension three ($N = 1$)

Dimension three is special in the sense that there is no shooting parameter. The anomalous exponent β is retrieved from either (14) or (16), where the accuracy depends on the length of the interval $[0, L]$ on which (11) is solved. This is also compared with those by both direct simulation of the blowup dynamic for (1) followed by data fitting and numerical renormalization group (RG) calculation performed in [5], in Table 1. The computation time is at most a few seconds for the ODE system while at least a few hours for direct simulation or numerical RG.

Methods	L	$\beta(n=3)$	$\beta(n=5)$	$\beta(n=7)$	$\beta(n=9)$
Shooting	10^2	1.580957	1.593860	1.574476	1.602537
Shooting	10^3	1.582976	1.598702	1.596328	1.607854
Shooting	10^4	1.583092	1.602900	1.598753	1.609265
Numerical RG [5]	400	1.582889	1.599152	1.604324	1.629743
Direct Computation [5]	N/A	1.582226	1.598044	1.606732	1.623508

Table 1: Comparison of the computed anomalous exponents β from different methods in different dimensions.

3.2. Higher dimensions ($N \geq 2$)

The solution to the system (11) may not exist on the whole domain with certain shooting parameters when the denominator $r - z_{2N+1}$ in the first equation in (11) changes sign. In this case the assumption of the weak dependence of $z_{2i}(L)$ with $i \geq 1$ on z_1 is not valid. Therefore the variation (20) is true only on part of the parameter space. This is is shown in

Figure 1(b) for dimension seven. The solution ceases to exist for shooting parameter $\mathbf{s}$ on the upper right region, where the gradient field is not defined. However, once the initial guess $\mathbf{s}^0$ is in the basin of attraction, it alway converges to a neighborhood of the unique fixed point. Numerical experiments indicate that the initial guess can be chosen as alternating zeros and positive numbers, such as $\mathbf{s}^0 = (0, C, 0, C, \cdots)$, for C positive and large. The choice of $C = 2$ works for any test cases up to dimension fifteen. The numerical results are presented in Table 1, compared with those obtained from much slower computation of the full partial different equation. Because of the sensitive dependence of the solution $\mathbf{z}(L)$ on the shooting parameter $\mathbf{s}$, the exponent β calculated using this shooting method is less accurate in higher dimensions.

4. Conclusion

We find the exponent (and the profile) for the self-similar solution of the aggregation equation in odd dimensions. Evidence is clear that we have an exact second-kind similar solution. However, the shooting method proposed here relies on reducing the problem to coupled local equations in odd dimensions only. An interesting open problem is to develop a full theory for the nonlocal problem in general dimensions and for general power-law kernels.

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