A unifying retinex model based on non-local differential operators

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ABSTRACT

In this paper, we present a unifying framework for retinex that is able to reproduce many of the existing retinex implementations within a single model. The fundamental assumption, as shared with many retinex models, is that the observed image is a multiplication between the illumination and the true underlying reflectance of the object. Starting from Morel's 2010 PDE model for retinex, where illumination is supposed to vary smoothly and where the reflectance is thus recovered from a hard-thresholded Laplacian of the observed image in a Poisson equation, we define our retinex model in similar but more general two steps.

First, look for a filtered gradient that is the solution of an optimization problem consisting of two terms: The first term is a sparsity prior of the reflectance, such as the TV or H1 norm, while the second term is a quadratic fidelity prior of the reflectance gradient with respect to the observed image gradients. In a second step, since this filtered gradient almost certainly is not a consistent image gradient, we then look for a reflectance whose actual gradient comes close.

Beyond unifying existing models, we are able to derive entirely novel retinex formulations by using more interesting non-local versions for the sparsity and fidelity prior. Hence we define within a single framework new retinex instances particularly suited for texture-preserving shadow removal, cartoon-texture decomposition, color and hyperspectral image enhancement.

Keywords: Retinex, non-local operators, reflectance, illumination normalization, contrast enhancement, dynamic range compression, shadow detection, shadow removal, cartoon-texture decomposition

1. INTRODUCTION

The relative robustness of the human visual system with respect to challenging illumination is remarkable. In many modern imaging and vision applications a comparable invariance to the conditions of illumination would be very desirable. Popular examples are the white balance correction in consumer photo cameras or contrast enhancement in high dynamic range imaging. More challenging examples involve illumination invariance in recognition and remote sensing tasks.

Retinex is a theory on the human visual perception, introduced and pioneered mainly by Edwin Land.¹⁻³ It was an attempt to explain how the human visual system, as a combination of processes supposedly taking place both in the retina and the cortex, is capable of adaptively coping with illumination that varies spatially both in intensity and color.

The fundamental observation is the insensitivity of human visual perception with respect to a slowly varying illumination on a Mondrian-like scene. Indeed, the underlying true reflectance ratio can be recovered by multiplying all intensity ratios at the sharp patch transitions along a path connecting these two patches. Land and McCann have built electronic circuits that reproduce this behavior.²

Today, in image processing, the retinex theory has been implemented in various different flavors, each particularly adapted to specific tasks, including color balancing, contrast enhancement, dynamic range compression and shadow removal in consumer electronics and imaging, bias field correction in medical imaging or even illumination normalization, e.g. for face detection. Depending on the application, the various retinex assumptions are given different importance and resulting implementations vary significantly.

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Here, we start by providing a short review of some of the many retinex flavors that currently exist, in section 2. We will then in section 3 recall some definitions and notions from non-local differential operators, and introduce a few new concepts. Based on these non-local differential operators both kernel-based and center-surround retinex can be expressed as variational models. From there, we will propose a unifying, non-local framework for retinex in section 4. Our proposed model takes the shape of a generalized fidelity to thresholded-gradient problem, and we will show how this can reproduce results of other state-of-the-art retinex models in section 5. Beyond simply reproducing existing results, we will explore some of the new degrees of freedom of the proposed unifying framework, as shown in section 6. We will discuss and conclude our framework in section 7.

2. A SHORT REVIEW OF RETINEX IMPLEMENTATIONS

2.1 Original Retinex algorithm

Land formalized the reflectance ratios, by summing thresholded log-ratios over continuous paths between two pixels.⁴ He defines the relative reflectance of pixel i to j as:

$$R(i,j) = \sum_{k} \delta_{\tau} \log \frac{I_{k+1}}{I_k} \tag{1}$$

where δ_{τ} denotes hard thresholding. The average relative reflectance at *i* is then estimated as

$$\bar{R}(i) = \mathbb{E}_{j}[R(i,j)] = \frac{1}{N} \sum_{j=1}^{N} R(i,j)$$
(2)

However, "the ultimate purpose is to describe any area by relating its reflectance to a single, standard, high reflectance somewhere in the Mondrian or to several equally high reflectances".² Instead of localizing the highest reflectance in a preprocessing step, which seemed biologically unplausible, it was proposed to estimate the maximum reflectance directly while performing the sequential sum along each path. Indeed, whenever the intermediate sequential sum from *j* up to I_{k+1} , i.e. the relative reflectance of I_{k+1} to *j*, becomes positive—equivalent to a sequential product bigger than 1—, one has reached a new maximum reflectance, and the sequential sum is reset as of there, with I_{k+1} as new reference. Due to the presence of the thresholding operator, the final reference pixel does not necessarily coincide with the brightest pixel along the path. For a mathematical definition and analysis of this reset mechanism, see Ref. 5.

There has been quite some debate in literature about the respective role and importance of both threshold and reset in the Retinex, including McCann himself.⁶ The only consensus seems to be that there is no consensus, and we do not wish to enter this debate here. The criterion will nonetheless serve us dividing the many retinex implementations in two broad classes: threshold-based versus reset-based. A third class of implementations is based on an alternative technique proposed by Land as well, which determines lightness as ratio of the local intensity compared to the average intensity of its immediate (circular) surroundings, without neither thresholding, nor reset.⁷ A forth class, finally, extracts the reflectance and illumination information variationally, by optimizing different energy functionals.

2.2 Threshold-based Retinex implementations (PDE)

In 1974, Horn proposed a mathematical alternative to the Retinex algorithm that differs substantially in form.⁸ He essentially stripped the Retinex algorithm down to a smoothness prior on the illumination field, and thus to a thresholding on intensity derivatives. He poses the problem of recovering the underlying reflectance R, which multiplied by the illumination B resulted in the observed intensity I:

$$I(x,y) = B(x,y)R(x,y)$$
(3)

$$i(x,y) = b(x,y) + r(x,y)$$
 (4)

where $i = \log(I)$ etc. Since the illumination *b* is supposed to be varying smoothly, the spatial derivatives of the observed intensity are mostly due to edges in the reflectance *r*. However, he realized that first order derivatives are directional in the two-dimensional case of images, and that the lowest order isotropic derivatives are found in the scalar Laplacian operator:

 Δb will be finite everywhere, while Δr will be zero except at each edge separating regions.⁸ Therefore, discarding the finite parts of the observed intensity Laplacian is supposed to yield the Laplacian of the reflectance (Poisson equation):

$$\Delta r = \delta_{\tau} \Delta i \tag{5}$$

A tight mathematical connection between Land's and Horn's computations, on the basis of Green's formula, has been shown in work by Hurlbert.⁹ A fully discrete alternative to Horn's convolution and inversion scheme was proposed by Marr.¹⁰ Very recently, Horn's model has been strongly backed up by a much more recent paper by Morel,¹¹ where the authors show a very tight connection between Horn's Laplacian thresholding and Land's original, resetless Retinex algorithm. Indeed, "*if* the Retinex paths are interpreted as symmetric random walks, then Retinex is equivalent to a Neumann problem for a linear Poisson equation".¹¹ The main difference between Horn and Morel concerns the argument of the hard thresholding operator: while Horn thresholds the scalar Laplacian, Morel thresholds the components of the gradient prior to computing their divergence. De facto, Morel thus effectively solves an L2-gradient fitting problem:

$$\hat{r} = \arg\min\left\{\|\nabla r - \delta_{\tau} \nabla i\|_{2}^{2}\right\}$$
(6)

We refer to this model as L2-retinex. Note that reconstruction from thresholded gradient has earlier been proposed by Blake.¹²⁻¹⁴ More recently, the L1-equivalent thresholded gradient-fidelity Retinex has been proposed: The *L1*-retinex minimizes the isotropic L1-distance.¹⁵

2.3 Reset-based Retinex implementations (Random walk)

Moving away from thresholding and relying purely on the reset mechanism, Frankle and McCann have patented their Retinex algorithm.¹⁶ The Frankle-McCann algorithm replaces sequential products along paths by pairwise pixel ratios sampled along discrete spirals. Long-distance interactions are computed first, then the sampling progressively approaches the center pixel while decreasing the spacing. At each step, the lightness estimate is updated with a ratio-product-reset-average operation.¹⁷ More recent variants of the algorithm mainly involve multiresolution image pyramids,^{17,18} different sampling patterns,^{19,20} or ratio modifiers.²¹ Provenzi et al. replace the path-based sampling pattern by a repeated sampling through random sprays.²² Indeed, if the threshold is removed from the Retinex formulation, then the reset reduces the relative reflectance, computed using a specific path, to the ratio of central pixel and brightest pixel along that same path.⁵ Therefore, many paths become redundant, and the maxima can be sampled more efficiently.

Beyond, the (white-patch) random spray retinex was combined with a (gray-world) model used for automatic color equalization (ACE).^{23,24} Eventually, the random spray sampling was replaced by a kernel, representing the sampling density of the random spray in the limit case:^{25,26}

$$R(i) = \sum_{j:I(j) \ge I(i)} w(i,j) f\left(\frac{I(i)}{I(j)}\right) + \sum_{j:I(j) < I(i)} w(i,j)$$
(7)

where w(i, j) is the kernel, representing the probability density of picking a pixel j in the neighborhood of i.²⁵ Note that, here again, we find the ratio modifier f previously introduced by Sobol.²¹

2.4 Center-surround Retinex implementations

A simple alternative to threshold/reset based Retinex algorithms was proposed by Land based on findings of lateral inhibition.⁷ The alternative consists in determining the local lightness (reflectance) as the ratio between local intensity and an average of its close surroundings. The fundamental idea is again that the low-frequency components are due to illumination, while the high-frequency details are features in the reflectance.

Only 10 years later, the idea was picked up and formulated as single- and multi-scale center-surround Retinex.^{27–29} The single-scale retinex is given by

$$R(i) = \log I(i) - \log \left[F * I\right](i) \tag{8}$$

where F is a Gaussian kernel, and multi-scale retinex is simply the combination of different single-scale Retineces. Changing the order of log and Gaussian convolution in the single scale retinex amounts to homomorphic filtering

$$R(i) = i(i) - [F * i](i)$$
(9)

which in turn can be identified as a special case of (resetless) kernel-Retinex, with the kernel $w(i, j) \equiv F$ and ratio modifier $f \equiv \log$:

$$R(i) = \sum_{j} w(i,j) \log\left(\frac{I(i)}{I(j)}\right) = i(i) - \sum_{j} w(i,j)i(j)$$
(10)

2.5 Variational Retinex

A whole family of variational Retinex models handles the regularity priors on the reflectance and illumination parts of the Retinex decomposition in a more explicit way. First, the variational framework by Kimmel introduces competing Gaussian smoothness priors on both the illumination and reflectance fields, as well as a quadratic fidelity prior between illumination and observed intensity:³⁰

$$\min_{b} \left\{ \int_{\Omega} |\nabla b|^2 + \alpha (b-i)^2 + \beta |\nabla b - \nabla i|^2 dx dy \right\} \quad \text{s.t.} \quad b \ge i, \quad \langle \nabla b, \vec{n} \rangle = 0 \text{ on } \partial\Omega.$$
(11)

Here, we rewrite the problem slightly, optimizing for the reflectance rather than the illumination, by substituting according to the coherence condition i = b + r:

$$\min_{r} \left\{ \|\nabla r - \nabla i\|_{2}^{2} + \alpha \|r\|_{2}^{2} + \beta \|\nabla r\|_{2}^{2} \right\} \quad \text{s.t.} \quad r \le 0, \quad \langle \nabla r, \vec{n} \rangle = 0 \text{ on } \partial\Omega.$$
(12)

This form makes clear that variational Retinex is an optimization between reflectance gradient fidelity and some sparsity penalties.

Subsequently, variations of this variational Retinex model have been proposed, mainly involving different norms for the fidelity and sparsity terms, and dropping the asymmetry constraint $r \le 0$. First, Ma and Osher have dropped a few terms and replace Gaussian smoothness of the reflectance by a TV-prior.³¹ As a complication, instead of the local TV prior, they also make use of non-local total variation. Further, Ng and Wang introduce an L2-fidelity prior between reflectance and intensity.³² Chen et al. have used a TV-L1-based variational Retinex approach, which they call logarithmic total variation (LTV), for illumination normalized face detection.³³ At this point it is worthwile noting, that both the L2- and L1-Retinex^{11,15} have a threshold-free variational equivalent. Indeed, the hard threshold on the intensity gradient can be seen as a contraction of an L0-sparsity prior on the gradients of the reflectance:

$$\min_{r} \left\{ \|\nabla r - \delta_{\tau} \nabla i\|_{2}^{2} \right\} = \min_{r} \left\{ \left\| \nabla r - \arg\min_{\vec{q}} \left\{ \|\vec{q} - \nabla i\|_{2}^{2} + \tau^{2} \|\vec{q}\|_{0} \right\} \right\|_{2}^{2} \right\}$$
(13)

which is a relaxed version of the more complicated problem

$$\min_{r} \left\{ \|\vec{q} - \nabla i\|_{2}^{2} + \tau^{2} \|\vec{q}\|_{0} \right\} \quad \text{s.t.} \quad \nabla r = \vec{q}$$
(14)

We will make use of this relaxation to retro-fit other variational models into a threshold based Poisson-problem.

3. NON-LOCAL DIFFERENTIAL OPERATORS (BASIC DEFINITIONS)

In this section, we recall and give a few definitions of non-local differential operators,³⁴ which we need in order to cast existing kernel-based Retinex methods into a variational framework, and based on which we will propose our unifying Retinex framework.

3.1 Products and norms

DEFINITION 3.1. To begin with, we require appropriate inner products. For scalars $i: \Omega \to \mathbb{R}$, we choose:

$$\langle i,j \rangle := \int_{\Omega} i(x)j(x)dx,$$
 (15)

which is the common L2 inner product. Accordingly, we introduce the following inner product for vectors $\vec{v} : \Omega \to \Omega \times \Omega$:

$$\langle \vec{u}, \vec{v} \rangle := \int_{\Omega \times \Omega} u(x, y) v(x, y) dx dy.$$
(16)

DEFINITION 3.2. The associated L2 norms are respectively for scalars $i : \Omega \to \mathbb{R}$:

$$\|i\|_{2} := \sqrt{\langle i, i \rangle} = \sqrt{\int_{\Omega} i(x)^{2} dx},$$
(17)

and for vectors $\vec{v} : \Omega \to \Omega \times \Omega$:

$$\|\vec{v}\|_2 := \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{\int_{\Omega \times \Omega} v(x, y)^2 dx dy}.$$
(18)

DEFINITION 3.3. Similarly, the L_1 -norm of the vector \vec{v} , $\|\vec{v}\|_1 : \Omega \times \Omega \to \mathbb{R}$, is defined as

$$\|\vec{v}\|_1 := \int_{\Omega \times \Omega} |v(x, y)| dx dy.$$
⁽¹⁹⁾

DEFINITION 3.4. Let *w* be a non-negative weighting function and \vec{v} a vector. The weighted L_0 -"norm" of the vector \vec{v} , $\|\vec{v}\|_{0,w} : \Omega \times \Omega \to \mathbb{R}$, is defined as

$$\|\vec{v}\|_{0,w} := \langle w, 1 - \delta(\vec{v}) \rangle = \int_{\Omega \times \Omega} w(x, y) (1 - \delta(v(x, y))) dx dy$$
⁽²⁰⁾

where δ is the dirac distribution.

DEFINITION 3.5. Finally, pointwise multiplication is written for scalars i and j as

$$(i \cdot j)(x) := i(x)j(x), \quad x \in \Omega$$
(21)

and for vectors \vec{u} and \vec{v} as:

$$(\vec{u} \cdot \vec{v})(x, y) := u(x, y)v(x, y), \quad x, y \in \Omega$$
(22)

3.2 Differential operators

DEFINITION 3.6. Let $\Omega \in \mathbb{R}^n$, $x \in \Omega$, i(x) be a real function $i : \Omega \to \mathbb{R}$. We define the non-local gradient of this function as the vector of all partial derivatives, $\nabla_w i : \Omega \to \Omega \times \Omega$:

$$(\nabla_w i)(x,y) := \sqrt{w(x,y)}(i(y) - i(x)), \quad x, y \in \Omega$$
(23)

for some non-negative weights w(x, y).

DEFINITION 3.7. The associated divergence of a vector $\vec{v} \in \Omega \times \Omega$, namely $\operatorname{div}_w \vec{v} : \Omega \times \Omega \to \Omega$, is then defined as the negative adjoint under the above inner products:

$$\left\langle \nabla_{w}i,\vec{v}\right\rangle = \left\langle i,-\operatorname{div}_{w}\vec{v}\right\rangle,\tag{24}$$

The expression for the divergence is easily found as

$$(\operatorname{div}_{w}\vec{v})(x) := \int_{\Omega} \sqrt{w(x,y)}v(x,y) - \sqrt{w(y,x)}v(y,x)dy.$$
(25)

DEFINITION 3.8. The non-local Laplacian, $\Delta_w i : \Omega \to \Omega$ is defined as the composition of non-local divergence and non-local gradient:

$$(\Delta_w i)(x) := (\operatorname{div}_w(\nabla_w i))(x) = \int_{\Omega} (w(x, y) + w(y, x))(i(y) - i(x))dy.$$
(26)

LEMMA 3.9. Let $w_s(x,y)$ be a symmetric weighting function, i.e. $\forall x, y \in \Omega : w_s(x,y) = w_s(y,x)$. This restriction simplifies the expressions of both the divergence and associated Laplacian:

$$(\operatorname{div}_{w_s}\vec{v})(x) = \int_{\Omega} \sqrt{w_s(x,y)} (v(x,y) - v(y,x)) dy,$$
(27)

and

$$(\Delta_{w_s}i)(x) = (\operatorname{div}_{w_s}(\nabla_{w_s}i))(x) = 2\int_{\Omega} w_s(x,y)(i(y) - i(x))dy,$$
(28)

where the Laplacian now differs from the regular graph Laplacian by a factor 2.

3.3 Filtered gradients

DEFINITION 3.10. Be $f : \mathbb{R} \to \mathbb{R}$ a real-valued distortion function applied to the finite differences. We define filtered non-local gradients, $\nabla_{w,f} i : \Omega \to \Omega \times \Omega$, as the quasi-gradients obtained as follows:

$$(\nabla_{w,f}i)(x,y) := \sqrt{w(x,y)}f(i(y) - i(x)), \quad x, y \in \Omega$$
⁽²⁹⁾

DEFINITION 3.11. We call $\Delta_{w,f}$ the filtered non-local Laplacian obtained by applying the (regular) divergence to filtered gradients

$$(\Delta_{w,f})(x) := (\operatorname{div}_{w}(\nabla_{w,f}i))(x) = \int_{\Omega} w(x,y) f(i(y) - i(x)) - w(y,x) f(i(x) - i(y)) dy.$$
(30)

LEMMA 3.12. Let f_a be an anti-symmetric real-valued function, i.e. $f_a(z) = -f_a(-z)$ and choose the weights $w_s(x,y) = w_s(y,x)$ symmetrically. The associated filtered non-local Laplacian Δ_{w_s,f_a} is given by:

$$(\Delta_{w_s, f_a})(x) = 2 \int_{\Omega} w_s(x, y) f_a(i(y) - i(x)) dy.$$
(31)

3.4 Filtered gradients as minimizers

Let *w* be a non-negative weighting function. We look for a vector \vec{q}_{L0} which is L_0 sparse as weighted by *w*, while the quasi-gradient $\sqrt{w} \cdot \vec{q}_{L0}$ remains close to the observed gradients $\nabla_w i$. This is the solution of the following optimization problem:

$$\vec{q}_{L0} = \arg\min_{\vec{q}} \left\{ \lambda^2 \|\vec{q}\|_{0,w} + \|\sqrt{w} \cdot \vec{q} - \nabla_w i\|_2^2 \right\}$$
(32)

and it is found as component-wise hard-thresholding applied to the non-local finite differences:

$$\vec{q}_{L0}(x,y) = S^h_{\lambda}(i(y) - i(x)), \quad \text{where} \quad S^h_{\tau}(z) = \begin{cases} 0 & |z| \le \tau \\ z & \text{otherwise} \end{cases}$$
(33)

The quasi-gradient $\sqrt{w} \cdot \vec{q}_{L0}$ is an instance of filtered non-local gradient with $f_a = S_{\lambda}^h$:

$$\left(\sqrt{w} \cdot \vec{q}_{L0}\right)(x, y) = \left(\nabla_{w, f_a} i\right)(x, y) = \sqrt{w(x, y)} f_a(i(y) - i(x)) \tag{34}$$

Similarly, we find the following equivalences:

$$\vec{q}_{TV} = \operatorname*{arg\,min}_{\vec{q}} \left\{ 2\lambda \| \sqrt{w} \cdot \vec{q} \|_1 + \| \sqrt{w} \cdot \vec{q} - \nabla_w i \|_2^2 \right\}$$
(35)

$$\left(\sqrt{w} \cdot \vec{q}_{TV}\right)(x, y) = \sqrt{w(x, y)} S^{s}_{\lambda}(i(y) - i(x)), \quad \text{where} \quad S^{s}_{\tau}(z) = \begin{cases} z + \tau & z < -\tau \\ 0 & |z| \le \tau \\ z - \tau & z > \tau \end{cases}$$
(36)

and

$$\vec{q}_{H1} = \arg\min_{\vec{a}} \left\{ \lambda \| \sqrt{w} \cdot \vec{q} \|_{2}^{2} + \| \sqrt{w} \cdot \vec{q} - \nabla_{w} i \|_{2}^{2} \right\}$$
(37)

$$\left(\sqrt{w} \cdot \vec{q}_{H1}\right)(x, y) = \sqrt{w(x, y)} S^{u}_{\lambda}(i(y) - i(x)), \quad \text{where} \quad S^{u}_{\tau}(z) = \frac{z}{1 + \tau}$$
(38)

Finally, TV-enhanced quasi-gradients can be found as follows:

$$\vec{q}^{TV} = \arg\min_{\vec{q}} \left\{ -2\lambda \|\sqrt{w} \cdot \vec{q}\|_1 + \|\sqrt{w} \cdot \vec{q} - \nabla_w i\|_2^2 \right\}$$
(39)

$$\left(\sqrt{w} \cdot \vec{q}^{TV}\right)(x, y) = \sqrt{w(x, y)} S^{s}_{-\lambda}(i(y) - i(x)), \quad \text{where} \quad S^{s}_{-\tau}(z) = \begin{cases} z + \tau & z > 0\\ 0 & z = 0\\ z - \tau & z < 0 \end{cases}$$
(40)

4. NON-LOCAL RETINEX

4.1 Closing the gap between kernel and variational retinex

We have already mentioned that the homomorphic filtering retinex can be rewritten as a Gaussian-kernel $w_g(x, y)$ based computation of the following form:

$$r(x) = i(x) - \sum_{y} w_g(x, y)i(y) = -\sum_{y} w_g(x, y)(i(y) - i(x))$$
(41)

provided that the Gaussian kernel is normalized, i.e. $\sum_{y} w_g(x, y) = 1$. The second sum now clearly identifies with our definition of non-local Laplacian, and we may thus also write:

$$r(x) + \frac{1}{2}\Delta_{w_g}i(x) = 0$$
(42)

This is the Euler-Lagrange equation corresponding to the following variational model:

$$\min_{r} \left\{ \left\| \nabla_{w_{g}} r - \nabla_{w_{g}} i \right\|_{2}^{2} - \left\| \nabla_{w_{g}} r \right\|_{2}^{2} + 2 \left\| r \right\|_{2}^{2} \right\}$$
(43)

Similarly, in their award-winning model, Bertalmio and colleagues have used their kernel-based lightness estimate together with a gray-world prior and a fidelity constraint to build a "perceptually inspired variational framework" for image enhancement.^{25,26} Their anti-symmetrized kernel-based Retinex has a variational formulation, which is very close to the ACE model,³⁵ namely:

$$\min_{R} \left\{ \int_{\Omega} \left[\alpha \left(R(x) - \frac{1}{2} \right)^2 + \beta \left(R(x) - I(x) \right)^2 \right] dx + 2C_{\sqrt{w}}^{\frac{\min}{\max}}(R) \right\}$$
(44)

where $C_{\sqrt{w}}^{\frac{\min}{max}}(R)$ is a contrast function. For particular, but reasonable choices of contrast function, the contrast term can be shown to be equivalent to

$$C_{\sqrt{w}}^{\frac{\min}{\max}}(R) \equiv -\int_{\Omega \times \Omega} \sqrt{w(x,y)} \left| f_a\left(r(y) - r(x)\right) \right| dx dy = -\left\| \nabla_{w,f_a} r \right\|_1$$
(45)

In particular, f_a may be the identity. Thus, we rewrite the perceptual contrast enhancement in terms of non-local derivatives as follows:

$$\min_{r=\log(R)} \left\{ \alpha \|R - 1/2\|_{2}^{2} + \beta \|R - I\|_{2}^{2} - 2 \|\nabla_{w}r\|_{1} \right\}$$
(46)

where the first term represents the gray-world prior, the second is a fidelity term with respect to the observed intensity, and the contrast term increases non-local TV of the reflectance.

Table 1. Filtered non-local gradient-fidelity based approximations to existing Retinex models. Both Poisson PDE¹¹ and L1-Retinex¹⁵ employ gradient filtering natively. For the other methods, the filtered gradient reproduces a gradient sparsity term actually present in the original model $(L_0 \rightarrow S_{\lambda}^h, TV \rightarrow S_{\lambda}^s, H1 \rightarrow S_{\lambda}^u)$.

Model	Norm p	Weights <i>w</i>	Filter <i>f</i> _{<i>a</i>}	Additional terms
Poisson PDE ¹¹	$\overline{L_2}$	local	$\overline{S^h_\lambda}$	_
L1-Retinex ¹⁵	L_1	local	S^h_{λ}	_
TV-Retinex ³¹	L_2	local	S^s_λ	_
Variational Retinex ³⁰	L_2	local	S^u_λ	$\alpha \ r\ _2^2 \qquad (r \le 0)$
TV-Retinex ³²	L_2	local	S^s_{λ}	$\beta \ r-i\ _2^2$
$TV-L_1^{33}$	L_1	local	_	$\alpha \ r\ _2^2 \qquad (\alpha \ r\ _1)$
Random walk/Kernel based ²⁶	L_2	Gaussian	$S^{s}_{-\lambda}$	$\alpha \ r\ _2^2 + \beta \ r - i\ _2^2$

4.2 Proposed model

So far we have seen that all Retinex models have a variational equivalent, potentially through the use of non-local differential operators. Even more, these variational counterparts all share a very similar structure: the energy typically comprises one or two fidelity terms (image and/or its gradient), as well as sparsity priors or alternatively, through negation, enhancement terms.

Also, we have shown that this type of variational problem can be retrofitted into a Horn/Morel-style gradient-fidelity problem, potentially adding further terms. In particular, we have shown in the previous section, how different gradient sparsity and fidelity terms translate into differing thresholding functions.

In general, we want to tackle the retinex problem in a two step approach:

1) We realize that the reflectance obeys both to some gradient sparsity priors and some gradient fidelity priors. In a first step, we thus look for an optimal quasi-gradient that best satisfies those two constraints. This quasi-gradient is obtained as filtered gradient of the observed image $\nabla_{w,f}i$. The sparsity and gradient fidelity terms will determine the exact filter function *f* to be used.

2) We fit the gradient of a reflectance to the quasi-gradient determined in the first step, while possibly minimizing some additional terms:

$$\hat{r} = \arg\min_{r} \left\{ \|\nabla_{w}r - \nabla_{w,f}i\|_{p}^{p} + \alpha \|r\|_{2}^{2} + \beta \|r - i\|_{2}^{2} \right\}$$
(47)

5. RESULTS I: RELATIONS TO EXISTING MODELS

The first results section is dedicated to demonstrate the unifying power of the proposed non-local two-step Retinex model.

5.1 Model correspondences

In the following paragraphs, we want to show how the existing Retinex implementations can be reproduced in our proposed non-local Retinex model. The different correspondences are summarized in table 1. In all these instances, we transform the gradient sparsity term into a corresponding gradient filter.

5.1.1 Poisson

The PDE version of Retinex¹¹ can be derived exactly from the L_2 -version p = 2 of the proposed Retinex model, under local weights w_l and gradient thresholding $f_a = S_{\lambda}^h$. Indeed:

$$\hat{r} = \operatorname*{arg\,min}_{r} \left\{ \|\nabla_{w_l} r - \nabla_{w_l, f_a} i\|_2^2 \right\} \quad (L_2 \text{-retinex})$$

$$\tag{48}$$

implies the Euler-Lagrange equations

$$(\Delta \hat{r})(x,y) = (\Delta_{w_l,f_a} i)(x,y) \quad x,y \in \Omega.$$
 (Poisson PDE Retinex) (49)

5.1.2 L1-retinex

The next close relative of the proposed non-local retinex model is its local L1 predecessor, L1-retinex.¹⁵ The closest match to L1-retinex in the proposed framework is obtained if we choose the weights $w_l(x, y)$ such as to reproduce the well-known local finite differences differential operators, gradient filtering $f_a = S_{\lambda}^h$, and with p = 1:

$$\hat{r} = \arg\min_{r} \left\{ \|\nabla_{w_l} r - \nabla_{w_l, f_a} i\|_1 \right\} = \arg\min_{r} \left\{ \int_{\Omega} \sum_{k=1}^d \left| \nabla_k r(x) - S_{\lambda}^h(\nabla_k i(x)) \right| dx \right\}$$
(50)

which is the anisotropic L1-distance for local gradient fidelity.

5.1.3 TV regularized Retinex

In Ref. 31, the authors propose to solve directly for an image, whose gradient is close to the observed gradient in L2, while minimizing isotropic TV^{*}:

$$\hat{r} = \operatorname*{arg\,min}_{r} \left\{ \|\nabla r - \nabla i\|_{2}^{2} + 2\lambda \||\nabla r|_{2}\|_{1} \right\}, \quad (\text{TV retinex})$$
(51)

A similar model can be obtained through the proposed general Retinex model by employing soft-shrinkage gradient filtering, $f_a = S_{\lambda}^s$, to which the according potential is recovered:

$$\hat{r} = \arg\min\left\{\|\nabla_w r - \nabla_{w,f_a} i\|_2^2\right\}$$
(52)

Again, the main difference is the use of anisotropic TV through gradient filtering in the proposed framework.

5.1.4 H1+L2 regularized

The variational Retinex model by Kimmel et al.³⁰ can be rewritten exactly as

$$\hat{r} = \arg\min_{r} \left\{ \|\nabla r - \nabla i\|_{2}^{2} + \alpha \|r\|_{2}^{2} + \lambda \|\nabla r\|_{2}^{2} \right\} \quad \text{s.t.} \quad r \le 0 \quad (\text{H1/L2 Retinex})$$
(53)

There, the authors motivate the L_2 term mainly as "a regularization of the problem that makes it better conditioned", and they state that "in practice this penalty term should be weak [...] and α should therefore be very small." The constraint r < 0 corresponds to the reset in the original Retinex theory. The constraint and L_2 norm together push the reflectance close to white.

We may find a similar problem within the proposed framework, where we choose uniform gradient scaling $f_a = S_{\lambda}^u$ and omit the clipping constraint:

$$\hat{r} = \arg\min_{r} \left\{ \left\| \nabla_{w_{l}} r - \nabla_{w_{l}, f_{a}} i \right\|_{2}^{2} + \alpha \left\| r \right\|_{2}^{2} \right\}$$
(54)

5.1.5 TV+L2 regularized

Recently, a mixture of TV regularized and Kimmel's variational approach was proposed.³² This model essentially boils down to:

$$\hat{r} = \arg\min_{r} \left\{ \|\nabla r - \nabla i\|_{2}^{2} + \beta \|r - i\|_{2}^{2} + 2\lambda \|\nabla r\|_{1} \right\} \quad (\text{TV/L2 Retinex})$$
(55)

Again, we may approximate this model with a similar energy based on similarity to filtered gradients, with $f_a = S_{\lambda}^s$:

$$\hat{r} = \arg\min_{r} \left\{ \left\| \nabla_{w_{l}} r - \nabla_{w_{l}, f_{a}} i \right\|_{2}^{2} + \beta \left\| r - i \right\|_{2}^{2} \right\}$$
(56)

5.1.6 TV-L1

The "logarithmic total variation" (LTV) model was suggested for extraction of illumination invariant features for face recognition.³³ It is defined as an TV- L_1 based on the logarithmic input image and its logarithmic illumination:

$$\hat{r} = \arg\min_{r} \{ \|\nabla r - \nabla i\|_{1} + \alpha \|r\|_{1} \} \quad (\text{TV-}L_{1})$$
(57)

Its equivalent in the proposed framework is found by relaxing the second term to an L2-norm, i.e. TV-L2 retinex.

^{*}In an extension, they use non-local TV as sparsity constraint as well.

5.1.7 Bertalmío

To approximate the perceptually inspired Retinex model through our proposed general framework, we complete the TVenhancing variational model by a gradient fidelity:

$$\hat{r} = \operatorname*{arg\,min}_{r} \left\{ \alpha \|r\|_{2}^{2} + \beta \|r - i\|_{2}^{2} + \|\nabla_{w}r - \nabla_{w}i\|_{2}^{2} - 2\lambda \|\nabla_{w}r\|_{1} \right\} \quad (-\mathrm{TV} + \mathrm{L2\,Retinex})$$
(58)

This is essentially homomorphic filtering with TV in place of H1. Again, we may now substitute by incorporating the TV-enhancement term as an input-gradient filter $f_a = S^s_{-\lambda}$:

$$\hat{r} = \arg\min_{r} \left\{ \left\| \nabla_{w} r - \nabla_{w, f_{a}} i \right\|_{2}^{2} + \alpha \left\| r \right\|_{2}^{2} + \beta \left\| r - i \right\|_{2}^{2} \right\}$$
(59)

5.2 The Logvinenko illusion

We have applied the whole range of retinex "modes" retrofitted above to existing retinex implementations to a single common test image extracted from the Logvinenko illusion pattern.³⁶ The test image is shown in Fig. 1a). The illusion consists of the following: due to the suggested smoothly varying lighting, the oblique grey diamonds of the upper row appear darker than the diamonds of the lower row. However, as shown in the adjacent Fig. 1b), their actual intensity is exactly equal. In this example, the retinex model is expected to separate the smooth shading from the rough checkerboard-like reflectance, thereby truly rendering the two rows of diamonds at different reflectances.

The first model, L2-retinex,¹¹ produces the standard result in Fig. 1c). It can be clearly seen that in particular the lower row of diamonds is not recovered completely flat, since the illumination is not smooth everywhere. The related L1-retinex¹⁵ in Fig. 1d) suffers from very similar artifacts. In Fig. 1e) we show the results of our model with parameters set to correspond to TV-regularized retinex,³¹ resulting in less artifacts. Adding an L2 fidelity-constraint ($\beta > 0$), as in Ref. 32, injects more of the initial shading into the estimated reflectance, see Fig. 1f).

The TV-L1-inspired model³³ is in our case a TV-L2 model for illumination recovery, where the TV-sparsity of the extracted illumination is tuned by the parameter α . It is clearly appreciated in Fig. 1g–j) that the impact of the parameter is quite severe, with higher values corresponding to the output desired for illumination invariant feature extraction. The choice of parameters inspired by Kimmel's retinex formulation yields the output shown in Fig. 1k–m), which corresponds well to the behavior expected from.³⁰ The parameter α controls the degree of dynamic range compression applied, i.e. the dominance of local contrast enhancement. Finally, in Fig. 1n–p) we provide the output produced by model parameters mimicking Bertalmío's perceptually inspired retinex.²⁵ Here, the unshrinking of the gradients has the unpleasant effect of amplifying pixel noise.

6. RESULTS II: NEW PERSPECTIVES

Beyond reproducing existing retinex models, our proposed framework also has the potential to yield new results thanks to its generalizing power. In the next sections, we explore a few new possibilities offered by choosing new sets of parameters, in particular based on p = 0 gradient fidelity, with applications to shadow detection and removal, and cartoon-texture decomposition.

6.1 L0 gradient fidelity

In Fig. 1 we have shown a series of decomposition results obtained with different model configurations. The best results in terms of piecewise constant reflectance versus illumination have been achieved with the basic hard thresholding models (L2- and L1-retinex), as well as the soft-thresholding based TV-retinex. However, all these models suffer from artifacts of illumination estimation at the edges and corners of the flat diamonds, where illumination smoothness is not a stringent enough prior. Therefore, we propose seeking for further illumination gradient penalty by choosing p = 0, corresponding to L0 gradient fidelity (as opposed to TV- or H1-sparsity of the illumination). In Fig. 2 we show a few results where we make use of TV-regularization of the reflectance (soft thresholding). In particular in combination with Gaussian kernel weights, the decomposition exhibits less artifacts than previous results, see Fig. 2b).



Figure 1. Logvinenko illusion and different retinex decompositions. a) Input image. b) Demonstration of the illusion: despite the appearances, the "horizontal squares" actually have equal intensity. Retinex is supposed to reproduce this illusion of intensity difference. c) Reflectance and illumination recovered using the L2-retinex (hard thresholding, p = 2, $\alpha = \beta = 0$). d) L1-retinex (hard thresholding, p = 1, $\alpha = \beta = 0$). e) TV-regularized retinex (soft thresholding, p = 2, $\alpha = \beta = 0$). f) Ng-Wang-like retinex (soft thresholding, p = 2, $\alpha = 0$, $\beta = 0.0015$). g-j) TV-L2 retinex (no thresholding, p = 1, $\alpha > 0$, $\beta = 0$). k-m) Kimmel-like retinex (gradient scaling, p = 2, $\alpha > 0$, $\beta = 0$). n-p) Bertalmio-like retinex (Gaussian kernel weights, gradient unshrinkage, p = 2, $\alpha > 0$, $\beta = 0.002$). g-p) Where applicable, α drastically tunes the amount of dynamic range compression.



Figure 2. Logvinenko illusion and new L0-based retinex decompositions. Soft thresholding, p = 0, $\alpha = \beta = 0$. a) Local weights. b) Narrow Gaussian kernel weights, $\lambda = 0.33$. c) Wider Gaussian kernel weights, $\lambda = 0.8$.



Figure 3. Adelson checker illusion. a) Input image. b) Demonstration of the illusion: despite the appearances, the squares A and B actually have equal intensity. Retinex is supposed to reproduce this illusion of intensity difference by removing the shadows. c) Reflectance and illumination recovered using the novel L0-based retinex. Hard thresholding $\lambda = 0.15$, p = 0, $\alpha = 0.04$. d) L2-retinex with dynamic range compression has too smooth illumination (hard thresholding $\lambda = 0.025$, p = 2, $\alpha = 0.01$). e-g) L2-retinex without dynamic range compression. The drop shadow is removed nicely, but the cylinder shading is not.

6.2 Shadows in artificial images: the Adelson checker illusion

Another instance, where H1 smoothness (p = 2) or TV-regularity (p = 1) constraints are not sparse enough priors for the illumination field, is the Adelson checker illusion. We show a grayscale image of the illusion image in Fig. 3a): the squares A and B appear to be of different intensity, for the human visual system corrects actual intensity by perceived shading. However, as shown in the adjacent demonstration, the diamonds truly have identical intensity. Here again, the role of retinex is to separate the shading from the underlying reflectance. However, while the reflectance is expected to be piecewise constant, the illumination has both smooth parts and sharp transitions. The sharp transitions are not sufficiently accounted for under simple L2 hard-thresholding, as shown in Fig. 3e–g); the estimated illumination always turns out too smooth. Even additional dynamic range compression cannot entirely fix the issues, as now parts of the checkerboard's reflectance also appear in the illumination, see Fig. 3d). Thanks to the new possibilities of the proposed framework, however, the problem is rather nicely solved using a combination of L0 gradient fidelity, hard thresholding, and slight dynamic range compression, as shown in Fig. 3c).

6.3 Shadow detection in natural images

Shadow removal from a single (natural) image plays an important role in many computer vision algorithms. Most methods are based on a two-step procedure: first detect shadows, and then reconstruct shadow-free images. Shadow detection can be based on features such as intensity, gradients or texture, and even make use of supervision or training data.^{37–40} Once



Figure 4. **Shadow detection results**. We compare the shadow detection results (illumination output) of our proposed model against the recently published results (blue mask) from Guo et al.^{38,39} The results of the first row are very comparable, while we believe the examples of the second row are in favor of the proposed model. Indeed, our illumination output may be "multilevel" rather than just binary, and therefore better reflect the different nuances of shade in natural images (pole). On the other hand, our approach is less subject to local artifacts and produces more coherent shadow estimates.

the shadow regions have been reliably detected, several techniques aim at reconstructing shadow-free images, through matting, inpainting, or Poisson editing.^{40,41}

Here, we explore the applications of the proposed unified retinex model for single step shadow detection and removal from a single image. We propose to use the L0 gradient fidelity criterion combined with dynamic range compression, without any gradient thresholding. The unfiltered L0 gradient fidelity is a strong prior on illumination gradient sparsity, while the dynamic range compression tends to take large intensity modulations out of the reflectance, and balance the mean intensities of inside- and outside-shadow regions. Our model can detect shadows in monochromatic and color images. We show a few example results in Fig. 4. We believe that the proposed model can largely compete with the recent state-of-the-art shadow detection scheme proposed Guo et al.^{38,39}

Moreover, our model also provides a shadow-free reflectance estimate at the same time. However, in most natural scenes, the actual border between shaded and unshaded regions is rather smooth, called the penumbra, which is due to the spatial extent of the light source. Hence, the estimated shadow boundary in the proposed model is consistently overly sharp, and the estimated shadow-free reflectance image includes artifacts, see Fig. 5c). This problem can partially be tackled by smoothing the estimated illumination field in post-processing, as shown in Fig. 5d). A noticeable difference in texture is still visible, however, due to the missing specular highlights in the shadowed region, exclusively lit by ambient light.

If the images are treated as color images, however, a few shortcomings of the simple shadow-removal model become obvious, beyond the penumbra-issue. In Fig. 5e) we show the output of retinex being applied to the lightness channel in HSV-space only. Since the shadowed region was lit by (sky-blueish) ambient light only, compared to warmer direct sun light, the colorcast after intensity correction becomes really striking. If, in contrast, we perform retinex on all three RGB channels independently, the colorcast can be successfully avoided, see Fig. 5f). However, since the three channels are not coupled, the respective shadow-boundaries differ slightly, creating local color-artifacts.

6.4 Cartoon-texture decomposition

The separation of an image into a piecewise regular component (cartoon) and its high-frequency parts (texture) is generally referred to as cartoon-texture decomposition.^{42–44} Now, if we give even more importance to dynamic range compression, then our proposed L0 gradient-fidelity based retinex model can be used to this very same end. Indeed, the "reflectance" will now only contain the texture of the image, whereas all larger scale intensity patches will be attributed to illumination (cartoon part). In Fig. 6 we show results of cartoon-texture decompositions of two natural images.

7. DISCUSSION AND CONCLUSIONS

In this paper, we have contributed a unifying framework for retinex, based on non-local differential operators. Our framework deals with the reflectance-illumination decomposition problem in two steps. The interest of such a two step procedure



Figure 5. **Shadow removal results. a)** Input image. **b)** Recently published results from Guo et al.^{38,39} **c)** Reflectance of input image reduced to grayscale. The sharp boundary of the detected shadow region creates artifacts in the penumbra. **d)** The artifacts are removed by smoothing the estimated illumination in post-processing. **e)** If the retinex model is applied only to the V-channel of the color image in HSV-space, then strong colorcast becomes apparent, due to different lighting color for direct and ambient light. **f)** The colorcast is avoided by correcting all three RGB channels (colorbalancing). However, local artifacts appear due to inconsistent shadow region boundaries in the three individual channels.



Figure 6. Cartoon-texture decomposition. For important α , the L0 model separates texture (reflectance) from cartoon (illumination).

is manifold: First, each step, i.e. thresholding the input gradient, followed by a gradient fitting is relatively simple to compute, compared to the non-compacted variational model. Further, the computational tools required to solve the gradient reconstruction step become independent of the gradient sparsity imposed. Finally, it can be shown that this two step procedure is fully compatible with Marr's theory of lightness computation in the (primate) retina.¹⁰ Also, it particularly neatly fits into Wilson-Cowan equations, modeling large-scale activity in cortical neural populations.^{45,46}

The proposed framework is able to reproduce some fundamental retinex models exactly,^{11,15} provides relaxed approximations to most others,^{30–32} and matches a few by substituting some of their terms.^{25,33} Beyond, yet other retinex models have already been shown to be equivalent to the models we refer to in this manuscript.⁹

Beyond reproducing the existing retinex models in a single variational framework, our proposition also opens the way to new retinex flavors by choosing different sets of parameters and filters. In particular, in this manuscript, we introduce L0 gradient fidelity based retinex, which provides interesting results for shadow detection and removal, as well as cartoon-texture decomposition.

Future work will focus on the exploration of different weighting functions. In particular, we may use different weighting functions for the gradient sparsity and fidelity terms of the first step, allowing us to have non-stationary (spatially varying) thresholds. This could have immediate benefits regarding color-retinex, where such conditional thresholding has already been proposed,^{47–49} or hyperspectral illumination suppression.⁵⁰

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