# A new class of incomplete Riemann solvers based on uniform rational approximations to the absolute value function

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**Abstract**

In this paper we propose a new class of incomplete Riemann solvers, based on approximations in the $L^\infty$-norm to the absolute value function in $[-1,1]$ by means of rational functions, for the numerical approximation of the solution of hyperbolic systems of conservation laws. The main idea relies on the construction of a numerical approximation to the viscosity matrix by using an appropriate rational real function $R(x)$, that approximates the function $|x|$ uniformly in $[-1,1]$, evaluated at the Jacobian of the fluxes of the hyperbolic system computed at some average value (for example, Roe averages). In addition to the Jacobians of the fluxes we shall use either the maximum in absolute value of the characteristic speeds in each cell or an upper bound of them. Thus, the resulting approximate Riemann solver is incomplete in the sense that we do not use the complete spectral decomposition of the Jacobian. Moreover, the new class of Riemann solvers consists of a hierarchy of schemes running from the more dissipative to the less dissipative ones, and having as limiting case a Roe-like scheme. According to the order of the approximation of the generating rational function used, the degree of dissipation can be dosed for particular applications. We study different rational approximations: Newman-type functions, iterative generated Halley functions, and also Chebyshev polynomial approximants. We test our basic algorithms for different initial value Riemann problems for ideal gas dynamics (HD) and magnetohydrodynamics (MHD) to observe their behavior with respect to challenging scenarios in numerical simulations, including some standard numerical pathologies (e. g., heat conduction, postshock oscillations and overheating) and the formation of compound waves in ideal MHD. We also examine our proposed schemes, by computing the numerical approximation of different initial value problems for nonconservative multilayer shallow water equations, where it has been observed that intermediate waves can be properly captured for an appropriate degree of approximation of the generating rational function used. Our numerical tests indicate that the proposed schemes are robust, running stable and accurate with a satisfactory time step restriction (CFL constant), and the computational cost is more advangeous with respect to schemes that use a complete spectral decomposition of the Jacobians.

*Keywords:* Hyperbolic systems, nonconservative products, incomplete Riemann solvers, Roe methods, Euler equations, ideal Magnetohydrodynamics, multilayer shallow water equations

## 1. Introduction

A possible way to classify the great number of approximate Riemann solvers for hyperbolic systems is to label them as *complete* or *incomplete*, depending if all the characteristic fields in the exact Riemann problem are considered or not. Among the class of complete Riemann solvers, Roe's method ([18]) is one of the most

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well-known, having been applied to a very large number of physical problems. However, in the cases in which analytic expressions for the eigenstructure are not available, Roe's scheme may be computationally expensive. For that reason, in certain situations it may be preferable to consider incomplete Riemann solvers, for which only part of the eigenstructure is needed, as Rusanov, Lax-Friedrichs, HLL, etc. In these cases, the lack of resolution of internal waves in complex scenarios may be an important drawback.

In [7] the authors introduced a family of incomplete Riemann solvers for both conservative and nonconservative hyperbolic systems denoted as PVM (Polinomial Viscosity Matrix), which are defined in terms of viscosity matrices based on polynomial evaluations of a given Roe matrix or the Jacobian of the fluxes at some other average value. The advantage of these methods relies on the fact that no spectral decomposition of the Roe matrix is needed, but only some information about the eigenvalues. An obvious consequence is that PVM methods are computationally simpler and faster than Roe's method. Besides, PVM methods could also be applied when the complete spectral structure is not known or it is hard to compute. A number of well-known schemes in the literature can be reinterpreted as PVM schemes, for example, Rusanov, Lax-Friedrichs, HLL, FORCE or GFORCE. Indeed, Roe's solver can also be viewed as a PVM method with associated polynomial related to the absolute value function; of course, the construction of such polynomial makes use of the spectral structure of the Roe matrix, so the implementation of Roe's method as a PVM scheme does not have any computational advantage.

On the other hand, stability requirements imply that the graph of the polynomial defining a PVM method must be over the graph of the absolute value function that, in a certain sense, is linked to Roe's method. Moreover, it can be observed that the behavior of a PVM scheme will be closer to that of Roe's method as its basis polynomial is closer to the absolute value function in the uniform norm. This fact suggests the idea of using accurate approximations to $|x|$ that lead to PVM schemes giving similar results to Roe's method, but with a much smaller computational cost. Following this line, a new PVM scheme is proposed in this paper, which is based on modified *Chebyshev* polynomials providing optimal uniform approximations to $|x|$.

As it is well-known ([14]), the order of approximation to $|x|$ can be greatly improved by using rational functions instead of polynomials. This leads to the core idea of this paper, that consists in following the strategy of construction of PVM methods but using proper rational functions to build the associated viscosity matrices. The resulting scheme will be denoted, of course, as RVM (Rational Viscosity Matrix). Two different families of rational approximations will be considered in this work. The first one is based on *Newman*-type approximations ([3, 4, 14]), that interpolate $|x|$ at certain properly chosen nodes. Numerical experiments show that Newman approximants of eighth-degree give as good results as Roe's method with savings of about one half of computational time. The second family of rational approximations are based on *Halley's* third-order method for finding roots. Halley-based RVM schemes, that are constructed recursively, provide a hierarchy of methods for which the amount of numerical dissipation can be estimated depending on the order of approximation of the basis rational functions. This allows the possibility, currently under investigation, of designing adaptive RVM schemes in which the order of approximation of the basis rational functions is locally determined by the degree of dissipation of the numerical fluxes. Thus, a low-order rational approximation could be applied in smooth parts of the solution, while its complex features could be computed using a higher order approximation.

It is important to point out that the presented RVM methods constitute a class of general-purpose Riemann solvers, that are constructed using a Roe matrix for the flux of the hyperbolic system and an estimate of its spectral radius, without making use of the spectral decomposition of the Roe matrix. The approximation of the viscosity matrix by means of functional evaluation of the Roe matrix, using rational uniform approximations to the absolute value function, allows to design a family of first-order schemes in which the numerical dissipation is directly related to the chosen order of approximation. Thus, RVM methods can take into account the internal waves in a more precise way that more dissipative standard schemes. As an additional advantage, no entropy-fix is needed in the presence of sonic points, as long as the rational functions do not cross the origin.

The proposed RVM first-order schemes are intended to be used as the basis for constructing higher order methods. However, that possibility is not explored here, as our purpose is to analyze the behavior of the first-order schemes in several complex scenarios related to the Euler, ideal Magnetohydrodynamics and multilayer shallow water equations, frequently appearing in the applications. Extensions to higher order methods, combined with the above mentioned adaptive strategy, will be the topic of a future work.

The paper is organized as follows. In Section 2, basic concepts regarding PVM schemes are reviewed. RVM schemes are introduced in Section 3, that constitutes the core of the paper. Several applications to the Euler and ideal Magnetohydrodynamics equations are presented in Section 4. The extension of RVM methods to nonconservative systems is explained in Section 5, and a number of tests related to multilayer shallow water equations are performed in Section 6. Finally, some conclusions are drawn in Section 7.

## 2. A review of PVM methods

In this section some basic facts about the construction of PVM methods ([7]) are reviewed, which will form the basis to introduce RVM methods in Section 3. In particular, the role of the absolute value function and its relationship with Roe's method is clarified. Furthermore, a new kind of PVM methods based on Chebyshev polynomials are introduced.

Let us consider a hyperbolic system of conservation laws,

$$\partial_t w + \partial_x F(w) = 0, \tag{2.1}$$

where $w(x,t)$ takes values on an open convex set $\mathcal{O} \subset \mathbb{R}^N$ and $F\colon \mathcal{O} \to \mathbb{R}^N$ is a smooth flux function. We are interested in the numerical solution of the Cauchy problem for (2.1) by means of a class of finite volume methods of the form

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}(D^+_{i-1/2} + D^-_{i+1/2}), \tag{2.2}$$

where $w_i^n$ denotes the approximation to the average of the exact solution at the cell $I_i = [x_{i-1/2}, x_{i+1/2}]$ at time $t^n = n\Delta t$ (in what follows, the dependence on time will be dropped unless necessary). The numerical fluxes are assumed to be defined by

$$D^{\pm}_{i+/2} = \frac{1}{2}\big(F(w_{i+1}) - F(w_i) \pm Q_{i+1/2}(w_{i+1} - w_i)\big), \tag{2.3}$$

where $Q_{i+1/2}$ denotes a numerical *viscosity matrix*. Thus, different numerical methods can be designed depending on the choice of the viscosity matrix. For example, Roe's method corresponds to

$$Q_{i+1/2} = |A_{i+1/2}|,$$

where $A_{i+1/2}$ is a Roe matrix for system (2.1).

The idea behind the PVM methods introduced in [7] is to consider viscosity matrices resulting from the polynomial evaluation of a Roe matrix $A_{i+1/2}$, that is,

$$Q_{i+1/2} = P_r^{i+1/2}(A_{i+1/2}),$$

where $P_r(x)$ is a polynomial of degree $r$. If the polynomial has the form

$$P_r^{i+1/2}(x) = \sum_{j=0}^{r} \alpha_j^{i+1/2} x^j, \tag{2.4}$$

then the numerical fluxes can be rewritten as

$$D^{\pm}_{i+1/2} = \pm\frac{\alpha_0^{i+1/2}}{2}(w_{i+1} - w_i) + \sum_{j=1}^{r} \frac{\delta_{j1} \pm \alpha_j^{i+1/2}}{2} A^{j-1}_{j+1/2}(F(w_{i+1}) - F(w_i)),$$

where $\delta_{j1}$ is Kronecker's delta.

The stability of the scheme is strongly related to the definition of the polynomial $P_r^{i+1/2}(x)$. In particular, let $\lambda_{1,i+1/2} < \lambda_{2,i+1/2} < \cdots < \lambda_{N,i+1/2}$ be the eigenvalues of $A_{i+1/2}$ and assume that an usual CFL condition holds:

$$\frac{\Delta t}{\Delta x}\max_{i,j} |\lambda_{j,i+1/2}| = \nu \leq 1. \tag{2.5}$$

Then the scheme (2.2)-(2.3) is $L^\infty$-stable if $P_r^{i+1/2}(x)$ verifies the following condition ([7]):

$$\nu \frac{\Delta x}{\Delta t} \geq P_r^{i+1/2}(x) \geq |x|, \quad \forall\, x \in [\lambda_{1,i+1/2}, \lambda_{N,i+1/2}],\ \forall\, i \in \mathbb{Z}. \tag{2.6}$$

A number of well-known schemes can be redefined as PVM schemes, e. g., Roe, Rusanov, Lax-Friedrichs, HLL, FORCE, GFORCE, etc. (see [7]).

The stability condition (2.6) states that the curve determined by the polynomial $P_r(x)$ must be over the graph of the absolute value function, which is related to Roe's method. It is also worth noticing that the best our polynomial approach to $|x|$ is, the closer the corresponding PVM scheme will be to Roe's method. Furthermore, it is interesting to construct such polynomial approximations without making use of the complete spectral structure of the problem, as Roe's method does, in order to improve the computational efficiency of the scheme.

The above considerations lead to the idea of using accurate approximations to $|x|$, resulting in PVM schemes that are close to Roe's method but with smaller computational cost. This approach could also be used in problems in which the complete spectral structure is not known or hard to compute.

In what follows, a new PVM scheme based on *Chebyshev polynomials*, which provide optimal approximations to the absolute value function, is proposed. The Chebyshev series of $|x|$, which is known to converge uniform and absolutely in the interval $[-1,1]$, is given by

$$|x| = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{4}{\pi} \frac{(-1)^{k+1}}{(2k-1)(2k+1)} T_{2k}(x), \quad x \in [-1,1],$$

where the Chebyshev polynomials of even degree $T_{2k}(x)$ are recursively defined as

$$T_0(x) = 1, \quad T_2(x) = 2x^2 - 1, \quad T_{2k}(x) = 2T_2(x)T_{2k-2}(x) - T_{2k-4}(x).$$

It is not difficult to see that the polynomial of degree $2p$ given by

$$\tau_{2p}(x) = \frac{2}{\pi} + \sum_{k=1}^{p} \frac{4}{\pi} \frac{(-1)^{k+1}}{(2k-1)(2k+1)} T_{2k}(x), \quad x \in [-1,1], \tag{2.7}$$

verifies the equality

$$\| |x| - \tau_{2p}(x) \|_\infty = \frac{2}{\pi} \frac{1}{2p+1}.$$

Thus, following the classical results of Bernstein ([1]), the order of approximation of $\tau_{2p}(x)$ to $|x|$ is optimal in the $L^\infty(-1,1)$ norm. Moreover, the recursive definition of the polynomials $T_{2k}(x)$ provides an explicit and efficient way to compute $\tau_{2p}(x)$.

Notice that $\tau_{2p}(x)$ do not strictly satisfy the stability condition (2.6): this is shown in Figure 1, where $\tau_{2p}(x)$ has been drawn for $p = 2, 3, 4$. This drawback can be avoided by substituing $\tau_{2p}(x)$ by $\tau_{2p}(x) + \varepsilon$, where $\varepsilon$ is chosen as the minimum value such that $\tau_{2p}(x) + \varepsilon$ fulfills condition (2.6) (see Section 3.1). However, in the numerical experiments performed, no differences between both approaches have been found.

Finally, in order to evaluate the Chebyshev approximation (2.7) in a given matrix $A$, the following expression is considered:

$$P_{2p}(A) = |\lambda_{\max}| \tau_{2p}\left( \frac{1}{|\lambda_{\max}|} A \right) \approx |A|, \tag{2.8}$$

where $\lambda_{\max}$ is the eigenvalue of $A$ with maximum absolute value. The corresponding PVM scheme will be denoted as PVM-Chebyshev-$2p$.

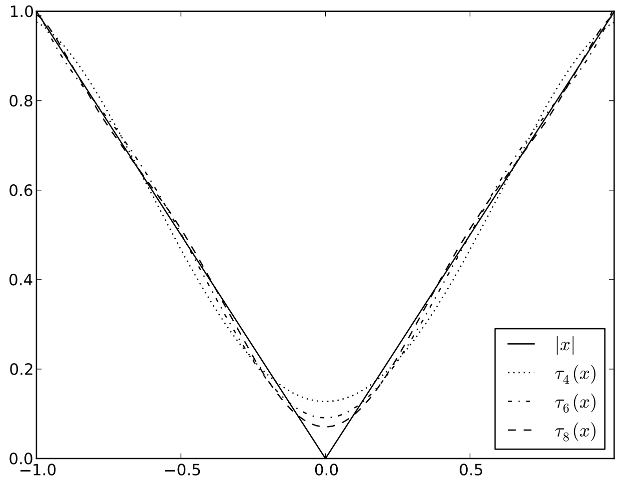


Figure 1: The Chebyshev approximations $\tau_{2p}(x)$ for $p = 2, 3, 4$.

## 3. RVM methods

This section, which constitutes the core of the paper, is dedicated to the introduction of RVM methods. As it was stated in the previous section, there exists a relationship between the quality of the polynomial approximations to the absolute value function and the behavior of the associated PVM schemes with respect to Roe's method. The basic idea behind the RVM methods consists in replacing the polynomial approximants by rational ones which, in general, give more precise approximations to the absolute value function in the uniform norm. Specifically, two families of approximats are considered, based on Newman- and Halley-type rational functions.

As it was shown by Bernstein ([1]), uniform approximations to the absolute value function by polynomials of degree $r$ are at most of order $\mathcal{O}(r^{-1})$. The Chebyshev approximations (2.7) are thus optimal in this sense. On the other hand, Newman ([14]) demonstrated that the order of approximation can be greatly improved by using rational functions. Based on this remark, a class of methods of the form (2.2)-(2.3) with viscosity matrix given by

$$Q_{i+1/2} = R^{i+1/2}(A_{i+1/2}),$$

is introduced, where $R^{i+1/2}(x)$ is a rational approximation to $|x|$. The resulting scheme will be called a RVM (Rational Viscosity Matrix) method. Under the CFL condition (2.5), the $L^\infty$-stability of a RVM method is assured by the condition

$$\nu \frac{\Delta x}{\Delta t} \geq R^{i+1/2}(x) \geq |x|, \quad \forall\, x \in [\lambda_{1,i+1/2}, \lambda_{N,i+1/2}],\ \forall\, i \in \mathbb{Z}, \tag{3.1}$$

as the proof of stability remains valid here (see [7]). Of course, the modification (2.8) is applied for matricial evaluation of $R^{i+1/2}(x)$.

Two different ways of constructing $R^{i+1/2}(x)$ are proposed here, the first one based on Newman-type functions and the second one relying on iterative processes. Both possibilities are further explored in the following sections.

*3.1. RVM schemes based on Newman approximations*

For a given $r \geq 4$, consider a set of distinct points in $(0,1]$, $X = \{0 < x_1 < \cdots < x_r \leq 1\}$, and construct the polynomial

$$p(x) = \prod_{k=1}^{r} (x + x_k).$$

The *Newman rational function* associated to the set $X$ is then defined by

$$R_r(x) = x\,\frac{p(x) - p(-x)}{p(x) + p(-x)}.$$

It is easy to see that $R_r(x)$ interpolates $|x|$ at the points $\{-x_r, \ldots, -x_1, 0, x_1, \ldots, x_r\}$. Also notice that for even $r$ the numerator and denominator of $R_r(x)$ are of degree $r$.

The uniform rate of approximation of $R_r(x)$ to $|x|$ depends on the choice of the set of nodes $X$. The three following possibilities will be considered here:

- $x_k = \xi^k$, where $\xi = \exp(-r^{-1/2})$. This choice corresponds to Newman's original definition ([14]) and provides an exponential rate of approximation, of the form $\mathcal{O}(\exp(-c\sqrt{r}))$, $c > 0$.
- $x_k = \cos(\pi(2k-1)/(4r))$, which are the Chebyshev nodes in $[0,1]$. As it was demonstrated in [4], the exact order of approximation is $\mathcal{O}(1/r \log r)$.
- $x_k = \sin^2(\pi(2k-1)/(4r))$, the adjusted Chebyshev nodes. These points are obtained by adjusting the Chebyshev roots $\cos(\pi(2k-1)/(2r))$ in $[-1,1]$ to the interval $[0,1]$. As it was proved in [3], the order of approximation is $\mathcal{O}(r^{-2})$ in this case.

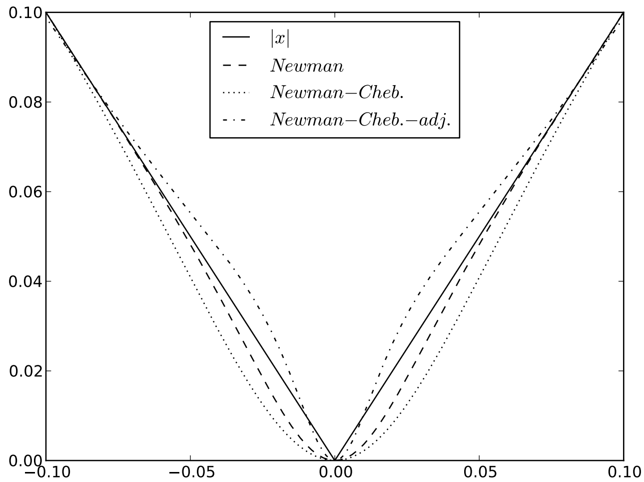


Figure 2: behavior of the Newman approximations $R_8(x)$ for $x \in [-0.1, 0.1]$.

The three types of Newman approximations are shown in Figure 2 for $r = 8$ (only values corresponding to $x \in [-0.1, 0.1]$ have been plotted, as the differences with respect to $|x|$ are not noticeable in the picture outside that interval). As the stability condition (3.1) is not fulfilled in any case, a *modified approximation* of the form $R_r^{\varepsilon}(x) = R_r(x) + \varepsilon$ should be considered instead. The value of $\varepsilon$ computed as

$$\varepsilon = \max\{|R_r(x^*) - |x^*||\colon R_r'(x^*) = 1,\ x^* \in [0,1]\}$$

guarantees that $R_r^\varepsilon(x)$ satisfies (3.1). A comparison between $R_r(x)$ and $R_r^\varepsilon(x)$ can be seen in Figure 3. The differences between using $R_r(x)$ or $R_r^\varepsilon(x)$ are particularly noticeable in the presence of sonic points: in this case, $R_r^\varepsilon(x)$ should be used to avoid entropy-violating solutions (see Section 6.1). On the other hand, when sonic points are not present in the solution, no significative differences between using $R_r(x)$ or $R_r^\varepsilon(x)$ in the RVM scheme have been detected in the numerical tests.

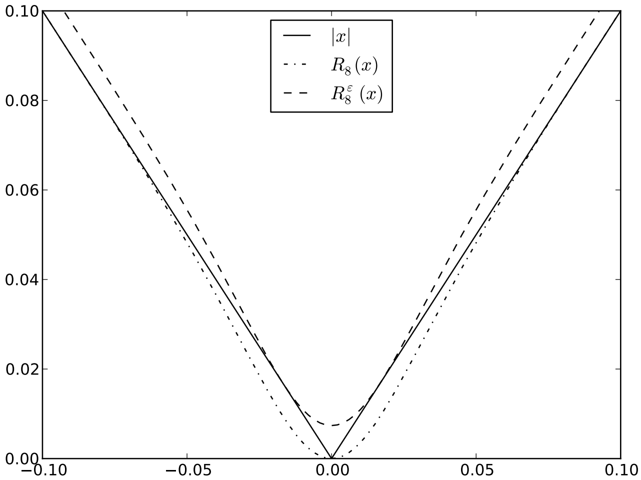


Figure 3: Comparison between $R_8(x)$ and $R_8^\varepsilon(x)$ for $x \in [-0.1, 0.1]$, using the Newman nodes. In this case $\varepsilon \approx 7.37e-3$.

The RVM method corresponding to a Newman-type approximation based on $R_r^\varepsilon(x)$ with Newman's nodes will be denoted as RVM-Newman-$r$.

### *3.2. RVM schemes based on iterative approximations*

Given a point $\bar{x} \in [-1, 1]$, its absolute value $|\bar{x}|$ can be viewed as the positive root of $f(x) = x^2 - \bar{x}^2$. Thus, it is possible to approximate $|\bar{x}|$ by means of an iterative method for finding roots, such as Newton's method. A more precise choice is given by the cubic *Halley's method*, which is defined as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k) - \dfrac{1}{2} f''(x_k) \dfrac{f(x_k)}{f'(x_k)}}.$$

In our particular case, the method has the following form:

$$x_{k+1} = x_k \, \frac{x_k^2 + 3\bar{x}^2}{3x_k^2 + \bar{x}^2}.$$

Taking $x_0 = 1$ as initial guess, Halley's method is well-defined and converges to $\bar{x}$ (see [5]). Moreover, the following estimate holds:

$$|x_{k+1} - \bar{x}| \leq \sum_{j=k+1}^{\infty} \frac{2}{3^{j+1}}. \tag{3.2}$$

The Halley rational approximations to $|x|$ are recursively defined as

$$H_{r+1}(x) = H_r(x) \, \frac{H_r(x)^2 + 3x^2}{3H_r(x)^2 + x^2}, \quad H_0(x) = 1. \tag{3.3}$$

Notice that the degrees of the numerator and denominator of $H_r(x)$ are both equal to $3^r - 1$.

It can be easily verified that $H_r(x)$ satisfies the stability condition (3.1) without further modifications. Besides, (3.2) and the fact that the maximum error occurs at $x = 0$ imply the equality

$$\|H_r(x) - |x|\|_\infty = \frac{1}{3^r},$$

which gives a measure on how precise the approach to $|x|$ is. Figure 4 shows the functions $H_r(x)$ for $r = 3, 4, 5$.

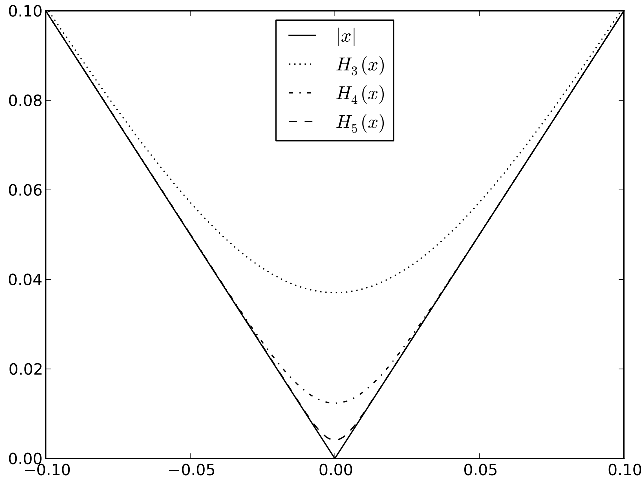


Figure 4: Halley rational approximations $H_r(x)$ in the interval $[-0.1, 0.1]$, for $r = 3, 4, 5$.

The RVM method associated to the Halley approximation $H_r(x)$ will be denoted as RVM-Halley-$r$.

## 4. Applications to the Euler and ideal Magnetohydrodynamics equations

In this section we test the performances of the RVM schemes introduced in Section 3 when they are applied to some challenging Riemann problems related to the Euler and ideal magnetohydrodynamics equations.

The one-dimensional ideal magnetohydrodynamics (MHD) system of equations has the following form:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v_x) = 0, \\ \partial_t(\rho v_x) + \partial_x(\rho v_x^2 + P^* - B_x^2) = 0, \\ \partial_t(\rho v_y) + \partial_x(\rho v_x v_y - B_x B_y) = 0, \\ \partial_t(\rho v_z) + \partial_x(\rho v_x v_z - B_x B_z) = 0, \\ \partial_t B_x = 0, \\ \partial_t B_y + \partial_x(v_x B_y - v_y B_x) = 0, \\ \partial_t B_z + \partial_x(v_x B_z - v_z B_x) = 0, \\ \partial_t E + \partial_x\big(v_x(E + P^*) - B_x(v_x B_x + v_y B_y + v_z B_z)\big) = 0, \end{cases} \tag{4.1}$$

where $\rho$ represents the mass density, $(v_x, v_y, v_z)$ and $(B_x, B_y, B_z)$ are the velocity and magnetic fields, and $E$ is the total energy. If $q$ and $B$ denote the magnitudes of the velocity and magnetic fields, the total energy can be expressed as

$$E = \frac{1}{2}\rho q^2 + \frac{1}{2}B^2 + \rho\varepsilon,$$

where the specific internal energy $\varepsilon$ is related to the hydrostatic pressure $P$ through the equation of state $P = (\gamma - 1)\rho\varepsilon$, $\gamma$ being the adiabatic constant. The total pressure $P^*$ is then defined as $P + P_M$, where $P_M = \frac{1}{2}B^2$ is the magnetic pressure. Notice that system (4.1) can be written in the form (2.1) with

$$w = \begin{pmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ B_x \\ B_y \\ B_z \\ E \end{pmatrix}, \qquad F(w) = \begin{pmatrix} \rho v_x \\ \rho v_x^2 + P^* - B_x^2 \\ \rho v_x v_y - B_x B_y \\ \rho v_x v_z - B_x B_z \\ 0 \\ v_x B_y - v_y B_x \\ v_x B_z - v_z B_x \\ v_x(E + P^*) - B_x(v_x B_x + v_y B_y + v_z B_z) \end{pmatrix}.$$

Let us define $(b_x, b_y, b_z) = (B_x, B_y, B_z)/\sqrt{\rho}$, $b^2 = b_x^2 + b_y^2 + b_z^2$, and the acoustic sound speed $a = \sqrt{\gamma P/\rho}$. The Alfven speed is given by $c_a = |b_x|$ and the fast and slow waves, $c_f$ and $c_s$, are defined as

$$c_{f,s}^2 = \frac{1}{2}\big(a^2 + b^2 \pm \sqrt{(a^2+b^2)^2 - 4a^2 b_x^2}\big).$$

The eight characteristic velocities of system (4.1) are then

$$\lambda_1 = u - c_f,\ \lambda_2 = u - c_a,\ \lambda_3 = u - c_s,\ \lambda_4 = \lambda_5 = u,\ \lambda_6 = u + c_s,\ \lambda_7 = u + c_a,\ \lambda_8 = u + c_f,$$

where the characteristic fields associated to $\lambda_{1,8}$, $\lambda_{3,6}$, $\lambda_{2,7}$ and $\lambda_{4,5}$ are called, respectively, the fast, slow, Alfven and entropy waves. The spectral structure of system (4.1) is further analyzed in [2, 19]; in particular, the system admits a complete set of eigenvectors.

A Roe matrix for system (4.1) was originally presented in [2] for the case $\gamma = 2$. Instead, the extension introduced in [6] is considered here, as it is valid for arbitrary values of $\gamma$. This Roe matrix will be used in order to construct PVM and RVM schemes in the numerical experiments that follow. In particular, Roe's method will be compared with PVM-Chebyshev-8 (based on an eighth degree polynomial), RVM-Newman-8 (based on the quotient of eighth degree polynomials; for the ease of implementation, their coefficients are presented in the Appendix) and RVM-Halley-$r$ for $r = 1, 2, 3, 4, 5$ (based on the quotiens of polynomials with degrees 2, 8, 26, 80 and 242 respectively, which are recursively defined). Moreover, comparisons with the HLL and FORCE methods will also be presented.

On the other hand, the Euler equations of gas dynamics can be directly obtained from (4.1) assuming that the magnetic field vanishes: $B_x = B_y = B_z = 0$. The flexibility of the PVM and RVM schemes allows to solve the Euler equations using the same MHD code by simply setting the magnetic field to zero in the initial conditions. This is the form in which the tests involving the Euler equations have been performed.

### *4.1. Numerical heat conduction in a stationary contact discontinuity*

The purpose of this test is to measure the effect of the *numerical heat conduction* (NHC) in the proposed RVM schemes. The NHC measures the amount of numerical difussion in contact discontinuities, which may cause erroneous heating across the discontinuity.

The following initial conditions for the Euler equations are considered:

$$(\rho, v_x, P) = \begin{cases} (1, 0, 1) & \text{for } x \le 0.5, \\ (2, 0, 1) & \text{for } x > 0.5, \end{cases}$$

and $\gamma = 1.4$. The solution of this problem consists of a stationary contact discontinuity located at $x = 0.5$. This test was first proposed in [12]; see also [11] for a discussion on the topic of NHC.

The problem has been solved in the interval $[0, 1]$ using 200 grid points, with $\Delta t/\Delta x = 0.4$, until two different final times $t = 1$ (short) and $t = 4$ (long). The results are shown in Figure 5. By design, Roe's method perfectly captures the solution, as it has zero dissipation in this case. As it can be observed, the NHC for RVM-Newman-8 and RVM-Halley-4 methods is considerably smaller than for HLL or FORCE methods. Moreover, the PVM-Chebyshev-8 scheme also provides quite good results.

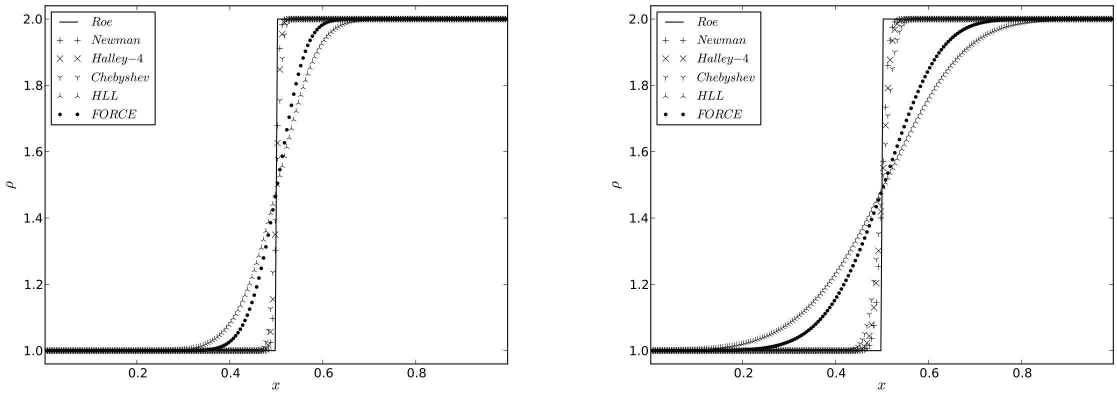


Figure 5: Density smoothing due to the NHC mechanism. Left: $t = 1$; right: $t = 4$.

### 4.2. *Overheating error in two colliding slabs*

Following [11, 12], the following initial conditions for the Euler equations are considered:

$$(\rho, v_x, P) = \begin{cases} (1, 1, 0.1) & \text{for } x \leq 0.5, \\ (1, -1, 0.1) & \text{for } x > 0.5, \end{cases}$$

and $\gamma = 5/3$. In this case, the original jump in velocity generates two shock waves that propagate in opposite directions from the center of the domain, while the gas remains at rest in between. Numerically, a pathology known as *overheating* occurs on most standard schemes: there is an error in the density around the shock point that is compensated by an excessive value of the internal energy, thus leading to an overheating of the fluid. The overheating error is of the order of $\mathcal{O}(1)$ independently of the discretization.

Figure 6 shows the density component calculated at time $t = 0.4$ in the interval $[0, 1]$ using 200 grid points and $\Delta t/\Delta x = 0.1$. Both RVM-Halley-4 and RVM-Newman-8 methods provide quite accurate results, while HLL or FORCE schemes do not properly capture the behavior of the solution around the shock point.

The problem becomes harder for lower values of the pressure (cold gas). The results obtained for $P = 10^{-3}$ are drawn in Figure 7. Similar comments as in the previous case apply here.

### 4.3. *A slowly moving shock wave*

A well-known deficiency of most Godunov-type schemes is the generation of unphysical oscillations downstream nearly stationary shocks, that cannot be effectively damped by the dissipation mechanism of the applied scheme (see [11] for a discussion).

An example of slowly moving shock is generated by the following Riemann problem for the Euler equations, proposed in [17]:

$$(\rho, v_x, P) = \begin{cases} (3.86, -0.81, 10.33) & \text{for } x \leq 0.1, \\ (1, -3.44, 1) & \text{for } x > 0.1, \end{cases}$$

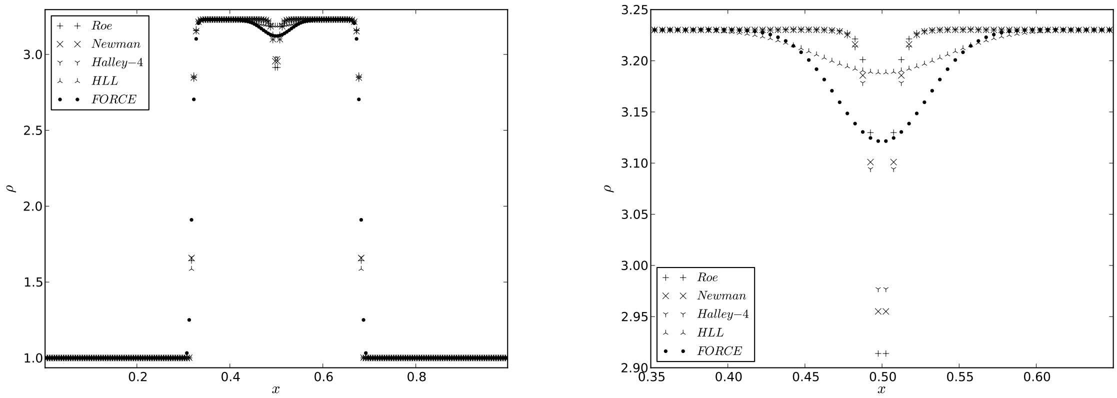


Figure 6: Collision of two equal strength shocks with $P = 0.1$. Left: density; right: zoom of the collision zone.

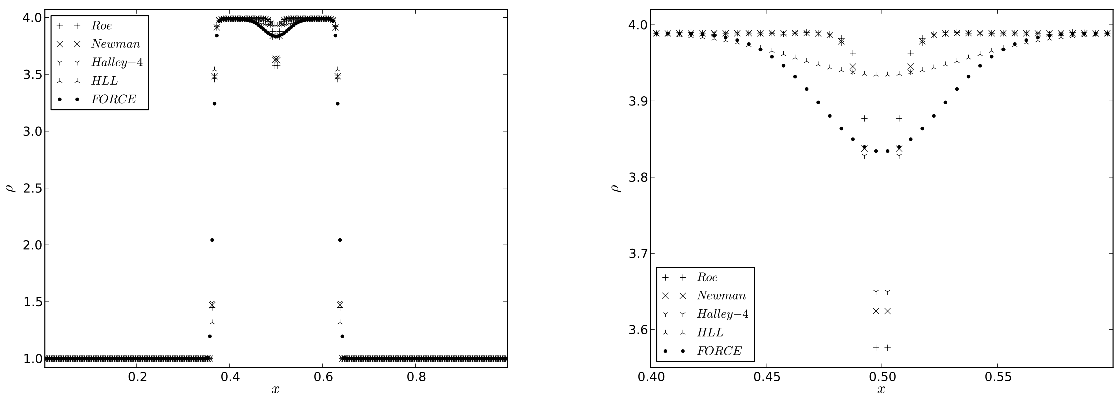


Figure 7: Collision of two equal strength shocks with $P = 10^{-3}$. Left: density; right: zoom of the collision zone.

with $\gamma = 1.4$. The problem has been solved in the interval $[0, 1]$ using 200 grid points and $\Delta t/\Delta x = 0.1$. The results obtained at time $t = 4$ are presented in Figure 8. As it can be noticed, the amplitude of the oscillations behind the shock are more pronounced for the less dissipative methods, that is, Roe's, RVM-Newman-8 and RVM-Halley-4. On the contrary, the FORCE scheme provides a better resolution of the shock.

Figure 9 shows the behavior of several RVM-Halley-$r$ methods on the top part of the shock. It can be observed that the amplitude of the oscillations grows as $r$ increases; indeed, for $r = 1$ the oscillations dissapear. This fact suggests the idea of designing adaptive RVM-Halley methods that control the amount of dissipation in terms of the order of approximation of the base rational functions. This idea is currently under investigation.

### *4.4. Brio-Wu shock tube problem*

Consider the Riemann problem for the MHD system (4.1) with initial data

$$(\rho, v_x, v_y, v_z, B_x, B_y, B_z, P) = \begin{cases} (1, 0, 0, 0, 0.75, 1, 0, 1) & \text{for } x \leq 0, \\ (0.125, 0, 0, 0, 0.75, -1, 0, 0.1) & \text{for } x > 0, \end{cases}$$

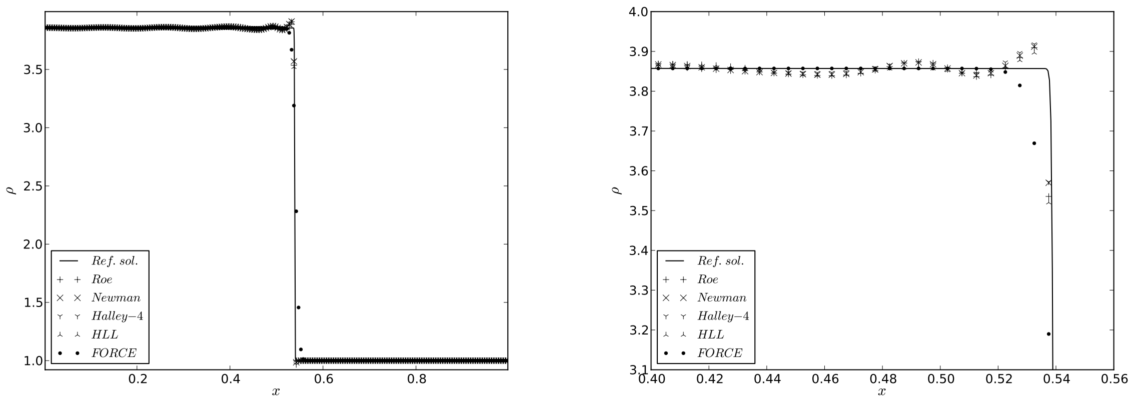


Figure 8: A slowly moving shock wave. Left: density; right: zoom of the top part of the shock.

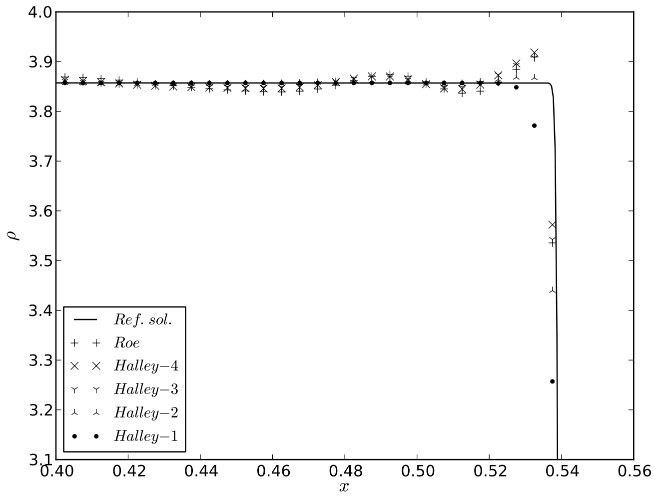


Figure 9: Slowly moving shock: zoom of the top part of the shock computed with RVM-Halley-$r$ for $r = 1, 2, 3, 4$.

and $\gamma = 2$. This test was proposed in [2] to show the formation of a compound wave consisting of an intermediate shock followed by a slow rarefaction wave. For each variable, the solution consists of five constant states separated by a left-moving fast rarefaction wave, a slow compound wave, a contact discontinuity, a right-moving slow shock and a right-moving fast rarefaction wave.

The problem has been solved until time $t = 0.2$ in the interval $[-1, 1]$ with 800 grid points and CFL number 0.8. It is found that the best results are provided by the RVM-Newman-8 scheme. The results can be seen in Figure 10, where the reference solution has been computed using Roe's method with 20000 points. Figure 11 shows the approximations to the density compound wave obtained with several methods. A comparison between the RVM-Halley-$r$ methods for $r = 1, 2, 3, 4, 5$ is presented in Figure 12. Finally, relative CPU times are shown in Table 1, where Roe's method has been taken as reference time.

It can be concluded that the best results are obtained with the RVM-Newman-8 method, being comparable to those calculated with Roe's scheme, followed by PVM-Chebyshev-8 and RVM-Halley-2 (in the case

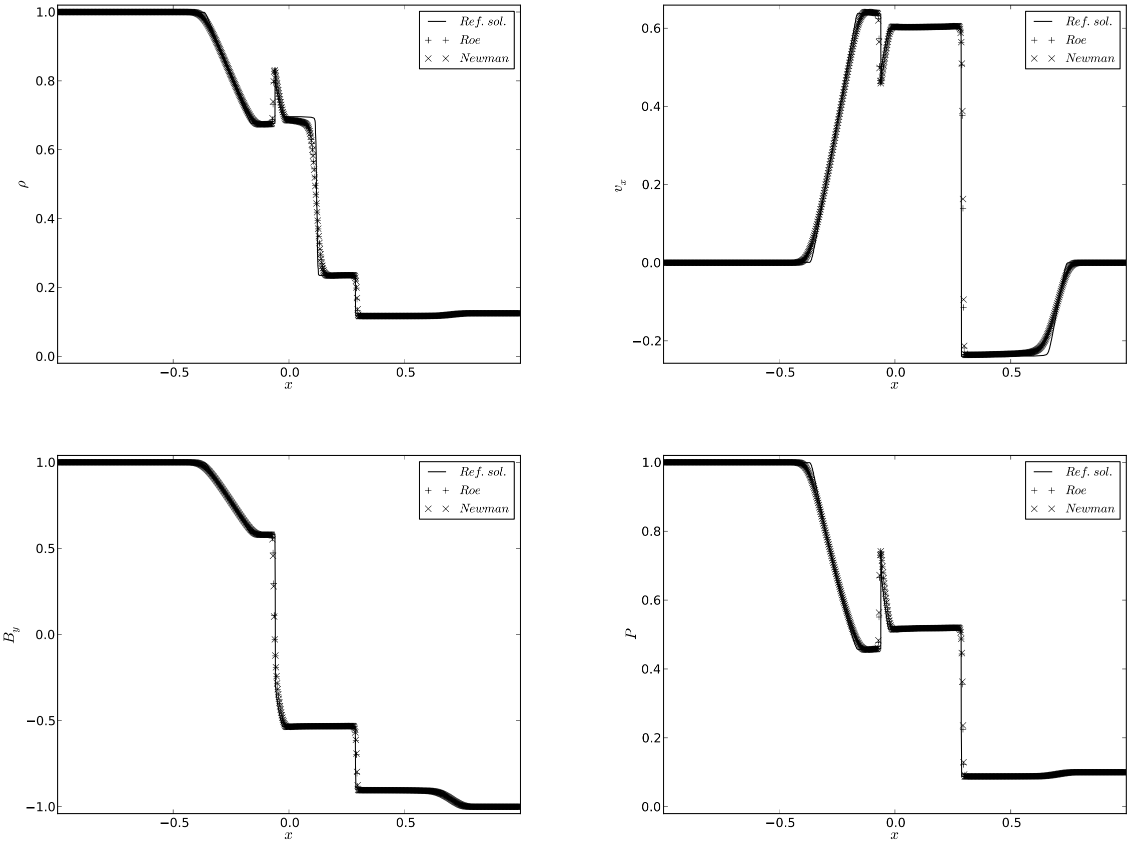


Figure 10: Solutions of the Brio-Wu shock tube problem 4.4. From top to bottom: $\rho$, $v_x$, $B_y$ and $P$.

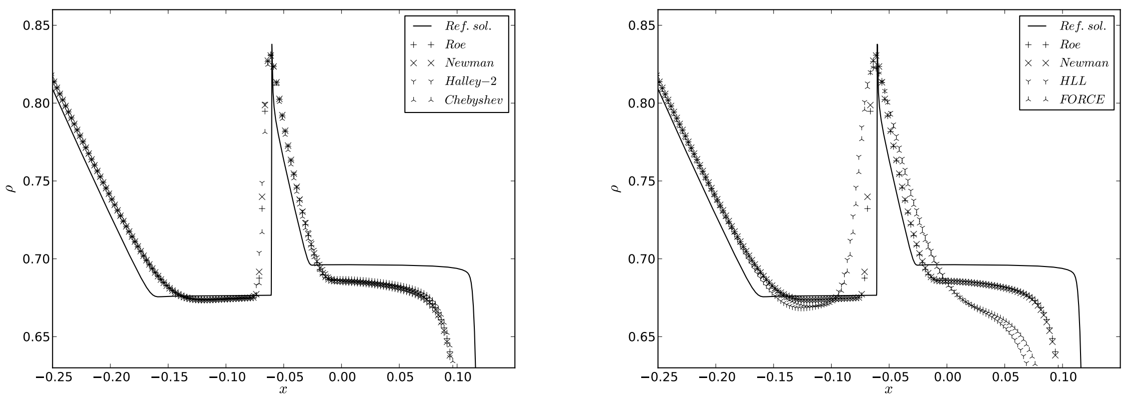


Figure 11: Closer view of the compound wave in test 4.4.

of RVM-Halley methods, no improvements in the solutions are seen for greater values of $r$: see Figure 12),

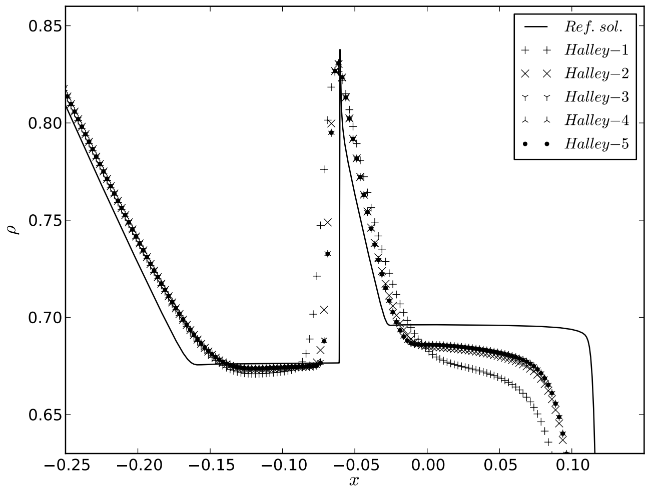


Figure 12: Density compound wave in test 4.4: comparison of the solutions obtained with the RVM-Halley-$r$ methods for $r = 1, 2, 3, 4, 5$.

| Methods | Relative CPU times |
|---|---|
| Roe | 1.0 |
| RVM-Newman-8 | 0.38 |
| RVM-Halley-1 | 0.31 |
| RVM-Halley-2 | 0.48 |
| RVM-Halley-3 | 0.65 |
| RVM-Halley-4 | 0.82 |
| RVM-Halley-5 | 1.0 |
| PVM-Chebyshev-8 | 0.27 |
| HLL | 0.08 |
| FORCE | 0.1 |

Table 1: Relative CPU times with respect to Roe's method for test problem 4.4.

which provide similar results. In any case, the solutions are much better than those computed with the HLL and FORCE schemes.

### *4.5. High Mach shock tube problem*

This problem was presented in [2] to test the robustness of the numerical schemes for high Mach number flows. The initial conditions are

$$(\rho, v_x, v_y, v_z, B_x, B_y, B_z, P) = \begin{cases} (1, 0, 0, 0, 0, 1, 0, 1000) & \text{for } x \leq 0, \\ (0.125, 0, 0, 0, 0, -1, 0, 0.1) & \text{for } x > 0, \end{cases}$$

and we take $\gamma = 2$. The Mach number of the right-moving shock is 15.5. The problem has been solved in $[-1, 1]$ using 200 grid points, CFL coefficient 0.8 and final time $t = 0.012$. The results are plotted in Figure 13. Again, a reference solution computed using Roe's method with 20000 points has been considered.

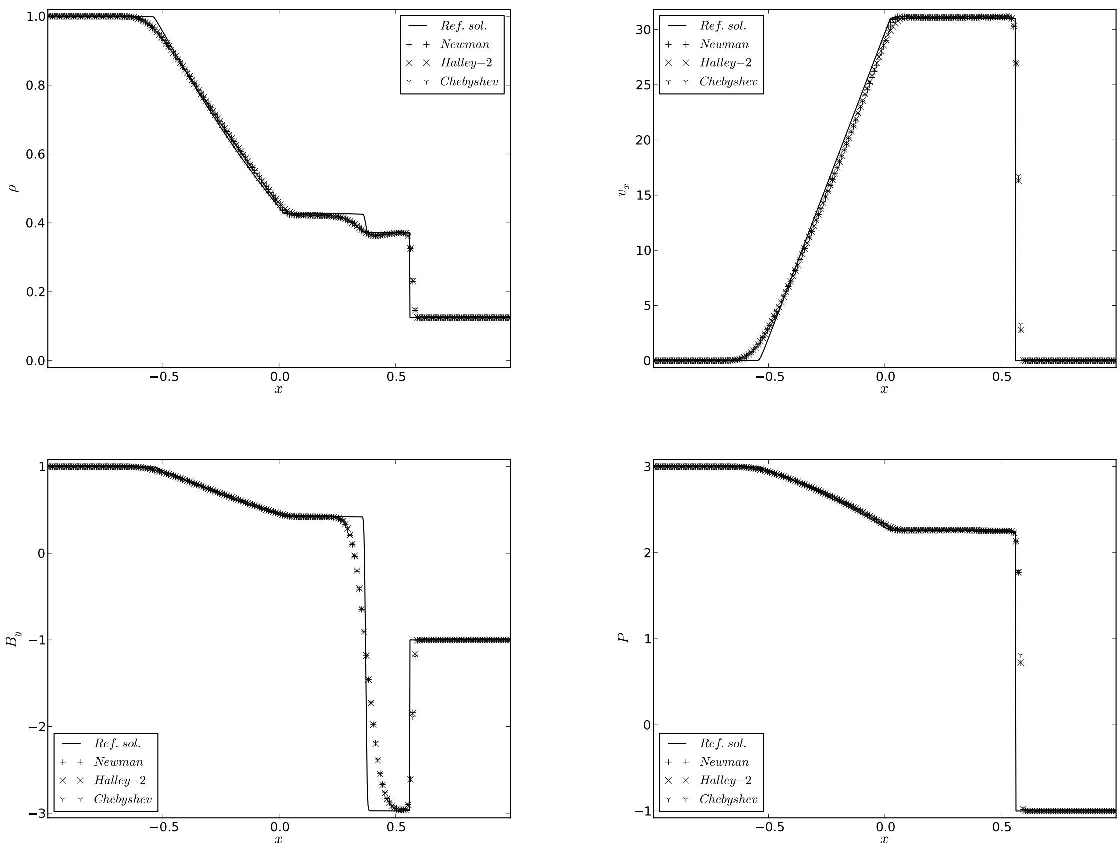


Figure 13: Results for the high Mach shock tube problem 4.5. From top to bottom: $\rho$, $v_x$, $B_y$ and $P$.

*4.6. Non-planar Riemann problem*

A non-planar Riemann problem with solution containing two strong rotational waves was proposed in [20]. The initial conditions are given by

$$(\rho, v_x, v_y, v_z, B_x, B_y, B_z, P) = \begin{cases} (1.7, 0, 0, 0, 1.1, 1, 0, 1.7) & \text{for } x \leq 0, \\ (0.2, 0, 0, 1.4968909, 1.1, \cos\beta, \sin\beta, 0.2) & \text{for } x > 0, \end{cases}$$

where $\beta = 2.3$. Notice that although the problem has an unique solution, the initial conditions are close to initial conditions for which the problem admits non-unique solutions (see [20]). Figure 14 shows the solution computed in the interval $[-1, 1.5]$ with 800 grid points, CFL number 0.8, $\gamma = 5/3$ and final time $t = 0.4$.

## 5. The nonconservative case

In this section it is shown how the RVM schemes can be extended to systems of conservation laws including source terms and nonconservative products, following the guidelines in [7]. The inspiring idea consists in writing them as general nonconservative hyperbolic systems to which the theory of path-conservative schemes ([15]) is applied, thus providing an automatic treatment of the source and nonconservative terms.

Let us consider a hyperbolic system of conservation laws with source terms and nonconservative products,

$$\partial_t w + \partial_x F(w) + B(w)\partial_x w = G(w)\partial_x H, \tag{5.1}$$

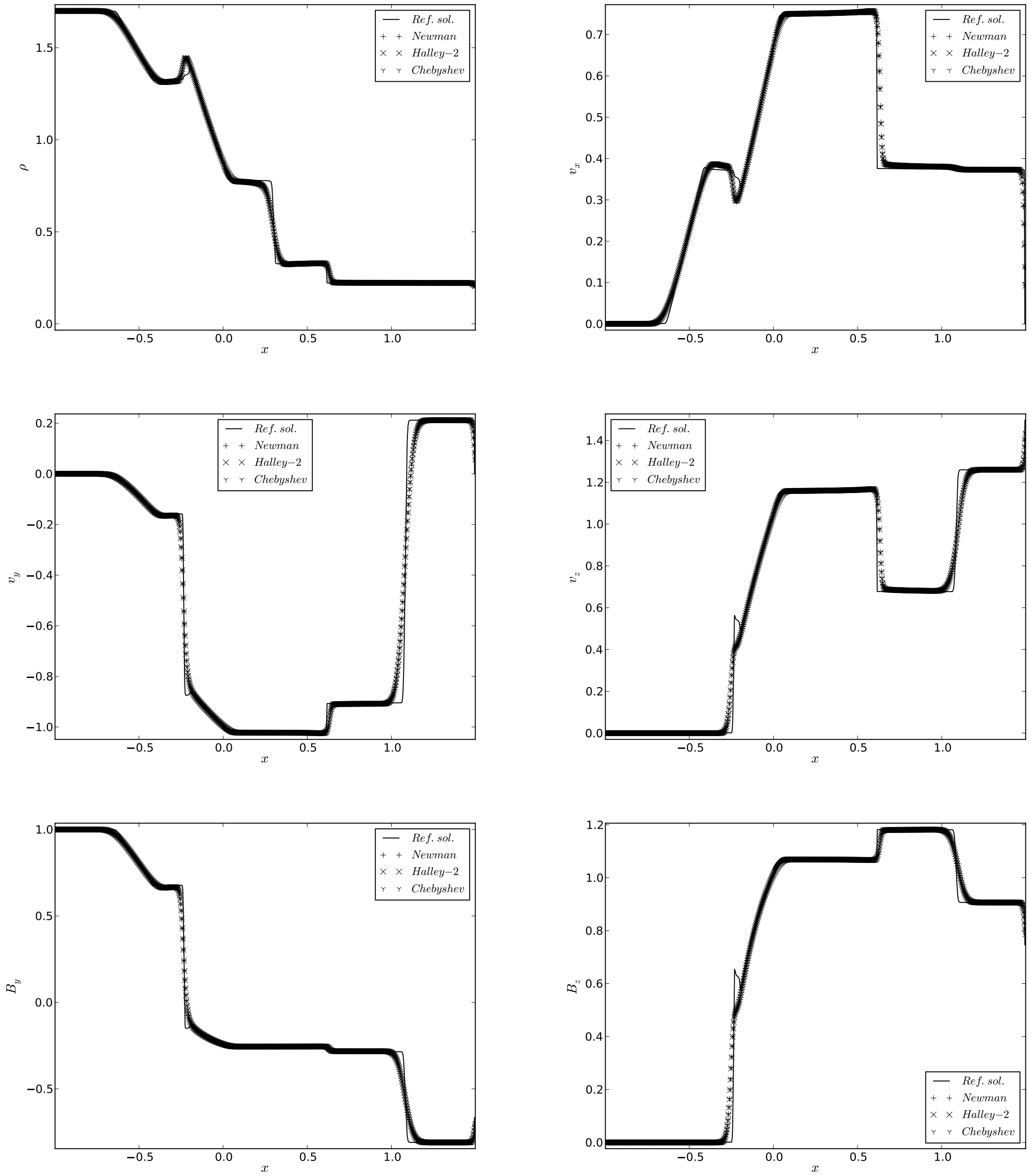


Figure 14: Results for the non-planar Riemann problem 4.6. From top to bottom: $\rho$, $v_x$, $v_y$, $v_z$, $B_y$ and $B_z$.

where $w(x,t)$ takes values on an open convex set $\mathcal{O} \subset \mathbb{R}^N$, $F \colon \mathcal{O} \to \mathbb{R}^N$ is a smooth flux function, $B \colon \mathcal{O} \to \mathcal{M}_N(\mathbb{R})$ is a smooth matricial function, and $G \colon \mathcal{O} \to \mathbb{R}^N$ and $H \colon \mathbb{R} \to \mathbb{R}$ are given functions. Introducing the trivial equation $\partial_t H = 0$, system (5.1) can be rewritten as

$$\partial_t W + \mathcal{A}(W)\partial_x W = 0, \tag{5.2}$$

where $W$ is the augmented vector

$$W = \begin{pmatrix} w \\ H \end{pmatrix} \in \Omega = \mathcal{O} \times \mathbb{R} \subset \mathbb{R}^{N+1}$$

and the matrix $\mathcal{A}(W)$ is given by

$$\mathcal{A}(W) = \begin{pmatrix} A(w) & -G(w) \\ 0 & 0 \end{pmatrix},$$

with $A(w) = J(w) + B(w)$, $J(w)$ being the Jacobian $\frac{\partial F}{\partial w}(w)$.

The definition of weak solutions for system (5.2) strongly depends on the choice of a Lipschitz family of paths $\Phi(s; W_L, W_R)$ joining arbitrary states $W_L$ and $W_R$ in the phase space $\Omega$. The interested reader is referred to [13] for a complete presentation of the related theoretical issues. For a chosen family of paths $\Phi$, the concept of Roe matrix can be extended to that of *Roe linearization* ([21]), which is a function $\mathcal{A}_\Phi : \Omega \times \Omega \to \mathcal{M}_{N+1}(\mathbb{R})$ verifying:

- $\mathcal{A}_\Phi(W_L, W_R)$ has $N+1$ distinct real eigenvalues, for every $W_L, W_R \in \Omega$.
- $\mathcal{A}_\Phi(W, W) = \mathcal{A}(W)$, for each $W \in \Omega$.
- For any $W_L, W_R \in \Omega$,

$$\mathcal{A}_\Phi(W_L, W_R)(W_R - W_L) = \int_0^1 \mathcal{A}(\Phi(s; W_L, W_R)) \frac{\partial \Phi}{\partial s}(s; W_L, W_R)\, ds.$$

Following [16], a Roe linearization for system (5.2) can be constructed as

$$\mathcal{A}_\Phi(W_L, W_R) = \begin{pmatrix} A_\Phi(W_L, W_R) & -G_\Phi(W_L, W_R) \\ 0 & 0 \end{pmatrix}.$$

where:

- $A_\Phi(W_L, W_R) = \mathcal{L}(w_L, w_R) + B_\Phi(W_L, W_R)$, $\mathcal{L}(w_L, w_R)$ being a Roe matrix for the flux $F$ in the usual sense, that is,

$$\mathcal{L}(w_L, w_R)(w_R - w_L) = F(w_R) - F(w_L).$$

- $B_\Phi(W_L, W_R)$ is a matrix verifying

$$B_\Phi(W_L, W_R)(W_R - W_L) = \int_0^1 B(\Phi(s; W_L, W_R)) \frac{\partial \Phi_w}{\partial s}(s; W_L, W_R)\, ds,$$

  where $\Phi = (\Phi_w, \Phi_H)^t$.

- $G_\Phi(W_L, W_R)$ is a vector satisfying

$$G_\Phi(W_L, W_R)(H_R - H_L) = \int_0^1 G(\Phi(s; W_L, W_R)) \frac{\partial \Phi_H}{\partial s}(s; W_L, W_R)\, ds.$$

The choice of a Roe linearization $\mathcal{A}_\Phi$ allows to define a finite volume scheme for solving (5.2):

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}(\mathcal{D}^+_{i-1/2} + \mathcal{D}^-_{i+1/2}), \tag{5.3}$$

with numerical fluxes given by

$$\mathcal{D}^\pm_{i+1/2} = \frac{1}{2}(\mathcal{A}_\Phi(W_i^n, W_{i+1}^n) \pm \mathcal{Q}_\Phi(W_i^n, W_{i+1}^n)). \tag{5.4}$$

The viscosity matrix $\mathcal{Q}_\Phi(W_L, W_R)$ is defined as

$$\mathcal{Q}_\Phi(W_L, W_R) = \begin{pmatrix} Q_\Phi(W_L, W_R) & -Q_\Phi(W_L, W_R)A_\Phi^{-1}(W_L, W_R)G_\Phi(W_L, W_R) \\ 0 & 0 \end{pmatrix},$$

where $Q_\Phi(W_L, W_R)$ is a numerical viscosity matrix for $A_\Phi(W_L, W_R)$. Notice that (5.3)-(5.4) is a *path-conservative* scheme as defined in [15].

Returning to the original unknown $w$, the numerical scheme can be written as

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}(D^+_{i-1/2} + D^-_{i+1/2}), \tag{5.5}$$

with numerical fluxes given by

$$\begin{aligned} D^\pm_{i+1/2} = \frac{1}{2}\big(&F(w_{i+1}) - F(w_i) + B_{i+1/2}(w_{i+1} - w_i) - G_{i+1/2}(H_{i+1} - H_i) \\ &\pm Q_{i+1/2}(w_{i+1} - w_i - A^{-1}_{i+1/2}G_{i+1/2}(H_{i+1} - H_i)\big), \end{aligned} \tag{5.6}$$

where $C_{i+1/2} = C_\Phi(W_i, W_{i+1})$ for $C \in \{B, G, A, Q\}$ (again, time dependence has been dropped). Notice that the term

$$Q_{i+1/2}A^{-1}_{i+1/2}G_{i+1/2}(H_{i+1} - H_i)$$

in (5.6) can be interpreted as the upwinding part of the source term discretization, and it has no sense if some eigenvalue of $A_{i+1/2}$ vanishes. A way to deal with this kind of resonant problems has been proposed in [7].

The definition of PVM schemes for system (5.1) follows the same guidelines as in the conservative case (Section 2), that is, the viscosity matrix $Q_{i+1/2}$ is taken as

$$Q_{i+1/2} = P_r^{i+1/2}(A_{i+1/2}),$$

where $P_r^{i+1/2}(x)$ is a polynomial of degree $r$ satisfying the stability conditon (2.6). If $P_r^{i+1/2}(x)$ has the form (2.4), the numerical fluxes can be written as

$$\begin{aligned} D^\pm_{i+1/2} = &\pm \frac{\alpha_0^{i+1/2}}{2}(w_{i+1} - w_i - A^{-1}_{i+1/2}G_{i+1/2}(H_{i+1} - H_i)) \\ &+ \sum_{j=1}^r \frac{\delta_{j1} \pm \alpha_j^{i+1/2}}{2} A^{j-1}_{j+1/2}(F(w_{i+1}) - F(w_i) + B_{i+1/2}(w_{i+1} - w_i)) \\ &- \sum_{j=1}^r \frac{\delta_{j1} \pm \alpha_j^{i+1/2}}{2} A^{j-1}_{j+1/2}G_{i+1/2}(H_{i+1} - H_i). \end{aligned}$$

It is worth noticing that the different choices of $P_r^{i+1/2}(x)$ considered in Section 2 provide natural extensions of Roe, Rusanov, Lax-Friedrichs, HLL, FORCE and GFORCE schemes to the nonconservative case.

The extension of RVM schemes to the nonconservative case is completely analogous, just choosing

$$Q_{i+1/2} = R^{i+1/2}(A_{i+1/2}),$$

where $R^{i+1/2}(x)$ is a given rational function satisfying (3.1). Both Newman and Halley rational approximations will be considered here.

## 6. Applications to multilayer stratified shallow flows

The behavior of RVM schemes in the nonconservative case is tested in this section, where the multilayer shallow water equations have been considered as a representative model including both source and nonconservative coupling terms.

The equations governing a multilayer stratified shallow flow are given by ([10])

$$\begin{cases} \partial_t h_j + \partial_x q_j = 0, \\ \partial_t q_j + \partial_x \left( \dfrac{q_j^2}{h_j} + \dfrac{g}{2} h_j^2 \right) + g h_j \partial_x \big( z_b + \displaystyle\sum_{k>j} h_k + \sum_{k<j} \dfrac{\rho_k}{\rho_j} h_k \big) = 0, \end{cases}$$

for $j = 1, \ldots, m$, where $m$ is the number of layers; $h_j$ denotes the fluids depths; $q_j = h_j u_j$ are the discharges, $u_j$ being the velocities; $z_b(x)$ represents the topography; $g$ is the gravity constant; $\rho_j$ denotes the density at the $m$-th layer, with $0 < \rho_1 \leq \cdots \leq \rho_m$. Notice than that index $j = 1$ corresponds to the upper layer and $j = m$ to the lower one.

The system can be written in the nonconservative form (5.1) by taking

$$w = (w_1, \ldots, w_m)^t, \quad w_j = (h_j, q_j)^t, \quad F(w) = (F_1, \ldots, F_m)^t, \quad F_j = \left( q_j, \frac{q_j^2}{h_j} + \frac{g}{2} h_j^2 \right)^t,$$

$$G(w) = (g h_1, \ldots, g h_m)^t, \quad H = H_{\text{ref}} - z_b(x),$$

where $H_{\text{ref}}$ is a constant reference height; the matrix $B(w)$ has components $B_{ij}(w)$ defined, for $i, j = 1, \ldots, 2m$, by

$$B_{ij}(w) = \begin{cases} \dfrac{\rho_j}{\rho_i} g h_i & \text{for } j = 1 + 2k, \ k = 0, \ldots, i-2, \\ g h_i & \text{for } j = 1 + 2k, \ k = i, \ldots, m-1, \\ 0 & \text{otherwise.} \end{cases}$$

### *6.1. Stationary transcritical flux with shock for the bilayer shallow water system*

The purpose of this test is to study the steady-state convergence to a solution with a shock for the bilayer shallow water system. The initial condition consists in an internal dam-break problem over a non-flat bottom defined by

$$z_b = \frac{1}{2} e^{-x^2}, \quad x \in [-5, 5].$$

Specifically, initial conditions $q_1(x, 0) = q_2(x, 0) = 0$ and

$$h_1(x, 0) = \begin{cases} 0.48 & \text{for } x < 0, \\ 0.5 & \text{for } x \geq 0, \end{cases} \qquad h_2(x, 0) = 1 - h_1(x, 0) - z_b(x),$$

have been taken. The stationary state is reached by imposing open wall boundary conditions. The ratio of densities of this experiment has been chosen as $\rho_1 / \rho_2 = 0.99$.

The numerical solutions have been computed until time $t = 100$ using 200 grid points and CFL number 0.9. Table 2 shows the relative CPU times with respect to Roe's method. Figure 15 shows the results obtained with Roe's and RVM-Newman-8 methods, where the reference solution has been computed using Roe's method with 3200 points. A comparison between RVM-Newman-8, HLL and FORCE schemes is shown in Figure 16; as it can be seen, both HLL and FORCE schemes do not resolve the shock properly. A closer view of the shock computed with several methods is presented in Figure 17.

It should be noticed that, as no entropy-fix has been applied, Roe's method introduces an artificial shock at the critical point; this also happens if $R_r(x)$ is used instead of $R_r^\varepsilon(x)$ in the RVM-Newman-8 scheme (see Section 3.1). On the contrary, the transition is well resolved by PVM-Chebyshev-8, RVM-Newman-8 and RVM-Halley-$r$ methods. As it happened in the experiments concerning the equations of ideal MHD (sections 4.4-4.6), the best results are obtained with the RVM-Newman-8 scheme. On the other hand, in the present case it is necessary to consider RVM-Halley-$r$ methods with $r \geq 4$ to achieve satisfactory results: see Figure 18.

| Methods | Relative CPU times |
|---|---|
| Roe | 1.0 |
| RVM-Newman-8 | 0.55 |
| RVM-Halley-1 | 0.46 |
| RVM-Halley-2 | 0.67 |
| RVM-Halley-3 | 0.89 |
| RVM-Halley-4 | 1.10 |
| RVM-Halley-5 | 1.32 |
| PVM-Chebyshev-8 | 0.41 |
| HLL | 0.16 |
| FORCE | 0.18 |

Table 2: Relative CPU times with respect to Roe's method for test problem 6.1.

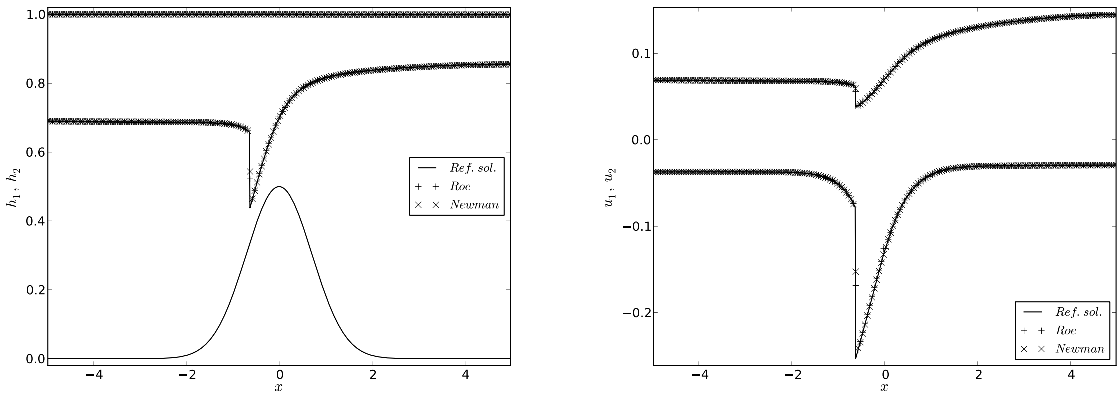


Figure 15: Results for test 6.1. Left: free surface, interface and bottom; right: velocities.

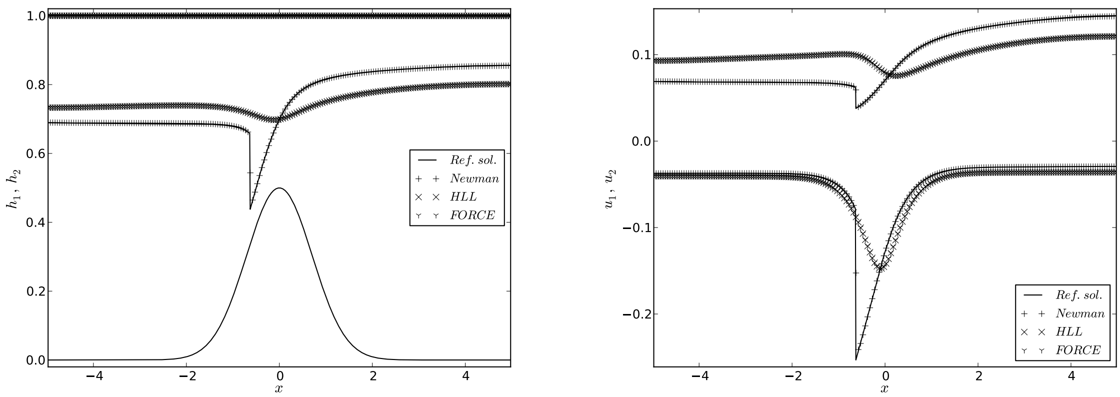


Figure 16: Solutions of test 6.1: comparison between RVM-Newman-8, HLL and FORCE.

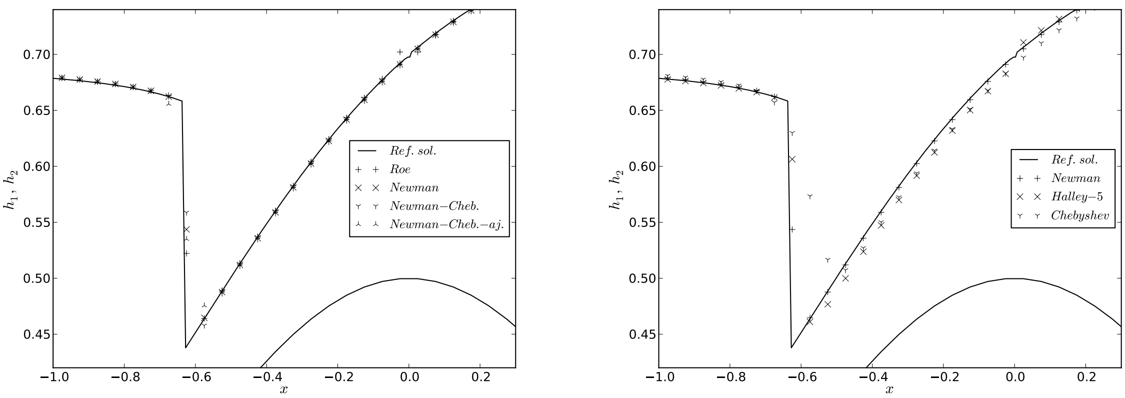


Figure 17: Closer view of the shock at the interface in test 6.1. Left: comparison between three versions of RVM-Newman; right: comparison between PVM-Chebyshev-8, RVM-Newman-8 and RVM-Halley-5.

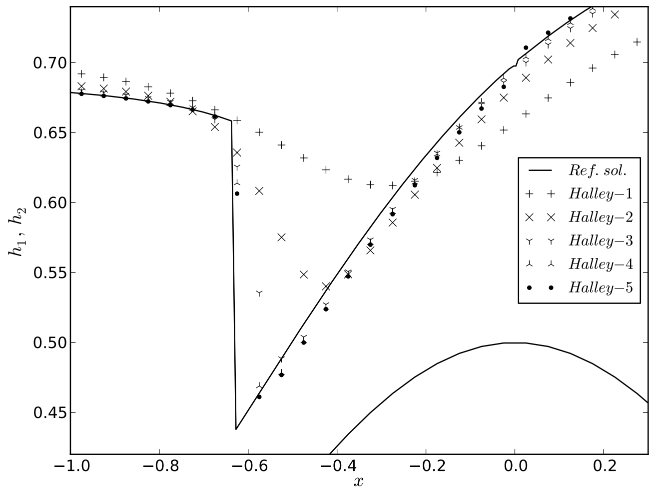


Figure 18: Closer view of the shock in test 6.1: solutions obtained with the RVM-Halley-$r$ schemes for $r = 1, 2, 3, 4, 5$.

### *6.2. Internal dam break for a multilayer shallow water system*

In this test a double internal dam-break problem for the four-layer ($m = 4$) model is considered. The initial conditions are given by

$$h_1(x,0) = \begin{cases} 0.9 & \text{for } x < 0, \\ 0.1 & \text{for } x \geq 0, \end{cases} \quad h_2(x,0) = 1 - h_1(x,0), \quad h_3(x,0) = h_1(x,0), \quad h_4(x,0) = h_2(x,0),$$

and $q_1(x,0) = q_2(x,0) = q_3(x,0) = q_4(x,0) = 0$, for $x \in [-5,5]$. Open wall boundary conditions have been imposed. The ratios of densities have been taken as $\rho_1/\rho_4 = 0.85$, $\rho_2/\rho_4 = 0.9$ and $\rho_3/\rho_4 = 0.95$. The results can be directly compared with those presented in [10], where this test was proposed.

Figure 19 shows the solutions obtained at $t = 5$ with 200 grid points and CFL number 0.9. As it can be seen, the best results are obtained with RVM-Newman-8 and RVM-Halley-5 schemes, being comparable to

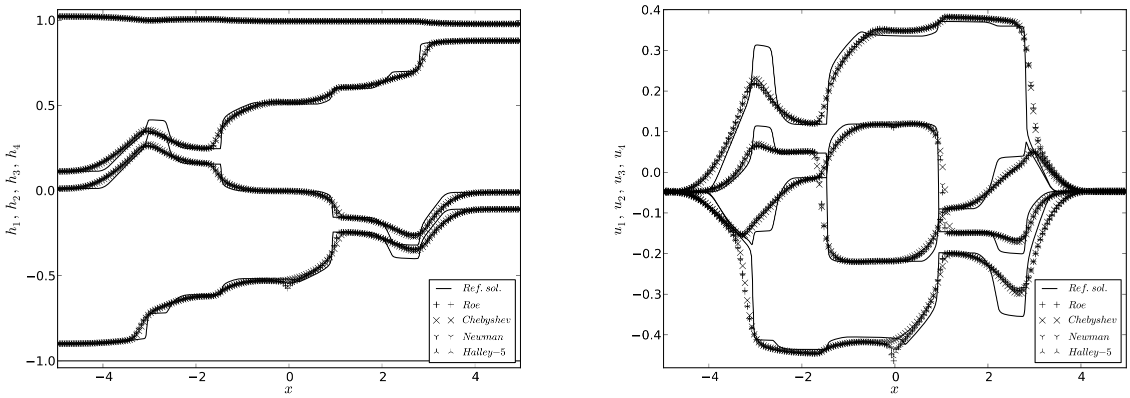


Figure 19: Test 6.2. Left: free surface and interfaces; right: velocities.

those produced by Roe's method except at $x = 0$, where the latter presents a small oscillation at the lower interface near $x = 0$. The relative CPU times with respect to Roe's method are 0.8 and 0.3 for, respectively, RVM-Halley-5 and RVM-Newman-8.

## 7. Conclusions

A new family of first-order Riemann solvers for general conservative and nonconservative hyperbolic systems has been introduced. These methods, denoted as RVM, are defined in terms of viscosity matrices computed by functional evaluations of the Jacobian of the fluxes at some average value (e. g., Roe averages), using rational uniform approximations to the absolute value function in $[-1, 1]$. In addition to the Jacobians of the fluxes, only the maximum in absolute value of the characteristic speeds in each cell or an upper bound of them is needed. Thus, the resulting approximate Riemann solver is incomplete in the sense that we do not use the complete spectral decomposition of the Jacobian. Moreover, no entropy-fix is needed for treating with sonic points.

The new class of RVM Riemann solvers consists of a hierarchy of schemes ranging from the more dissipative to the less dissipative ones, and having as limiting case a Roe-like scheme. Depending on the order of the approximation of the generating rational function used, the degree of dissipation can be dosed for particular applications. Two different types of RVM schemes have been proposed, based on Newman-type and iteratively generated Halley-type rational approximations to the absolute value function. Moreover, a method based on Chebyshev polynomial approximations has also been considered.

Different initial value Riemann problems for ideal gas dynamics and magnetohydrodynamics have been considered to examine the behavior of RVM schemes with respect to challenging scenarios in numerical simulations, including some standard numerical pathologies (e. g., heat conduction, postshock oscillations and overheating) and the formation of compound waves in ideal MHD. On the other hand, numerical approximations of several initial value problems for nonconservative multilayer shallow water equations have been carried out. It has been observed that intermediate waves can be precisely captured for an appropriate degree of approximation of the generating rational function used. The numerical tests indicate that the proposed schemes are robust, running stable and accurate with a satisfactory time step restriction, and the computational cost is more advangeous with respect to schemes that use a complete spectral decomposition of the Jacobians. Thus, RVM methods provide a serious alternative to Roe's scheme when approximating time-dependent solutions in which the spectral decomposition is computationally expensive. Furthermore, following the ideas in [8, 9] it is possible to use RVM methods as basis for constructing higher order methods for multidimensional systems. This will be the topic of a research work in progress.

## Appendix

The RVM-Newman-8 method has been applied in the numerical experiments. For the ease of implementation, the explicit form of the function $R_8^\varepsilon(x)$ is detailed in this appendix.

A simple calculation shows that the rational function $R_8(x)$ can be written as

$$R_8(x) = \frac{a_8x^8 + a_6x^6 + a_4x^4 + a_2x^2}{x^8 + b_6x^6 + b_4x^4 + b_2x^2 + b_0}.$$

If the original definition of the nodes is considered (see Section 3.1), then

$$\begin{aligned} a_8 &= 3.15936173596092 & b_6 &= 4.00790208450847 \\ a_6 &= 2.66037513232789 & b_4 &= 1.00920540531312 \\ a_4 &= 0.223933399698289 & b_2 &= 0.0283967795936465 \\ a_2 &= 0.0018842014579903 & b_0 &= 0.0000502000298516861 \end{aligned}$$

while the value of the parameter $\varepsilon$ is given by

$$\varepsilon = 0.0073705383650891.$$

On the other hand, if the Chebyshev nodes are used then

$$\begin{aligned} a_8 &= 5.10114861868916 & b_6 &= 11.0108586149772 \\ a_6 &= 13.0528938096911 & b_4 &= 9.21524750769325 \\ a_4 &= 3.91883716338631 & b_2 &= 0.961801777180106 \\ a_2 &= 0.120551892275778 & b_0 &= 0.00552427172801991 \end{aligned}$$

with

$$\varepsilon = 0.0125760117893106.$$

Finally, the following values correspond to the adjusted Chebyshev nodes:

$$\begin{aligned} a_8 &= 4.0 & b_6 &= 6.5 \\ a_6 &= 5.5 & b_4 &= 2.578125 \\ a_4 &= 0.65625 & b_2 &= 0.08203125 \\ a_2 &= 0.00390625 & b_0 &= 0.000030517578125 \end{aligned}$$

and

$$\varepsilon = 0.00203846963093366.$$

In practice, however, the differences found between the three versions of the method are not noticeable.

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