TEICHMÜLLER EXTREMAL MAP OF MULTIPLY-CONNECTED DOMAINS USING BELTRAMI HOLOMORPHIC FLOW

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Abstract. A numerical method for computing the Teichmüller extremal map between multiply-connected domains is presented. Given two multiply-connected domains, there exists a unique Teichmüller map(T-Map) between them minimizing the conformality distortion. The extremal T-Map can be considered as the "most conformal" mapping between multiply-connected domains. In this paper, we propose an iterative scheme to obtain the extremal T-Map using the Beltrami holomorphic flow (BHF). The BHF procedure iteratively adjusts the initial map based on a sequence of Beltrami coefficients, which are complex-valued function defined on the source domain. It produces a sequence of quasi-conformal maps, which converges to the T-Map minimizing the conformality distortion. We test our method on synthetic data together with real human face data. Results show that our algorithm computes the extremal T-Map between multiply-connected domains accurately and efficiently.

Key words. Teichmüller map, extremal map, multiply-connected, Beltrami holomorphic flow, Beltrami coefficient, quasiconformal map.

1. Introduction. Establishing meaningful mappings between different domains is an important research topic in many fields. Applications can be found in different areas such as registration, shape analysis and grid generation. Conformal mapping has been widely used to establish good one-to-one correspondence between different domains, since it preserves the local geometry well. According to the Riemann mapping theorem, conformal mappings between simply-connected domains always exist. However, this fact is not true for multiply-connected domains. Given two multiply-connected domains with different conformal modules, there is generally no conformal mapping between them. In this case, it is usually desirable to obtain a mapping that minimizes the conformality distortion. Every mapping is associated with a unique Beltrami coefficient(BC), which is a complex-valued function μ_f defined on the source domain. The BC, μ_f, measures the deviation of the mapping from a conformal map. Given two multiply-connected domains Ω_1 and Ω_2, there exists a unique map $f : \Omega_1 \to \Omega_2$, called the Teichmüller extremal map (extremal T-Map), minimizing the L^∞ norm of the BC. Therefore, the extremal T-Map can be considered as the "most conformal" mapping between multiply-connected domains, which is a natural extension of conformal mappings.

In this work, our goal is to numerically solve the following mathematical problem:

$$f^* = \mathbf{argmin}_{f:\Omega_1\to\Omega_2}\{||\mu_f||_\infty\} \tag{1.1}$$

such that $f^*(\partial\Omega_1) = \partial\Omega_2$ (boundary condition).

We present in this paper a numerical method to compute the extremal T-Map between arbitrary multiply-connected domains. The domains of interest can either be planar domains or surfaces embedded in $\mathbb{R}^3$. We propose an iterative algorithm to obtain the extremal T-Map using the Beltrami holomorphic flow (BHF). The BHF procedure iteratively adjusts the initial map, based on a sequence of Beltrami coefficients. It produces a sequence of quasi-conformal maps, which converges to the desired extremal T-Map. Numerical experiments have been carried out on synthetic data together with real human face data. Results show that our algorithm computes T-Map between multiply-connected domains accurately and efficiently.

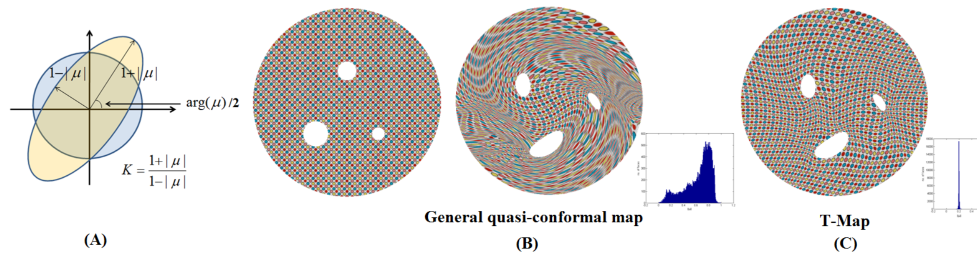

Fig. 1.1. *(A) illustrate how the conformality distortion under a quasi-conformal map can be determined by μ. (B) shows a general quasi-conformal map visualized by texture mapping. Conformality distortion is not uniform. (C) shows a Teichmüller map, whose conformality distortion is uniform everywhere.*

The paper is organized as follows. In section 2, we review some previous works closely related to the current work. In section 3, we describe some basic mathematical background necessary for explaining this work. In section 4, we formulate the mathematical problem in details. Our proposed algorithm to compute the extremal T-Map will be discussed in section 5. The detailed numerical implementation of our proposed model will be explained in Section 6. In section 7, we show the experimental results to demonstrate the effectiveness of the proposed method. A concluding remark will be given in section 8.

2. Related work. Teichmüller extremal maps are closely related to conformal maps. Simply-speaking, a Teichmüller extremal map is the optimal quasi-conformal map that is closest to conformal. The computation of conformal maps have been extensively studied [4, 5, 7, 2, 8, 6]. For example, Hurdal et al. [8] proposed to compute the conformal parameterizations using circle packing and applied it to registration of human brains. Gu et al. [5, 7, 6] proposed to compute the conformal parameterizations of Riemann surfaces for registration using harmonic energy minimization and holomorphic 1-forms. Later, the authors proposed the curvature flow method to compute conformal parameterizations of high-genus surfaces onto their universal covering spaces [16, 17, 18]. Conformal registration is advantageous for it preserves the local geometry well.

In real world situations, mappings are usually quasi-conformal, which induce bounded amount of conformality distortion. Numerical quasi-conformal maps have also been widely studied. Lui et al. [11] proposes to compute quasi-conformal registration between hippocampal surfaces which matches geometric quantities (such as curvatures) as much as possible. A method called the Beltrami Holomorphic flow is used to obtain the optimal Beltrami coefficient associated to the registration [10, 9, 13, 22]. Beltrami coefficient has been applied to represent general surface homeomorphisms, which is comparatively easier to manipulate than 3D coordinate functions. Using Beltrami representation, compression of surface maps has been proposed [9], which can be applied for video compression [13]. Wei et al. [15] also proposes to compute quasi-conformal mapping for feature matching face registration. The Beltrami coefficient associated to a landmark points matching parameterization is approximated. However, either exact landmark matching or the bijectivity of the mapping cannot be guaranteed, especially when very large deformations occur. In order to compute quasi-conformal mapping from the Beltrami coefficients effectively. Quasi-Yamabe method is introduced, which applies the curvature flow method to compute the quasi-conformal mapping [12]. The algorithm can deal with surfaces with general topologies. Later, extremal quasi-conformal mapping, which minimizes conformality distortion has been proposed. Zorin et al. [21] proposes a least square algorithm to compute mapping between connected domains with given Dirichlet condition defined on the whole boundaries. The extremal mapping is obtained by minimizing a least square Beltrami energy, which is non-convex. The algorithm can obtain an extremal mapping when initialization is carefully chosen. However, the convergence to the global minimum cannot be guaranteed. Recently, Lui et al. [23] proposed to compute the unique Teichmüller map between simply-connected Riemann surfaces of finite type. The proposed algorithm was applied for landmark-based surface registration.

3. Overview of quasi-conformal geometry. In this section, we describe some basic mathematical concepts relevant to our algorithms. For details, we refer the readers to [3, 14].

A surface S with a conformal structure is called a *Riemann surface*. Given two Riemann surfaces M and N, a map $f: M \to N$ is *conformal* if it preserves the surface metric up to a multiplicative factor called the *conformal factor*. A generalization of conformal maps is the *quasi-conformal* maps, which are orientation preserving homeomorphisms between Riemann surfaces with bounded conformality distortion, in the sense that their first order approximations takes small circles to small ellipses of bounded eccentricity [3]. Mathematically, $f\colon \mathbb{C} \to \mathbb{C}$ is quasi-conformal provided that it satisfies the Beltrami equation:

$$\frac{\partial f}{\partial \overline{z}} = \mu(z)\frac{\partial f}{\partial z}. \tag{3.1}$$

for some complex valued function μ satisfying $||\mu||_\infty < 1$. μ is called the *Beltrami coefficient*, which is a measure of non-conformality. μ_f measures how far the map is deviated from a conformal map. $\mu \equiv 0$ if and only if f is conformal. Infinitesimally, around a point p, f may be expressed with respect to its local parameter as follows:

$$\begin{aligned} f(z) &= f(p) + f_z(p)z + f_{\overline{z}}(p)\overline{z} \\ &= f(p) + f_z(p)(z + \mu(p)\overline{z}). \end{aligned} \tag{3.2}$$

Obviously, f is not conformal if and only if $\mu(p) \neq 0$. Inside the local parameter domain, f may be considered as a map composed of a translation to $f(p)$ together with a stretch map $S(z) = z + \mu(p)\overline{z}$, which is postcomposed by a multiplication of $f_z(p)$, which is conformal. All the conformal distortion of $S(z)$ is caused by $\mu(p)$. $S(z)$ is the map that causes f to map a small circle to a small ellipse. From $\mu(p)$, we can determine the angles of the directions of maximal magnification and shrinking and the amount of them as well. Specifically, the angle of maximal magnification is $\arg(\mu(p))/2$ with magnifying factor $1 + |\mu(p)|$; The angle of maximal shrinking is the orthogonal angle $(\arg(\mu(p)) - \pi)/2$ with shrinking factor $1 - |\mu(p)|$. Thus, the Beltrami coefficient μ gives us all the information about the properties of the map (see Figure 1.1).

The maximal dilation of f is given by:

$$K(\phi) = \frac{1 + ||\mu||_\infty}{1 - ||\mu||_\infty}. \tag{3.3}$$

Quasiconformal mapping between two Riemann surfaces R_1 and R_2 can also be defined. Instead of the Beltrami coefficient, the *Beltrami differential* has to be used.

A Beltrami differential $\mu(z)\frac{\overline{dz}}{dz}$ on the Riemann surface R_1 is an assignment to each chart (U_α, ϕ_α) of an L_∞ complex-valued function μ_α, defined on local parameter z_α such that

$$\mu_\alpha(z_\alpha)\frac{d\overline{z_\alpha}}{dz_\alpha} = \mu_\beta(z_\beta)\frac{d\overline{z_\beta}}{dz_\beta}, \tag{3.4}$$

on the domain which is also covered by another chart (U_β, ϕ_β), where $\frac{dz_\beta}{dz_\alpha} = \frac{d}{dz_\alpha}\phi_{\alpha\beta}$ and $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$.

An orientation preserving diffeomorphism $f : R_1 \to R_2$ is called quasi-conformal associated with $\mu(z)\frac{\overline{dz}}{dz}$ if for any chart (U_α, ϕ_α) on R_1 and any chart (V_β, ψ_β) on R_2, the mapping $f_{\alpha\beta} := \psi_\beta \circ f \circ f_\alpha^{-1}$ is quasi-conformal associated with $\mu_\alpha(z_\alpha)\frac{d\overline{z_\alpha}}{dz_\alpha}$.

4. The mathematical formulation of the problem.

4.1. The extremal problem. Given two multiply-connected domains or surfaces Ω_1 and Ω_2, both with $n + 1$ boundaries. Denote the boundaries of Ω_1 and Ω_2 by $\{\gamma_0, \gamma_1, ..., \gamma_n\}$ and $\{\gamma'_0, \gamma'_1, ..., \gamma'_n\}$ respectively. Conformal maps between arbitrary multiply-connected domains generally do not exist. One might be interested in studying extremal quasi-conformal mappings, which are extremal in the sense of minimizing the $||\cdot||_\infty$ over all Beltrami differentials corresponding to quasi-conformal mappings between Ω_1 and Ω_2. The idea of extremality is to make $K(f)$ as small as possible such that f is as "nearly conformal" as possible. Extremal mapping always exists but needs not be unique.

Let $f : \Omega_1 \to \Omega_2$ be a quasi-conformal mapping between Ω_1 and Ω_2. Assume that f satisfies the boundary condition: $f(\gamma_i) = \gamma'_i$ for all i. Note that the point-wise correspondence between the boundaries is not required. f is said to be an *extremal mapping* if for any quasi-conformal mapping $h : \Omega_1 \to \Omega_2$ satisfying the boundary condition,

$$K(f) \le K(h) \tag{4.1}$$

It is called *uniquely* extremal if the inequality (4.1) is strict [19, 20].

Note that an extremal mapping is not unique for general cases. According to equation (3.3), $K(f)$ is minimum if and only if $||\mu(f)||_\infty$ is minimized. The extremal problem can therefore be expressed as finding $f^* : \Omega_1 \to \Omega_2$ that solves:

$$f^* = \mathbf{argmin}_{f\in\mathcal{A}}\{||\mu_f||_\infty\} \tag{4.2}$$

where $\mathcal{A} = \{f : \Omega_1 \to \Omega_2 : f(\gamma_i) = \gamma'_i \text{ for } 0 \le i \le n\}$.

The extremal map is closely related to another type of mapping, called the *Teichmüller map (T-Map)*. Simply-speaking, a T-Map is a quasi-conformal map with uniform conformality distortion. Mathematically, a quasi-conformal map f is said to be a Teichmüller mapping associated with an integrable holomorphic function $\varphi : \Omega_1 \to \mathbb{C}$ if its associated Beltrami coefficient is of the form:

$$\mu(f) = k\frac{\overline{\varphi}}{|\varphi|} \tag{4.3}$$

for some constant $0 \le k < 1$ and integrable holomorphic function $\varphi \ne 0$. The Beltrami coefficient of this form is said to be of *Teichmüller type*.

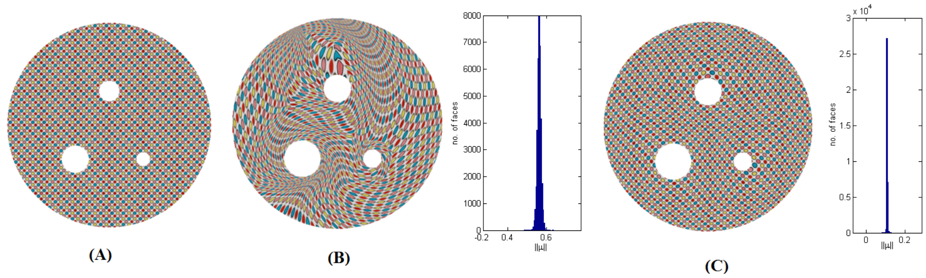

Fig. 4.1. *Teichmüller maps with different maximal dilations. (A) and (B) shows two circle domains with three holes. (B) shows a Teichmüller map, visualized by texture mapping. Its BC norm is equal to 0.58. (C) shows the extremal Teichmüller map, whose BC norm (=0.11) is minimum over all possible Teichmüller map.*

Figure 1.1(B) and (C) shows the difference between a general quasi-conformal map and a Teichmüller map. (B) shows a general quasi-conformal map visualized by texture mapping. The small circles on (A) are mapped to small ellipses on (B) with different eccentricity (see the histogram of the norm of its Beltrami coefficient). (C) shows a T-Map visualized by texture mapping. The small circles on (A) are mapped to small ellipses on (B) with uniform eccentricity everywhere. As we can see from the histogram, the norm of the Beltrami coefficient accumulates at 0.3.

In general, there are many Teichmüller maps between two multiply-connected domains with the same topology. In particular, given the boundary correspondence $h : \partial\Omega_1 \to \partial\Omega_2$ satisfying $h'(e^{i\theta}) \neq 0$ and $|h''(e^{i\theta})| < \infty$, there exists a unique Teichmüller map f between Ω_1 and Ω_2 [20]. Also, f is uniquely extremal for its boundary values.

Theorem 4.1. *Let Ω_1 and Ω_2 be multiply-connected domains with the same topology and $h : \partial\Omega_1 \to \partial\Omega_2$. Suppose $f : \Omega_1 \to \Omega_2$ is a Teichmüller map with a quadratic differential of finite norm, with $f|_{\partial\Omega_1} = h$. Then f is uniquely extremal for its boundary values.*

Proof. Suppose g is an extremal extension of h. Let the Beltrami coefficient of f be $\mu_f = k\frac{\overline{\varphi}}{\varphi}$. Since f and g agrees on their boundaries, the following inequality holds [20]:

$$\begin{aligned} &\int_{\Omega_1} \frac{\left(|\alpha|^2 - |\beta|^2\right) + (1-|\mu_f|)(|\alpha| - \mathbf{Re}(\frac{\bar{\beta}\alpha}{|\alpha|}))}{(1+|\mu_f|)(1-|\beta^2|)}|\varphi| \\ &\qquad \leq \mathbf{Re}\int_{\Omega_1} \frac{\bar{\alpha}}{\alpha}\left(|\varphi| - \frac{\mu_f}{|\mu_f|}\varphi\right)\frac{(1-\bar{\beta}\alpha)(\alpha-\beta)}{(1-|\mu_f|^2)(1-|\beta^2|)} \end{aligned} \tag{4.4}$$

where $\alpha = \mu_{f^{-1}} \circ f$; $\beta = \mu_{g^{-1}} \circ g$. Since $\mu_f = k\frac{\overline{\varphi}}{\varphi}$, the right-hand side of (4.4) vanishes. Hence, $\alpha = \beta$. This implies: $\mu_{f^{-1}} = \mu_{g^{-1}}$. Since f^{-1} and g^{-1} has the same boundary values, namely, h, we have $f^{-1} = g^{-1}$. Thus, f is the unique extremal map satisfying the boundary values h. ∎

Therefore, with different boundary value $h : \partial\Omega_1 \to \partial\Omega_2$, different Teichmüller map can be obtained (see Figure 4.1). We denote the collection of all possible Beltrami coefficients of Teichmüller type associated with quasi-conformal maps between Ω_1 and Ω_2 by $T(\Omega_1)$. In other words,

$$T(\Omega_1) := \{\nu : \Omega_1 \to \mathbb{C} : \nu = k\frac{\bar{\varphi}}{|\varphi|}, 0 \leq k < 1, \int_{\Omega_1} |\varphi| < \infty\} \tag{4.5}$$

Our goal is to look for the optimal $\nu^* := k^* \frac{\bar{\varphi^*}}{|\varphi^*|} \in T(\Omega_1)$ whose $||\nu^*||_\infty$ $(= k^*)$ is minimized over $T(\Omega_1)$. It turns out that ν^* is the unique minimizer. It is also the Beltrami coefficient associated with the unique extremal map between Ω_1 and Ω_2 (see Figure 4.1). This is guaranteed by the following theorem.

Theorem 4.2. *Let Ω_1 and Ω_2 be multiply-connected domains with the same topology. There exists a unique extremal map $f : \Omega_1 \to \Omega_2$ satisfying the boundary condition: $f(\gamma_i) = \gamma_i'$ for all i. Also, f is a Teichmüller map associated with an integrable holomorphic quadratic function on Ω_1.*

Proof. By the compactness argument, there exists an extremal map $f_{ext} : \Omega_1 \to \Omega_2$ with $f(\gamma_i) = \gamma_i'$ for all i such that $||\mu(f_{ext})||_\infty = \inf_{f:\Omega_1\to\Omega_2}\{||\mu(f)||_\infty\} := k$. Let $h = f_{ext}|_{\partial\Omega_1}$. We proceed to prove that f_{ext} is unique and is a Teichmüller map.

By Theorem 4.1, there exists a unique Teichmüller map $f : \Omega_1 \to \Omega_2$ such that $f|_{\partial\Omega_1} = h$. f is the unique extremal map for the boundary value h. Hence, $f_{ext} = f$. Hence, all extremal map between Ω_1 and Ω_2 must be a Teichmüller map.

Now, suppose $g : \Omega_1 \to \Omega_2$ is another extremal map. Since $g^{-1} \circ f$ is homotopic to identity, we conclude that either there exists a set of positive measure on Ω_1 for which $|\mu(g)(z)| > k$ or $\mu(g) = \mu(f)$. Since g is extremal, $||\mu(g)||_\infty = k$. Hence, $\mu(g) = \mu(f)$ is of Teichmüller type. □

4.2. Variational formulation of the extremal problem. In this section, we give a variational formulation of the extremal problem. The T-Map can then be computed through optimization techniques.

Recall that a Teichmüller extremal mapping is extremal in the sense of minimizing the $||\cdot||_\infty$ over all Beltrami differentials. According to the Teichmüller theory, the unique extremal map between multiply-connected domains is a Teichmüller map. Therefore, our goal is to look for a T-Map minimizing the conformality distortion. The extremal problem can then be formulated as follows:

$$f^* = \mathbf{argmin}_{f:\Omega_1\to\Omega_2} E_1(f) = \mathbf{argmin}_{f:\Omega_1\to\Omega_2}\{||\mu(f)||_\infty\} \tag{4.6}$$

subject to:

- $f^*(\partial\Omega_1) = \partial\Omega_2$ (boundary condition);
- $\mu(f^*) = k\frac{\overline{\varphi}}{\varphi}$ for some constant $0 \leq k < 1$ and integrable holomorphic function $\varphi : \Omega_1 \to \mathbb{C}$ $(\varphi \neq 0)$.

However, minimizing $E_1(f)$ with respect to the space of diffeomorphisms between Ω_1 and Ω_2 is difficult. In fact, let $f = f_1 + if_2$, the minimization problem can be expanded as follows:

$$f = \mathbf{argmin}_f\{||\mu(f)||_\infty\} = \mathbf{argmin}_f\{||\frac{\partial f/\partial \overline{z}}{\partial f/\partial \overline{z}}||_\infty\} \tag{4.7}$$

subject to $f(\partial\Omega_1) = \Omega_2$ and $\mu(f) = k\frac{\overline{\varphi}}{\varphi}$ for some constant $0 \leq k < 1$ integrable holomorphic function $\varphi : \Omega_1 \to \mathbb{C}$ $(\varphi \neq 0)$.

In order to minimize the above constrained minimization problem effectively, we propose to reformulate the energy functional to define it over the space of all Beltrami coefficients:

$$(\nu, f) = \mathbf{argmin}_{\nu:D_1\to\mathbb{C}} E_2(\nu) := \mathbf{argmin}_{\nu:\Omega_1\to\mathbb{C}}\{||\nu||_\infty\} \tag{4.8}$$

subject to:

- $\nu = \mu(f)$ and $||\nu||_\infty < 1$;
- $\nu = k\frac{\overline{\varphi}}{\varphi}$ for some constant $0 \le k < 1$ and holomorphic function $\varphi : D_1 \to \mathbb{C}$;
- $f(\partial\Omega_1) = \Omega_2$ (boundary condition).

Minimizing E_2 with respect to BCs subject to the constraints is advantageous since the diffeomorphic property of the mapping can be easily controlled. Every diffeomorphism is associated with a smooth Beltrami coefficient $\mu(f)$. $\mu(f)$ measures the bijectivity (1-1 and onto) of f. In fact, $\mu(f)$ is related to the Jacobian $J(f)$ of f by the following formula:

$$|J(f)|^2 = |\frac{\partial f}{\partial z}|^2(1 - |\mu(f)|^2) \tag{4.9}$$

Therefore, the map f is bijective if $|\mu(f)|$ is everywhere less than 1. When solving the minimization problem (4.8), the bijectivity of the mapping in each iterations can be ensured by enforcing $||\nu||_\infty < 1$. Our goal is to look for a sequence of $\{\nu_n\}_{n=1}^\infty$ converging to the optimal BC, ν^*, which corresponds to our desired Teichmüller extremal map.

In section 5, we describe a numerical iterative scheme to obtain such a sequence.

5. Proposed algorithm. We describe our proposed method to compute the T-Map in this section.

5.1. Beltrami holomorphic flow(BHF). The numerical computation of a T-Map is equivalent to finding its associated Beltrami coefficient(BC). As BC varies, its associated quasi-conformal map varies and vice versa. We first examine the relationship between the variation of BCs and their associated quasi-conformal map.

Let $f^\mu : \Omega_1 \to \Omega_2$ be a quasi-conformal map, whose BC is $\mu : \Omega_1 \to \mathbb{C}$. Assume μ varies by ω, and assume its associated quasiconformal map $f^{\mu+\omega}$ varies by $\vec{V}$. In other words, $f^{\mu+\omega}(z) = f^\mu(z) + \vec{V}(z)$. Obviously, $\vec{V}$ depends on ν. In fact, if $f^{\mu+t\omega}(z) = f^\mu(z) + \vec{V_t}(z)$ $(t \in \mathbb{C})$, then $\vec{V_t}(z)$ depends holomorphically on $t \in \mathbb{C}$. We call the flow from f^μ to $f^{\mu+t\omega} = f^\mu(z) + \vec{V_t}(z)$ the *Beltrami holomorphic flow* (BHF) from μ to $\mu + t\omega$ [10, 9, 13, 22]. In particular, $\vec{V}(z) = \vec{V_1}(z)$.

We shall develop an algorithm to obtain $\vec{V}$. Let $\nu = \mu + \omega$. Our problem can be simply put as finding the variation $\vec{V}$ as μ changes to ν. Hence, $f^\nu = f^\mu + \vec{V}$.

Theorem 5.1. *Let f^μ and f^ν be the quasi-conformal maps with Beltrami coefficients $\mu : \Omega_1 \to \mathbb{C}$ and $\nu : \Omega_1 \to \mathbb{C}$ respectively. Suppose $f^\nu = f^\mu + \vec{V}$. Let $\mathcal{A}$ be the differential operator defined by $\mathcal{A} := \frac{\partial}{\partial\bar{z}} - \nu\frac{\partial}{\partial z}$. Then:*

$$\mathcal{A}\vec{V} = -\mathcal{A}f^\mu \tag{5.1}$$

Proof. Since f^ν is the quasi-conformal map with Beltrami coefficient $\nu : \Omega_1 \to \mathbb{C}$, $\frac{\partial f^\nu}{\partial \bar{z}} = \nu \frac{\partial f^\nu}{\partial z}$. Equivalently,

$$\mathcal{A} f^\nu = (\frac{\partial}{\partial \bar{z}} - \nu \frac{\partial}{\partial z}) f^\nu = 0. \tag{5.2}$$

Now, since $f^\nu = f^\mu + \vec{V}$, we obtain

$$\mathcal{A} f^\nu = \mathcal{A}(f^\mu + \vec{V}) \Longrightarrow 0 = \mathcal{A}(f^\mu + \vec{V}) \tag{5.3}$$

Hence, $\mathcal{A}\vec{V} = -\mathcal{A} f^\mu$ as required. □

In other words, finding $\vec{V}$ is equivalent to solving the partial differential equation (5.1) subject to the boundary condition that

$$f^\nu + \vec{V}(\partial\Omega_1) = \partial\Omega_2 \tag{5.4}$$

Using Theorem 5.1, we propose to iteratively deform f^μ to f^ν. More specifically, our goal is to obtain a sequence of quasi-conformal maps $\{f_n\}_{n=1}^\infty$ such that $f_0 = f^\mu$ and $f_\infty = f^\nu$. To do this, we first set $f_0 = f^\mu$. We then approximate the solution of Equation (5.1) with the boundary constraint using the least square method to obtain $\vec{V}_0$. We get a new quasi-conformal map $f_1 := f_0 + \vec{V}_0$. Suppose at the n^{th} iteration, we have the quasi-conformal map f_n with Beltrami coefficient ν_n. We then approximate the solution of Equation (5.1) by putting $\mu = \nu_n$ to obtain $\vec{V}_n$. Set $f_{n+1} := f_n + \vec{V}_n$. A sequence of quasi-conformal maps $\{f_n\}_{n=1}^\infty$is obtained, whose Beltrami coefficients converge to ν. We call such a process to deform f^μ to f^ν iteratively the *Beltrami holomorphic flow (BHF)* from μ to ν, and denote it by: **BHF**$(\mu \to \nu)$.

The iterative scheme to obtain the Beltrami holomorphic flow can be described as follows:

Algorithm 6.1 : *(Beltrami holomorphic flow)*
Input : *$f^\mu : \Omega_1 \to \Omega_2$ with Beltrami coefficient μ*
Output : *Sequence of quasi-conformal maps $\{f_n\}_{n=1}^\infty$*

1. *Set $f_0 = f^\mu$. Solve Equation (5.1) to obtain $\vec{V}_0$;*
2. *Given f_n, compute $\nu_n := \mu(f_n)$; solve Equation (5.1) by putting $\mu = \nu_n$ to obtain $\vec{V}_n$; Set $f_{n+1} := f_n + \vec{V}_n$;*
3. *If $||\nu_{n+1} - \nu_n|| \geq \epsilon$, repeat step 2. Otherwise, stop the iteration.*

5.2. Iteration scheme for computing T-Maps. Our goal is to obtain a sequence of Beltrami coefficients $\{\nu_n\}_{n=1}^\infty$ converges to the optimal ν^* associated to the desired T-Map f^*.

Given an initial map $f_0 : \Omega_1 \to \Omega_2$ such that $f_0(\partial\Omega_1) = \partial\Omega_2$, let $\nu_0 = \mu(f_0)$ be the Beltrami coefficient associated with f_0. We proceed to iteratively adjust ν_0 to solve the optimization problem (4.8).

Recall that our optimal ν^* must be of Teichmüller type. That is, $\nu^* \in T(\Omega_1)$ where

$$T(\Omega_1) := \{\nu : \Omega_1 \to \mathbb{C} : \nu = k\frac{\bar{\varphi}}{|\varphi|}, 0 \leq k < 1, \int_{\Omega_1} |\varphi| < \infty\} \tag{5.5}$$

To find the desired T-Map, our strategy is to apply an iterative minimization scheme over the space of $T(\Omega_1)$ to minimize $E_2(\nu) = ||\nu||_\infty$. Firstly, we descend ν

to minimize E_2 over the space $\mathfrak{B}(\Omega_1)$ of all Beltrami coefficients. We then project ν into $T(\Omega_1)$. A sequence of Beltrami coefficients can be obtained, whose supreme norms monotonically decreases. According to Theorem 4.2, the minimizer exists and is unique. Hence, the sequence converges to an optimal Beltrami coefficient ν^* associated to our desired Teichmüller extremal map f^*.

The algorithm can be described more specifically as follows. Given ν_0, we first project it into the space of $T(\Omega_1)$. To do this, we normalize ν_0 by an averaging operator:

$$\mathcal{N}(\nu) = (\frac{\int_{\Omega_1} |\nu| d\Omega_1}{A(\Omega_1)})e^{i\theta} \tag{5.6}$$

where $\nu = e^{i\theta}$ and $A(\Omega_1) =$ area of Ω_1.

We can then obtain a quasi-conformal map $g := \mathbf{BHF}(\nu_\mathbf{0} \to \mathcal{N}(\nu_\mathbf{0})))$, whose Beltrami coefficient is given by ν. Note that ν is generally not in $T(\Omega_1)$. We repeat the process by updating g by $g := \mathbf{BHF}(\nu \to \mathcal{N}(\nu))$. We stop the process until $|\nu \to \mathcal{N}(\nu)| < \epsilon$. Eventually, we obtain a Teichmüller map $g : \Omega_1 \to \Omega_2$, whose Beltrami coefficient $\nu \in T(\Omega_1)$ is closest to ν_0. We call this process the *projection* of ν_0 into the space of $T(\Omega_1)$, and denote it by $(\nu, g) = \mathcal{P}(\nu_0)$.

Such a projection is guaranteed to exist and its supreme norm must be less than $||\nu_0||_\infty$, according to Theorem 4.1. The convergence of the above process of obtaining a T-Map, which is homotopic to ν_0, can be verified by an argument of harmonic energy minimization with respect to a special metric [24]. Simply speaking, each $\mu \in T(\Omega_1)$ induces a metric on Ω_1. A unique harmonic map between Ω_1 and Ω_2 with respect to this metric exists, which is associated with a harmonic energy $E_{harmonic}$. The function $E_{harmonic} : \mathfrak{B}(\Omega_1) \to \mathbb{R}^+$ is convex. The minimizer is the unique Beltrami coefficient associated with the T-Map homotopic to ν_0. The minimizer can then be obtained by a standard descent method.

Now, to minimize $E_2(\nu) = ||\nu||_\infty$, we perform a damping operation on ν. That is, we diffuse ν through

$$\frac{\partial \nu}{\partial t} = -\nu \tag{5.7}$$

We diffuse ν over a finite time over $\mathfrak{B}(\Omega_1)$, and denote the damping operation on ν by $\mathcal{D}(\nu)$. It reduces $||\nu||_\infty$.

A new map $g_1 : \Omega_1 \to \Omega_2$ can then be obtained by BHF with the given ν_0:

$$g_1 := \mathbf{BHF}(\nu_\mathbf{0} \to \mathcal{P}(\mathcal{D}(\nu_\mathbf{0}))) \tag{5.8}$$

Let $\mu_1 := \mu(g_1)$. We project μ_1 into the space of $T(\Omega_1)$ to get $\nu_1 := \mathcal{P}(\mu_1)$ and Teichmüller map f_1 associated to ν_1. The iteration continues until it converges to the optimal ν^* associated to the desired extremal T-Map f^*. According to Theorem 4.2, the minimizer uniquely exists and is of Teichmüller type.

Therefore, given $f_n : \Omega_1 \to \Omega_2$ whose Beltrami coefficient is ν_n, we adjust ν_n and f_n as follows:

$$\begin{aligned} g_{n+1} &:= \mathbf{BHF}(\nu_\mathbf{n} \to \mathcal{D}(\nu_\mathbf{n})) \\ \mu_{n+1} &:= \mu(g_{n+1}) \\ (\nu_{n+1}, f_{n+1}) &:= \mathcal{P}(\mu_{n+1}) \end{aligned} \tag{5.9}$$

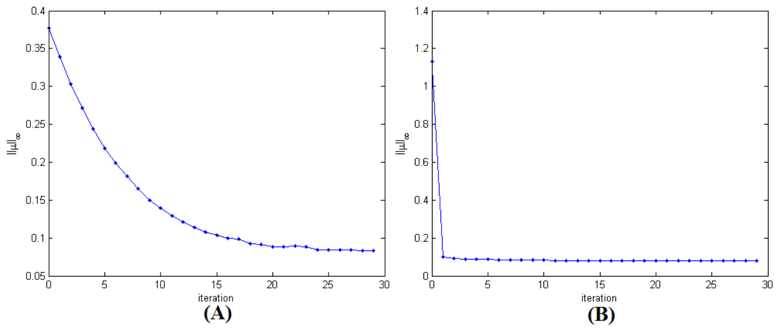

FIG. 5.1. *Rate of convergency under different initializations. (A) shows the supreme norm of ν_n versus iterations under an arbitrary initialization. The iterative scheme takes about 25 iterations to converge. (B) shows the supreme norm of ν_n versus iterations under the initialization described in section 5.3. The iterative scheme get to the optimal Beltrami coefficient in about 1 or 2 iteration.*

As a result, we obtain a sequence of Beltrami coefficients $\{\nu_n\}_{n=1}^{\infty}$ converges to the optimal ν^* associated to the desired T-Map f^*, which solves the optimization problem (4.8). In summary, the iterative scheme for computing T-Maps can be described as follows:

Algorithm 6.2 : *(Iteration scheme for computing T-Maps)*
Input : *Multiply-connected domains Ω_1 and Ω_2 of the same topology*
Output : *Optimal Beltrami coefficient ν^* and the T-Map f^**

1. *Obtain an initial map $f_0 : \Omega_1 \to \Omega_2$ with $f_0(\partial\Omega_1) = \partial\Omega_2$. Set $\nu_0 = \mu(f_0)$;*
2. *Given ν_n, compute $g_{n+1} := \mathbf{BHF}(\nu_\mathbf{n} \to \mathcal{D}(\nu_\mathbf{n}))$ and $\mu_{n+1} := \mu(g_n)$; Project μ_{n+1} into $T(\Omega_1)$ to obtain $(\nu_{n+1}, f_{n+1}) := \mathcal{P}(\mu_{n+1})$;*
3. *If $||\nu_{n+1} - \nu_n|| \geq \epsilon$, repeat step 2. Otherwise, stop the iteration.*

5.3. Initialization of the iterative scheme. In practice, the iterative scheme to obtain the T-Map converges fast if a good initialization is chosen.

We propose to obtain an initial map f_0 whose Beltrami coefficient is closest to $\mu = 0$ in the least square sense, using Beltrami holomorphic flow. We first compute a harmonic map $h : \Omega_1 \to \Omega_2$ between Ω_1 and Ω_2 with arbitrary boundary correspondence. Let μ_h be the Beltrami coefficient of h. We then obtain an initial map f_0 by $f_0 = \mathbf{BHF}(\mu_\mathbf{h} \to \mu \equiv \mathbf{0})$.

Experimental results show that with this initialization, the iterative scheme converges very fast. As shown in Figure 5.1(A), we show the supreme norm of ν_n versus iterations under an arbitrary initialization. The iterative scheme takes about 25 iterations to converge. In 5.1(B), we perform the same experiment but using the initialization introduced in this subsection. The iterative scheme get to the optimal Beltrami coefficient in about 1 or 2 iterations. Despite that different initializations are used, the iterative scheme converge to the same extremal map. It illustrates that the extremal map is unique. In all experiments we have done, our iterative scheme gets to the extremal map in about 1 or 2 iterations if the proposed initialization is used.

6. Numerical implementation. In this section, we will explain in detail the numerical implementation of the algorithms proposed in section 5.

In practice, multiply-connected 2D domains or surfaces in $\mathbb{R}^3$ are usually represented discretely by triangular meshes. Suppose K_1 and K_2 are two meshes with the same topology representing Ω_1 and Ω_2. We define the set of vertices on K_1 and K_2 by $V^1 = \{v_i^1\}_{i=1}^n$ and $V^2 = \{v_i^2\}_{i=1}^n$ respectively. Similarly, we define the set of triangular faces on K_1 and K_2 by $F^1 = \{T_j^1\}_{j=1}^m$ and $F^2 = \{T_j^2\}_{j=1}^m$. Our goal is to look for a piecewise linear homeomorphism between K_1 and K_2 that approximates the Teichmüller extremal mapping between Ω_1 and Ω_2.

6.1. Implementation details of BHF. The major step in computing the Beltrami holomorphic flow as described in Algorithm 6.1 is to solve equation (5.1). We first discretize the operator $\mathcal{A}$ in equation (5.1). Let $f = (u + \sqrt{-1}v) : K_1 \to K_2$. To compute $\mathcal{A}$, we simply need to approximate the partial derivatives at every face T. We denote them by $D_x f(T) = D_x u + \sqrt{-1}D_x v$ and $D_y f(T) = D_y u + \sqrt{-1}D_y v$ respectively. Note that f is piecewise linear. The restriction of f on each triangular face T can be written as:

$$f|_T(x,y) = \begin{pmatrix} a_T x + b_T y + r_T \\ c_T x + d_T y + s_T \end{pmatrix} \tag{6.1}$$

Clearly, $D_x u(T) = a_T$, $D_y u(T) = b_T$, $D_x v(T) = c_T$ and $D_y v(T) = d_T$. Now, the gradient $\nabla_T f := (D_x f(T), D_y f(T))^t$ on each face T can be computed by solving the linear system:

$$\begin{pmatrix} \vec{v}_1 - \vec{v}_0 \\ \vec{v}_2 - \vec{v}_0 \end{pmatrix} \nabla_T \tilde{f}_i = \begin{pmatrix} \frac{\tilde{f}_i(\vec{v}_1) - \tilde{f}_i(\vec{v}_0)}{|\vec{v}_1 - \vec{v}_0|} \\ \frac{\tilde{f}_i(\vec{v}_2) - \tilde{f}_i(\vec{v}_0)}{|\vec{v}_2 - \vec{v}_0|} \end{pmatrix}, \tag{6.2}$$

where $[\vec{v_0}, \vec{v_1}]$ and $[\vec{v_0}, \vec{v_2}]$ are two edges on T. By solving equation 6.2, a_T, b_T, c_T and d_T can be obtained. Hence on each face T,

$$\nabla_T \tilde{f}_i = \frac{1}{2A} \sum_{j=1}^{3} \tilde{f}_i(\vec{v}_j)\vec{s}_j, \tag{6.3}$$

where A is the area of T and

$$\begin{aligned} \vec{s}_1^T &= \vec{n} \times (\vec{v}_3 - \vec{v}_2) \\ \vec{s}_2^T &= \vec{n} \times (\vec{v}_1 - \vec{v}_3) \\ \vec{s}_3^T &= \vec{n} \times (\vec{v}_2 - \vec{v}_1), \end{aligned} \tag{6.4}$$

where $\vec{n}$ is the unit normal of T. On each face T, let $\nu(T) \equiv \nu_T$. Using the relations $\frac{\partial}{\partial z} = (D_x - \sqrt{-1}D_y)/2$ and $\frac{\partial}{\partial \bar{z}} = (D_x + \sqrt{-1}D_y)/2$, the operator $\mathcal{A}$ can be discretized on each face T as follows:

$$\mathcal{A}\tilde{f}_i(T) = \frac{1}{4A}(1 - \nu_T, \sqrt{-1} + \sqrt{-1}\nu_T) \sum_{j=1}^{3} \tilde{f}_i(\vec{v}_j)\vec{s}_j^T. \tag{6.5}$$

Note that the right hand side of the above equation is linear in every $\tilde{u}_i(\vec{v}_j)$ and $\tilde{v}_i(\vec{v}_j)$, $j = 1, 2, 3$. Hence, the above discretization of $\mathcal{A}$ transforms (5.1) into a linear system of $\vec{V}(\vec{v}_i)$, $i = 1, \cdots, n$. Let $\vec{V}(\vec{v}_i) = (P_i, Q_i)^t$ and $f^\mu(\vec{v}_i) = u_i + \sqrt{-1}v_i$, then for each face T_j, $j = 1, \cdots, m$, we have

$$\begin{aligned}&\frac{1}{4Area(T_j)}(1-\nu_{T_j}, \sqrt{-1}+\sqrt{-1}\nu_{T_j})\sum_{i=1}^{3}(P_{T_j(i)}+\sqrt{-1}Q_{T_j(i)})\vec{s}_i^{T_j}\\ =&-\frac{1}{4Area(T_j)}(1-\nu_{T_j}, \sqrt{-1}+\sqrt{-1}\nu_{T_j})\sum_{i=1}^{3}(u_{T_j(i)}+\sqrt{-1}v_{T_j(i)})\vec{s}_i^{T_j},\end{aligned} \tag{6.6}$$

where $T_j(i), i = 1, 2, 3$, is such that $T_j = [\vec{v}_{T_j(1)}, \vec{v}_{T_j(2)}, \vec{v}_{T_j(3)}]$.

Secondly, the boundary constraint (5.4) can be approximated by a linear constraint, so that the least square method can be applied to solve the problem. For each boundary vertex $\vec{v}_i \in \partial\Omega_1$, we only require $\vec{V}(\vec{v}_i)$ to be parallel to the tangent of $\partial\Omega_2$ at $f^\mu(\vec{v}_i)$ for each boundary vertex $\vec{v}_i \in \partial\Omega_1$,. That is, if $\vec{V}(\vec{v}_i) = (P_i, Q_i)^t$ and $(a_i, b_i)^t$ is the direction of the tangent, then

$$b_i P_i - a_i Q_i = 0, \tag{6.7}$$

which is a linear constraint. The linear system (6.6) together with the constraint (6.7) may be overdetermined. Therefore we solve the system by least square method. However, it is equivalent to solve a non-singular linear system, so the system can be solved effectively. For each iteration of Algorithm 6.1, $\vec{V}_n(\vec{v}_i)$ is solved as above. Set $\tilde{f}_{n+1}(\vec{v}_i) := f_n(\vec{v}_i) + \vec{V}_n(\vec{v}_i)$. For each boundary vertex $\vec{v}_i$, it is not necessary that $\tilde{f}_{n+1}(\vec{v}_i) \in \partial\Omega_2$. Nevertheless, when $||\nu||_\infty$ is small enough, $\tilde{f}_{n+1}(\vec{v}_i)$ will not be far away from $\partial\Omega_2$. Hence we can project $\tilde{f}_{n+1}(\vec{v}_i)$ onto $\partial\Omega_2$ and obtain the solution $f_{n+1}(\vec{v}_i)$ such that $f_{n+1}(\partial\Omega_1) = \partial\Omega_2$, i.e.

$$f_{n+1}(\vec{v}_i) := \mathbf{argmin}_{f^* \in \partial\Omega_2} \|\tilde{f}_{n+1}(\vec{v}_i) - f^*\|_2. \tag{6.8}$$

6.2. Implementation details of the iterative scheme. The main operators involved in the iterative scheme proposed in section (5.2) are: $\mathbf{BHF}(\mu \to \nu)$, $\mathcal{L}$ and $\mathcal{P}$. The numerical implementation of $\mathbf{BHF}(\mu \to \nu)$ was described in the last subsection. We now describe the numerical implementation of $\mathcal{L}$ and $\mathcal{P}$ in detail.

Recall that the Laplace smooth $\mathcal{L}(\nu)$ diffuses ν through $\frac{\partial \nu}{\partial t} = -\nu$. In the discrete case, we define the damping operator as follows:

$$\mathfrak{D}(\nu)(T) := \nu(T) - \epsilon\nu(T) \tag{6.9}$$

where T is a triangular face of K_1, $\epsilon > 0$.

Another operator is the projection operator $\mathcal{P}(\nu)$, which projects ν into the space of BCs of Teichmüller type $T(\Omega_1)$. In the discrete case, the projection is defined as follows: is defined as follows:

$$\mathcal{P}(\nu)(T) := \left(\frac{\sum_{T\in \text{ all faces of } K_1} |\nu|(T)}{\text{No. of faces of } K_1}\right)\frac{\nu(T)}{|\nu(T)|} \tag{6.10}$$

where T is a triangular face of K_1, Nbhd(T) is the set of neighborhood faces of T and $|\text{Nbhd}(T)|$ is the number of neighborhood faces in the set Nbhd(T).

7. Experiments. We have tested our proposed algorithms on synthetic data together with real 3D surface data obtain from the 3D scanner. All experiments are carried out on a laptop with an Intel Core i7 2.10 GHz CPU and 12GB RAM.

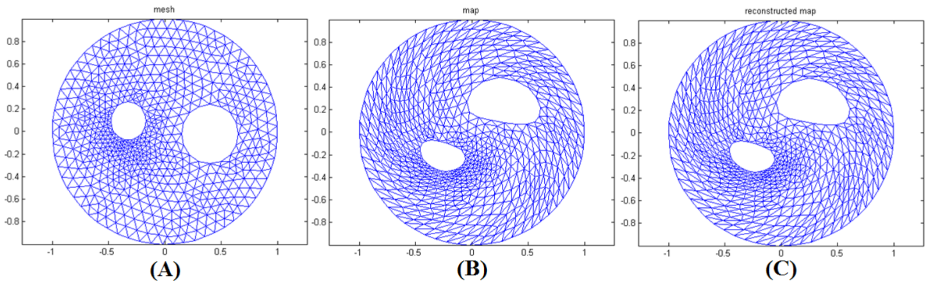

FIG. 7.1. *(A) and (B) shows two triply-connected domains. A quasi-conformal map f between (A) and (B) is given and its associated Beltrami coefficient is μ_f. (C) shows the reconstructed map obtained from μ_f using BHF.*

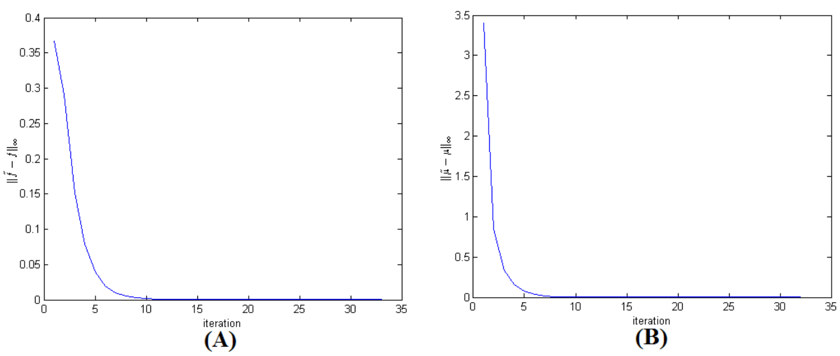

FIG. 7.2. *(A) and (B) show the error in coordinates $||f - f_n||_\infty$ and error in BC $||\mu_f - \mu(f_n)||_\infty$ versus iterations respectively of the experiment in Figure 7.1.*

7.1. Performance of BHF. We first examine the performance of BHF to iteratively flow a map to another quasi-conformal map with the prescribed Beltrami coefficient(BC). In Figure 7.1, we apply the BHF to get a quasi-conformal map with prescribed BC between two triply-connected domains. (A) and (B) shows two triply-connected domains. Given a quasi-conformal map f between (A) and (B), we obtain a Beltrami coefficient μ_f corresponding to f. Using BHF, we can reconstruct the map f from μ_f. (C) shows the reconstructed map, which closely resembles to the original one. Figure 7.2(A) and (B) show the error in coordinates $||f - f_n||_\infty$ and error in BC $||\mu_f - \mu(f_n)||_\infty$ versus iterations respectively. Both converge to 0 quickly in less than 10 iterations.

We repeat the experiment to compute the quasi-conformal map between two circle domains with three holes using BHF. Figure 7.5(A) and (B) shows two circle domains with three holes. We compute the Beltrami coefficient μ_f corresponding to a given map f. Using BHF, we reconstruct f from μ_f. (C) shows the reconstructed map, which closely resembles to the original one. Figure 7.7 shows the error in coordinates $||f - f_n||_\infty$ and error in BC $||\mu_f - \mu(f_n)||_\infty$ versus iterations. Again, both converge to 0 quickly in less than 10 iterations.

7.2. T-Maps between 2D multiply-connected domains. We test our proposed iterative scheme to compute the Teichmüller extremal map on synthetic 2D multiply-connected domains. In Figure 7.5, we compute the Teichmuller extremal

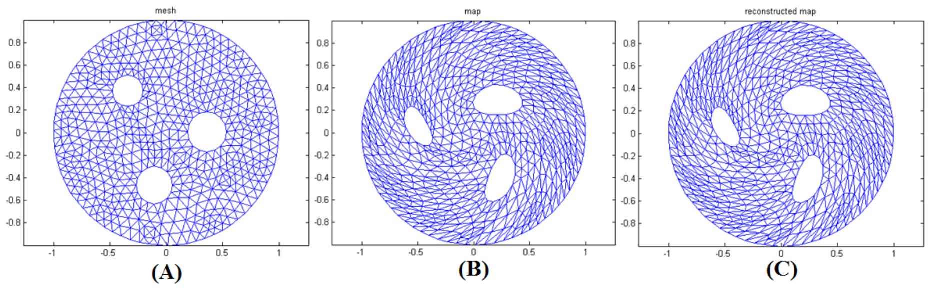

FIG. 7.3. *(A) and (B) shows two circle domains with three holes. A quasi-conformal map f between (A) and (B) is given and its associated Beltrami coefficient is μ_f. (C) shows the reconstructed map obtained from μ_f using BHF.*

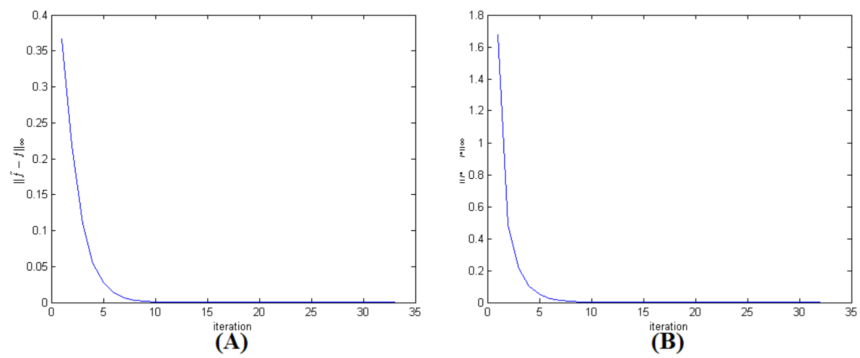

FIG. 7.4. *(A) and (B) show the error in coordinates $||f - f_n||_\infty$ and error in BC $||\mu_f - \mu(f_n)||_\infty$ versus iterations respectively of the experiment in Figure 7.5.*

map between two circle domains with three holes. (A) and (B) shows the two multiply-connected domains. The obtained extremal T-Map is visualized using texture mapping. The small circles on (A) are mapped to small ellipses on (B) under the extremal T-Map, with the uniform eccentricity. In (C), we show the histogram of the norm of the BC. It accumulates at 0.53, meaning that the conformality distortion is uniform over the whole domain. It means the computed extremal map is indeed a Teichmüller map.

Our proposed algorithm automatically determines the optimal boundary correspondence such that the conformality distortion is minimized. Figure 7.6 shows the boundary correspondence for each boundary component, plotted as a monotonic function from $[0, 2\pi]$ to $[0, 2\pi]$.

Figure 7.7(A) shows $||\mu(f_n)||_\infty$ versus iterations. (B) shows the zoom-in of (A). The plots show that our proposed algorithm iteratively minimize the supreme norm of BC. The optimal Beltrami coefficient is associated to our desired extremal map, which is a Teichmüller map.

We also test the algorithm on synthetic circle domains with more holes. Figure 7.8 shows the Teichmüller extremal map between two multiply-connected domains with 5 holes obtained by our proposed method. The histogram of the BC norm is shown in (C), which means that the conformality distortion is uniform everywhere. In Figure 7.9, we test our method on circle domains with 9 holes. (B) shows the obtained Teichmüller extremal map visualized by texture mapping. The histogram

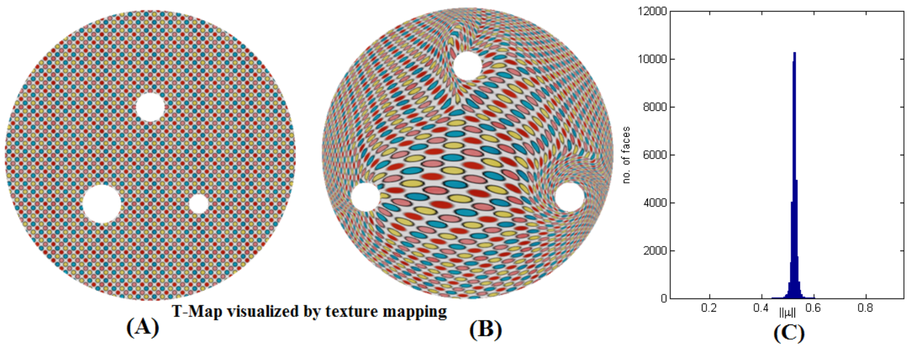

Fig. 7.5. *(A) and (B) shows the two multiply-connected domains with three holes. The obtained extremal T-Map is visualized by texture mapping. The small circles on (A) are mapped to small ellipses on (B) under the extremal T-Map, with the same eccentricity. (C) shows the histogram of the norm of the BC.*

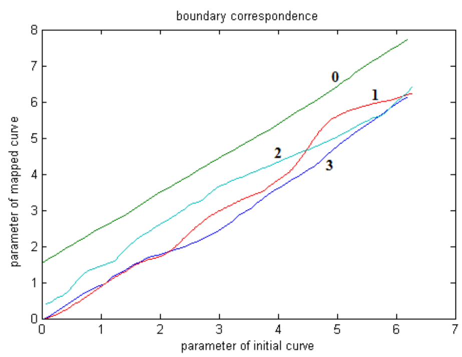

Fig. 7.6. *This figure shows the boundary correspondence for each boundary component, plotted as a monotonic function from* $[0, 2\pi]$ *to* $[0, 2\pi]$.

of the BC norm is shown in (C), which means the obtained map is indeed a T-Map. This example demonstrates the effectiveness of our algorithm even on complicated domains with many holes.

Furthermore, our method can be applied to any multiply-connected domains with arbitrary shapes (not restricted to circle domains). Figure 7.10 shows the computed Teichmüller extremal map between two multiply-connected domains of arbitrary shapes with two holes, visualized by texture mapping. Again, the histogram of the BC norm (as shown in (C)) shows that the conformality distortion is uniform everywhere, meaning that the extremal map is indeed of Teichmüller type.

7.3. T-Maps between multiply-connected surfaces. The proposed method can be easily extended to compute extremal T-Map between general multiply-connected surfaces through conformal parameterization. We test the proposed method to compute extremal map between multiply-connected 3D human faces. We also apply the algorithm to obtain the extremal parameterization of the human face onto a simple user-defined parameter domain, which is important for grid generation.

Figure 7.11(A) and (B) shows two multiply-connected human faces. The extremal T-Map between them is computed, which is visualized using texture mapping. The

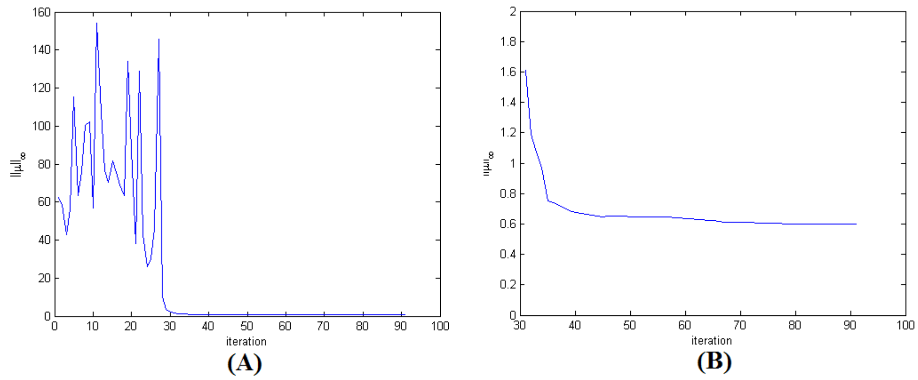

FIG. 7.7. *(A) shows $||\mu(f_n)||_\infty$ versus iterations of the experiment in Figure 7.5. (B) shows the zoom-in of (A).*

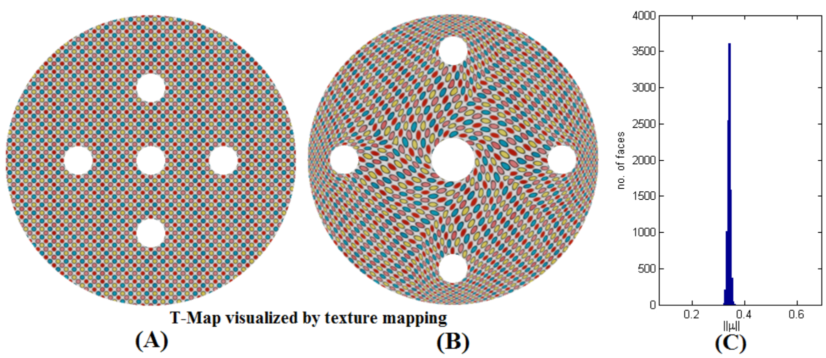

FIG. 7.8. *(A) and (B) shows the two multiply-connected domains with five holes. The obtained extremal T-Map is visualized by texture mapping. The small circles on (A) are mapped to small ellipses on (B) under the extremal T-Map, with the uniform eccentricity. (C) shows the histogram of the norm of the BC.*

small circles on (A) are mapped to small ellipses on (B) under the T-Map, with the same eccentricity. In (C), we show the histogram of the norm of the BC. It accumulates at 0.21, meaning that the conformality distortion is uniform everywhere. Figure 7.12(A) shows the supreme norm of BC, $||\mu(f_n)||_\infty$ in each iterations. (B) shows the zoom-in of (A). Again, the plots show that our proposed algorithm iteratively minimizes the supreme norm of BC. The optimal Beltrami coefficient is associated to our desired extremal map between the two multiply-connected human faces.

In Figure 7.13, we compute the extremal parameterization of the multiply-connected human faces with three holes. (A) shows a simple user-defined parameter domain. Using our algorithm, an extremal map parameterizing the human face onto the simple parameter domain with least conformality distortion can be obtained. On the simple parameter domain, structured grids can easily obtained as shown in (A). Using the extremal T-Map, we map the structured grid onto the multiply-connected human face as shown in (B). (C) shows the histogram of the BC norm which illustrates that the obtained map is indeed of Teichmüller type. In (D), we map the checkerboard texture on the parameter domain onto the human face using the T-Map. This example demonstrates that the Teichmüller extremal map can be applied for grid generation

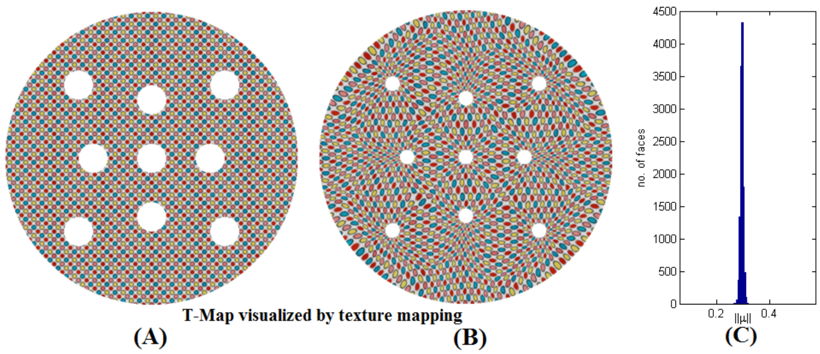

Fig. 7.9. *(A) and (B) shows the two multiply-connected domains with nine holes. The obtained extremal T-Map is visualized by texture mapping. The small circles on (A) are mapped to small ellipses on (B) under the extremal T-Map, with the uniform eccentricity. (C) shows the histogram of the norm of the BC.*

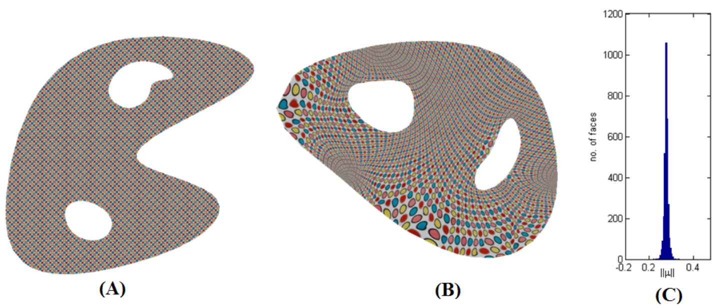

Fig. 7.10. *(A) and (B) shows the two multiply-connected domains of arbitrary shapes with two holes. The obtained extremal T-Map is visualized by texture mapping. The small circles on (A) are mapped to small ellipses on (B) under the extremal T-Map, with the uniform eccentricity. (C) shows the histogram of the norm of the BC.*

on multiply-connected domains or surfaces.

8. Conclusion. In this paper, we present a numerical method to compute the Teichmüller extremal map between arbitrary multiply-connected domains. The domains of interest can either be planar domains or surfaces embedded in $\mathbb{R}^3$. Given two multiply-connected domains with boundaries, there exists a unique Teichmüller map(T-Map) between them minimizing the conformality distortion. The T-Map can be considered as the "most conformal" mapping between multiply-connected domains. In this work, we propose an iterative algorithm to obtain the T-Map using the Beltrami holomorphic flow (BHF). The BHF procedure iteratively adjusts the map, based on a sequence of complex-valued functions converging to an optimal Beltrami coefficient associated to the desired T-Map. It produces a sequence of quasi-conformal maps, which converges to the T-Map minimizing the conformality distortion. We test our proposed algorithms on synthetic 2D multiply-connected domains together with real 3D human faces. Experimental results show that our algorithm computes T-Map between multiply-connected domains accurately and efficiently.

In the future, we will extend our algorithm to compute Teichmüller extremal map between high-genus surfaces and between surfaces represented by point clouds.

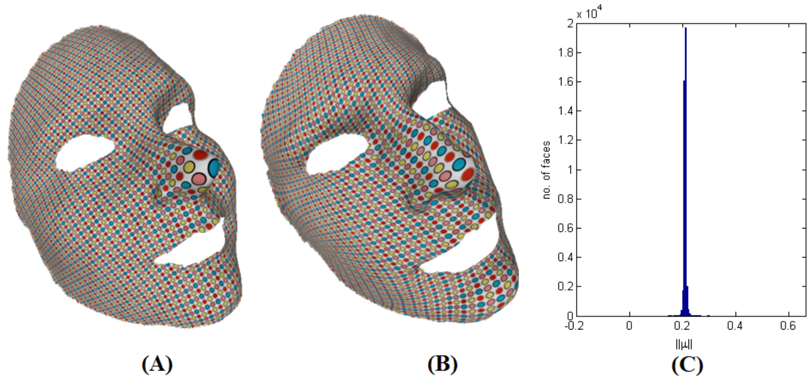

Fig. 7.11. *A) and (B) shows two multiply-connected human faces. The extremal T-Map between them is computed, which is visualized using texture mapping. (C) shows the histogram of the norm of the BC.*

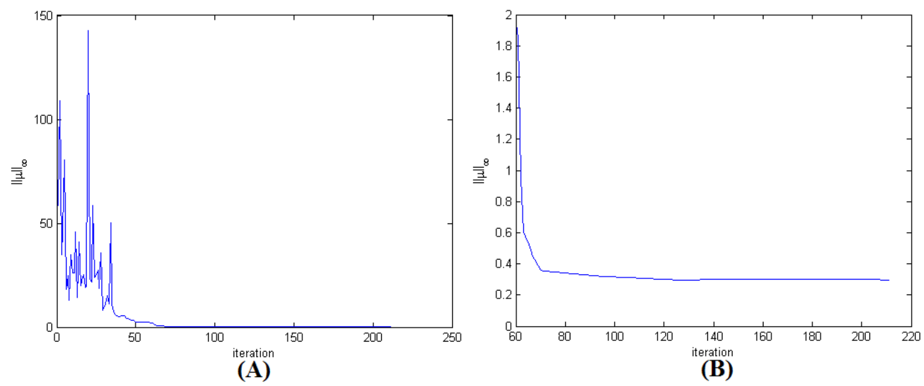

Fig. 7.12. *(A) shows the supreme norm of BC, $||\mu(f_n)||_\infty$ in each iterations of the experiment in Figure 7.11. (B) shows the zoom-in of (A).*

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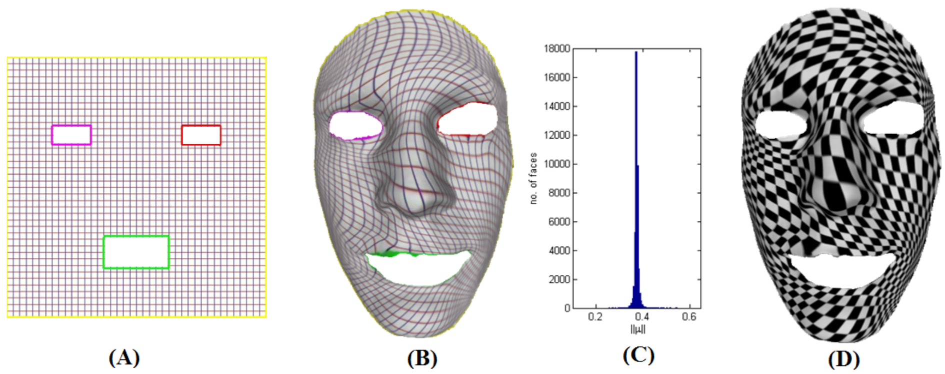

Fig. 7.13. *The extremal parameterization of the multiply-connected human faces with three holes. (A) shows a simple user-defined parameter domain. On the simple parameter domain, structured grids can easily obtained as shown in (A). In (B), the structured grid is mapped onto the multiply-connected human face using the extremal T-Map. (C) shows the histogram of the BC norm which illustrates that the obtained map is indeed of Teichmüller type. In (D), the checkerboard texture on the parameter domain is mapped onto the human face using the T-Map.*

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