A Fast Global Optimization-Based Approach to Evolving Contours with Generic Shape Prior

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Abstract-In this paper, we propose a new global optimizationbased approach to contour evolution, with or without the novel variational shape constraint that imposes a generic star shape using a continuous max-flow framework. In theory, the proposed continuous max-flow model provides a dual perspective to the reduced continuous min-cut formulation of the contour evolution at each discrete time frame, which proves the global optimality of the discrete time contour propagation. The variational analysis of the flow conservation condition of the continuous max-flow model shows that the proposed approach does provide a fully time implicit solver to the contour convection PDE, which allows a large time-step size to significantly speed up the contour evolution. For the contour evolution with a star shape prior, a novel variational representation of the star shape is integrated to the continuous max-flow-based scheme by introducing an additional spatial flow. In numerics, the proposed continuous max-flow formulations lead to efficient duality-based algorithms using modern convex optimization theories. Our approach is implemented in a GPU, which significantly improves computing efficiency. We show the high performance of our approach in terms of speed and reliability to both poor initialization and large evolution step-size, using numerous experiments on synthetic, real-world and 2D/3D medical images.

Index Terms—Image segmentation, active contour, level set, convex optimization, primal-dual optimization

I. INTRODUCTION

M ANY applications of image processing and computer vision can be modeled and solved by the evolution of contours, where the given 2-D curve or 3-D surface C^{-1} gradually propagates to objects of interest in images subject to a priori image information and optimization criteria. One of the well-known techniques is the edge-based active contour introduced by Kass, Witkin and Terzopoulos [1]. Following [1], Kichenassamy *et al.* [2] and Casselles *et al.* [3] proposed to evolve the contour C by gradually minimizing a geometrical function, i.e. the edge weighted length

$$\min_{\mathcal{C}} \int_{\partial \mathcal{C}} g(s) \, ds \tag{1}$$

where g(s) is given by some image indicator function. Many practical experiments show the effectiveness of such theory of gradually evolving the contour to the objects' boundary. However, only limited local image edge information is considered in such contour propagation process, where the evolving

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 1 In this paper, we use C to represent the contour or its enclosed region, when there is no misunderstanding.

contour can be easily trapped in an undesirable minimum due to false local edge information, image noise or texture. Moreover, the performance of the computational result is highly sensitive to the choice of the initial contour: a poor initialization results in an unsatisfactory result.

On the other hand, the region-based active contours [4], [5], [6] incorporate image information associated to the regions inside and outside of the evolving contour, especially some energies using the global statistical image features [5], [7], [8] or shapes [9], [10], [11]. These methods are robust to image noise and intensity inhomogeneities, therefore overcome the major drawbacks of the standard edge-based active contour approaches. The level set method introduced by Osher & Sethian [12] provides an efficient way to embed region-based information into the contour propagation process with the ability to change topology of the contour. We refer the readers to [13], [6] for a wide spectrum of publications about level set methods. However, the level set-based methods rely on a local optimization scheme by explicitly solving the associated convection PDE for which the CFL condition restricts the discrete propagation time step-size [6]. Moreover, the standard level set motion driven by the mean-curvature requires an accurate approximation to the non-smooth high-order derivatives [4], [14], which results in a highly complicated numerical implementation and, in turn, constrains the step-size of each iteration to be small enough to achieve numerical stability of convergence. In general, such relatively small time step-size limits efficiency of the level set methods.

A. Motivations

Some interesting global optimization approaches to contour evolution were proposed, which are different from the local optimization based active contour schemes in both theory and algorithmic implementation.

Boykov *et al.* [15] proposed the distance map w.r.t. the current contour as the cost functions of propagating the contour inside or outside of the current contour and formulate the discrete time contour evolution as a classical min-cut problem. The global optimality of the computed min-cut solution implies that the new contour position is globally optimal. In fact, the same theory was first introduced and studied by [16], [17] in early 90s, which proved that, given the outer force f and mean-curvature κ , the mean-curvature flow problem:

$$\partial_t \mathcal{C} = -\kappa + f \tag{2}$$

can be approximated iteratively by the time-discrete solution $C_t \rightarrow C_{t+h}$, as $h \rightarrow 0$, and C_{t+h} minimizes the following

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energy [17]:

$$\min_{\mathcal{C}} \int_{\partial \mathcal{C}} ds + \int_{\mathcal{C} \triangle \mathcal{C}_t} \frac{1}{h} \operatorname{dist}(x, \mathcal{C}_t) dx - \int_{\mathcal{C}} f dx. \quad (3)$$

It should be noted that Chambolle [18] also studied the mean-curvature driven motion (2) of contours with f(x) = 0 and used the distance function as the displacement reference at each discrete time evolution. Chambolle [18] showed that at each discrete time frame, the next contour position C_{t+h} can be obtained by the zero level set of the total-variation regularized signed distance w.r.t. C_t , such that for any pixel x at the computed boundary of C_{t+h} , whose projection on the boundary of C_t is x_t , one has

$$x = x_t - h\kappa(x)\mathbf{n}(x_t), \qquad (4)$$

where $\mathbf{n}(x_t)$ is the unit outward normal to C_t at x_t . (4) exactly corresponds to the equation of mean-curvature motions (2) when $h \to 0$. Bresson & Chan [19] extended the theoretical work of [18] to the case of geodesic contour evolution along with region-based forces. Chambolle & Darbon [20] followed the same theory proposed in [18] and developed an efficient graph cut-based numerical solver.

Motivated by the works of [16], [17], [15], [18], we propose a global optimization-based approach using a continuous maxflow framework to efficiently evolve a contour to its globally optimal position at each discrete time frame. More precisely, given the current contour C_t at time t, we propagate C_t to its new position C_{t+h} at the next time frame t + h where the new contour C_{t+h} optimizes the following energy globally and exactly:

$$\min_{\mathcal{C}\in\mathcal{L}} \left\{ \int_{\mathcal{C}^+} c^+(x) \, dx + \int_{\mathcal{C}^-} c^-(x) \, dx \right\} + \int_{\partial\mathcal{C}} g(s) \, ds \quad (5)$$

where \mathcal{L} is some constraint set related to the prior information on contours, e.g. the star shape constraint discussed in this paper; C^+ and C^- are the two distinct regions associated to deformations: region expansion and region shrinkage (see Sec. II for details). We demonstrate that (5) can be equivalently reformulated as a spatially continuous min-cut problem [21]; then solved globally and exactly by means of convex relaxation, for which efficient numerical solvers exist [22], [23], [24]. In this paper, we focus on the continuous max-flow based approach proposed by Yuan *et al.* [24], and develop new results in both theory and algorithm together with its novel integration of the star shape prior.

B. Contributions

We summarize our main contributions along with major differences from previous works as follows:

We develop a new global optimization-based approach to the contour evolution, with or without a novel variational shape constraint to the star shape discussed in Sec. III-A, using the proposed continuous max-flow framework. More precisely, we formulate evolving the given contour at each discrete time frame by achieving the minimum cost of region changes, which can be identically modeled as a spatially continuous min-cut problem with or without the proposed variational star shape constraint. To this end, we introduce the continuous max-flow model which provides an elegant dual perspective to the convex relaxed continuous min-cut formulation and proves the global optimality of the computed continuous min-cut solution by means of convex relaxation, i.e. the contour can be moved to its globally optimal position at each discrete time. The optimality analysis of the flow conservation condition of the continuous max-flow model, i.e. flow-in is pixel-wise balanced by flow-out, proves that the proposed continuous max-flow based approach does provide a fully time implicit solver to the contour convection PDE, which allows a large time-step size to significantly speed up the contour evolution. The revealed connection between the time implicit contour propagation PDE and the classical flow conservation constraint is new in theory. For the evolution of the star shaped contours, a novel variational representation of the star shape is introduced, which can be integrated to the continuous max-flow based approach to contour propagation by simply introducing an additional spatial flow. Likewise, the global optimality of such star shaped contour evolution at each discrete time frame can also be demonstrated under the proposed continuous max-flow perspective.

In numerics, the introduced continuous max-flow models can be directly applied to derive efficient duality-based algorithms using modern convex optimization theories, where the nonlinear total-variation function and the star shape constraint in the continuous min-cut problems are properly encoded by the projections to simple convex sets respectively. Such continuous max-flow based algorithms can easily be implemented in a modern parallel computing platform, e.g. a GPU etc, which greatly speeds up the algorithms and achieves a high computing performance in practice. We show the performance of the proposed global optimization-based contour evolution approaches, in terms of efficiency and reliability to both poor initialization and large evolution step-size through numerous experiments on synthetic images, real-world images, and 2D/3D medical images.

In contrast to the conventional local optimization-based contour evolution schemes, our proposed global optimizationbased approach has the following advantages: it evolves the contour in a fully time implicit way and allows large evolution step-size, which significantly reduce the total number of discrete time propagation steps and speeds up the contour evolution; only the first-order derivatives appear in the global optimization-based contour evolution procedure at each discrete time step, which greatly simplifies the numerical implementation; in addition, its inherent flow-maximization style enables the integration of prior information, e.g. the variational shape descriptor (star shape) in this paper.

Comparing to the graph cut-based contour evolution method proposed in [15], we formulate and solve the contour evolution in the spatially continuous setting by the proposed continuous max-flow based approach. Our method avoids metrication artifacts that are present in graph cut-based approaches [25], [26], see Fig. 3(c) in Sec. V-A, and can obtain a sub-pixel accuracy of the contour position. In particular, a variational analysis of the global optimality to the proposed continuous max-flow model shows the close connection between the fully time implicit scheme to contour convection and the classical flow conservation condition. According to the authors' knowledge, this is not available by the theory of graph cuts. The continuous max-flow algorithms implemented in a GPU achieves a better numerical performance in terms of efficiency and memory load, which is demonstrated by the experiment, shown in in Sec. V-A, of the 3D surface motion driven by its meancurvature.

Finally, we address the differences of our approach from other interesting works proposed by [18] and [19], where [18] formulates mean-curvature driven evolution using the zero level set of the total-variation regularized signed distance function, and [19] extends it to region-based geodesic contour evolution. Our proposed continuous max-flow-based approach is distinct from such works in both theory and numerics; essentially, our approach is based on a flow-maximization framework which allows coupling of the generic shape constraint to the contour evolution such as the star shape prior studied in this paper. According to our understanding, this is not achievable by the methods proposed in [18] and [19].

II. GLOBAL OPTIMALITY OF CONTOUR EVOLUTION: A CONTINUOUS MAX-FLOW PERSPECTIVE



Fig. 1. 1(a): Contour evolution; 1(b): spatially continuous flow-maximization configuration; 1(c): contour movement; from x_t to x, the distance is $\phi(x)$.

In this section, we first introduce the variational principle of achieving the minimum cost of label/region changes to the discrete time evolution of contours, which can be formulated as the min-cut problem with the spatially continuous setting, i.e. the continuous min-cut problem. With helps of the introduced continuous max-flow model, we demonstrate the proposed continuous min-cut model of contour evolution can be solved globally and exactly by means of convex relaxation, i.e. at each discrete time frame, the current contour can be moved to its globally optimal position. Moreover, through the variational analysis of the continuous max-flow model, the classical flow conservation condition implies that the computed global optimum of the formulated continuous min-cut problem does provide a fully time implicit evolution scheme to the contour convection PDE.

A. Global Optimization Based Contour Evolution

1) Variational Model: For the given contour C_t at time t and the new contour C_{t+h} at the next time frame t + h, let $u_t(x) \in \{0, 1\}$ be the labeling function of the enclosed region of C_t such that

$$u_t(x) := \begin{cases} 1, & \text{where } x \text{ is inside } \mathcal{C}_t \\ 0, & \text{otherwise} \end{cases}$$
(6)

and $u_{t+h}(x) \in \{0,1\}$ is the labeling function of \mathcal{C}_{t+h} at t+h.

Inspired by the work [17], we define the cost functions $c^+(x)$ and $c^-(x)$ w.r.t. two distinct regions C^+ and C^- , see Fig. 1, where:

- C⁺ indicates the region expansion w.r.t. C_t: for ∀x ∈ C⁺, it is initially outside C_t at time t, and 'jumps' to be inside C_{t+h} at t + h; for such a 'jump', it pays the cost c⁺(x).
- C⁻ indicates the region shrinkage w.r.t. Ct: for ∀x ∈ C⁻, it is initially inside Ct at t, and 'jumps' to be outside Ct+h at t+h; for such a 'jump', it pays the cost c⁻(x).

We propose to propagate the contour C_t to its new position C_{t+h} with the minimum total cost of such 'jumps' or region changes; which amounts to minimizing the following energy:

$$\min_{\mathcal{C}} \int_{\mathcal{C}^+} c^+(x) \, dx \, + \, \int_{\mathcal{C}^-} c^-(x) \, dx \, + \, \int_{\partial \mathcal{C}} g(s) \, ds \,, \quad (7)$$

where $g(s) = g(|\nabla I(s)|), \forall s \in \partial C$, is defined by the image edge indicator function² and encodes the energy of weighted length (geodesic active contour) [3].

Clearly, when $c^+(x)$ and $c^-(x)$ are set to be the distance between x to the current contour C_t , i.e. $\operatorname{dist}(x, C_t)$, and g(x) = 1 for $\forall x \in \Omega$, the proposed variational model (7) coincides with the mean-curvature driven contour evolution equation (3) with vanishing outer force f(x) = 0, proposed in [16], [17], [15]. In general, the cost functions $c^+(x)$ and $c^-(x)$ are data-associated, depending on the specified application: for example, $c^+(x)$ and $c^-(x)$ can be defined through the firstorder variation of the distribution matching function, e.g. the Bhattacharyya distance (see Sec. V for more details).

2) Spatially Continuous Min-Cut Model: Now we show (7) can be equally formulated as a spatially continuous min-cut problem. To achieve this, we define the cost functions $D_1(x)$ and $D_2(x)$:

$$D_1(x) := \begin{cases} c^-(x), & \text{where } x \in \mathcal{C}_t \\ 0, & \text{otherwise} \end{cases}$$
(8)

and

$$D_2(x) := \begin{cases} c^+(x), & \text{where } x \notin \mathcal{C}_t \\ 0, & \text{otherwise} \end{cases}$$
(9)

which can be interpreted by the energies w.r.t. label assignments such that:

- For any pixel x at time t+h, to assign the label 0 to x, it either pays the cost c⁻(x) for x ∈ C_t whose initial label is 1 and changes to be 0 at t + h, or pays 0 for x ∉ C_t whose initial label is 0 and does not change at t+h. This exactly gives the function D₁(x).
- For any pixel x at time t+h, to assign the label 1 to x, it either pays the cost c⁺(x) for x ∉ C_t whose initial label is 0 and changes to be 1 at t + h, or pays 0 for x ∈ C_t whose initial label is 1 and does not change at t+h. This exactly gives the function D₂(x).

²In this paper, we consider

$$g(|\nabla I(x)|) = \gamma_1 + \gamma_2 \exp(-\gamma_3 |\nabla I(x)|)$$

where $\gamma_{1,2,3} > 0$.

In other words, for the pixels which stay with their original labels, no cost is paid; costs only occur to the pixels which change their status.

By such label assignment costs $D_1(x)$ and $D_2(x)$, the proposed optimization model (7) to contour evolution can be equally reformulated as the continuous min-cut problem:

$$\min_{u(x)\in\{0,1\}} \langle 1-u, D_1 \rangle + \langle u, D_2 \rangle + \int_{\Omega} g(x) \left| \nabla u \right| \, dx \,, \quad (10)$$

where the last weighted total-variation function formulates the weighted length term in (7) [27], [28].

3) Convex Relaxation and Continuous Max-Flow Model: Now we study the convex relaxation model to (10), i.e.

$$\min_{u(x)\in[0,1]} \langle 1-u, D_1 \rangle + \langle u, D_2 \rangle + \int_{\Omega} g(x) \left| \nabla u \right| \, dx \quad (11)$$

where the binary constraint $u(x) \in \{0, 1\}$ in (10) is relaxed to $u(x) \in [0, 1]$ such that (11) is reduced to a convex optimization problem. We call (11) the *convex relaxed min-cut model*.

[21], [29] proved that, for $g(x) = \alpha$ is constant, the convex relaxed min-cut model (11) solves the binary constrained continuous min-cut problem (10) globally and exactly. In this work, we study the exactness of the general case of (11), for which g(x) is not constant, through its dual formulation, i.e. the continuous max-flow model.

To motivate the continuous max-flow model proposed in [30], we first introduce the spatially continuous flow configuration (see Fig. 1(b) for illustration): two extra nodes s and t are added as the source and sink terminals; we link s to each pixel $x \in \Omega$ and link x to t, see Fig. 1(b); we also define the source flow $p_s(x)$ which is directed from s to x, the sink flow $p_t(x)$ which directed from $\forall x \in \Omega_t$ to t; moreover, there is the spatial flow p(x) around x.

With such flow settings, the *continuous max-flow model* is proposed by maximizing the total flow from the source s:

$$\max_{p_s, p_t, p} \int_{\Omega_s} p_s \, dx \tag{12}$$

subject to the constraints on flow capacity and conservation:

$$p_s(x) \le D_1(x), \ p_t(x) \le D_2(x), \quad \forall x \in \Omega;$$
 (13)

$$|p(x)| \le g(x), \quad \forall x \in \Omega; \tag{14}$$

$$\operatorname{div} p(x) - p_s(x) + p_t(x) = 0, \quad \forall x \in \Omega.$$
(15)

As shown in [30] and [24], through simple variational analysis, we can prove

Proposition 1: The continuous max-flow model (12) is dual to the convex relaxed min-cut formulation (11).

Proof: By introducing the multiplier function u(x) to the linear equality of the flow conservation condition (15), we have the equivalent primal-dual formulation to the continuous max-flow model (12):

$$\min_{u} \max_{p_s, p_t, p} \int_{\Omega} p_s(x) \, dx + \langle u, \operatorname{div} p - p_s + p_t \rangle \tag{16}$$

subject to the flow capacity conditions (13) and (14).

Rearranging (16) gives

$$\min_{u} \max_{p_s, p_t, p} \langle 1 - u, p_s \rangle + \langle u, p_t \rangle + \langle u, \operatorname{div} p \rangle$$
(17)

subject to the flow capacity constraints (13) and (14).

By [31], we see that the min and max operations can be interchanged; hence we can first maximize (17) over the flow variables p_s , p_t and p. Following the variational analysis in [30] and [24], the maximization of (17) over $p_s(x) \le D_1(x)$ and $p_t(x) \le D_2(x)$ essentially gives rise to

$$u(x) \in [0,1], \quad \forall x \in \Omega,$$

the maximization over the spatial flow p(x) follows

$$\max_{p(x)|\leq g(x)} \langle u, \operatorname{div} p \rangle = \int_{\Omega} g(x) |\nabla u| \, dx$$

By the above formulations, the maximization of (17) over $p_s(x)$, $p_t(x)$ and p(x) amounts to the convex relaxed min-cut problem (11). Therefore, Prop. 1 is proved.

4) Globally Optimal Contour Evolution: The duality/equivalence between the continuous max-flow model (12) and the convex relaxed min-cut model (11) proposed in Prop. 1 implies that we can solve (11) by tackling the associated maximum flow model (12), which actually lays down the basis of the introduced efficient algorithms in Sec. IV.

Moreover, with helps of (12), we can also prove the exactness of the convex relaxed min-cut model (11) such that:

Proposition 2: Let $u^*(x)$ be one global optimum of (11), its thresholding $u^{\ell}(x) \in \{0, 1\}$:

$$u^{\ell}(x) = \begin{cases} 1, & \text{when } u^*(x) > \ell \\ 0, & \text{when } u^*(x) \le \ell \end{cases}, \quad (18)$$

for any $\ell \in [0,1)$, solves the original binary-constrained continuous min-cut problem (10) globally and exactly, i.e. the function $u^{\ell}(x) \in \{0,1\}$ indicates the thresholded level set S^{ℓ} which provides the global optimum contour to (7).

Proof: Let (p_s^*, p_t^*, p^*) and u^* be the optimal primaldual pair of the equivalent primal-dual model (16). Therefore, (p_s^*, p_t^*, p^*) optimizes the continuous max-flow problem (12) and $u^*(x)$ optimizes its dual problem (11); both problems have the same energy by [31], such that the minimum energy of (11) equals to

$$\int_{\Omega} p_s^* \, dx \, .$$

For the max-flow problem (12), the flow conservation condition (15) is satisfied, i.e.

$$\left(\operatorname{div} p^* - p_s^* + p_t^*\right)(x) = 0, \quad \forall x \in \Omega.$$
(19)

For any given $\ell \in [0, 1)$, let S^{ℓ} be the thresholded level set of $u^*(x)$ and $u^{\ell}(x) \in \{0, 1\}$ be its indicator function by (54).

For any pixel $x \in \Omega \setminus S^{\ell}$, i.e. where $u^*(x) \leq \ell < 1$, it is easy to see, by the variation of u, that to reach the maximum of (17) over $u^*(x)$, we have

$$p_s^*(x) = D_1(x), \quad \forall x \in \Omega \setminus S^\ell;$$
 (20)

and, for any pixel $x \in S^{\ell}$, i.e. $u^*(x) > \ell \ge 0$, we have

$$p_t^*(x) = D_2(x), \quad \forall x \in S^\ell.$$
(21)

Hence, by the flow conservation condition (19), we have

$$p_s^*(x) = \left(\operatorname{div} p^* + D_2\right)(x), \quad \forall x \in S^\ell.$$
 (22)

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The proof of Prop. 1 shows that the energy of the maxflow model (12) is equivalent to the energy of the primal-dual model (17), in turn, the energy of the convex relaxed min-cut (11). Therefore, by (20) and (22), the total energy of (10) for each level set S^{ℓ} , is

$$\int_{\Omega} p_s^* dx = \int_{\Omega \setminus S^\ell} D_1 dx + \int_{S^\ell} \left(D_2 + \operatorname{div} p^* \right) dx$$
$$= \int_{\Omega \setminus S^\ell} D_1 dx + \int_{S^\ell} D_2 dx + \int_{\partial S^\ell} g(s) ds$$

where we observe

$$\int_{S^{\ell}} \operatorname{div} p^* dx = \int_{\partial S^{\ell}} g(s) \, ds.$$

Therefore, we have

$$\int_{\Omega} p_s^* dx = \left\langle 1 - u^{\ell}, D_1 \right\rangle + \left\langle u^{\ell}, D_2 \right\rangle + \int_{\Omega} g(x) \left| \nabla u^{\ell} \right| dx$$

where the max-flow energy at the left-hand also gives the minimum energy of (11). In other words, the binary function $u^{\ell}(x) \in \{0, 1\}$, which is the indicator function of the region S^{ℓ} , solves the nonconvex min-cut problem (10) globally, i.e. S^{ℓ} provides a global optimum to (7)! Hence Prop. 2 is proved.

B. Fully Time Implicit Contour Evolution

Observe (19)-(21) given in the proof of Prop. 2, we have

Proposition 3: For the global optimum $u^*(x)$ of (11) and any $\ell \in [0, 1)$, let $u^{\ell}(x) \in \{0, 1\}$

$$u^{\ell}(x) = \begin{cases} 1, & \text{when } u^*(x) > \ell \\ 0, & \text{when } u^*(x) \le \ell \end{cases}$$
(23)

be the indicator function of the thresholded level set $S^{\ell} \subset \Omega$; then at each pixel $x \in \partial S^{\ell}$, we have

$$g(x)\kappa^{\ell}(x) + \nabla g(x) \cdot \left(\frac{\nabla u^{\ell}}{|\nabla u^{\ell}|}\right) - D_1(x) + D_2(x) = 0 \quad (24)$$

where

$$\kappa^{\ell}(x) = \operatorname{div}\left(\frac{\nabla u^{\ell}}{|\nabla u^{\ell}|}\right).$$

Proof: Through the facts (19)-(21), for each pixel x at the boundary ∂S^{ℓ} , we have

$$\operatorname{div} p^*(x) - D_1(x) + D_2(x) = 0.$$

Actually, div $p^*(x)$ is the first-order variation of the functional $\int g(x) |\nabla u| dx$ and p^* maximizes the functional $\int p(x) \nabla u dx$ over $|p(x)| \leq g(x)$ which implies at each pixel $x \in \partial S^{\ell}$

$$p^*(x) = g(x) \frac{\nabla u^\ell}{|\nabla u^\ell|}.$$

Then

$$\operatorname{div} p^*(x) = g(x)\kappa^{\ell}(x) + \nabla g(x) \cdot \left(\frac{\nabla u^{\ell}}{|\nabla u^{\ell}|}\right)$$

where

$$\kappa^{\ell}(x) = \operatorname{div}\left(\frac{\nabla u^{\ell}}{|\nabla u^{\ell}|}\right).$$

Therefore, Prop. 3 is proved.

Through Prop. 3, at each globally optimal position, each pixel at the contour satisfies (24) which provides the dynamics of contour evolution during each discrete time frame. In the following section, we apply (24) to analyze the movement of contours at each discrete time frame and, by (24), we show the computed globally optimal contour C from $u^{\ell}(x) \in \{0, 1\}$ actually provides a fully time implicit evolution scheme. As

shown in the proof, it is interesting to see that (24) is directly related to the flow conservation condition (15) of the proposed continuous max-flow model (12)!

Now we apply (24) to analyze some typical types of contour evolutions in image processing.

1) Mean-Curvature Driven Contour Evolution: For the current contour C_t at time t, let g(x) = 1, both $c^-(x)$ and $c^+(x)$ be linear to the distance from x to its boundary ∂C_t , i.e.

$$c^{-}(x) = c^{+}(x) = \operatorname{dist}(x, \partial \mathcal{C}_{t})/h \tag{25}$$

where h is the discrete time gap.

By (8) and (9), we set

$$D_1(x) := \begin{cases} \operatorname{dist}(x, \partial \mathcal{C}_t)/h, & \text{where } x \in \mathcal{C}_t \\ 0, & \text{otherwise} \end{cases}$$
(26)

and

$$D_2(x) := \begin{cases} \operatorname{dist}(x, \partial \mathcal{C}_t)/h, & \text{where } x \notin \mathcal{C}_t \\ 0, & \text{otherwise} \end{cases}$$
(27)

Then in view of Prop. 2 and Prop. 3, we have the following corollary

Corollary 4: Given g(x) = 1 and the region-deformation cost functions $c^+(x)$ and $c^-(x)$ as (25), for each discrete time frame from t to t + h, the current contour C_t can evolve to its globally optimal position C_{t+h} . Moreover, at each pixel $x \in \partial C_{t+h}$, its motion satisfies:

$$h\kappa(x) + \phi(x) = 0 \tag{28}$$

where $\phi(x)$ is the signed distance function such that

$$\phi(x) = \begin{cases} -\operatorname{dist}(x, \partial \mathcal{C}_t), & \text{where } x \in \mathcal{C}_t \\ \operatorname{dist}(x, \partial \mathcal{C}_t), & \text{otherwise} \end{cases}$$
(29)

The proof of the first part is obtained from Prop. 2. The meancurvature motion equation (28) can be proved using (24) with g(x) = 1 and (26)-(27).

The signed distance function $\phi(x)$ (29) measures how far the given pixel $x_t \in \partial C_t$ moves to its new position at $x \in \partial C_{t+h}$ along its outward normal $\mathbf{n}(x_t)$, see Fig. 1(c). Then it follows from (28) that

$$x = x_t - h\kappa(x)\mathbf{n}(x_t).$$

By this fact, we see that the computed new contour C_{t+h} is not only globally optimal, but also providing a fully time implicit scheme to the mean-curvature contour motion. 2) Geodesic Contour Evolution: For the geodesic contour evolution, where g(x) is given by some image edge indicator function, let the region deformation cost functions $c^+(x)$ and $c^-(x)$ be given by the associated distance function (25) which defines the min-cut costs $D_1(x)$ and $D_2(x)$, see (26)-(27).

Then we have

Corollary 5: Given the image edge indicator function g(x)and the region-deformation cost functions $c^+(x)$ and $c^-(x)$ as (25), for each discrete time frame from t to t + h, the current contour C_t can evolve to its globally optimal position C_{t+h} . Moreover, at each pixel $x \in \partial C_{t+h}$, its motion satisfies:

$$h\left(g(x)\kappa(x) + \nabla g(x) \cdot \left(\frac{\nabla u}{|\nabla u|}\right)\right) + \phi(x) = 0$$
 (30)

where $\phi(x)$ is the signed distance (29) and $\frac{\nabla u}{|\nabla u|}$ is the outward normal vector at x.

The proof of Cor. 5 directly follows from Prop. 2 and Prop. 3.

As the analysis in Sec.II-B1, (30) provides the implicit timediscrete geodesic motion equation such that for the given pixel $x_t \in \partial C_t$, it moves to its new position at $x \in \partial C_{t+h}$ from the time instance t to t + h along its outward normal $\mathbf{n}(x_t)$ and

$$x = x_t - h\left(g(x)\kappa(x) + \nabla g(x) \cdot \left(\frac{\nabla u}{|\nabla u|}\right)\right)\mathbf{n}(x_t).$$

3) Active Contour with Region-Based Force: In practice, the contour is driven by some region-based information besides the curvature function. For example, Chan and Vese [4] proposed the difference between the intensity models of inside and outside regions, i.e.

$$f(x) = \tau \left((\mu_{in} - I(x))^2 - (\mu_{out} - I(x))^2 \right)$$
(31)

where μ_{in} and μ_{out} provide the mean intensity values of inside and outside of the current contour C_t and $\tau > 0$ is the weight parameter.

For the contour evolution with such a region based force f(x), given the image edge indicator function g(x), let the region deformation cost functions $c^+(x)$ and $c^-(x)$ be

$$c^{+}(x) = \left(\operatorname{dist}(x, \partial \mathcal{C}_{t}) + f(x)\right)/h \tag{32}$$

and

$$c^{-}(x) = \left(\operatorname{dist}(x, \partial \mathcal{C}_{t}) - f(x)\right)/h \tag{33}$$

where h is the discrete time gap. Correspondingly, $c^+(x)$ and $c^-(x)$ define the min-cut costs $D_1(x)$ and $D_2(x)$ through (8) and (9).

Then we have

Corollary 6: Given the image edge indicator function g(x) and the region-based force f(x), we define the regiondeformation cost functions $c^+(x)$ and $c^-(x)$ by (32)-(33). For each discrete time frame from t to t + h, the current contour C_t can evolve to its globally optimal position C_{t+h} . Moreover, at each pixel $x \in \partial C_{t+h}$, its motion satisfies:

$$h\left(g(x)\kappa(x) + \nabla g(x) \cdot \left(\frac{\nabla u}{|\nabla u|}\right)\right) - f(x) + \phi(x) = 0 \quad (34)$$

where $\phi(x)$ is the signed distance (29) and $\frac{\nabla u}{|\nabla u|}$ is the outward normal vector at x.

As the analysis in Sec.II-B1, (34) provides the implicit timediscrete geodesic motion equation with the region force f(x), such that for the given pixel $x_t \in \partial C_t$, it moves to its new position at $x \in \partial C_{t+h}$ from the time instance t to t+h along its outward normal $\mathbf{n}(x_t)$ and

$$x = x_t - \left(h\left(g(x)\kappa(x) + \nabla g(x) \cdot \left(\frac{\nabla u}{|\nabla u|}\right)\right) - f(x)\right)\mathbf{n}(x_t).$$

III. CONTOUR EVOLUTION WITH PRIOR CONSTRAINTS



Fig. 2. 2(a) shows some examples of star shape, which are generated by the superformula [32]: http://en.wikipedia.org/wiki/Superformula. 2(b) illustrates the definition of the star shape prior.

In this section, we describe the contour evolution with the shape prior constraint, more specifically, the generic starshape prior. We first formulate the star-shape prior by a novel variational constraint and introduce it to the continuous mincut model discussed in Sec. II. Likewise, we introduce the new continuous max-flow model which is dual to the corresponding convex relaxation min-cut problem. In this regard, the star shape constraint in the continuous min-cut formulation is integrated to the proposed continuous max-flow model by introducing an extra spatial flow. We prove the proposed optimization model of star shaped contour evolution can be solved globally and exactly by convex relaxation, i.e. at each discrete time frame, the current contour can be moved to its globally optimal position. Moreover, the flow conservation condition to the new continuous max-flow model reveals that the global optimum of the continuous min-cut with star shape prior provides a fully time implicit solver to the star shaped contour convection PDE.

A. Star Shape Prior

The star shape was first proposed by [33], which is defined with respect to a center point o (see Fig. 2(b)): An object has a star shape if for any point x inside the object, all points on the straight line between the center o and x also lie inside the object; in other words, the object boundary can only pass any radial line starting from the origin o one single time. In fact, the star shape is a generic shape prior that properly models a wide spectrum of shapes, while effectively rules out all the inconsistent segments. Some examples of the star shapes are illustrated by Fig. 2(a) which are generated by the so-called superformula [32]. Veksler [33] proposed the star shape prior for segmentation using graph cuts. In this work, we propose a new variational formulation of the star-shape prior: for the center point o, let $d_o(x)$ be the distance map with respect to o and $e(x) = \nabla d_O(x)$; we, therefore, define the star shape prior as follows:

$$\nabla u(x) \cdot e(x) \ge 0, \quad \forall x \in \Omega.$$
 (35)

There is a similar definition of the star shape prior proposed by Strekalovskiy & Cremers [34]. We claim ours is different from theirs since the distance map is not constrained with respect to any single center point, which can also be defined w.r.t. any special marked area to achieve a even more generic shape prior.

In essence, the star shape prior (35) models a customized subset of the simply connected regions, where the holes and disjoint regions are exactly ruled out. For example, for each of the two regions: the hole and disjoint area (illustrated by Fig. 2(b)), there always exist some points which violate (35); see the red point x where the angle of the two vectors $\nabla u(x)$ and e(x) is obtuse, hence $\nabla u(x) \cdot e(x) < 0$; the same for the red point y, where $\nabla u(y) \cdot e(y) < 0$.

B. Global Optimization to Star Shaped Contour Evolution

Now we consider the contour evolution subject to the starshape prior: let \mathcal{L} be the set of all the contours with a star shape; like in Sec. II-A1, we define the costs $c^+(x)$ and $c^-(x)$ w.r.t. region expansions and shrinkages; to this end, we propose to propagate the contour C_t to its new position C_{t+h} with the minimum total cost of region changes, while the contour still keeps a star shape. Observe (7), this amounts to

$$\min_{\mathcal{C}\in\mathcal{L}} \int_{\mathcal{C}^+} c^+(x) \, dx + \int_{\mathcal{C}^-} c^-(x) \, dx + \int_{\partial\mathcal{C}} g(s) \, ds \,. \quad (36)$$

Likewise, with helps of the label assignment costs (8)-(9), the variational model (36) to the star-shaped contour evolution can be equally reformulated as

$$\min_{u(x)\in\{0,1\}} \langle 1-u, D_1 \rangle + \langle u, D_2 \rangle + \int_{\Omega} g(x) \left| \nabla u \right| \, dx \,. \quad (37)$$

subject to the star shape constraint

$$\nabla u(x) \cdot e(x) \ge 0, \quad \forall x \in \Omega.$$
 (38)

The optimization problem (37) gives rise to the *continuous min-cut model* constrained by a star shape prior (38).

In this section, we show the binary constrained combinatorial optimization problem (37), subject to the shape constraint (38), can be solved globally and exactly by means of convex relaxation.

We first formulate the associated convex relaxation model of (37) along with the star shape constraint as follows:

$$\min_{u(x)\in[0,1]} \langle 1-u, D_1 \rangle + \langle u, D_2 \rangle + \int_{\Omega} g(x) |\nabla u| \, dx \quad (39)$$

subject to

$$\nabla u(x) \cdot e(x) \ge 0, \quad \forall x \in \Omega,$$

where the binary constraint $u(x) \in \{0, 1\}$ in (37) is replaced by the convex set $u(x) \in [0, 1]$ such that (39) gives a convex optimization problem subject to a variational linear constraint. 1) Continuous Max-Flow Model: To study the convex relaxation model (39), we introduce a new continuous max-flow formulation and show its duality to (39).

With this respect, we apply a similar flow configuration as in Sec. II-A3 [30], [24]; and we add an extra spatial flow q(x), besides the spatial flow p(x), around each pixel $x \in \Omega$ such that:

$$q(x) = \lambda(x)e(x)$$
, where $\lambda(x) \ge 0$. (40)

The direction of such a spatial flow q(x) is along the same direction of the given reference vector e(x) at each pixel x.

In consequence, we have 4 flows passing each pixel $x \in \Omega$: $p_s(x)$, $p_t(x)$, p(x) and $q(x) = \lambda(x)e(x)$. Similar to the flow constraints (13)-(15) of the continuous max-flow model (12), we propose the flow capacity constraints and conservation condition to such flows, and, therefore, introduce the new continuous max-flow formulation as follows:

$$\max_{p_s, p_t, p, \lambda} \int_{\Omega} p_s(x) \, dx \tag{41}$$

subject to

$$|p(x)| \le g(x), \quad \lambda(x) \ge 0, \quad \forall x \in \Omega;$$
 (42)

$$p_s(x) \le C_s(x), \quad p_t(x) \le C_t(x), \quad \forall x \in \Omega;$$
 (43)

$$\left(\operatorname{div}\left(p+\lambda e\right)-p_{s}+p_{t}\right)(x) = 0, \quad \forall x \in \Omega.$$
 (44)

For the continuous max-flow model (41), (42) and (43) are the constraints on flow capacities; (44) is the flow conservation condition at each pixel $x \in \Omega$, where the extra spatial flow $q(x) = \lambda(x)e(x), \lambda(x) \ge 0$, provides an expanding flow field w.r.t. the origin point *o*, and it applies to make the segmentation result fitting the star-shape prior.

(44) is different to the conventional flow conservation condition in (15), which can be formulated as

$$\operatorname{div} p - p_s + p_t \in -\operatorname{div} S \tag{45}$$

where S be the convex set such that

$$S := \{ s(x) = \lambda(x)e(x), \ \lambda(x) \ge 0, \ \forall x \in \Omega \}.$$

$$(46)$$

In fact, (45) is a relaxation to the classical flow conservation constraint (15) which requires all the passing flows are exactly balanced at each pixel.

Through similar analysis given in the proof of Prop. 1, we can prove

Proposition 7: The continuous max-flow model (41) is dual to the convex relaxation problem (39).

Proof: Introduce the multiplier function u(x) to the linear equality of the flow conservation condition (44), we thus have the equivalent primal-dual formulation to (41):

$$\min_{u} \max_{p_s, p_t, p, \lambda} \int_{\Omega} p_s(x) \, dx + \langle u, \operatorname{div} \left(p + \lambda e \right) - p_s + p_t \rangle$$
(47)

subject to the flow capacity conditions (42) and (43).

The optimization of (47) is equivalent to

$$\min_{u} \max_{p_s, p_t, p, \lambda} \langle 1 - u, p_s \rangle + \langle u, p_t \rangle + \langle u, \operatorname{div} p \rangle + \langle u, \operatorname{div}(\lambda e) \rangle$$
(48)

subject to (42) and (43). We see that the min and max operations of (48) can be interchanged (see [31]); hence we

can first maximize (48) over the flow variables p_s , p_t , p and λ . Following a similar analysis in [30] and [24], the maximization over p_s and p_t essentially gives rise to

$$u(x) \in [0,1], \quad \forall x \in \Omega,$$

the maximization over the spatial flow p follows

$$\max_{|p(x)| \le g(x)} \langle u, \operatorname{div} p \rangle = \int_{\Omega} g(x) |\nabla u| \, dx$$

Given the assumption $\partial u = 0$ at the boundary, we have

$$\langle u, \operatorname{div}(\lambda e) \rangle = - \langle \lambda, \nabla u \cdot e \rangle;$$

then the maximization of the last term in (48) over the spatial variable $\lambda(x) \ge 0$ is

$$\nabla u(x) \cdot e(x) \ge 0, \quad \forall x \in \Omega.$$

Therefore, Prop. 7 is proved.

2) Globally Optimal Star-Shaped Contour Evolution: The duality or equivalence between the continuous max-flow model (41) and the convex relaxation model (39) indicates that we can solve (39) by computing the associated maximum flow (41). With helps of the proposed continuous max-flow model (41), we can further prove

Proposition 8: Let $u^*(x)$ be one global optimum of (39), its thresholding $u^{\ell}(x) \in \{0, 1\}$:

$$u^{\ell}(x) = \begin{cases} 1, & \text{when } u^*(x) > \ell \\ 0, & \text{when } u^*(x) \le \ell \end{cases},$$
(49)

for any $\ell \in [0,1)$, solves the original binary-constrained continuous min-cut problem (37) globally and exactly.

Proof: Let $(p_s^*, p_t^*, p^*, \lambda^*)$ and u^* be the optimal primaldual pair of the equivalent primal-dual model (47), which means that $(p_s^*, p_t^*, p^*, \lambda^*)$ optimizes the max-flow problem (41) and $u^*(x)$ optimizes its dual problem (39); both problems have the same energy (see [31]) such that the minimum energy of (39) equals to the maximum total flow

$$\int_{\Omega} p_s^* \, dx \, .$$

For the max-flow problem (41), the flow conservation condition (44) is satisfied, i.e.

$$\left(\operatorname{div}\left(p^* + \lambda^* e\right) - p_s^* + p_t^*\right)(x) = 0, \quad \forall x \in \Omega.$$
 (50)

Let S^{ℓ} be any level set of u^* thresholded by $\ell \in (0, 1]$ and $u^{\ell}(x) \in \{0, 1\}$ be its indicator function.

For any pixel $x \in \Omega \setminus S^{\ell}$, i.e. where $u^*(x) < \ell \leq 1$, it is easy to see that through the variation of u(x), we have

$$p_s^*(x) = D_1(x), \quad \forall x \in \Omega \setminus S^\ell;$$
 (51)

and, for any pixel $x \in S^{\ell}$, i.e. $u^*(x) \ge \ell > 0$, we have

$$p_t^*(x) = D_2(x), \quad \forall x \in S^\ell.$$
(52)

By (50), we have

$$p_s^*(x) = \left(\operatorname{div}\left(p^* + \lambda^* e\right) + D_2\right)(x), \quad \forall x \in S^\ell.$$
 (53)

The proof of Prop. 7 indicates that the energy of the maxflow model (41) is equivalent to the energy of the primal-dual model (48) and also the energy of (39). Therefore, by (51) and (53), the total energy of (37) for each level set S^{ℓ} , is

$$\begin{split} \int_{\Omega} p_s^* \, dx &= \int_{\Omega \setminus S^\ell} D_1 \, dx + \int_{S^\ell} \left(D_2 + \operatorname{div} \left(p^* + \lambda^* e \right) \right) dx \\ &= \int_{\Omega \setminus S^\ell} D_1 \, dx + \int_{S^\ell} D_2 \, dx + \int_{\partial S^\ell} g(s) \, ds \\ &- \int_{\partial S^\ell} \lambda^* e(s) \, ds \end{split}$$

where

$$e(s) = \nabla u^{\ell}(x) \cdot e(x) \ge 0, \quad x \in \partial S^{\ell}$$

and by simple variational analysis, we also have

$$\int_{\partial S^\ell} \lambda^* e(s) \, ds \, = \, 0 \, .$$

Therefore, we have

$$\int_{\Omega} p_s^* dx = \int_{\Omega \setminus S^{\ell}} D_1 dx + \int_{S^{\ell}} D_2 dx + \int_{\partial S^{\ell}} g(s) ds dx$$

which is also the minimum energy of (39), i.e. the convex relaxation of the binary constrained optimization problem (37). In other words, the binary function $u^{\ell}(x) \in \{0, 1\}$, which is the indicator function of S^{ℓ} , solves the nonconvex star-shape constrained min-cut problem (37) globally. Prop. 8 is proved.

C. Discrete Time Evolution of Star-Shaped Contours

Observe (50)-(52) in the proof of Prop. 8, we have

Proposition 9: For the global optimum $u^*(x)$ of (11) and any $\ell \in [0, 1)$, let $u^{\ell}(x) \in \{0, 1\}$

$$u^{\ell}(x) = \begin{cases} 1, & \text{when } u^*(x) > \ell \\ 0, & \text{when } u^*(x) \le \ell \end{cases}$$
(54)

be the indicator function of the thresholded level set $S^{\ell} \subset \Omega$; then at each pixel $x \in \partial S^{\ell}$, we have

$$\left(g\kappa^{\ell} + \nabla g \cdot \left(\frac{\nabla u^{\ell}}{|\nabla u^{\ell}|}\right) - D_1 + D_2 + v^{\ell}\right)(x) = 0 \quad (55)$$

where

$$\kappa^{\ell}(x) = \operatorname{div}\left(\frac{
abla u^{\ell}}{|
abla u^{\ell}|}
ight), \quad v^{\ell}(x) = \operatorname{div}(\lambda^{*}(x)e(x)).$$

Its proof is similar as the one given in Prop. 3.

In contrast to the motion equation (24) at each discrete time frame, the star shape prior introduces an additional 'speed' term v^{ℓ} to the contour evolution equation (55), which makes the next contour C_{t+h} be consistent to the shape constraint. v^{ℓ} is also called the *star-shape speed* in this paper. Now we apply (24) to analyze the movement of contours at each discrete time frame.

By Prop. 8, we observe that (37) can be solved exactly and globally, which means the contour with the star shape constraint can evolve to its globally optimal position at each time frame. Moreover, through Prop. 9, each pixel at the globally optimal contour satisfies (55). To this end, (55) provides the dynamics of contour evolution at each discrete time. Similar to Cor. 4-6, we can apply (55) to analyze the evolution of contours with the additional star-shape speed which forces the consistency to the star shape prior.

For the mean curvature driven contour evolution proposed in Sec. II-B1, we have

Corollary 10: Given g(x) = 1 and the region-deformation cost functions $c^+(x)$ and $c^-(x)$ as (25), for each discrete time frame from t to t + h, the current contour C_t can evolve to its globally optimal position C_{t+h} , while both contours C and C_{t+h} are consistent to the star shape. Moreover, at each pixel $x \in \partial C_{t+h}$, its motion satisfies:

$$h\kappa(x) + \phi(x) + v(x) = 0 \tag{56}$$

where $\phi(x)$ is the signed distance (29) and v(x) the optimal star-shape speed.

For the geodesic contour evolution proposed in Sec. II-B2, we have

Corollary 11: Given the image edge indicator function g(x)and the region-deformation cost functions $c^+(x)$ and $c^-(x)$ as (25), for each discrete time frame from t to t + h, the current contour C_t can evolve to its globally optimal position C_{t+h} , while both contours C and C_{t+h} are consistent to the star shape. Moreover, at each pixel $x \in \partial C_{t+h}$, its motion satisfies:

$$h\left(g(x)\kappa(x) + \nabla g(x) \cdot \left(\frac{\nabla u}{|\nabla u|}\right)\right) + \phi(x) + v(x) = 0 \quad (57)$$

where $\phi(x)$ is the signed distance (29), $\frac{\nabla u}{|\nabla u|}$ is the outward normal vector at x and v(x) the optimal star-shape speed.

For the region-based contour evolution proposed in Sec. II-B3, we have

Corollary 12: Given the image edge indicator function g(x) and the region-based force f(x), we define the regiondeformation cost functions $c^+(x)$ and $c^-(x)$ by (32)-(33). For each discrete time frame from t to t + h, the current contour C_t can evolve to its globally optimal position C_{t+h} , while both contours C and C_{t+h} are consistent to the star shape. Moreover, at each pixel $x \in \partial C_{t+h}$, its motion satisfies:

$$\left(h\left(g\kappa + \nabla g \cdot \frac{\nabla u}{|\nabla u|}\right) - f + \phi + v\right)(x) = 0 \qquad (58)$$

where $\phi(x)$ is the signed distance function (29), $\frac{\nabla u}{|\nabla u|}$ is the outward normal vector at x and v(x) the optimal star-shape speed.

In view of the signed distance function $\phi(x)$, (56)-(58) provide the time implicit implementation of the respective contour evolution, together with the star-shape speed.

IV. CONTINUOUS MAX-FLOW BASED ALGORITHMS

In Sec. II-A3 and Sec. III-B, we observe that the proposed continuous max-flow models (12) and (41) provide a powerful tool to analyze the respective continuous min-cut models (10) and (37). In addition to this, another advantage to utilize the proposed continuous max-flow models is that they result in numerically simple and efficient algorithms using modern convex optimization theories, see [35] and [36]. In the following subsections, we will notice that, in the continuous max-flow-based algorithms, the nonlinear total-variation function and the associated star shape constraint are encoded by projections to

some simple convex sets. Moreover, the continuous max-flow based algorithms can be easily implemented in the parallel computing system, e.g. a GPU, which significantly speeds up numerical computation.

A. Continuous Max-Flow Algorithm to Contour Evolution

Through Prop. 1 and Prop. 2, for the contour evolution without shape prior (7), the continuous min-cut problem (10) can be globally and exactly solved by its corresponding continuous max-flow model (12). The corresponding continuous max-flow-based algorithm can be found in [24], which is based on the classical augmented Lagrangian method [37], [35]. Here, we list the main steps of the continuous max-flow based algorithm:

Let $R(p_s, p_t, p)$ be the flow residue function given by

$$R(p_s, p_t, p) = \operatorname{div} p - p_s + p_t,$$
 (59)

 $L(p_s, p_t, p, u)$ be the Lagrangian function of the continuous max-flow problem (12):

$$L(p_s, p_t, p, u) := \int_{\Omega} p_s(x) \, dx + \langle u, R(p_s, p_t, p) \rangle ,$$

and $L_c(p_s, p_t, p, u)$ be the augmented Lagrangian function such that:

$$L_c(p_s, p_t, p, u) := L(p_s, p_t, p, u) - \frac{c}{2} \|R(p_s, p_t, p)\|^2$$

where c > 0 is constant.

At each k-th iteration, the continuous max-flow based algorithm consists of projection-descent steps of flows p_s , p_t and p such that

$$p^{k+1} = \operatorname{Proj}_{|p(x)| \le g(x)} \left(p^k + \alpha \nabla \left(R(p_s^k, p_t^k, p^k) - \frac{u^k}{c} \right) \right);$$

$$p_s^{k+1} = \min \left(D_1(x), \left(R(0, p_t^k, p^{k+1}) - \frac{u^k}{c} \right)(x) \right);$$

$$p_t^{k+1} = \min \left(D_2(x), \left(\frac{u^k}{c} - R(p_s^{k+1}, 0, p^{k+1}) \right)(x) \right);$$

together with a simple update in the labeling function u

$$u^{k+1} = u^k - c R(p_s^{k+1}, p_t^{k+1}, p^{k+1});$$
(60)

where $\alpha > 0$ is the projection step-size chosen by [38].

B. Continuous Max-Flow Algorithm to Star-Shaped Contour Evolution

For the contour evolution with the star-shape prior (36), the continuous min-cut problem (37) can be globally and exactly solved by its corresponding continuous max-flow model (41), through Prop. 7 and Prop. 8. The continuous max-flow algorithm to the star-shaped contour evolution proposed in the following part is slightly different from the continuous max-flow-based algorithm shown in Sec. IV-A, where an extra flow adjustment step for $\lambda(x)$ is considered at each iteration.

Let $R(p_s, p_t, p, \lambda)$ be the flow residue function such that

$$R(p_s, p_t, p, \lambda) = \operatorname{div}(p + \lambda e) - p_s + p_t, \qquad (61)$$

 $L(p_s, p_t, p, \lambda, u)$ be the Lagrangian function of the continuous max-flow problem (41):

$$L(p_s, p_t, p, \lambda, u) := \int_{\Omega} p_s(x) \, dx + \langle u, R(p_s, p_t, p, \lambda) \rangle ,$$

and $L_c(p_s, p_t, p, \lambda, u)$ be the augmented Lagrangian function:

$$L_c(p_s, p_t, p, \lambda, u) := L(p_s, p_t, p, \lambda, u) - \frac{c}{2} \left\| R(p_s, p_t, p, \lambda) \right\|^2$$

where c > 0 is constant.

At each k-th iteration, our algorithm consists of projectiondescent steps in flows p_s , p_t , p and λ such that

$$p^{k+1} = \operatorname{Proj}_{|p(x)| \le g(x)} \left(p^k + \alpha \nabla \left(R(p_s^k, p_t^k, p^k, \lambda^k) - \frac{u^k}{c} \right) \right);$$

$$p_s^{k+1} = \min \left(D_1(x), \left(R(0, p_t^k, p^{k+1}, \lambda^k) - \frac{u^k}{c} \right)(x) \right);$$

$$p_t^{k+1} = \min \left(D_2(x), \left(\frac{u^k}{c} - R(p_s^{k+1}, 0, p^{k+1}, \lambda^k) \right)(x) \right);$$

$$\lambda^{k+1} = \max \left(0, \lambda^k(x) + \beta^k(x) \right);$$

together with a simple update in the labeling function u

$$u^{k+1} = u^k - c R(p_s^{k+1}, p_t^{k+1}, p^{k+1}, \lambda^{k+1}); \qquad (62)$$

where $\alpha > 0$ is the projection step-size chosen by [38] and $\beta(x)$ is related to $|e(x)|^2$ as discussed in the following part.

1) Optimization of $\lambda(x)$: To see the optimization over λ at each k-th iteration, let

$$F^{k} = \frac{u^{k}}{c} - \left(\operatorname{div} p^{k+1} - p_{s}^{k+1} + p_{t}^{k+1}\right).$$

Then the optimization over λ at each k-th iteration reduces to the following constrained minimization problem

$$\min_{\lambda \ge 0} \left\| \operatorname{div}(\lambda e) - F^k \right\|^2.$$
(63)

To optimize (63), we first apply the gradient descent step

$$\lambda^{k+1/2}(x)e(x) = \lambda^k(x)e(x) + \theta \nabla \big(\operatorname{div}(\lambda^k e) - F^k\big)(x)$$

where $\theta > 0$ is the step-size.

We product both sides of the above equation by the vector e(x), then divide by $|e(x)|^2$; which results

$$\lambda^{k+1/2}(x) = \lambda^k(x) + \beta^k(x)$$

where

$$\beta^{k}(x) = \frac{\theta \nabla (\operatorname{div}(\lambda^{k} e) - F^{k})(x) \cdot e(x)}{|e(x)|^{2}} \, .$$

 $\lambda^{k+1}(x)$ is computed by the projection of $\lambda^{k+1/2}$ to $\lambda(x) \geq 0$ such that

$$\lambda^{k+1} = \max\left(0, \lambda^{1+2}(x)\right)$$

V. EXPERIMENTS

In this section, we illustrate the numerical performance of the proposed continuous max-flow-based contour evolution approach, together with comparisons to graph-cuts and level-sets. Experiments are conducted to the image segmentation tasks on synthetic images, real-world images and 2D/3D medical images. Moreover, we demonstrate the effectiveness of the star-shape prior to image segmentation. In all the experiments, we implement the proposed continuous max-flow algorithms in a parallel computing platform, i.e. NVidia CUDA GPU (NVIDIA Corp., Santa Clara, CA); the user interface and the computation of other image processing are implemented on Matlab (Natick, MA). All computations are performed on a Windows desktop with Intel QuadCore CPU (2.83 GHz) and NVIDIA GTX 580 GPU (NVIDIA Corp., Santa Clara, CA).

The convergence of algorithms is evaluated by the absolute mean of the labeling functions $u^{k+1}(x)$ and $u^k(x)$ for the two sequential contours, such that

err. =
$$\frac{\left\|u^{k+1}(x) - u^k(x)\right\|_{\ell_1}}{|\Omega|}$$
,

where the ℓ_1 norm is defined as $\|\cdot\|_{\ell_1} = \int_{\Omega} |\cdot| dx$.

In view of the proposed continuous max-flow algorithms (see also (60) and (62)), such error evaluation boils down to the measurement of either the average absolute flow residue or the average absolute duality gap which is meaningful in the optimization theory, i.e.

err. =
$$c \frac{\left\| \operatorname{div} p^{k+1} - p_s^{k+1} + p_t^{k+1} \right\|_{\ell_1}}{|\Omega|}$$

for the continuous max-flow algorithm without shape constraints, and

rr. =
$$c \frac{\left\| \operatorname{div} \left(p^{k+1} + \lambda^{k+1} e \right) - p_s^{k+1} + p_t^{k+1} \right\|_{\ell_1}}{|\Omega|}$$

for the continuous max-flow algorithm with the star-shape prior.

A. Validation and Comparisons

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Fig. 3. Experiment of mean-curvature driven 3D surface evolution (volume size: $150 \times 150 \times 150$ voxels). (a) The radius plot of the 3D ball evolution driven by its mean-curvature flow, which is computed by the proposed continuous max-flow algorithm; its function is theoretically $r(t) = \sqrt{C-2t}$. (b) The computed 3D ball at one discrete time frame, which fits a perfect 3D ball shape. This is in contrast to (c), the computation result by graph cut [15] with a 3D 26-connected graph. The computation time of the continuous max-flow algorithm for each discrete time evolution is around 1 sec., which is faster than the graph cut method (120 sec.).

1) Synthetic Experiment: Evolution by Mean-Curvature Flow: Fig. 3 shows the experiment of evolving a 3D ball by the mean-curvature flow (3D volume: $150 \times 150 \times 150$ voxels), where the cost functions are set as (26) and (27). The radius of the initial ball is 56. For each discrete time-frame, the proposed continuous max-flow algorithm takes approx. 1 sec. to converge with a high-accuracy of err. $\leq 10^{-5}$. Fig. 3(a) shows the radius plot of the computed 3D ball sequence using the continuous max-flow algorithm, whose shape fits the theoretical result $r(t) = \sqrt{C - 2t}$. Fig. 3(b) demonstrates the 3D ball at one time instance during its evolution. We repeat the experiment of the mean-curvature driven motion of a 3D ball using graph-cut [15]³, where a 26-connected graph is used. It takes around 120 sec. for each discrete time evolution, which is slower than the continuous max-flow based algorithm. The memory usage for such a 26-connected 3D graph cut is about 4.2G bytes, higher than the memory load of the implemented continuous max-flow algorithm (less than 0.2G bytes). In addition, Fig. 3(c) shows one computed result during its evolution, for which the metrication effects are visible comparing to the perfect 3D ball computed by the continuous max-flow algorithm. For the graph-cut method, more neighbours can be used to reduce such visible metrication artifacts shown in Fig. 3(c). But it is expected that higher memory load and longer computation could be taken.



(d) Iter. #2, $h = 10^5$ (e) Iter. #22, h = 500 (f) Iter. #8, $h = 10^3$

Fig. 4. Experiments of contour evolution with various large time step sizes. The contour evolves with the region-based force f(x) as in (31), for the given gray-scale image (860 × 645 pixels). The two mean values μ_{in} and μ_{out} are updated after each discrete time evolution. The contour starts with the same initialization (see Fig. 4(a)), but different discrete time gaps h = 500, 10^3 and 10^5 . Fig. 4(b) - Fig. 4(d) show the contours after 2 outer iterations, with the time gap h = 500, 10^3 and 10^5 respectively. Clearly, the larger discrete time gap leads to the bigger change of the contour during each outer iteration. For the smallest time step h = 500, the contour evolution stops at the final result (see Fig. 4(e)) after 22 outer iterations . For $h = 10^3$, the contour evolution stops at its result (see Fig. 4(f)) after 8 evolution steps.

2) Reliability to Large Discrete Time-Step: As discussed in Sec. II-B3, the optimum computed by the continuous max-flow algorithm globally and exactly solves the contour evolution with the region force. In fact, it provides a time-implicit solver, through Cor. 6, such that larger discrete time-step is allowed.

In this experiment, the contour evolves with the regionbased force f(x) defined as in (31); it starts with the same initialization, as shown in Fig. 4(a), but different discrete time

³The graph cut experiment is based on the code downloaded from the website: http://vision.csd.uwo.ca/code/maxflow-v3.01.zip.

gaps h = 500, 10^3 and 10^5 are applied. After each discrete time evolution, i.e. each outer iteration, the two mean values μ_{in} and μ_{out} are updated. Fig. 4(b) - Fig. 4(d) show the contours after 2 outer iterations, with the time gap h = 500, 10^3 and 10^5 respectively. Actually, for the extremely large time gap $h = 10^5$, the contour evolution stops at the reasonable result after only 2 outer iterations. Clearly, larger discrete time gap leads to bigger changes of the contour during each outer iteration, hence faster convergence to the final result. For h = 500, the contour evolution stops at the final result (see Fig. 4(e)) after 22 outer iterations . For $h = 10^3$, the contour evolution stops at its result (see Fig. 4(f)) after 8 evolution steps. This experiment shows the continuous max-flow based algorithm is reliable to the chosen discrete time gap h of the contour evolution, even with a very large value of h.



(d) Result for init. 1 (e) Result for init. 2 (f) Result for init. 3

Fig. 5. Experiments of contour evolution with various initialization conditions. The data cost functions for the continuous min-cut problem are defined as in Sec. II-B3, for the given gray-scale image (860×645 pixels). Three different initializations, see Fig. 5(a) - Fig. 5(c), are applied in the experiments. A relatively large time step $h = 10^5$ is used. All the experiments stop at nearly the same meaningful positions, see Fig. 5(d) - Fig. 5(f), even if the initial contour, e.g. Fig. 5(a) - Fig. 5(b), is far from the final result. The initial contour shown in Fig. 5(a) takes 3 outer iterations to stop at its final contour shown in Fig. 5(d); the initial contour shown in Fig. 5(b) takes 2 outer iterations to stop at its final contour shown in Fig. 5(e); the initial contour shown in Fig. 5(c) takes 2 outer iterations to stop at its final contour in Fig. 5(f).

3) Reliability to Initialization: Now we show the proposed continuous max-flow-based contour evolution approach is reliable to poor initializations. Similar to the previous experiment, we evolve the contour with a region-based force f(x) given by (31). The data cost functions for the continuous min-cut problem are defined as in Sec. II-B3, where the proposed continuous max-flow algorithm is employed to compute the global optimum at each contour evolution iteration. Three different initial contours are applied in the experiments, see Fig. 5(a) - Fig. 5(c), and a relatively large time step $h = 10^5$ is applied. The contours finally stop at nearly the same meaningful positions, as shown in Fig. 5(d) - Fig. 5(f). Although the initial contour is far from the final results as shown in Fig. 5(a) and Fig. 5(b), the proposed approach can still stop at its final reasonable position.

The initial contour shown in Fig. 5(a) takes 3 outer iterations to stop at its final contour shown in Fig. 5(d); the initial contour shown in Fig. 5(b) takes 2 outer iterations to stop at

els

its final contour shown in Fig. 5(e); the initial contour shown in Fig. 5(c) takes 2 outer iterations to stop at its final contour shown in Fig. 5(f).





(a) Size: 860×645 pix-

(b) Iter. # 500



(d) Initial position

SPLS



(e) Iter. # 1000

(f) Iter. # 1000

(c) Iter. # 500



Fig. 6. Comparison of the proposed continuous max-flow-based contour evolution approach to the classical level-set methods: the narrow band level-set method (NBLS) [4] and the sparse field level-let method (SPLS) [39]. All the methods start with the same initialization contour (see Fig. 6(d)). The proposed approach converges to the final result after 2 outer iterations within 2.4 sec. (1.0 sec. for the continuous max-flow solver and 1.4 sec. for computing the cost); in contrast, the two level set methods take more than 1700 iterations for convergence (time costs are shown in the bottom table). Clearly, the proposed continuous max-flow-based approach significantly improves the contour evolution in terms of computation time and the total number of iterations. Moreover, for the proposed method, the contour of NBLS (see Fig. 6(j)) comparing to the result of NBLS (see Fig. 6(k)) and SPLS (see Fig. 6(j)), where the inside circle of the small box is clearly cropped by the proposed method but not by NBLS and SPLS.

331.0

1800

Matlab

4) Comparisons to Level Set Methods: We compare the proposed continuous max-flow-based contour evolution approach to the classical level-set method, where two level set implementations: the narrow band level set method (NBLS) [4] and the sparse field level-set method (SPLS) [39], are used for comparisons. All the methods start with the same initialization contour (see Fig. 6(d)). The proposed continuous max-flow-

based method converges to the final result within 2.4 sec. in total after just 2 outer iterations: 1.0 sec. for the continuous max-flow solver and 1.4 sec. for computing the cost; in contrast, the two level set methods take more than 1700 outer iterations for convergence. The computation results shown in the bottom table of Fig. 6 demonstrate the proposed continuous max-flow-based approach significantly improves the efficiency of contour evolution in terms of computation time and the total number of iterations.

Moreover, for the proposed approach, the contour finally evolves to a better result (see Fig. 6(j)) comparing to the result of NBLS (see Fig. 6(k)) and SPLS (see Fig. 6(l)): the inside circle of the small box is successfully segmented by the proposed global optimization-based method but not by NBLS and SPLS, and also some visible artifacts appear in the final results of NBLS and SPLS.

B. Applications to Image Segmentation



(d) Sample seeds

(e) Initial position

(f) Iter. # 2

Fig. 7. Experiments of gray-scale image segmentation. **1st Row:** For the image with 321×481 pixels, the initial contour just starts at the green region shown in Fig. 7(a). After 13 outer iterations, the contour stops at its final result shown by Fig. 7(c) (around 12.7 sec. in total: 1.7 sec. for the continuous max-flow solver and 11 sec. for computing the costs). **2nd Row:** For the segmentation of the zebra image with 250×167 pixels. The initial contour is given in Fig. 7(f) (around 0.65 sec. in total: 0.05 sec. for the continuous max-flow solver and 0.6 sec. for computing the costs).

1) Image Segmentation by Histogram Matching: The intensity or color histogram of the specified image objects provides a global and robust clues to segment meaningful objects in images. Let $z \in Z$ be the photometric values, $q_{in}(z)$ and $q_{out}(z)$ be the probability density functions (PDFs) of the foreground and background regions, i.e. the two regions inside and outside the segmentation contour, which can be obtained through the sampled seeds. We segment the image by finding the regions C_{in} or C_{out} , whose PDFs best match the given PDFs $q_{in}(z)$ and $q_{out}(z)$ respectively. Let $p_{in}(u, z)$ and $p_{out}(u, z)$ be the estimated PDFs of the inside and outside regions C_{in} or C_{out} , where u(x) gives the labeling function of C_{in} . The PDFs $p_i(u, z)$, i = in, out, can be estimated by

the Parzen method [40] such that

$$p_i(u,z) = \frac{\int_{\Omega} K(z - I(x)) \, u \, dx}{\int u \, dx}, \quad i = in, out,$$

where $K(\cdot)$ is the Gaussian kernel function $K(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-x^2/2\sigma^2)$. In these experiments, a Bhattacharyya distance [8] is em-

In these experiments, a Bhattacharyya distance [8] is employed for matching PDFs of the two regions inside and outside of the segmentation contour, which amounts to the Bhattacharyya-based histogram-matching energy formulated as follows

$$-\sum_{z\in\mathcal{Z}}\sqrt{p_{in}(u,z)\,q_{in}(z)} - \sum_{z\in\mathcal{Z}}\sqrt{p_{out}(1-u,z)\,q_{out}(z)}\,.$$
(64)

The energy function (64) is highly non-linear and non-convex, which can not be directly optimized in a single step. Similar to the common level-set method, we gradually evolve a contour C to its best position such that its labeling function u(x)minimizes (64). To this end, we compute the region-based force f(x) by means of the first-order variation⁴ of the energy function (64) and define the corresponding cost functions as (32) and (33). In this section, we conduct segmentation experiments on both gray-scale images and color images.

Fig. 7 shows the experiments of gray-scale image segmentation, where the intensity PDFs are matched. The prior PDFs of the inside and outside regions are computed using the user-input seeds, shown in Fig. 7(a) and Fig. 7(d): green for foreground and red for background. For the first experiment, illustrated by the images of the first row of Fig. 7, the initial contour is given by the foreground seeds, i.e. the green region. The contour stops at its final result (see Fig. 7(d)) after 13 discrete time evolutions (around 12.7 sec. in total: 1.7 sec. for the continuous max-flow solver and 11 sec. for computing the costs). The second experiment starts its initial contour (see Fig. 7(d)) and stops at its final contour (see Fig. 7(f)) after 2 outer iterations (around 0.65 sec. in total: 0.05 sec. for the continuous max-flow solver and 0.6 sec. for computing the costs).

Fig. 8 shows the experiments of color image segmentation, where PDFs of color distribution are matched. Similar to the experiments of gray-scale image segmentation, the prior PDFs of the inside and outside regions are computed using the user-input seeds: green for foreground and red for background (see the images at the left side columns of Fig. 8). Here the color PDFs are generated with 128 bins for each color channel. For the challenging experiments shown in Fig. 8(d), Fig. 8(f) and Fig. 8(h), where the PDFs of foreground and background are highly overlapped with each other, the proposed global optimization based contour evolution method finds quite reasonable results. Its numerical efficiency can be demonstrated in the table of Fig. 8.

2) Non-Parametric Texture Image Segmentation: Fig. 9 demonstrates the experiment of the non-parametric texture image segmentation by the global optimization based contour evolution, where the Bhattacharyya distance is directly used to



Fig. 9. Experiment of non-parametric texture image segmentation: for the gray-scale zebra image with 481×321 pixels, PDFs are evaluated over the texture feature proposed by Houhou *et al* [41]. The contour starts at the position shown in Fig. 9(a) and stops at the final result shown in Fig. 9(c) after 20 outer iterations (6 sec. in total: 1 sec. for the continuous max-flow solver and 5 sec. for computing the costs).

measure the inside and outside texture PDFs [8]. The texture PDFs are generated to the texture feature proposed by Houhou *et al* [41]. The contour starts at the position shown in Fig. 9(a) and stops at the final result shown in Fig. 9(c) after 20 outer iterations (6 sec. in total: 1 sec. for the continuous max-flow solver and 5 sec. for computing the costs).

C. Medical Imaging Applications



Fig. 10. Experiments of 2D medical image segmentation: **1st Row:** 2D liver CT image segmentation $(512 \times 512 \text{ pixels})$; left image shows the sampled seeds and the initial contour starts at the foreground sampled region (green); the final contour stops after 5 outer iterations and is shown in the right image (4.9 sec. in total: 1.5 sec. for the continuous max-flow solver and 3.4 sec. for the cost computation). **2nd Row:** 2D Prostate US image segmentation (800×523 pixels); left image shows the sampled seeds and the initial contour starts at the foreground sampled region (green); the final contour stops after 10 outer iterations and is shown in the right image (7.8 sec. in total: 2.7 sec. for the continuous max-flow solver and 5.1 sec. for the cost computation). **3rd Row:** 2D Prostate MR image segmentation (262×216 pixels); left image shows the sampled seeds and the initial contour starts at the foreground sampled region (green); the final contour stops after 5 outer iterations and is shown in the right image ($1.6 \text{ sec. in total: } 0.3 \text{ sec. for the continuous max-flow solver and 1.3 sec. for the continuous max-flow solver and 1.5 sec. for the contin$

1) 2D Medical Image Segmentation: We conduct three experiments of 2D medical image segmentation. The results



Fig. 8. Experiments of color image segmentation: for each experiment, the sampled seeds are shown in the left picture (green: foreground, red: background); the initial contour is shown in the middle picture; the final contour is illustrated in the right picture. The table shows the details of image size, computation time and total number of outer iterations for all the experiments.

are shown in Fig. 10: 2D liver computed tomography (CT) segmentation (1st row), 2D end-fire transrectal prostate ultrasound (US) segmentation (2nd row), 2D T2-weighted prostate magnetic resonance (MR) segmentation (3nd row). Like the experiments for grayscale image segmentation in Sec. V-B, matching the intensity distributions of foreground and background is used to drive the contour evolution. Both US and MR image segmentation are challenging: the US image often has weak boundaries of the objects and the intensity distributions of the MR image objects are typically inhomogeneous. As illustrated by these experiment results, the proposed global optimization-based contour evolution algorithm can locate the object's boundaries efficiently and successfully, even when the image quality is poor.

In general, for the 2D liver CT image segmentation (1st row of Fig. 10): left image shows the sampled seeds where the foreground sampled region (green) is used as the initial contour; the final contour stops (shown in the right image) after 5 outer iterations (4.9 sec. in total: 1.5 sec. for the continuous max-flow solver and 3.4 sec. for the cost computation). For the 2D Prostate US image segmentation (2nd row of Fig. 10): left image shows the sampled seeds where the foreground sampled

region (green) is used as the initial contour; the final contour (shown in the right image) stops after 10 outer iterations (7.8 sec. in total: 2.7 sec. for the continuous max-flow solver and 5.1 sec. for the cost computation). For the 2D Prostate MR image segmentation (3rd row of Fig. 10); left image shows the sampled seeds and the initial contour starts at the foreground sampled region (green); the final contour (shown in the right image) stops after 5 outer iterations (1.6 sec. in total: 0.3 sec. for the continuous max-flow solver and 1.3 sec. for the cost computation).



Fig. 11. Experiment of 3D carotid MR image segmentation (size: $161 \times 144 \times 111$ voxels): (a) 3D initial contour obtained by region growing; (b) 3rd outer iteration (final result); (c) the result for one 2-D slice. The approach takes 34 sec. in total: 8 sec. for the continuous max-flow solver and 26 sec. for the cost computation.

2) 3D Carotid MRI Segmentation: Fig. 11 shows the segmentation of the 3D carotid artery from a T1-weighted 3D carotid MR image. The initial guess for the 3D segmentation is first computed using the region growing algorithm which is initialized by a single sample seed. The algorithm locates the final contour by 3 outer iterations (34 sec. in total: 8 sec. for the continuous max-flow solver and 26 sec. for the cost computation by Matlab). In comparison to previous methods [42], much fewer sampled seeds are necessary to input by the user, and the proposed approach can also find the correct 3D carotid artery boundary much faster.

D. Experiments with Star Shape Prior

1) Real-world Image Segmentation: The star shape prior implicitly prefers an object with a topologically simple boundary. This is not only helpful for the real-world image segmentation, but also for most medical imaging objects, such as the prostate etc. In most cases, the star shaper prior helps the initial contour locate the final boundary with much fewer outer iterations comparing to the approach without star shape prior. Fig. 12 shows the experiments of real-world image segmentation with the star shape prior, which illustrate the effectiveness of the star shape prior. In all experiments, only one outer iteration is needed to stop the contour at its final position. The details of computation time are listed below Fig. 12. Moreover, with the star shape prior, accurate image segmentation can be computed with fewer user inputs.

2) Medical Image Segmentation: Fig. 13 shows two experiments of medical image segmentation with the star shape constraint: the 2D brachial artery ultrasound (US) image segmentation and the 2D prostate US image segmentation.



(d) Iter. # 1 (e) Iter. # 1 (f) Iter. # 1

Fig. 12. Experiments of real-world image segmentation with the star shape prior. **1st Column:** the original image $(481 \times 321 \text{ pixels})$ and sampled seeds are shown in the upper image; the computed contour after 1 outer iteration and the center of star shape are shown in the bottom image (2 sec. in total: 0.9 sec. for the continuous max-flow solver and 1.1 sec. for the cost computation). **2nd Column:** the original image (481×321 pixels) and sampled seeds are shown in the upper image; the computed contour after 1 outer iteration and the center of star-shape are shown in the bottom image (1.9 sec. in total: 0.7 sec. for the continuous max-flow solver and 1.2 sec. for the cost computation). **3rd Column:** the original image (513×371 pixels) and sampled seeds are shown in the upper image; the computed contour after 1 outer iteration and the center of star-shape are shown in the bottom image (2.3 sec. in total: 0.8 sec. for the continuous max-flow solver and 1.5 sec. for the computation).



Fig. 13. Experiments of medical image segmentation with star shape prior. 1st Row: 2D brachial artery ultrasound (US) image segmentation (808×408) pixels); the sampled seeds are shown in the left side image and the initial contour starts at the foreground sampled region (green); the middle image shows the final contour computed without the star shape prior; the right side image shows the final contour (after 2 outer iterations) computed with the star shape prior; the total computation time for the star-shaped contour evolution is 4.7 sec. in total: 3.6 sec. for the continuous max-flow solver and 1.1 sec. for the cost computation. 2nd Row: 2D prostate (US) image segmentation $(617 \times 380 \text{ pixels})$; the sampled seeds is shown in the left side image and the initial contour starts at the foreground sampled region (green); the middle image shows the final contour computed without the star shape prior; the right side image shows the final contour (after 3 outer iterations) computed with the star shape prior; the total computation time for the star-shaped contour evolution is 3.6 sec. in total: 2.6 sec. for the continuous max-flow solver and 1.0 sec. for the cost computation.

The segmentation of US images is challenging due to the fact that there often exist weak image boundaries and image speckles, which bias image segmentation algorithms to the wrong position. Fig. 13(b)and Fig. 13(e) show the computation result without the star shape prior, which are clearly not the correct locations of respective organs. Fig. 13(c)and Fig. 13(f) show the computed final contour in the presence of the star

shape prior. In these experiments, the star shape constraint improves the segmentation results by accurately locating the object boundaries.

VI. CONCLUSION

In this work, we described a global optimization-based approach to the contour evolution, with or without a generic varational shape prior, using the proposed continuous maxflow framework. It provides an efficient and reliable way to gradually propagate a contour to objects of interest in images, where we show that the contour can be evolved to its globally optimal position at each discret time frame by casting it as a spatially continuous min-cut problem. The proposed continuous max-flow model provides an elegant dual perspective to the reduced continuous min-cut formulation of the contour evolution at each discrete time frame. It can be used to prove a global optimality of the discrete time contour propagation with or without the star shape prior. The variational analysis of the classical pixel-wise flow conservation constraint, i.e. the flow-in is balanced by flow-out, shows the global optimum of the proposed approach does provide a fully time implicit solver to the contour convection PDE, where a large time-step size is allowed to significantly speed up the contour evolution. We also integrate a novel variational representation of the star shape to the continuous max-flow-based scheme by simply introducing an additional spatial flow, which is applied to study the star-shaped contour evolution.

The proposed continuous max-flow models directly lead to new efficient duality-based algorithms through modern convex optimization theories, which can be easily implemented in a GPU and significantly speed up the computation. Experiment results on synthetic images, real-world images, and 2D/3D medical images show the high-performance of the proposed continuous max-flow-based contour evolution approaches in terms of efficiency and reliability to both poor initialization and large evolution step-size.

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