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On the Mathematical Foundations of Computational Photography

Does the *flutter shutter* work better at night?

Yohann Tendero · Jean-Michel Morel

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Abstract *Flutter shutter* (coded exposure) cameras allow to extend indefinitely the exposure time for uniform motion blurs. Recently, Tendero et al. [42] proved that for a fixed known velocity v the gain of a *flutter shutter* in terms of root means square error (RMSE) cannot exceeds a 1.1717 factor compared to an optimal *snapshot*. The aforementioned bound is optimal in the sense that this 1.1717 factor can be attained.

However, this disheartening bound is in direct contradiction with the recent results by O. Cossairt, M. Gupta and S. Nayar [8]. Our first goal in this paper is to resolve mathematically this discrepancy. An interesting question was raised by the authors of [8]. They state that the "gain for computational imaging is significant only when the average signal level J is considerably smaller than the read noise variance σ_r^2" [8, page 5]. In other words, according to [8] a *flutter shutter* would be more efficient when used in low light conditions e.g. indoor scenes or at night. Our second goal is to prove that this statement is based on an incomplete camera model and that a complete mathematical model disproves it.

To do so we propose a general *flutter shutter* camera model that includes photonic, thermal[1] and additive[2] (sensor readout, quantification) noises of finite variances. Our analysis provides exact formulae for the mean square error of the final deconvolved image. It also allows us to confirm that the gain in terms of RMSE of *any* flutter shutter camera is bounded from above by 1.1776 when compared to an optimal *snapshot*. The bound is uniform with respect to the observation conditions and applies for any fixed known velocity. Incidentally, the proposed formalism and its consequences also apply to e.g. the Levin et al. *motion-invariant photography* [17,18] and variant [6]. In short, we bring mathematical proofs to the effect of contradicting the claims of [8]. Lastly, this paper permits to point out the kind of optimization needed if one wants to turn the *flutter shutter* into a useful imaging tool.

Keywords Computational photography · Motion blur · Flutter shutter · Motion-invariant photography · Coded exposure · Efficiency · Mean square error · Signal to noise ratio · Shannon-Whittaker interpolation.

Y. Tendero
LTCI, CNRS, Télécom ParisTech, Université Paris-Saclay, 75013, Paris, France, E-mail: yohann.tendero@telecom-paristech.fr. Part of this work was done at the UCLA Math. dept. Yohann Tendero is partially supported on the grant ONR No: N000141410683.

J.-M. Morel
CMLA, École Normale Supérieure de Cachan 61, Av du Président Wilson 94235 Cachan, France. Jean-Michel Morel is supported by the Office of Naval Research under grant N00014-97-1-0839, the European Research Council under advanced grant Twelve labours, Centre National d'Etudes Spatiales, and Project ASTRID 2012 STEREO, ANR-12-ASTR-0035.

1 Introduction

Digital cameras count at each pixel sensor the number of photons emitted by the observed landscape during an interval of time called exposure time. The photon count follows a Poisson random variable. Its mean is the ideal (noiseless) pixel value. The difference between

[1] The amplifier noise may also be mentioned as another source of background noise, which can be included w.l.o.g. in the thermal noise.

[2] The additive (sensor readout) noise may contain other components such as e.g. reset noise and quantization noise. We include them w.l.o.g. in the readout.

the mean and the observed sensor photon count is called (shot) noise. With a passive camera there is no control over the landscape photon emission. Thus, with a passive camera the only way to safely increase the signal to noise ratio (SNR) is to accumulate more photons by increasing the exposure time. Yet, when the scene or the camera move during the exposition process the resulting images are degraded by motion blur. A motion blur is not invertible as soon as the support of its kernel exceeds two pixels. Obtaining a longer exposure time without the effects of motion blur has therefore been long considered as one of the core problems of photography.

For uniform motion blurs a setup that solves this dilemma was proposed in [1,2,3,27,28,29]. The authors propose to attach a *flutter shutter* to a camera. The shutter interrupts the photon flux on sub-intervals of the exposure time and permits to get an invertible motion blur kernel. Numerically a *flutter shutter* is described by a binary shutter sequence -or *flutter shutter code*- giving the intervals where the photon flux is interrupted. If the *flutter shutter code* is well chosen, a *flutter shutter* can guarantee the invertibility of any uniform motion blur. This fact has received a mathematical proof in [42]. Thus, for arbitrarily high velocities or equivalently for arbitrary long exposures one can invert the blur by a simple deconvolution. Therefore, many more photons can be sensed by the camera. This seems to imply that the *flutter shutter* is a revolutionary tool that should equip all cameras. Yet, can a *flutter shutter* indefinitely increase the SNR by an increased exposure time? We proved that the answer is no [42]. More precisely, given a landscape moving in uniform translation at a known velocity v, the gain of a *flutter shutter* in terms of root mean square error (RMSE) cannot exceed a 1.1717 factor compared to an optimal *snapshot*. This bound is optimal in the sense that the 1.1717 factor can be attained but requires an infinite exposure time.

However, in [8] Cossairt, Gupta and Nayar claim that the maximal gain of any *flutter shutter* camera cannot exceed $\sqrt{1+\frac{\sigma_r^2}{J}}$ where σ_r^2 is the variance of an additive Gaussian sensor readout noise, and J the average signal. Thus, [8] contradicts [42] when $\sigma_r^2 = 0$. In [8], Cossairt et al. measure the gain in terms of RMSE with respect to a *snapshot*. In the sequel, we shall follow that path and also measure the gain of the *flutter shutter* in terms of RMSE with respect to a *snapshot*. Moreover, in [8] the motion blur kernel (velocity v) is assumed to be known. In the sequel we will make the same assumption and always assume that the relative camera/landscape velocity v is known. The other main result in [8, page 5] is that the "gain for computational imaging is significant only when the average signal level J is considerably smaller than the read noise variance σ_r^2." In other words, according to [8] a *flutter shutter* would be more efficient when used at night.

To resolve these contradictions, this paper develops a general mathematical theory of *flutter shutter* cameras. It starts by carefully modelling the first steps of image acquisition by a light sensor. The model includes obscurity (thermal) as well as additive (sensor readout, quantization) and photon (Poisson) noises without any approximation. In [42] we considered only the Poisson noise and omitted the additive and thermal noises that are taken into account here. We need to include these new noise components to decide if (in particular) the additive noise can affect the gain of a *flutter shutter* as suggested by Cossairt, Gupta and Nayar in [8]. In [42] we distinguished two implementations of a *flutter shutter*. For an *analog flutter shutter* the aperture is operated by a physical shutter in front of the sensor. The "shutter" controls the percentage of photons emitted by the landscape that can reach the sensor. Instead, the *numerical flutter shutter* is a temporal filter. It takes a series of consecutive snapshots and combines them numerically. It therefore allows for arbitrary codes, that can be non-binary and even negative. We proved in [42] that the *numerical flutter shutter* beats the *analog flutter shutter*. For each *flutter shutter*: *analog* or *numerical*, a closed formula will be given for the mean square error of the estimated restored signal. In addition, the results of Cossairt, Gupta and Nayar [8] will be carefully analyzed, discussed and finally disproved by pointing out a model error. The formalism, the analysis, consequences and, in particular the performance bound, also apply to the *motion-invariant photography* [17,18] and variant [6]. Indeed, the *motion-invariant photography* has been proved to be mathematically equivalent to an *analog flutter shutter* [42].

1.1 Related work

There is a considerable literature on the problem of increasing the SNR of moving scenes. The question of the optimal exposure time using a conventional camera is investigated in [4]. This paper [4] considers the case of non-invertible blurs with supports larger than two pixels and needs a regularized deconvolution [10]. Following the *flutter shutter* literature we shall not adopt this perspective here. Only well-posed deconvolution strategies will be compared. Indeed, the main goal of the *flutter shutter* is to make motion blur deconvolution a *well-posed problem.*

In [37] the authors use a full multi-image framework acquiring a burst of sharp but noisy images. They recover a sharp image with an increased SNR. For a review on multi-image denoising we refer to [5]. The denoising-by-registration-and-accumulation strategy is the classic and most convincing competitor to *flutter shutter* strategies. Since they rely on delicate registration algorithms they are greedy in memory and computations. Indeed, many images are acquired and fused. By opposition, with a *flutter shutter* method only one image is recorded and deconvolved.

In [12] the authors reconstruct a movie from a single image using a temporally and spatially varying mask placed on the aperture. The mask helps encoding the spatio-temporal information.

In [11,26,35,45,46,47] the authors use hybrid or complex camera systems. They raise other problems such as an expensive computational cost or hardware issues that we shall not discuss further. These issues are out of the scope of this paper.

A compressive sensing *flutter shutter* camera was designed in [33]. The *flutter shutter* ensures that no information is lost by the motion blur, and the compressed sensing technique deals with the resolution gain. The compressed sensing technique is also used in [30] for spatio-temporal up-sampling. Alternatively the case of periodic events is investigated in [31].

As we mentioned, the simplest setup was proposed by Agrawal, Raskar et al. [1,2,3,27,28,29]. These authors proposed to modulate the photon flux that enters the camera by opening and closing the camera shutter according to a pseudo random binary sequence. When the motion is uniform the resulting blur kernel becomes invertible, with a non-vanishing Fourier transform. The visual result of an image acquired by *flutter shutter* is close to a stroboscopic image. For visual examples, we refer to [38] that also provides a peer-reviewed simulator of *flutter shutter* cameras. Nonetheless one can recover a neat image after deconvolution. In [14,19,25, 44] the authors use an active dynamic lighting pattern in place of the shutter to recreate a *flutter shutter* effect. In [20] the *flutter shutter* apparatus is applied to iris images and in [48] to bar-codes. In [36] the authors use a local deblurring user-driven scheme on a *flutter shutter* embedded camera to deal with spatially varying blurs caused by the presence of several velocities in the observed scene. In [32] the authors treat the question of denoising an image taken by a *flutter shutter* camera and suggest an user assisted estimation of the blur. Their conclusion is that the denoising should be applied both before and after deconvolution [32]. In [9] the authors treat the question of *a posteriori* motion estimation using a *flutter shutter*. (In the present paper we shall follow e.g. [1,2,3,6,8,17,18,27,28,29,42] and assume that the velocity v is known.) Another solution to get an invertible motion blur using only one image was found in [17,18]. The *motion-invariant photography* [17,18] of Levin et al. consists in accelerating the camera in the direction of the scene motion. Hence, an *a priori* knowledge of the motion direction is required. This approach is generalized in [6] to the case of unknown directions. Note that [6] uses two images instead of one. The *motion-invariant photography* produces an invertible kernel that is mathematically equivalent to an *analog flutter shutter* kernel as proved in [42] (this permits to suppress the burden of moving the camera and works for any direction motion as any *flutter shutter*). In [21] the *motion-invariant photography* is implemented using the lens of the camera.

Most of these works actually develop more complex hardware setups than the original *flutter shutter*. Nevertheless, it is clear that the common denominator is to obtain a sharp image and to increase the exposure time. The exact evaluation of the mean square error (MSE) is thus the core question of these camera designs. Furthermore, a fair MSE comparison should have systematically been performed with a classic *snapshot*. To the best of our knowledge, the only papers that considered such comparisons are [8] and [42]. In [42] only the photon (Poisson) noise is treated. By opposition Cossairt et al. [8] aim at treating the additive (sensor readout) noise as well as the signal Poisson noise. In [8] the authors used several model approximations to model a *flutter shutter* camera. Here we intend to avoid these approximations and to treat the three kinds of noise (Poisson, additive, thermal). Thus, the proposed formalism covers the study and noise sources considered in [8], while [42] considered only the Poisson (photon) noise. To the best of our knowledge, this paper is the first to propose a simple formula modelling *flutter shutter* cameras in whole generality, taking into account all noise sources inherent to photon sensing. While the development of the theory is relatively long it delivers few and compact closed formulas summarizing the theory. Following e.g. [8] we shall retain the RMSE after deconvolution as a performance criterion.

Plan of the paper

Section 2 models the first steps of an image acquisition by a sensor array, treating the three noises. This model is applied in section 3 to the *analog flutter shutter*, which copes with all the aperture codes proposed in the literature on the *flutter shutter*. The model of section 2 is applied in section 4 to the *numerical flutter shutter*, introduced and proved to beat the *analog flutter shutter* in [42]. For each of these *flutter shutter* apparatus, sections 3 and 4 give two explicit formulas for the MSE of the final deconvolved image. The results of [8] are carefully analyzed and discussed in section 5.

The annexes A to D contain proofs used throughout this paper. A glossary of notations is available in annex E. (In the rest of the text, Latin numerals refer to formulae in the glossary page 18.)

2 The general image acquisition model

Formalizing the *flutter shutter* requires an accurate continuous stochastic model of the photon capture by a sensor array. As argued in [1,2,3,6,8,17,18,27,28,29, 39,40,41,42,43] the formalization will be done without loss of generality (w.l.o.g.) in the case where the sensor array is 1D. Thus, the photographed object can be conceived as a "landscape" moving in a direction parallel to the sensor array. Let $\mathbf{P}_l : [0, +\infty) \times \mathbb{R}$ be a bidimensional Poisson process of intensity $l(t, x)$, $\forall (t, x) \in [0, +\infty) \times \mathbb{R}$ (here l is called landscape, t and x are the time and spatial positions, respectively). We assume that $l \in L^1_{loc}$. The value $l(x)$ can be thought as the photon emission intensity of the observed scene at a position $x \in \mathbb{R}$. A pixel sensor is a photon counter. The photon emission follows a Poisson distribution [8]. This means that the observation of a pixel sensor of unit length centered at $x \in \mathbb{R}$ using an exposure time of $\Delta t > 0$ is a Poisson random variable

$$\mathbf{P}_l\left([0, \Delta t] \times \left[x - \frac{1}{2}, x + \frac{1}{2}\right]\right) \sim \mathcal{P}\left(\int_0^{\Delta t}\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} l(t, y)dydt\right),$$

where Δt is the exposure time, $\left[x - \frac{1}{2}, x + \frac{1}{2}\right]$ represents the normalized sensor unit and $X \sim P$ means that a random variable X has law P. In other terms the probability to observe k photons coming from the landscape l seen at the position x on the time interval $[0, \Delta t]$ and using a normalized sensor is

$$\frac{\left(\int_0^{\Delta t}\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} l(t, y)dydt\right)^k \exp\left(-\int_0^{\Delta t}\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} l(t, y)dydt\right)}{k!}.$$

From now on we assume $l(t, x) = l(x - tv(t))$, and mainly $v(t) \equiv v$. For sampling purposes we assume that the theoretical landscape l is seen through an optical system with a point spread function (PSF) g. We assume that g, as usual for optical kernels, belongs to the Schwartz class $g \in S(\mathbb{R})$ and is $[-\pi, \pi]$ band-limited, i.e. $\hat{g}$ (see the definition of the Fourier transform **(xxiii)**) is supported in $[-\pi, \pi]$.

Definition 1 (*O*bservable landscape.)
We call *ideal landscape* the non-negative deterministic function $\widetilde{u} := \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} * g * l$, where g is the *point spread function* of the optical system.

Hereinafter, $*$ denotes the convolution **(viii)**. Since $g \in S(\mathbb{R})$ is $[-\pi, \pi]$ band-limited we deduce that $\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} * g$ belongs to $S(\mathbb{R})$ and is also $[-\pi, \pi]$ band-limited. Thus, $\widetilde{u} \in L^1_{loc}(\mathbb{R})$ and is $[-\pi, \pi]$ band-limited. In other words, the convolution with the PSF g allows the acquisition system to sample the deterministic function $\widetilde{u}$ at unit rate, i.e. using normalized pixel sensors of unit length. We shall denote by $\widetilde{u}(x)$ the ideal (noiseless) pixel landscape value at a pixel centered at x. Note that the landscape $\widetilde{u}$ contains in itself all spatial integrations required, from the PSF g and from the normalized pixel sensor.

Definition 2 (*Ideal acquisition system.*)
The image acquired by the *ideal* acquisition system before sampling, corresponds to samples of the Poisson process $\mathbf{P}_l$. Its intensity is the deterministic function $\widetilde{u}$ called ideal landscape (see definition 1). This ideal landscape $\widetilde{u}$ is related to the landscape $l \in L^1_{loc}(\mathbb{R})$ by

$$\begin{aligned}&\mathbf{P}_{g*l}\left([t_1, t_2] \times \left[x - \frac{1}{2}, x + \frac{1}{2}\right]\right)\\ &\sim \mathcal{P}\left(\int_{t_1}^{t_2}\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} (g * l)(y - vt)dydt\right)\\ &\sim \mathcal{P}\left(\int_{t_1}^{t_2} \widetilde{u}(x - vt)dt\right), \qquad (1)\end{aligned}$$

where $t_1 < t_2$ and $[t_1, t_2]$ is the time integration (or exposure) interval and $\left[x - \frac{1}{2}, x + \frac{1}{2}\right]$ represents a normalized pixel centered at position $x \in \mathbb{R}$. Recall that $\widetilde{u} \in L^1_{loc}(\mathbb{R})$ and is $[-\pi, \pi]$ band-limited.

A realistic acquisition system adds a landscape independent noise also known as obscurity noise (or dark noise or thermal noise). The sensor also adds a landscape and exposure time independent noise called "sensor readout" noise" or additive noise. This is formalized in the next definition.

Definition 3 (*Acquisition system with obscurity and additive (sensor readout) noises.*)
Each observation at a pixel centered at x is corrupted by an additive noise $\eta(x)$ called readout noise. We assume that $\mathbb{E}(\eta(x)) = 0$, that $\text{var}(\eta(x)) = \sigma_r < +\infty$ and that the obscurity noise has variance $\sigma_o^2 < +\infty$. Thus, from (1) we have

$$\begin{aligned}obs(x) &\sim \mathbf{P}_{g*l+\sigma_o^2}\left([t_1, t_2] \times \left[x - \frac{1}{2}, x + \frac{1}{2}\right]\right) + \eta(x)\\ &\sim \mathcal{P}\left(\int_{t_1}^{t_2} \left(\widetilde{u}(x - vt) + \sigma_o^2\right) dt\right) + \eta(x). \qquad (2)\end{aligned}$$

We further assume that $\widetilde{u}$ is such that the limit

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\widetilde{u}(x)|\, dx := \mu \in \mathbb{R}$$

exists and is finite. Note that since $\widetilde{u}$ is band-limited in particular we have $\widetilde{u} \in L^1_{loc}(\mathbb{R}) \cap L^2_{loc}(\mathbb{R})$. For our analysis it will be useful to consider the deterministic function defined by $u(x) := \widetilde{u}(x) - \mu$. We assume that $u = (\widetilde{u} - \mu) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Notice that, since $\widetilde{u}$ is $[-\pi, \pi]$ band-limited the deterministic function $\mathbb{R} \ni x \mapsto \widetilde{u}(x) + \sigma_o^2 = u(x) + \mu + \sigma_o^2$ is $L^1_{loc}(\mathbb{R})$ and enjoys a Fourier transform in $S'(\mathbb{R})$, the space of tempered distributions on $\mathbb{R}$, as the sum of constants and of the function $u \in L^1(\mathbb{R})$. Furthermore, the function $\widetilde{u}(x) + \sigma_o^2 = u(x) + \mu + \sigma_o^2$ is $[-\pi, \pi]$ band-limited as it inherits the cut-off frequency of the PSF $g \in S(\mathbb{R})$. In the sequel, $obs(x)$ will be obtained for $x \in \mathbb{Z}$. Hence, we further assume that the sequence of random variables $(\eta(n))_{n\in\mathbb{Z}}$ are mutually independent, identically distributed and obviously independent from the signal samples $\mathcal{P}\left(\int_{t_1}^{t_2} \left(u(n - vt) + \mu + \sigma_o^2\right) dt\right)$, where $n \in \mathbb{Z}$. This independence assumption represents no limitation for the physical model. Indeed, a photon is not caught twice and the additive (sensor readout) noise comes from an inaccurate reading of the pixel value. For any $k \in \mathbb{Z}$ let the random variables $\eta(k)$ be defined on $\eta(k) : (\Omega, \mathcal{T}_\eta, \mathbb{P}_\eta) \to (\mathbb{R}, \mathcal{B}or(\mathbb{R}))$ where, $\Omega := \mathbb{R}$ is the sample space, $\mathcal{B}or(\mathbb{R})$ is the Borel algebra on $\mathbb{R}$, $\mathcal{T}_\eta$ is chosen as the smallest algebra on Ω that makes $\eta(k)$ measurable, i.e $\mathcal{T}_\eta = \{\eta(k)^{-1}B : B \in \mathcal{B}or(\mathbb{R})\}$. The Poisson random variables are defined on $\mathcal{P}(\lambda) : (\mathbb{N}, \mathcal{T}_\mathbb{N}, \mathbb{P}_{Poisson(\lambda)}) \to (\mathbb{R}, \mathcal{B}or(\mathbb{R}))$, where $\mathcal{T}_\mathbb{N}$ is the smallest algebra that contains $\mathbb{N}$. Suppose that at a pixel centered at position x we have $\int_{t_1}^{t_2} \widetilde{u}(x - vt) + \sigma_o^2 dt = \lambda$. From (2), and for any $x \in \mathbb{R}$, $obs(x)$ is the sum of the independent Poisson and $\eta(x)$ random variables $\mathcal{P}(\lambda) + \eta(x) : (\Omega, \mathcal{T}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}or(\mathbb{R}))$. It is therefore measurable with the sigma-algebra $\mathcal{T} = \{(\mathcal{P}(\lambda) + \eta(k))^{-1}B : B \in \mathcal{B}or(\mathbb{R})\}$. In addition, $\mathbb{P}$ is such that $\mathbb{P}(\mathcal{P}(\lambda)+\eta(k) \leqslant x) = \mathbb{E}\big(F_{\mathcal{P}(\lambda)}(x - \eta(k))\big)$, where $F_{\mathcal{P}(\lambda)}$ is the cumulative distribution function of a Poisson random variable with intensity λ. Thus, $obs(x)$ is a measurable function (random variable).

When we talk of "almost sure" convergence, and "L^1 or L^2" convergence, we refer to an almost sure convergence in Ω, or convergence in $L^1(\Omega)$, $L^2(\Omega)$ respectively. To avoid any confusion we shall always write $L^1(\mathbb{R})$, $L^2(\mathbb{R})$ or $L^2_{loc}(\mathbb{R})$ without abbreviation when we refer to a convergence in Lebesgue spaces on $\mathbb{R}$.

Sampling, interpolation Because the optical kernel g provides a cutoff frequency, the function $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is $[-\pi, \pi]$ band-limited. Therefore, u can be sampled at unit rate. This means that from $u(n)$ observed for $n \in \mathbb{Z}$ we would deduce $u(x)$ for any $x \in \mathbb{R}$ by replacing obs by the function u in (3).

The discrete sensor observations, or samples, are random variables denoted by $obs(n)$ (see definition 3). The time series formed by the observed $obs(n)$ samples is only obtained for $n \in \mathbb{Z}$. In the sequel we shall convolve/deconvolve these observed samples. This means that a continuous model of $obs(\cdot)$ is required. In other words, the time series $obs(n)$ observed for $n \in \mathbb{Z}$ must be interpolated to $obs(x)$ for any $x \in \mathbb{R}$. Given a discrete array observation $obs(n)$ $n \in \mathbb{Z}$, its band-limited interpolate $obs(x)$, for $x \in \mathbb{R}$, is *formally* defined by the Shannon-Whittaker interpolation as

$$obs(x) = \sum_{n\in\mathbb{Z}} obs(n)\mathrm{sinc}(x - n), \tag{3}$$

where $\mathrm{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$. We shall prove the convergence of (3) with our observed samples $obs(n)$ $n \in \mathbb{Z}$ in section 3.1 for the *analog flutter shutter* and in section 4.1 for the *numerical flutter shutter*. Having defined the mathematical behavior of an imaging sensor we are now in position to formalize the *flutter shutter* method.

3 The *analog flutter shutter*

An *analog flutter shutter* (coded exposure) modulates temporally the photon flux that enters the camera. There are two different tools implementing a *flutter shutter* with a moving sensor or scene. The first technical possibility is to implement the *flutter shutter gain function* on the sensor as an optical (temporally varying) filter. For example, the Agrawal, Raskar et al. *flutter shutter* [1,2,3,27,28,29] consists in opening/closing the camera shutter on sub-intervals of the exposure time. Thus, for the Agrawal, Raskar et al. *flutter shutter* the *flutter shutter gain function* is binary and piecewise constant. Such a *flutter shutter gain function* can be described by a code called *flutter shutter code*. The mathematical generalization proposed in [42] allows for smoother, i.e. non-binary and/or non-piecewise constant *flutter shutter gain function* $\alpha(t)$. To be precise, the gain $\alpha(t)$ at time $t \in \mathbb{R}$ is defined as the proportion of photons that are allowed to travel to the sensor. Thus, only *non-negative* (actually in $[0, 1]$) functions $\alpha(t)$ are feasible. The device (roughly speaking a generalized shutter) controls the percentage of photons emitted by the landscape that can travel to the sensor. This modulation obviously takes place before the photons hit the sensor. This setup is called *analog flutter shutter*. Recall that the formalism of [42] does not include any obscurity or additive noise. Thus, the formalism of [42] does not permit to answer the question we shall treat here: Are there values of the variances of the obscurity σ_o^2 or additive σ_r^2 noises that make the *flutter shutter* more efficient?

Definition 4 (*A*nalog flutter shutter gain function.) Let $\alpha(t) \in [0, 1]$ be the gain used at time t. We call *analog flutter shutter gain function* any *non-negative* function $\alpha \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Let α be an *analog flutter shutter gain function.* From definition 3 we deduce that the observed sample at position n can be any realization of the random variable

$$\begin{aligned} obs(n) &\sim \mathcal{P}\left(\int_{-\infty}^{\infty} \alpha(t)\left(\tilde{u}(n-vt)+\sigma_o^2\right)dt\right)+\eta(n) \\ &\sim \mathcal{P}\left[\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)+\|\alpha\|_{L^1(\mathbb{R})}\left(\mu+\sigma_o^2\right)\right]+\eta(n). \quad (4)\end{aligned}$$

The random variables $obs(n)$ are observed only for $n \in \mathbb{Z}$. Indeed, by assumption the sensor collects samples at unit rate. Notice that since $u \in L^1(\mathbb{R})$ by Riemann-Lebesgue's theorem (see e.g. [16, proposition 2.1]) $\mathbb{R} \ni \xi \mapsto \hat{u}(\xi)$ is continuous and compactly supported since u is $[-\pi, \pi]$ band-limited. Thus, $\hat{u} \in L^1(\mathbb{R})$ and by Riemann-Lebesgue's theorem (see e.g. [16, proposition 2.1]) u is continuous and $\lim_{x\to\pm\infty} u(x) = 0$. Hence u is uniformly bounded. It follows that $\tilde{u}+\sigma_o^2 = u+\mu+\sigma_o^2$ is uniformly bounded. Thus, the convolution $\alpha * \left(\tilde{u} + \sigma_o^2\right)$ is everywhere well-defined and (4) makes sense.

Definition 5 Let α be an *analog flutter shutter gain function.* We call *analog flutter shutter* samples at position n of the landscape u at velocity v the random variable

$$obs(n) \sim \mathcal{P}\left[\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)+\|\alpha\|_{L^1(\mathbb{R})}\left(\mu+\sigma_o^2\right)\right]+\eta(n). \quad (5)$$

Recall that the random variables $obs(n)$ $n \in \mathbb{Z}$ are mutually independent (see page 5). Roughly, this is justified by the fact that a photon is not caught twice.

Remark 1 The above definition implies that we take the simplifying assumption that thermal photons are also filtered by the camera shutter. This means that our model is optimistic. Indeed, the MSE cannot but decrease by this assumption, and since we aim at lower bounds on the MSE, all of our conclusions herein are valid for a camera designed so that thermal photons are unaffected by the *flutter shutter.*

The observed samples of any *analog flutter shutter* are defined in definition 5. These samples $obs(n)$ are only observed for $n \in \mathbb{Z}$. Our main goal is to define and compute the average MSE of a restored signal estimating $\tilde{u}(x)$, for any $x \in \mathbb{R}$, from the observed random samples $obs(n)$ obtained for $n \in \mathbb{Z}$. This is needed if one wants to bring the discussion on the *flutter shutter* e.g., on its efficiency with respect to the observation conditions. Thus, our work-plan is

Work-plan:

Step1 Justify the mathematical feasibility of the Shannon-Whittaker interpolation

$$\text{“}obs(x) = \sum_{n\in\mathbb{Z}} obs(n)\operatorname{sinc}(x-n)\text{”}$$

for the stochastic samples $obs(n)$ (see definition 5). This requires some care because the underlying process is not wide sense stationary, the interpolation kernel is not $L^1(\mathbb{R})$ (hence not "stable") and strong convergence properties are needed to get closed formulae. This is done in section 3.1. The study yields proposition 2 on page 8.

Step2 Deduce the conditions on the *analog flutter shutter gain function* α (see definition 4) for the existence of an inverse filter γ. Define and prove the mathematical feasibility of a restored signal $\tilde{\mathrm{u}}_{\mathrm{est,ana}}(x)$ that estimates the ideal landscape $\tilde{u}(x)$ for any $x \in \mathbb{R}$. This is done in section 3.2. The study yields proposition 3 on page 10.

Step3 Give a closed and exact formula for the average MSE. The formula must allow for an immediate computation of the MSE for *any flutter shutter gain function.* This is needed if one wants to compare *any flutter shutter gain function* in terms of average MSE. This is treated in section 3.3. The study yields theorem 1 on page 11.

The next proposition will be useful for the sequel.

Proposition 1 *The observed samples of the analog flutter shutter satisfy, for any $n \in \mathbb{Z}$,*

$$\mathbb{E}\left(obs(n)\right) = \tfrac{1}{|v|}\left(\alpha\left(\tfrac{\cdot}{v}\right)*u\right)(n) + \left(\mu+\sigma_o^2\right)\|\alpha\|_{L^1(\mathbb{R})} \quad (6)$$

and

$$\operatorname{var}(obs(n)) = \tfrac{1}{|v|}\left(\alpha\left(\tfrac{\cdot}{v}\right)*u\right)(n) + \left(\mu+\sigma_o^2\right)\|\alpha\|_{L^1(\mathbb{R})} + \sigma_r^2. \quad (7)$$

Proof This is a direct consequence of the definition 5. □

Remark 2 Note that $\mathbb{Z} \ni n \mapsto \operatorname{var}\left(obs(n)\right)$ is uniformly bounded. Indeed, $u \in L^1(\mathbb{R})$ and is $[-\pi, \pi]$ band-limited. Thus, we have $\hat{u} \in L^1(\mathbb{R})$. Again by Riemann-Lebesgue's theorem (see e.g. [16, proposition 2.1]) we have that u is uniformly bounded. The result follows from $\alpha \in L^1(\mathbb{R})$, $L^1 * L^\infty \subset L^\infty$ and $\mu, \sigma_o^2, \sigma_r^2$ being finite constants by assumption.

We have defined the observed stochastic samples $obs(n)$ for every $n \in \mathbb{Z}$. Our first goal is to justify the Shannon-Whittaker interpolation (3) of the sequence of $obs(n)$, observed for $n \in \mathbb{Z}$, to a stochastic process $obs(x)$. Furthermore, $obs(x)$ will be deconvolved to get the final *flutter shutter* estimate $\tilde{\mathrm{u}}_{\mathrm{est,ana}}(x)$ and we also need to justify the mathematical feasibility of this deconvolved stochastic process. Lastly, we shall compute the MSE of $\tilde{\mathrm{u}}_{\mathrm{est,ana}}$ with respect to $\tilde{u}$. We shall treat in detail the definition of $obs(x)$ and deduce the completely analogous construction of $\tilde{\mathrm{u}}_{\mathrm{est,ana}}(x)$.

3.1 The interpolated observed signal $obs(x)$

This section tackles the step 1 of our work-plan. We give a rigorous proof of the Shannon-Whittaker interpolation of the stochastic observed samples $obs(n)$ obtained for $n \in \mathbb{Z}$. Formally, we wish to consider the Shannon-Whittaker interpolate (3) (page 5) of the stochastic sequence $obs(n)$ observed for $n \in \mathbb{Z}$ in order to get

$$obs(x) = \sum_{n=-\infty}^{\infty} obs(n)\mathrm{sinc}(x-n) \tag{8}$$

for any $x \in \mathbb{R}$. Usually this kind of work requires to assume that $\mathbb{E}\left(obs(n)obs(m)\right) = f(m-n)$ for some function f, i.e. the discrete L^2 random process is wide sense stationary (w.s.s.), in order to use a Wiener-Khinchin argument. However, here it is clear from theorem 1 that one cannot expect $obs(n)$ to be w.s.s. since the landscape u cannot be assumed to be constant. Furthermore, very strong convergence properties are needed because 1) we will need to convolve/deconvolve $obs(x)$ and prove the feasibility of these operations and 2) not only the mathematical feasibility is needed but also closed formulae are needed to analyze the *flutter shutter*.

The rest of section 3.1 is organized as follows. Section 3.1.1 proves that the series in equation (8) converges almost surely. Section 3.1.2 proves the quadratic mean convergence of (8). Section 3.1.3 gives closed formulae for the expectation and variance of (8). We recall that the study yields proposition 2 on page 8.

3.1.1 Almost sure convergence of (8)

Let us prove first that, for any $x \in \mathbb{R}$, the series in equation (8) converges almost surely. We shall use Kolmogorov's two series theorem (see e.g. [34, p. 386]). Thus, we need to evaluate, for any $x \in \mathbb{R}$, the series formed by the expectations and variances. On the one hand, for any $x \in \mathbb{R}$ we have

$$\sum_{n=-\infty}^{\infty}\mathbb{E}\left(obs(n)\mathrm{sinc}(x-n)\right)=\sum_{n=-\infty}^{\infty}\mathbb{E}(obs(n))\mathrm{sinc}(x-n) \tag{9}$$

$$=\sum_{n=-\infty}^{\infty}\left[\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)+\left(\mu+\sigma_o^2\right)\|\alpha\|_{L^1(\mathbb{R})}\right]\mathrm{sinc}(x-n) \tag{10}$$

$$=\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(x)+\left(\mu+\sigma_o^2\right)\|\alpha\|_{L^1(\mathbb{R})}\in\mathbb{R}, \tag{11}$$

where we use (6) in (10) and Poisson's formula **(xxv)** equation (57) in (11). Indeed, the function $\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u$ is in $L^1(\mathbb{R})$ and is $[-\pi,\pi]$ band-limited. The Shannon-Whittaker formula $\sum_n \mathrm{sinc}(x-n) = 1$ completes the justification of (11). Note that

$$\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(x)+\left(\mu+\sigma_o^2\right)\|\alpha\|_{L^1(\mathbb{R})}\in L^1_{loc}(\mathbb{R})\cap L^2_{loc}(\mathbb{R}).$$

Thus, from (9)-(11) the series of the expectations converges.

On the other hand, for the series of the variances, for any $x \in \mathbb{R}$ we have

$$\sum_{n=-\infty}^{\infty}\mathrm{var}(obs(n)\mathrm{sinc}(x-n))=\sum_{n=-\infty}^{\infty}\mathrm{var}(obs(n))\mathrm{sinc}^2(x-n) \tag{12}$$

$$\leqslant \sup_n \mathrm{var}\left(obs(n)\right)\sum_{n=-\infty}^{\infty}\mathrm{sinc}^2(x-n) < +\infty. \tag{13}$$

Indeed, by remark 2 on page 6 $\sup_n \mathrm{var}\left(obs(n)\right) < +\infty$ and the series $\sum_n \mathrm{sinc}^2(x-n)$ converges for any $x \in \mathbb{R}$, which justifies (13). Thus, from (12)-(13) the series of the variances converges. By assumption the random variables $obs(n)$ are independent and, for any $x \in \mathbb{R}$ the random variables $obs(n)\mathrm{sinc}(x-n)$ are independent too. Thus, from Kolmogorov's two series theorem (see e.g. [34, p. 386]) we obtain that, for any $x \in \mathbb{R}$, the series

$$\sum_{n=-\infty}^{\infty} obs(n)\mathrm{sinc}(x-n)$$

converges almost surely to a limit that we can therefore call $obs(x)$.

3.1.2 Quadratic mean convergence of (8)

For any $x \in \mathbb{R}$, we need to swap the sum and the expectation in (9) and the sum and the variance in (12) in order to provide closed formulae for the expectation and variance of the Shannon-Whittaker interpolate (8). Thus, we need to prove an L^2 (quadratic mean) convergence for any $x \in \mathbb{R}$.

The proof is based on the Cauchy criterion. Let us set

$$\widetilde{obs}(n) := obs(n) - \mathbb{E}(obs(n)),$$

so that, for any $n \in \mathbb{Z}$,

$$\mathbb{E}(\widetilde{obs}(n)) = 0;\ \mathrm{var}\left(\widetilde{obs}(n)\right) = \mathbb{E}(\widetilde{obs}(n)^2) = \mathrm{var}(obs(n)).$$

For any $x \in \mathbb{R}$, we need to justify the existence of the following infinite sum in L^2 (quadratic mean) sense

$$\widetilde{obs}(x) := \sum_{n=-\infty}^{\infty}\widetilde{obs}(n)\mathrm{sinc}(x-n) = \sum_{n=-\infty}^{\infty}\left[obs(n)-\mathbb{E}(obs(n))\right]\mathrm{sinc}(x-n).$$

To this aim, consider the finite sums

$$\tilde{o}^N(x) := \sum_{n=-N}^{N}\widetilde{obs}(n)\mathrm{sinc}(x-n).$$

This implies for $N \geqslant M$ and any $x \in \mathbb{R}$,

$$\mathbb{E}\left(\left|\tilde{o}^N(x) - \tilde{o}^M(x)\right|^2\right) = \sum_{M<|n|\leqslant N} \operatorname{var}\left(obs(n)\right)\operatorname{sinc}^2(x-n)$$
$$\leqslant \left(\sup_n \operatorname{var}\left(obs(n)\right)\right) \sum_{M<|n|\leqslant N} \operatorname{sinc}^2(x-n) < \epsilon$$

for M large enough. Indeed, by remark 2 on page 6 $\sup_n \operatorname{var}(obs(n)) < +\infty$ and the series $\sum_n \operatorname{sinc}^2(x-n)$ converges for any $x \in \mathbb{R}$. Thus, by the Cauchy criterion we deduce that, for any $x \in \mathbb{R}$, the series

$$\widetilde{obs}(x) = \sum_{n=-\infty}^{+\infty} \widetilde{obs}(n)\operatorname{sinc}(x-n)$$

converges in L^2 (in quadratic mean) to a limit that we can therefore call $\widetilde{obs}(x)$. We shall now prove that $\sum_{n=-N}^{N} obs(n)\operatorname{sinc}(x-n)$ converges in L^2.
For any $x \in \mathbb{R}$ the partial sums

$$\sum_{n=-N}^{N} \widetilde{obs}(n)\operatorname{sinc}(x-n) + \sum_{n=-N}^{N} \mathbb{E}(obs(n))\operatorname{sinc}(x-n)$$
$$= \sum_{n=-N}^{N} obs(n)\operatorname{sinc}(x-n)$$

converge in L^2 as the sum of the deterministic constant $\mathbb{E}(obs(x))$ and of the quadratic mean convergent series $\tilde{o}^N(x) = \sum_{n=-N}^{N} \widetilde{obs}(n)\operatorname{sinc}(x-n)$. Thus, we proved that, for any $x \in \mathbb{R}$, the series $\sum_{n=-\infty}^{\infty} obs(n)\operatorname{sinc}(x-n)$ converges almost surely and in quadratic mean.

3.1.3 Calculations of the expectation and variance of (8)

The convergence in quadratic mean implies the convergence of the two firsts moments (see [13, p. 158, Exercise 5.6 (a)-(b)]. Hence, for any $x \in \mathbb{R}$ we have

$$\mathbb{E}\left(\sum_{n=-N}^{N} obs(n)\operatorname{sinc}(x-n)\right) \xrightarrow{N\to\infty} \mathbb{E}(obs(x));$$

$$\mathbb{E}(obs(x)) = \mathbb{E}\left(\sum_{n=-\infty}^{\infty} obs(n)\operatorname{sinc}(x-n)\right), \tag{14}$$

and

$$\operatorname{var}\left(\sum_{n=-N}^{N} obs(n)\operatorname{sinc}(x-n)\right) \xrightarrow{N\to\infty} \operatorname{var}\left(obs(x)\right);$$

$$\operatorname{var}\left(obs(x)\right) = \operatorname{var}\left(\sum_{n=-\infty}^{\infty} obs(n)\operatorname{sinc}(x-n)\right). \tag{15}$$

Recall that L^2 convergence implies L^1 convergence by Cauchy-Schwartz. Therefore, we can swap the sum and the expectation in (14) and the sum and the variance in (15). Thus, combining (14), (9)-(11) and (15), (12)-(13), for any $x \in \mathbb{R}$, we have

$$\mathbb{E}\left(\sum_{n=-N}^{N} obs(n)\operatorname{sinc}(x-n)\right)$$
$$\xrightarrow{N\to\infty} \mathbb{E}(obs(x)) = \mathbb{E}\left(\sum_{n=-\infty}^{\infty} obs(n)\operatorname{sinc}(x-n)\right)$$
$$= \sum_{n=-\infty}^{\infty} \mathbb{E}(obs(n))\operatorname{sinc}(x-n)$$
$$= \left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right) * u\right)(x) + \left(\mu + \sigma_o^2\right)\|\alpha\|_{L^1(\mathbb{R})} \in \mathbb{R} \tag{16}$$

and

$$\operatorname{var}\left(\sum_{n=-N}^{N} obs(n)\operatorname{sinc}(x-n)\right)$$
$$\xrightarrow{N\to\infty} \operatorname{var}\left(obs(x)\right) = \operatorname{var}\left(\sum_{n=-\infty}^{\infty} obs(n)\operatorname{sinc}(x-n)\right)$$
$$= \sum_{n=-\infty}^{\infty} \operatorname{var}\left(obs(n)\right)\operatorname{sinc}^2(x-n) \in \mathbb{R}. \tag{17}$$

In addition, for any $x \in \mathbb{R}$, from (16) the random variable $obs(x)$ has finite expectation. Consequently, $obs(x)$ is finite almost surely. Moreover, from (12)-(13), (17) and remark 2 on page 6 we deduce that $\operatorname{var}(obs(x))$ is uniformly bounded. Thus, we have $\int_a^b \mathbb{E}(obs(x))^2 dx \leqslant C|b-a|$ and $obs(x) \in L^2_{loc}(\mathbb{R})$ almost surely and, by Cauchy-Schwartz, $obs(x) \in L^1_{loc}(\mathbb{R})$ almost surely. Hence, for any $x \in \mathbb{R}$ we have that $obs \in L^1_{loc}(\mathbb{R}) \cap L^2_{loc}(\mathbb{R})$ almost surely.

Thus, we obtain the existence result summarized in the following proposition.

Proposition 2 (Validity/Existence of $obs(x)$ the Shannon-Whittaker interpolation.)

For any $x \in \mathbb{R}$, the Shannon-Whittaker interpolated observation

$$obs(x) := \sum_{n=-\infty}^{+\infty} obs(n)\operatorname{sinc}(x-n)$$

of the discrete L^2 (has finite variance) random process $obs(n)$ almost surely belongs to $L^1_{loc}(\mathbb{R}) \cap L^2_{loc}(\mathbb{R})$. This $obs(x)$ is the almost surely convergent and quadratic mean convergent series,

$$obs(x) = \sum_{n=-\infty}^{+\infty} obs(n)\operatorname{sinc}(x-n).$$

Furthermore, from (16), *its expectation is*

$$\mathbb{E}(obs(x)) = \left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right) * u\right)(x) + \left(\mu + \sigma_o^2\right)\|\alpha\|_{L^1(\mathbb{R})} \ \forall x \in \mathbb{R} \quad (18)$$

and, from (17), *its variance is*

$$\text{var}\,(obs(x)) = \sum_{n=-\infty}^{\infty} \text{var}\,(obs(n))\,\text{sinc}(x-n)^2 \ \forall x \in \mathbb{R},$$

where $\text{var}(obs(n))$ *is given by proposition 1 equation* (7).

Note that proposition 2 is optimal in the sense that one cannot expect a stronger convergence. We shall now turn to the definition of the restored signal of an *analog flutter shutter* that we shall denote $\tilde{\mathbb{u}}_{\text{est,ana}}(x)$.

3.2 The interpolated restored signal $\tilde{\mathbb{u}}_{\text{est,ana}}(x)$

This section treats the step 2 of our work-plan that yields proposition 3 on page 10.

Recall that the existence of $obs(x)$ is given in proposition 2. Formally, we wish to consider $obs * \gamma(x)$ where γ will be the deconvolution filter that inverts the convolution by the *flutter shutter gain function* $\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)$ in proposition 2 equation (18). We need to prove the mathematical feasibility of this convolution, deduce the conditions on the *flutter shutter gain function* $\alpha(t)$ to guarantee that $\mathbb{E}(\gamma * obs(x)) = \tilde{u}(x)$ for any $x \in \mathbb{R}$.

Recall that the *flutter shutter gain function* defined in definition 4 satisfies $\alpha \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

We shall assume that $\hat{\alpha}(\xi v) \neq 0$ for any $\xi \in [-\pi, \pi]$. Under that condition the convolution $\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)\right) * u$ is invertible because u is $[-\pi,\pi]$ band-limited. Thus, provided $\hat{\alpha}(\xi v) \neq 0 \quad \forall \xi \in [-\pi,\pi]$ we can consider the inverse filter (defined by its inverse Fourier transform **(xxiii)**)

$$\gamma(x) := \mathcal{F}^{-1}\left(\frac{\mathbb{1}_{[-\pi,\pi]}(\xi)}{\hat{\alpha}(\xi v)}\right)(x), \quad (19)$$

that will be applied to the observed random function $obs(x)$ given in proposition 2. From the form of $\hat{\gamma}$ we deduce that

$$\gamma^2(x) \leqslant \frac{C}{1+x^2} \ \forall x \in \mathbb{R} \quad (20)$$

for some suitable constant C. Note that $\gamma \in L^2(\mathbb{R})$. Furthermore from (19) we have $\hat{\gamma}(\xi) = \frac{\mathbb{1}_{[-\pi,\pi]}(\xi)}{\hat{\alpha}(\xi v)}$ and therefore

$$\hat{\gamma}(0) = \frac{1}{\hat{\alpha}(0)} = \frac{1}{\int_{\mathbb{R}} \alpha(x)dx}. \quad (21)$$

We have

Lemma 1 (Inversion formula for the *flutter shutter*.) *Let* $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *be* $[-\pi,\pi]$ *band-limited. Then*

$$\sum_n \left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right) * u\right)(n)\gamma(x-n) = u(x),$$

for any $x \in \mathbb{R}$.

Proof See annex A on page 16 ▫

In order to get an unbiased estimate of $\tilde{u}(x)$ we would like to define

$$\tilde{\mathbb{u}}_{\text{est,ana}}(x) := obs * \gamma(x) - \sigma_o^2, \quad (22)$$

for any $x \in \mathbb{R}$.
(In the sequel we shall prove that $\mathbb{E}(\tilde{\mathbb{u}}_{\text{est,ana}}(x)) = \tilde{u}(x)$ for any $x \in \mathbb{R}$.) Yet, the convolution in (22) is undefined.

We shall now justify the feasibility of the convolution in (22), i.e the validity of (22).

It follows from its definition (19) that the inverse filter $\gamma(x)$ is $[-\pi,\pi]$ band-limited and that it is $C^\infty(\mathbb{R})$, bounded, and belongs to $L^2(\mathbb{R})$. Indeed its Fourier transform is compactly supported and bounded and therefore belongs to $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Yet, we only know from proposition 2 that $obs(x)$ is almost surely locally square integrable. Thus, we cannot simply use (22) for a definition, since the convolution is undefined. Consequently we shall, again, recur to a convergence argument for the definiteness of $\tilde{\mathbb{u}}_{\text{est,ana}}$. So we set

$$\begin{aligned}\tilde{\mathbb{u}}^N_{\text{est,ana}} : &= \sum_{-N}^{N} obs(n)\left(\text{sinc}(\cdot - n) * \gamma\right)(x) - \sigma_o^2 \\ &= \sum_{-N}^{N} obs(n)\gamma(x-n) - \sigma_o^2.\end{aligned}$$

Our goal is to establish a suitable convergence for these finite sums to

$$\tilde{\mathbb{u}}_{\text{est,ana}}(x) := \sum_{n=-\infty}^{\infty} obs(n)\gamma(x-n) - \sigma_o^2.$$

The proof follows exactly by the same arguments as for the construction of $obs(x)$ leading to proposition 2. Indeed, the decay of γ given in (20) is the same as the sinc.

We now compute the expected value and variance of the restored signal $\tilde{\mathbb{u}}_{\text{est,ana}}$. We have

$$\mathbb{E}\left(\tilde{\mathbb{u}}^N_{\text{est,ana}}(x)\right) = \sum_{-N}^{N} \mathbb{E}(obs(n))\gamma(x-n) - \sigma_o^2$$

$$= \sum_{-N}^{N}\left[\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right) * u\right)(n) + \left(\mu + \sigma_o^2\right)\|\alpha\|_{L^1(\mathbb{R})}\right]\gamma(x-n) - \sigma_o^2,$$

which implies by lemma 1, **(xxv)** equation (58) and (21) that

$$\mathbb{E}(\tilde{\mathrm{u}}^N_{\text{est,ana}}(x)) \xrightarrow{N\to\infty} u(x) + \mu = \tilde{u}(x),$$

for any $x \in \mathbb{R}$. From remark 2 on page 6 we have that $\sup_n \operatorname{var}(obs(n)) < \infty$. Since γ is $L^2(\mathbb{R})$, $[-\pi,\pi]$ band-limited we deduce that $\gamma^2 \in L^1(\mathbb{R})$ and $\widehat{\gamma^2}(\xi)$ is supported on $[-2\pi, 2\pi]$. Moreover, from (19) we have

$$\begin{aligned}\widehat{\gamma^2}(2\pi) = (\hat{\gamma} * \hat{\gamma})(2\pi) &= \int_{-\pi}^{\pi} \frac{\mathbb{1}_{[-\pi,\pi]}(2\pi - y)}{\hat{\gamma}(\pi v)\hat{\gamma}(2\pi - y)} dy \\ &= \frac{\mathbb{1}_{]-\pi,\pi[}(2\pi - y)}{\hat{\gamma}(\pi v)\hat{\gamma}(2\pi - y)} dy = 0.\end{aligned}$$

Indeed, the integrand is non-zero only on a set of zero Lebesgue measure. Similarly, $\widehat{\gamma^2}(-2\pi) = 0$. Thus, from Poisson's formula **(xxv)** we deduce that $\sum_n \widehat{\gamma^2}(2\pi n) = \widehat{\gamma^2}(0) = \sum_n \gamma^2(x-n) = \|\gamma\|^2_{L^2(\mathbb{R})}$. Hence, for any $x \in \mathbb{R}$ we have $\operatorname{var}(\tilde{\mathrm{u}}_{\text{est,ana}}(x)) \leqslant \sup_n \operatorname{var}(obs(n)) \|\gamma\|_{L^2(\mathbb{R})} < +\infty$.

Therefore, we proved

Proposition 3 (Validity/Existence of the deconvolution of $obs(x)$.)
The stochastic process

$$\tilde{\mathrm{u}}_{est,ana}(x) := \sum_{n=-\infty}^{\infty} obs(n)\gamma(x-n) - \sigma_o^2$$

is L^2 (has finite variance) and almost surely belongs to $L^1_{loc}(\mathbb{R}) \cap L^2_{loc}(\mathbb{R})$. Indeed, $\tilde{\mathrm{u}}_{est,ana}$ is the limit of $\sum_{n=-N}^{N} obs(n)\operatorname{sinc}(x-n)$ that converges almost surely and in quadratic mean. Furthermore, for any $x \in \mathbb{R}$ its expectation is

$$\mathbb{E}(\tilde{\mathrm{u}}_{est,ana}(x)) = \tilde{u}(x)$$

and its variance is

$$\operatorname{var}(\tilde{\mathrm{u}}_{est,ana}(x)) = \sum_{n=-\infty}^{\infty} \operatorname{var}(obs(n))(\gamma(x-n))^2 < \infty,$$

where $\operatorname{var}(obs(n))$ *is given by proposition 1 equation* (7).

The strong convergence of proposition 2 is kept. This is valid for any γ with sufficient decay. This proposition also implies that, for any $x \in \mathbb{R}$, the stochastic process $\tilde{\mathrm{u}}_{\text{est,ana}}(x)$ is an unbiased estimator of $\tilde{u}(x)$. Moreover, for any $x \in \mathbb{R}$ the ratio $\frac{\mathbb{E}(\tilde{\mathrm{u}}_{\text{est,ana}}(x))}{\sqrt{\operatorname{var}(\tilde{\mathrm{u}}_{\text{est,ana}}(x))}} > 0$ (usually called signal to noise ratio) is positive.

3.3 The average mean square error of $\tilde{\mathrm{u}}_{\text{est,ana}}$

The mathematical existence of $\tilde{\mathrm{u}}_{\text{est,ana}}(x)$ is given in proposition 3. We shall now turn to the step 3 of our work-plan justifying the existence of the limit of

$$\frac{1}{2T}\int_{-T}^{T} \mathbb{E}\left(|\tilde{\mathrm{u}}_{\text{est,ana}}(x) - \tilde{u}(x)|^2\right) dx$$

when $T \to +\infty$, and giving an explicit formula for this limit. This is needed to bring the discussion on the possible change of efficiency of the *flutter shutter* with respect to the obscurity σ_o^2 and additive (sensor readout) σ_r^2 noise variances and compared to an optimal *snapshot.*

The following lemma will be useful for the rest of the analysis.

Lemma 2 (Variance of the restored signal $\tilde{\mathrm{u}}_{\text{est,ana}}$.)
The variance of the restored signal $\tilde{\mathrm{u}}_{est,ana}$ defined in proposition 3 page 10 satisfies, for any $x \in \mathbb{R}$,

$$\begin{aligned}\operatorname{var}(\tilde{\mathrm{u}}_{est,ana}(x)) =& \sum_{n=-\infty}^{\infty}\left[\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right) * u\right)(n)(\gamma(x-n))^2\right] \\ &+ \|\gamma\|^2_{L^2(\mathbb{R})}\left(\|\alpha\|_{L^1(\mathbb{R})}(\sigma_o^2+\mu)+\sigma_r^2\right). \quad (23)\end{aligned}$$

In addition, the function $\mathbb{R} \ni x \mapsto \operatorname{var}(\tilde{\mathrm{u}}_{est,ana}(x))$ can be written as

$$\operatorname{var}(\tilde{\mathrm{u}}_{est,ana}(x)) = f_{ana}(x) + \|\gamma\|^2_{L^2(\mathbb{R})}\left(\|\alpha\|_{L^1(\mathbb{R})}(\sigma_o^2+\mu)+\sigma_r^2\right),$$

where

$$f_{ana}(\cdot) := \sum_{n=-\infty}^{\infty}\left[\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right) * u\right)(n)(\gamma(\cdot - n))^2\right] \in L^1(\mathbb{R}). \quad (24)$$

Proof See annex B page 17. ▫

We now calculate the average MSE of the *analog flutter shutter.* To this aim, consider any $T > 0$. From lemma 2 we deduce that

$$\begin{aligned}&\int_{-T}^{T} \operatorname{var}(\tilde{\mathrm{u}}_{\text{est,ana}}(x))\, dx \\ &= \int_{-T}^{T} f(x)dx + 2T\|\gamma\|^2_{L^2(\mathbb{R})}\left(\|\alpha\|_{L^1(\mathbb{R})}(\sigma_o^2 + \mu) + \sigma_r^2\right).\end{aligned}$$

Recall that, from proposition 3 $\mathbb{E}(\tilde{\mathrm{u}}_{\text{est,ana}}(x)) = \tilde{u}(x)$ for any $x \in \mathbb{R}$. Hence, we deduce that

$$\mathbb{E}\left(|\tilde{\mathrm{u}}_{\text{est,ana}}(x) - \tilde{u}(x)|^2\right) = \operatorname{var}(\tilde{\mathrm{u}}_{\text{est,ana}}(x)) \ \ \forall x \in \mathbb{R}.$$

Thus, we obtain

$$\begin{aligned}&\frac{1}{2T}\int_{-T}^{T}\operatorname{var}(\tilde{\mathrm{u}}_{\text{est,ana}}(x))dx = \frac{1}{2T}\int_{-T}^{T}\mathbb{E}\left(|\tilde{\mathrm{u}}_{\text{est,ana}}(x) - \tilde{u}(x)|^2\right)dx \\ &\xrightarrow{T\to\infty} \|\gamma\|^2_{L^2(\mathbb{R})}\left(\|\alpha\|_{L^1(\mathbb{R})}(\sigma_o^2 + \mu) + \sigma_r^2\right).\end{aligned}$$

Moreover, from its definition (19) and Plancherel's identity **(xxiii)** equation (55) γ satisfies

$$\|\gamma\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{|\hat{\alpha}(\xi v)|^2}\mathbb{1}_{[-\pi,\pi]}(\xi)d\xi.$$

Thus, we proved the following theorem.

Theorem 1 *For any known velocity v and any flutter shutter gain function α the average MSE of an analog flutter shutter satisfies*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbb{E}\left(|\tilde{u}(x)-\tilde{\mathbb{u}}_{est,ana}(x)|^2\right)dx$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\|\alpha\|_{L^1(\mathbb{R})}\left(\mu+\sigma_o^2\right)+\sigma_r^2}{|\hat{\alpha}(\xi v)|^2}\mathbb{1}_{[-\pi,\pi]}(\xi)d\xi.$$

For comparison with [8], the quantity J in [8] relates to μ, the notation σ_r^2 is the same as in [8] and σ_o^2 is not considered in [8]. Theorem 1 directly relates the average MSE with the *analog flutter shutter gain function* and the observations conditions, i.e. the additive (sensor readout) noise variance σ_r^2 and the obscurity noise variance σ_o^2.

To sum up, we defined the observed samples $obs(n)$ obtained for $n \in \mathbb{Z}$ (see definition 5 page 6). We defined the Shannon-Whittaker interpolated observation $obs(x)$ for any $x \in \mathbb{R}$ (see proposition 2 page 8) of the stochastic observed samples $obs(n)$ that are only obtained for $n \in \mathbb{Z}$. We defined the inverse filter and proved rigorously the existence of $\tilde{\mathbb{u}}_{\text{est,ana}}(x)$, i.e. the estimated landscape from the observed stochastic samples $obs(n)$ (see proposition 3 page 10). The estimated landscape $\tilde{\mathbb{u}}_{\text{est,ana}}(x)$ is designed to ensure that $\tilde{\mathbb{u}}_{\text{est,ana}}(x)$ is equal to the observed landscape $\tilde{u}(x)$ in expectation, i.e. $\mathbb{E}(\tilde{\mathbb{u}}_{\text{est,ana}}(x)) = \tilde{u}(x)$, for any $x \in \mathbb{R}$. Theorem 1 gives the average MSE of the *analog flutter shutter* for any *flutter shutter gain function* α, and any (finite) additive (sensor readout) and obscurity noise variances (observation conditions). Theorem 1 gives a closed formula with no approximation and taking into account all noise sources. This is needed to bring the discussion on the efficiency of the *flutter shutter.*

4 The *numerical flutter shutter*

As we have seen there are two different tools that implement a *flutter shutter*. The first one is the *analog flutter shutter* and is treated in section 3. The second one is the *numerical flutter shutter* [42]. The *numerical flutter shutter* is nothing but a temporal filter. It amounts to acquire with a camera a burst of L images using an exposure time $\Delta t > 0$. The k-th elementary image is assigned a numerical gain $\alpha_k \in \mathbb{R}$. The final observed image is obtained as the weighted sum of elementary images with weights $(\alpha_k)_{k\in\{0,\ldots,L-1\}}$. According to [15, 24] an image sensor can have a duty ratio of nearly 100% (the duty ratio is the ratio of light integration time over readout, storage, reset times - that is the percentage of useful time). Thus, a sensor can integrate light without interruption. This means that the *numerical flutter shutter* as it is described below, i.e. without "dead time" between two consecutive gains $\alpha_k \in \mathbb{R}$ is doable from a technological point of view. However, its interest seems limited: why not keeping all images instead of adding them all? One of the obvious reasons is compression/transmission bandwidth, particularly for e.g. Earth observation satellites or for cell phones. The motion blur due to a drift in satellite trajectory can be eliminated by a *numerical flutter shutter*. The *numerical flutter shutter* requires no additional transmission (or computational) burden. Only the sum is stored/transmitted like for the *analog flutter shutter.*

Hereinafter, it will be useful to associate with the *flutter shutter* its *code* $(\alpha_k)_{k\in\{0,\ldots,L-1\}}$, and its *flutter shutter gain function* defined by $\alpha(t) = \alpha_k$ for $t \in [k\Delta t, (k+1)\Delta t)$. From definition 3 we deduce that the k-th elementary image at a pixel at position n can be any realization of

$$\mathcal{P}\left(\int_{k\Delta t}^{(k+1)\Delta t} u(n-vt)+\mu+\sigma_o^2 dt\right)+\eta(k).$$

Definition 6 Let $(\alpha_k)_{k\in\{0,\ldots,L-1\}}$, where $\alpha_k \in \mathbb{R}$ be a *flutter shutter code*. We call *numerical flutter shutter* samples at position n of the landscape $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ at velocity v the random variable

$$obs(n)\sim\sum_{k=0}^{L-1}\alpha_k\left(\mathcal{P}\left(\int_{k\Delta t}^{(k+1)\Delta t}u(n-vt)+\mu+\sigma_o^2dt\right)+\eta(k)\right).$$

We call *flutter shutter gain function* the deterministic function

$$\alpha(t) = \sum_{k=0}^{L-1}\alpha_k\mathbb{1}_{[k\Delta t,(k+1)\Delta t)}(t). \tag{25}$$

The measurability of $obs(n)$ is obvious as it is a finite sum of measurable functions.

The observed samples of any *numerical flutter shutter* are defined in definition 6. These samples $obs(n)$ are only obtained for $n \in \mathbb{Z}$. Our main goal is to define and compute the average MSE of a restored signal estimating $\tilde{u}(x)$, for any $x \in \mathbb{R}$, from the observed samples $obs(n)$ obtained for $n \in \mathbb{Z}$. This is needed if one wants to bring the discussion on the *flutter shutter* e.g., on its efficiency with respect to the observation conditions.

Thus, our work-plan follows the same path as for the *analog flutter shutter* treated in section 3:

Work-plan:

Step1 Justify the mathematical feasibility of the Shannon-Whittaker interpolation

$$\text{“}obs(x) = \sum_{n\in\mathbb{Z}} obs(n)\text{sinc}(x-n)\text{”}$$

for the stochastic samples $obs(n)$ (see definition 6). This is done in section 4.1.

Step2 Deduce the conditions on α (see (25)) for the existence of an inverse filter γ. Define and prove the mathematical feasibility of a restored signal $\tilde{\mathfrak{u}}_{\text{est,num}}(x)$ estimating $\tilde{u}(x)$ for any $x \in \mathbb{R}$. This is done in section 4.2.

Step3 Compute the average MSE. This is theorem 2.

The next proposition will be useful for the sequel.

Proposition 4 *The observed samples of the numerical flutter shutter are such that, for $n \in \mathbb{Z}$*

$$\mathbb{E}\left(obs(n)\right) = \left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n) + \left(\mu+\sigma_o^2\right)\left(\int_{-\infty}^{\infty}\alpha(t)dt\right) \quad (26)$$

and

$$\text{var}(obs(n)) = \left(\frac{1}{|v|}\alpha^2\left(\frac{\cdot}{v}\right)*u\right)(n) + \|\alpha\|^2_{L^2(\mathbb{R})}\left(\frac{\sigma_r^2}{\Delta t}+\mu+\sigma_o^2\right). \quad (27)$$

Proof See annex C page 17. ▫

Remark 3 Note that there is a slight abuse of notation, the additive (sensor readout) noise for the k-th elementary image and at pixel n should be denoted by $\eta_n(k)$ and not directly $\eta(k)$. For a pixel at position n the (total) noise term is the sum $\sum_{k=0}^{L-1}\alpha_k\eta_n(k)$. However, since the $\eta_n(k)$ are identically distributed and mutually independent the sum $\sum_{k=0}^{L-1}\alpha_k\eta_n(k)$ can be seen w.l.o.g. as a unique $\eta(n)$. Its expectation is $\mathbb{E}\left(\sum_{k=0}^{L-1}\alpha_k\eta_n(k)\right) = 0$ since $\mathbb{E}(\eta_n(k)) = 0$. Its variance is $\sum_{k=0}^{L-1}\alpha_k^2\text{var}\left(\eta(k)\right) = \|\alpha\|^2_{L^2(\mathbb{R})}\frac{\sigma_r^2}{\Delta t}$.

Comparing (6) and (26) we notice that the *numerical* and *analog flutter shutter* have equal expectations. Indeed, $\alpha \geqslant 0$ for the *analog flutter shutter*. However, comparing (7) and (27) they differ by their variances. Thus, in general their MSE will differ.

4.1 The interpolated observed signal $obs(x)$

This section tackles the step 1 of our work-plan: the existence of "$obs(x) = \sum_{n=-\infty}^{+\infty} obs(n)\text{sinc}(x-n)$". We have defined the observed stochastic samples $obs(n)$ for any $n \in \mathbb{Z}$ (see definition 6). The Shannon-Whittaker interpolation of the sequence of $obs(n)$ by a stochastic process $obs(x) = \sum_{n=-\infty}^{+\infty} obs(n)\text{sinc}(x-n)$ follows from the *analog flutter shutter* case. We now give the sketch of the proof.

To prove the almost sure convergence we just notice that the calculations (9)-(11) (page 7) are almost the same. Indeed, comparing (6) and (26) we see that we just need to replace $\|\alpha\|_{L^1(\mathbb{R})}$ by $\int_{-\infty}^{+\infty}\alpha(t)dt$ (which is finite by (25)). The rest of the proof remains the same, the main ingredients being the band-limitedness of u and the use of the Poisson's summation formula that are both still valid. The calculations (12)-(13) are also valid for the *numerical flutter shutter*. Indeed, from (27) and by the same argumentation than the one in remark 2 (page 6), we deduce that $\mathbb{Z} \ni n \mapsto \text{var}(obs(n))$ is also uniformly bounded. The calculations of section 3.1.1 proving the quadratic mean convergence also hold true, the key argument being $\sup_n \text{var}(obs(n)) < +\infty$. The calculations of section 3.1.3 are still valid, the key arguments being the quadratic mean convergence and, again, the band-limitedness of u. Thus, a proposition similar to proposition 2 (page 8) holds true for the *numerical flutter shutter.*

4.2 The interpolated restored signal $\tilde{\mathfrak{u}}_{\text{est,num}}(x)$

This section tackles the step 2 of our work-plan and yields proposition 5.

From the Fourier transform of α we immediately deduce a necessary condition for the invertibility of *any* (*analog* or *numerical*) piecewise constant *flutter shutter*. Indeed, from (25) we deduce that, for any $\xi \in \mathbb{R}$,

$$\hat{\alpha}(\xi v) = \Delta t\text{sinc}\left(\frac{\xi v\Delta t}{2\pi}\right)e^{\frac{-i\xi v\Delta t}{2}}\sum_{k=0}^{L-1}\alpha_k e^{-ik\xi\Delta t}.$$

Since u is $[-\pi,\pi]$ band-limited is it necessary that $|v|\Delta t < 2$ for the convolution in (26) to be invertible for $[-\pi,\pi]$ band-limited functions. Moreover, we recall that *analog* and *numerical flutter shutters* have equal expectations. Thus, they share the same inverse filter defined in (22). Thus, the existence of $\tilde{\mathfrak{u}}_{\text{est,num}}(x) := obs*\gamma(x) - \sigma_o^2$ follows from the *analog flutter shutter* case. In particular, $\forall x \in \mathbb{R}$ we have $\mathbb{E}(\tilde{\mathfrak{u}}_{\text{est,num}}(x)) = \mathbb{E}(\tilde{\mathfrak{u}}_{\text{est,ana}}(x)) = \tilde{u}(x)$. This means that, for any $x \in \mathbb{R}$, $\tilde{\mathfrak{u}}_{\text{est,num}}(x)$ is also an unbiased estimator of $\tilde{u}(x)$. We have

Proposition 5 (Validity/Existence of $obs(x)$ the Shannon-Whittaker interpolation.)

The stochastic process

$$\tilde{\mathfrak{u}}_{est,num}(x) := \sum_{n=-\infty}^{\infty} obs(n)\gamma(x-n) - \sigma_o^2$$

is L^2 (has finite variance) and almost surely belongs to $L^1_{loc}(\mathbb{R}) \cap L^2_{loc}(\mathbb{R})$. Indeed, it is equal to the limit of $\sum_{n=-N}^{N} obs(n)\text{sinc}(x-n)$ that converges almost surely and in quadratic mean. Furthermore, for any $x \in \mathbb{R}$, its expectation is

$$\mathbb{E}(\tilde{\mathbb{u}}_{est,num}(x)) = \tilde{u}(x)$$

and its variance is

$$\text{var}\left(\tilde{\mathbb{u}}_{est,num}(x)\right) = \sum_{n=-\infty}^{\infty} \text{var}\left(obs(n)\right)\left(\gamma(x-n)\right)^2 < \infty,$$

where $\text{var}(obs(n))$ *is given by proposition 4 equation* (27).

4.3 The average mean square error of $\tilde{\mathbb{u}}_{\text{est,num}}$

This section tackles the step 3 of our work-plan. The study yields to theorem 2.

From proposition 5 we deduce that for any $x \in \mathbb{R}$ we have that $\mathbb{E}(\tilde{\mathbb{u}}_{\text{est,num}}(x)) = \tilde{u}(x)$. Thus, for any $x \in \mathbb{R}$ we have that

$$\mathbb{E}\left|\tilde{\mathbb{u}}_{\text{est,num}}(x) - \tilde{u}(x)\right|^2 = \text{var}\left(\tilde{\mathbb{u}}_{\text{est,num}}(x)\right).$$

We recall that *analog* and *numerical flutter shutters* differ by their variances. Therefore, their MSE differ.

We have

Theorem 2 *For any known velocity v and any flutter shutter gain function α the average MSE of a numerical flutter shutter satisfies*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathbb{E}\left(|u(x) - \mathbb{u}_{est,num}(x)|^2\right)dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\|\alpha\|^2_{L^2(\mathbb{R})}\left(\mu + \sigma_o^2 + \frac{\sigma_r^2}{\Delta t}\right)}{|\hat{\alpha}(\xi v)|^2} \mathbb{1}_{[-\pi,\pi]}(\xi)d\xi.$$

Proof See annex D page 17. □

Recall that the quantity J in [8] relates to μ, the notation σ_r^2 is the same as in [8] and σ_o^2 is not considered in [8].

Now, we are in a position verify rigorously the claim of [8] which states that the "gain for computational imaging is significant only when the average signal level J is considerably smaller than the read noise variance σ_r^2" [8, page 5].

5 Discussion, comparison with [8]

An investigation of the gain of a *flutter shutter* camera compared to a *snapshot* was proposed in [8]. More precisely, in [8] Cossairt, Gupta and Nayar provided an upper bound for the *gain* of a *flutter shutter*. The questions that are at stake are: 1) What is the image formation model of [8]? 2) What is the *flutter shutter* camera model of [8]? 3) Under which hypotheses does the upper bound $\sqrt{1+\frac{\sigma_r^2}{J}}$ [8, equation (11) page 5] hold true? Recall that in [42] Tendero et al. proved that for a fixed known velocity v the gain in terms of RMSE of the optimal *numerical flutter shutter* is of a factor of 1.1717 compared to an optimal *snapshot* when $\sigma_r^2 = 0$. However, this statement was proved only in the case without additive (sensor readout) noise, namely when $\sigma_r^2 = 0$. Thus, we must explain this discrepancy. 4) Answer the interesting question raised by the authors when they state that the "gain for computational imaging is significant only when the average signal level J is considerably smaller than the read noise variance σ_r^2" [8, page 5]. In other words, are there values of σ_o^2 or σ_r^2 that turn the *flutter shutter* into an efficient imaging tool?

In [8] the image model is discrete, finite and is written in terms of linear algebra. The image model considered in [8] is

$$\text{“}g = Hf + \eta\text{” [8, equation (1) page 3].} \tag{28}$$

The observed discrete signal is $g \in \mathbb{R}^N$ where N is the number of samples of a one-dimensional image. The ideal landscape is a discrete set of samples $f \in \mathbb{R}^N$. "The measurement matrix H is a circulant matrix, where each row of the matrix encodes the motion blur kernel" [8, section III, page 4] hence $H \in \mathcal{M}_{N\times N}(\mathbb{R})$. Since H is a circulant matrix [8] assumes periodic boundary conditions for the discrete convolution that underlies (28). Thus, Cossairt et al. implicitly assume that the moving scene f is itself periodic. The "noise vector η is assumed to be independently sampled from a zero mean Gaussian distribution $\mathcal{N}(0,\sigma^2)$"[8, section II, page 3]. In [8] Cossairt et al. consider "an affine noise model where there are two sources of noise, signal-independent read noise, and signal-dependent photon noise" [8, section II, page 3]. This means that $\text{var}(\eta) = \sigma^2 = J + \sigma_r^2$ where J is the "measured signal level [...] (in photons)" [8, section II, page 3] and σ_r^2 is the "signal-independent read noise" [8, section II, page 3]. The dependency of J with respect to H is stated as " Suppose the average signal level for the impulse camera is J, so that the total noise $\sigma_i^2 = J + \sigma_r$. The CI technique captures C times more light. Hence, the total noise is

$$\sigma_c^2 = CJ + \sigma_r\text{” [section II, page 4][8].} \tag{29}$$

The additive (sensor readout) noise does not depend on H. Furthermore, we have $H \in \mathcal{M}_{N\times N}([0,1])$ [8, section II, page 3]. In addition, he value C in (29) is defined as "the sum of elements in each row of the measurement matrix H" [8, section II, page 3]. Therefore, we deduce that [8] aims at modelling an *analog flutter shutter.*

Having seen what [8] aims at modelling we shall now turn to the two next steps namely the deconvolution and “gain” measurement definitions. From the observation g given by (28) the signal f is estimated by $H^{-1}g$ [8, equation (3), page 3]. Obviously this requires the matrix H to be invertible. This is similar to assuming that the *flutter shutter gain function* α satisfies $\hat{\alpha}(\xi v) \neq 0$ for any $\xi \in [-\pi, \pi]$, as we need in sections 3 and 4. The gain is defined in terms of RMSE and compared to an “[. . .] impulse imaging”, i.e. a *snapshot*

$$\text{“}G = \sqrt{\frac{\text{MSE}_{\text{snapshot}}}{\text{MSE}_{\text{flutter}}}}\text{”} \text{ [8, equation (5), page 3].} \quad (30)$$

In [8, section II, page 3] the authors state that “for impulse imaging, $H = Id$ (the identity matrix), and the camera measures the signal f directly.” This assertion is not true in general. As soon as the velocity satisfies $v \neq 0$ *any snapshot* requires a deconvolution. In other words there is no “impulse imaging” when photographing moving objects. Furthermore, a standard camera does not measure “f directly”, unless we assume that $v = 0$.

To sum up there are three main approximations used in [8]: 1) the landscape is supposed to be periodic, 2) the signal dependent noise J is assumed to be constant (and not Poissonian) in the image, 3) the code “$(1, 0, \cdots, 0)$” that corresponds to the so called “impulse imaging” [8, section II, page 3] provides a sharp image that requires no deconvolution. These approximations will be shortly proved fatal for the discussion on the *flutter shutter*, i.e. insufficient for a correct discussion of the properties of the *flutter shutter*. Having reviewed the approximations made in [8] we are now in the position to clarify the tightness of the upper bound $\sqrt{1 + \frac{\sigma_r^2}{J}}$ [8, equation (11) page 5]. To this aim, we need to define the “gain” of the method.

Following [8], we define the gain of a *flutter shutter* method, compared to a *snapshot*, by the ratio of their average RMSE formalized in the following definition.

Definition 7 (and proposition: comparative RMSE or gain of a *flutter shutter* method)
Given any *analog flutter shutter* with a *flutter shutter function* α and a *snapshot* using an exposure time Δt we call gain of the *flutter shutter* with respect to a *snapshot* the ratio

$$R := \sqrt{\frac{\text{MSE}_{\text{snapshot}}(\Delta t)}{\text{MSE}_{\text{flutter}}(\alpha)}} = \sqrt{\frac{\frac{1}{2\pi}\left(\|\mathbb{1}_{[0,\Delta t]}\|_{L^1(\mathbb{R})}\left(\mu + \sigma_o^2\right) + \sigma_r^2\right)\int_{-\pi}^{\pi}\frac{d\xi}{\left|\widehat{\mathbb{1}_{[0,\Delta t]}}(\xi v)\right|^2}}{\frac{1}{2\pi}\left(\|\alpha\|_{L^1(\mathbb{R})}\left(\mu + \sigma_o^2\right) + \sigma_r^2\right)\int_{-\pi}^{\pi}\frac{d\xi}{|\hat{\alpha}(\xi v)|^2}}}. \quad (31)$$

(The above formulae are taken from theorem 1, applied with $\alpha = \mathbb{1}_{[0,\Delta t]}$ for the MSE of the *snapshot.*)

We shall now check the claim in [8] which states that the “gain for computational imaging is significant only when the average signal level J is considerably smaller than the read noise variance σ_r^2” [8, page 5] is valid for the general case, i.e. for non-periodic observations and without approximations. We have

Theorem 3 (Refutation of [8])
Consider a scene u moving at velocity v. Then the gain R of any *analog flutter shutter, with respect to an optimal snapshot, is bounded independently of σ_o^2 and σ_r^2. An upper bound is*

$$R \leqslant \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\xi}{2\sin\left(\frac{\xi}{2}\right)}\right|^2 d\xi} < \sqrt{1.3864}\ (<1.1776). \quad (32)$$

Notice that the provided bound does not tend to $+\infty$ when the additive (sensor readout) noise variance $\sigma_r^2 \to +\infty$: the *flutter shutter* does not work significantly better at night or with “bad” sensors having a huge additive (sensor readout) noise σ_r^2. This is in contrast with the bound of [8]. As a matter of fact, things do not get better if either σ_o^2 or σ_r^2 become large.

Proof From definition 7 we shall provide an upper bound for R^2. To this aim, we shall first provide a lower bound for the average MSE of the *flutter shutter*. The upper bound for R^2 is then deduced from the average MSE of a well chosen *snapshot*.

From theorem 1 we deduce that the optimal *analog flutter shutter*, in terms of average MSE, if it exists, minimizes

$$\text{MSE}_{\text{flutter}}(\alpha) = \left(\|\alpha\|_{L^1(\mathbb{R})}\sigma_o^2 + \sigma_r^2\right)\int_{-\pi}^{\pi}\frac{1}{|\hat{\alpha}(\xi v)|^2}\frac{d\xi}{2\pi}. \quad (33)$$

Hence, we shall now exhibit a lower bound of (33). Since for any *analog flutter shutter gain function* we have $\alpha(t) \in [0, 1]$ we deduce that $\|\alpha\|_{L^1(\mathbb{R})} \geqslant \|\alpha\|^2_{L^2(\mathbb{R})}$. In addition, the function $f : (0, +\infty) \ni x \mapsto \frac{1}{x}$ is strictly convex and $\int_{-\pi}^{\pi}\frac{d\xi}{2\pi} = 1$. Therefore, from (33), by Jensen's inequality applied to $|\hat{\alpha}(\xi v)|^2$ with the function f we obtain

$$\text{MSE}_{\text{flutter}}(\alpha) \geqslant \left(\|\alpha\|^2_{L^2(\mathbb{R})}\left(\mu + \sigma_o^2\right) + \sigma_r^2\right)\frac{1}{\int_{-\pi}^{\pi}|\hat{\alpha}(\xi v)|^2\frac{d\xi}{2\pi}}. \quad (34)$$

From the strict convexity of f, we deduce that the equality case in (34) occurs when $\hat{\alpha}(\xi v) = C$ for all $\xi \in [-\pi, \pi]$ and a certain constant $C \in \mathbb{R}$. This implies that a lower bound of (33) can be obtained with $\widehat{\alpha^*}(\xi v) = C\mathbb{1}_{[-\pi,\pi]}(\xi)$ or equivalently $\widehat{\alpha^*}(\xi) = C\mathbb{1}_{[-\pi|v|,\pi|v|]}(\xi)$.

Thus, we have $\|\widehat{\alpha^*}\|^2_{L^2(\mathbb{R})} = 2\pi|v|C^2$, $\|\alpha^*\|^2_{L^2(\mathbb{R})} = |v|C^2$ (Plancherel) and from (33) we deduce that

$$\begin{aligned}\mathrm{MSE}_{\text{flutter}}(\alpha) &\geqslant \frac{1}{2\pi}\left(|v|C^2\left(\mu+\sigma_o^2\right)+\sigma_r^2\right)\int_{-\pi}^{\pi}\frac{d\xi}{C^2} \quad (35)\\ &= \frac{1}{2\pi}\left(|v|C^2\left(\mu+\sigma_o^2\right)+\sigma_r^2\right)\frac{2\pi}{C^2}\\ &= |v|\left(\mu+\sigma_o^2\right)+\frac{\sigma_r^2}{C^2} \quad (36)\\ &\geqslant |v|\left(\left(\mu+\sigma_o^2\right)+|v|\sigma_r^2\right). \quad (37)\end{aligned}$$

Indeed, for any $x \in \mathbb{R}$, we have that

$$\begin{aligned}\alpha(x) &= \mathcal{F}^{-1}\left(C\mathbb{1}_{[-\pi|v|,\pi|v|]}\right)(x) = \frac{C}{2\pi}\int_{-\pi|v|}^{\pi|v|}e^{ix\xi}d\xi\\ &= \frac{C}{2\pi}\left[\frac{e^{ix\xi}}{ix}\right]_{-\pi|v|}^{\pi|v|} = C\frac{\sin(\pi|v|x)}{\pi x}.\end{aligned}$$

Thus, when $|C| > \frac{1}{|v|}$ we have $|\alpha(0)| > 1$, which is not is not implementable with an *analog flutter shutter* (see definition 4 page 5). This means that choosing $C := \frac{1}{|v|}$ suffices to produce a lower bound for (36). Thus, using $C = \frac{1}{|v|}$ in (36) we get (37) and therefore proved that the average MSE of any *analog flutter shutter* is bounded from below by $|v|\left(\mu+\sigma_o^2+|v|\sigma_r^2\right)$. By definition, an optimal *snapshot* satisfies, if it exits,

$$\inf_{\Delta t\in\left(0,\frac{2}{|v|}\right)}\mathrm{MSE}_{\text{snapshot}}(\Delta t) \leqslant \mathrm{MSE}_{\text{snapshot}}(\Delta t), \quad (38)$$

for any $\Delta t \in \left(0, \frac{2}{|v|}\right)$. Indeed, invertible *snapshots* satisfy $|v|\Delta t < 2$.

In the one hand, for any invertible *snapshot* and any $\xi \in [-\pi,\pi]$ we have that

$$\widehat{\mathbb{1}_{[0,\Delta t]}}(\xi v) = \frac{2\sin\left(\frac{\xi v\Delta t}{2}\right)}{\xi v}e^{-i\frac{\xi v\Delta t}{2}} \neq 0. \quad (39)$$

Hence, combining definition7, equation (31) and (39) we deduce that for any $\Delta t \in \left(0, \frac{2}{|v|}\right)$ we have

$$\begin{aligned}&\mathrm{MSE}_{\text{snapshot}}(\Delta t) \quad (40)\\ &=\frac{1}{2\pi}\left(\Delta t\left(\mu+\sigma_o^2\right)+\sigma_r^2\right)\int_{-\pi}^{\pi}\frac{d\xi}{\left|\frac{2\sin\left(\frac{\xi|v|\Delta t}{2}\right)}{\xi v}\right|^2}\\ &= \left(\Delta t\left(\mu+\sigma_o^2\right)+\sigma_r^2\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\xi}{\left|\frac{2\sin\left(\frac{\xi|v|\Delta t}{2}\right)}{\xi v}\right|^2}\\ &= \left(\Delta t\left(\mu+\sigma_o^2\right)+\sigma_r^2\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\xi v}{2\sin\left(\frac{\xi|v|\Delta t}{2}\right)}\right|^2 d\xi\\ &= |v|\left(|v|\Delta t\left(\mu+\sigma_o^2\right)+|v|\sigma_r^2\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\xi}{2\sin\left(\frac{\xi|v|\Delta t}{2}\right)}\right|^2 d\xi. \quad (41)\end{aligned}$$

On the other hand, the choice $\Delta t := \frac{1}{|v|}$ satisfies $\Delta t \in \left(0, \frac{2}{|v|}\right)$ and is therefore admissible. Hence, from (40)-(41) we deduce that

$$\mathrm{MSE}_{\text{snapshot}}\left(\Delta t := \frac{1}{|v|}\right) = |v|\left(\mu+\sigma_o^2+|v|\sigma_r^2\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\xi}{2\sin\left(\frac{\xi}{2}\right)}\right|^2 d\xi. \quad (42)$$

Thus, combining (38) and (42) we obtain

$$\inf_{\Delta t\in\left(0,\frac{2}{|v|}\right)}\mathrm{MSE}_{\text{snapshot}}(\Delta t) \leqslant |v|\left(\mu+\sigma_o^2+|v|\sigma_r^2\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\xi}{2\sin\left(\frac{\xi}{2}\right)}\right|^2 d\xi. \quad (43)$$

Combining (35)-(37) and (43) we deduce (32). The numerical value in (32) was evaluated numerically. This concludes our proof.□

Theorem 3 means that for any known velocity v and observation conditions (the variances σ_o^2 and σ_r^2) the gain, in terms of RMSE, of any *flutter shutter* cannot be overwhelming provided the comparative *snapshot* is not too badly chosen. This obviously invalidates the conclusions of [8] whose approximations did not permit to bring a correct discussion of the *flutter shutter* method. Notice that the formula of theorem 1 (and its consequences) also applies to the Levin et al. *motion-invariant photography* and variant as well [6,17,18]. Indeed, [42] proved that the *motion-invariant photography* is mathematically equivalent to an *analog flutter shutter*. This means that the upper bound 1.1776 on the gain of theorem 3 also applies to e.g. [6,17,18].

6 Conclusion

This paper has modelled the stochastic photon acquisition of a moving landscape by a light sensor. By opposition to what was done in [42] the model that has been developed in this paper permits to cover the case of any additive (sensor readout, quantization) and obscurity noises of finite variances in addition to the photonian Poisson noise. In particular it covers the investigation of the *flutter shutter* of [8]. The mathematical formulation developed in this paper permits to formalize and analyze a general *flutter shutter* theory which includes the standard photography, the original Agrawal, Raskar et al. [1,2,3,27,28,29], the *motion-invariant photography* [6,17,18], the *analog* and the *numerical flutter shutter* introduced in [42]. Thus, to the best of our knowledge, this paper covers the entire literature on the *flutter shutter* and variants. For each of these methods, closed formulae provide the average MSE of the final image obtained after deconvolution. The formulae assume that the velocity is a known constant. In addition, we neglect the boundaries effects for the deconvolution. (This is in favor of the *flutter shutter* because the inverse filter of a *flutter shutter* has larger support than the inverse filter of a *snapshot*.)

Obtaining these closed formulae required a proof of strong convergence properties of the Shannon-Whittaker interpolation after convolution/deconvolution of stochastic samples that are not wide sense stationary. The construction is not limited to the *flutter shutter*.

The formalism used in [8] has been discussed. We pointed out the approximations used by the authors. We have given an explanation to the contradiction between the conclusions of [8] and [42] when there is no readout noise and disproved the conclusions of [8]. These contradictions can be led back to excessive simplifications in the acquisition model that lead [8] to misleading conclusions. More precisely 1) the upper bound $\sqrt{1+\frac{\sigma_r^2}{J}}$ [8, equation (11) page 5] where J is the mean photon emission and σ_r^2 is the additive (sensor readout) variance does not hold true and 2) the statement "gain for computational imaging is significant only when the average signal level J is considerably smaller than the read noise variance σ_r^2" [8, page 5] does not hold true.

The *gain* of a *flutter shutter* does not becomes arbitrarily large when the additive (sensor readout, quantization) noise increases or under poor lighting conditions. The gain is to be understood in terms of root mean square error and with respect to an optimal *snapshot*. It has been proved here that the gain of *any flutter shutter* cannot exceed a 1.1776 factor, even though the exposure time is infinite. This 1.1776 upper bound factor can be compared to the asymptotic bound of 1.1717 proved in [42]. Both bounds apply as soon as the motion has a fixed known velocity, which also is the set-up of [8]. Obviously, our theory and its consequences also applies to other *flutter shutter* variants e.g. [7,22,23].

We proved, contrarily to [8], that the observation conditions have very little impact on the gain of a *flutter shutter* camera. In other words, with a *flutter shutter* no specific observation condition such as: lighting, sensor quality, thermal noise permit a breakthrough in the image quality. Thus, to make the *flutter shutter* into an efficient acquisition tool one should turn to a more clever optimization of the *flutter shutter gain function*, e.g., the velocity is not a known constant.

A Proof of lemma 1 on page 9

For any $x \in \mathbb{R}$, we have

$$\sum_{n=-\infty}^{\infty}\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)\gamma(x-n)$$

$$=\int_{-\infty}^{+\infty}\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(y)\gamma(x-y)dy \tag{44}$$

$$=\left[\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)*\gamma\right](x) \tag{45}$$

$$=\mathcal{F}^{-1}\left[\mathcal{F}\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(\xi)\mathcal{F}(\gamma)(\xi)\right](x)$$

$$=\mathcal{F}^{-1}\left[\hat{\alpha}(\xi v)\hat{u}(\xi)\hat{\gamma}(\xi)\right](x) \tag{46}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{\alpha}(\xi v)\hat{u}(\xi)\hat{\gamma}(\xi)e^{ix\xi}d\xi \tag{47}$$

$$=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\hat{\alpha}(\xi v)\hat{u}(\xi)\hat{\gamma}(\xi)e^{ix\xi}d\xi \tag{48}$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{u}(\xi)e^{ix\xi}d\xi = u(x). \tag{49}$$

Poisson's formula **(xxv)** justifies (44). Indeed, consider

$$f_x(y)=\left[\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right](y)\gamma(x-y). \tag{50}$$

From Cauchy-Schwartz's inequality we have

$$\|f_x\|_{L^1(\mathbb{R})}\leqslant\left\|\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right\|_{L^2(\mathbb{R})}\|\gamma\|_{L^2(\mathbb{R})}.$$

Indeed, since $u\in L^1(\mathbb{R})$ by Young's inequality we have

$$\left\|\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right\|_{L^2(\mathbb{R})}\leqslant\left\|\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)\right\|_{L^2(\mathbb{R})}\|u\|_{L^1(\mathbb{R})} =\|\alpha\|_{L^2(\mathbb{R})}\|u\|_{L^1(\mathbb{R})}<+\infty,$$

because $\alpha\in L^2(\mathbb{R})$ and $u\in L^1(\mathbb{R})$. Since $\gamma\in L^2(\mathbb{R})$, from (50), we deduce that $f_x\in L^1(\mathbb{R})$. Moreover, we have

$$\hat{f}_x(\xi)=\left(\left(\hat{\alpha}\left(\cdot v\right)\hat{u}\left(\cdot\right)\right)*\hat{\gamma}\left(\cdot\right)e^{-ix\xi}\right)(\xi). \tag{51}$$

Since u is $[-\pi,\pi]$ band-limited and γ is $[-\pi,\pi]$ band-limited from its definition (19) we deduce that $\hat{f}_x(\xi)=0$ for all $\xi\in\mathbb{R}$ such that $|\xi|>\pi$. Yet, from Poisson's formula **(xxv)** this only implies that

$$\sum_n f_x(n)=\hat{f}_x(0)+\hat{f}_x(-2\pi)+\hat{f}_x(2\pi).$$

From (51), $f_x(y)$ is given by a convolution. Thus,

$$\hat{f}_x(2\pi)=\int_{\mathbb{R}}\hat{\alpha}(\xi v)\hat{u}(\xi)\hat{\gamma}(\xi-2\pi)e^{-ix\xi}d\xi =\int_{-\pi}^{\pi}\hat{\alpha}(\xi v)\hat{u}(\xi)\hat{\gamma}(\xi-2\pi)e^{-ix\xi}d\xi$$

since $\hat{u}$ is $[-\pi,\pi]$ band-limited. Since the integrand is non-zero on the set $\{\pi\}$, which has Lebesgue measure zero, we deduce that $\hat{f}_x(2\pi)=0$. Similarly we obtain $\hat{f}_x(-2\pi)=0$. The validity of (45)-(46) is, from the continuity of the $L^2(\mathbb{R})$ function $\mathbb{R}\ni x\mapsto\left[\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)*\gamma\right](x)$, obvious. Not less obvious is (47) from the inverse Fourier transform definition **(xxiii)**. The definition (19) of $\hat{\gamma}$ justifies (48). The fact that u is $[-\pi,\pi]$ band-limited and again the definition of $\hat{\gamma}$ (19) justifies (49) and completes our proof.

B Proof of lemma 2 on page 10

We first prove (23) then prove that $\mathrm{var}(\tilde{\mathfrak{u}}_{\mathrm{est,ana}}(\cdot))$ can be written as the sum of an $L^1(\mathbb{R})$ function and a constant. From proposition 3 on page 10, for any $x \in \mathbb{R}$, we have

$$\mathrm{var}(\tilde{\mathfrak{u}}_{\mathrm{est,ana}}(x)) = \sum_{n=-\infty}^{\infty} \mathrm{var}(obs(n))\,(\gamma(x-n))^2. \quad (52)$$

Therefore, combining proposition 1 (equation (7)) on page 6 and (52), for any $x \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathrm{var}(\tilde{\mathfrak{u}}_{\mathrm{est,ana}}(x))\\
&=\sum_{n=-\infty}^{\infty}\left[\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)+\left(\|\alpha\|_{L^1(\mathbb{R})}(\mu+\sigma_o^2)+\sigma_r^2\right)\right](\gamma(x-n))^2\\
&=\sum_{n=-\infty}^{\infty}\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)(\gamma(x-n))^2 \quad (53)\\
&+\left(\|\alpha\|_{L^1(\mathbb{R})}(\mu+\sigma_o^2)+\sigma_r^2\right)\sum_{n=-\infty}^{\infty}(\gamma(x-n))^2\\
&=\sum_{n=-\infty}^{\infty}\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)(\gamma(x-n))^2 \quad (54)\\
&+\|\gamma\|^2_{L^2(\mathbb{R})}\left(\|\alpha\|_{L^1(\mathbb{R})}(\mu+\sigma_o^2)+\sigma_r^2\right),
\end{aligned}$$

and (23) is proved. As a consequence, the function $\mathbb{R} \ni x \mapsto \mathrm{var}(\tilde{\mathfrak{u}}_{\mathrm{est,ana}}(x))$ can be written as

$$\mathrm{var}(\tilde{\mathfrak{u}}_{\mathrm{est,ana}}(x)) = f_{\mathrm{ana}}(x) + \|\gamma\|^2_{L^2(\mathbb{R})}\left(\|\alpha\|_{L^1(\mathbb{R})}(\mu+\sigma_o^2)+\sigma_r^2\right),$$

where $f_{\mathrm{ana}}(\cdot)$ is defined by (24) (page 10). It remains to justify that $f_{\mathrm{ana}} \in L^1(\mathbb{R})$. We have

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\left|\sum_{n=-\infty}^{\infty}\left[\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)(\gamma(x-n))^2\right]\right|dx\\
&\leqslant\int_{-\infty}^{+\infty}\sum_{n=-\infty}^{\infty}\left[\left|\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)\right|(\gamma(x-n))^2\right]dx\\
&=\sum_{n=-\infty}^{\infty}\left[\left|\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)\right|\int_{-\infty}^{+\infty}(\gamma(x-n))^2dx\right]\\
&=\sum_{n=-\infty}^{\infty}\left[\left|\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)\right|\|\gamma\|^2_{L^2(\mathbb{R})}\right]\\
&=\|\gamma\|^2_{L^2(\mathbb{R})}\sum_{n=-\infty}^{\infty}\left|\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)\right|\\
&=\|\gamma\|^2_{L^2(\mathbb{R})}\int_{-\infty}^{+\infty}\left|\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(x)\right|dx\\
&\quad\text{(by } \textbf{(xxv)} \text{ equation (57))}\\
&=\|\gamma\|_{L^2(\mathbb{R})}\left\|\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right\|_{L^1(\mathbb{R})}<+\infty \;(L^1*L^1\subset L^1 \text{ convolution}).
\end{aligned}$$

This concludes our proof.

C Proof of proposition 4 page 12

We first prove (26) then (27). From the *numerical flutter shutter* samples definition (definition 6 page 11), since, by assumption, $\mathbb{E}(\eta(n)) = 0$ we have

$$\begin{aligned}
\mathbb{E}(obs(n)) &= \sum_{k=0}^{L-1}\alpha_k\int_{k\Delta t}^{(k+1)\Delta t}u(n-vt)+\mu+\sigma_o^2dt\\
&=\int_0^{L\Delta t}\alpha(t)\left(u(n-vt)+\mu+\sigma_o^2\right)dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}(obs(n)) &= \int_0^{L|v|\Delta t}\frac{1}{|v|}\alpha\left(\frac{t}{v}\right)\left(u(n-t)+\mu+\sigma_o^2\right)dt\\
&=\left(\frac{1}{|v|}\alpha\left(\frac{\cdot}{v}\right)*u\right)(n)+(\mu+\sigma_o^2)\left(\int_{-\infty}^{\infty}\alpha(t)dt\right)
\end{aligned}$$

and (26) is proved. We now prove (27). From definition 6, since $\mathrm{var}(\eta(n)) = \sigma_r^2$ we have

$$\begin{aligned}
\mathrm{var}(obs(n))&=\sum_{k=0}^{L-1}\alpha_k^2\left(\int_{k\Delta t}^{(k+1)\Delta t}u(n-vt)+\mu+\sigma_o^2dt+\mathrm{var}(\eta(k))\right)\\
&=\left(\frac{1}{|v|}\alpha^2\left(\frac{\cdot}{v}\right)*(u+\mu+\sigma_o^2)\right)(n)+\sum_{k=0}^{L-1}\alpha_k^2\mathrm{var}(\eta(k))\\
&=\left(\frac{1}{|v|}\alpha^2\left(\frac{\cdot}{v}\right)*u\right)(n)+\|\alpha\|^2_{L^2(\mathbb{R})}(\mu+\sigma_o^2)+\sum_{k=0}^{L-1}\alpha_k^2\sigma_r^2\\
&=\left(\frac{1}{|v|}\alpha^2\left(\frac{\cdot}{v}\right)*u\right)(n)+\|\alpha\|^2_{L^2(\mathbb{R})}(\mu+\sigma_o^2)+\frac{\|\alpha\|^2_{L^2(\mathbb{R})}}{\Delta t}\sigma_r^2\\
&=\left(\frac{1}{|v|}\alpha^2\left(\frac{\cdot}{v}\right)*u\right)(n)+\|\alpha\|^2_{L^2(\mathbb{R})}\left(\mu+\sigma_o^2+\frac{\sigma_r^2}{\Delta t}\right)
\end{aligned}$$

and (27) is proved. This concludes our proof.

D Proof of theorem 2 page 13

The proof follows the same path as the proof of lemma 2 on Annex B: we first justify that $\mathrm{var}(\tilde{\mathfrak{u}}_{\mathrm{est,num}}(x))$ can be written as the sum of an $L^1(\mathbb{R})$ function and a constant. We then prove the formula in theorem 2.
From proposition 5 (page 12), we have that

$$\begin{aligned}
\mathrm{var}(\tilde{\mathfrak{u}}_{\mathrm{est,num}}(x)) &= \sum_{n=-\infty}^{\infty}\mathrm{var}(obs(n))\,(\gamma(x-n))^2\\
&=\sum_{n=-\infty}^{\infty}\left[\left(\frac{1}{|v|}\alpha^2\left(\frac{\cdot}{v}\right)*u\right)(n)+\|\alpha\|^2_{L^2(\mathbb{R})}\left(\frac{\sigma_r^2}{\Delta t}+\mu+\sigma_o^2\right)\right](\gamma(x-n))^2,
\end{aligned}$$

where the last equality is justified by proposition 4 page 12. The rest of the calculation is similar to (53)-(54) in annex B, and we obtain that

$$\mathrm{var}(\tilde{\mathfrak{u}}_{\mathrm{est,num}}(x)) = f_{\mathrm{num}}(x) + \|\alpha\|^2_{L^2(\mathbb{R})}\|\gamma\|^2_{L^2(\mathbb{R})}\left(\frac{\sigma_r^2}{\Delta t}+\mu+\sigma_o^2\right),$$

for any $x \in \mathbb{R}$, where

$$f_{\mathrm{num}}(\cdot) := \sum_{n=-\infty}^{\infty}\left[\left(\frac{1}{|v|}\alpha^2\left(\frac{\cdot}{v}\right)*u\right)(n)(\gamma(\cdot-n))^2\right] \in L^1(\mathbb{R}).$$

The proof that $f_{\mathrm{num}} \in L^1(\mathbb{R})$ is identical to the proof that $f_{\mathrm{ana}} \in L^1(\mathbb{R})$ given in appendix B. Indeed, $\alpha \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and therefore $\alpha^2 \in L^1(\mathbb{R})$. We now justify the formula in theorem 2. From proposition 5 page 12, for any $x \in \mathbb{R}$ we have $\mathbb{E}(\tilde{\mathfrak{u}}_{\mathrm{est,num}}(x)) = \tilde{u}(x)$ and therefore $\mathbb{E}|\tilde{\mathfrak{u}}_{\mathrm{est,num}}(x) - \tilde{u}(x)|^2 = \mathrm{var}(\tilde{\mathfrak{u}}_{\mathrm{est,num}}(x))$. Thus, by similar calculations to those in page 10, between lemma 2 and theorem 1 we obtain

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\mathbb{E}|\tilde{\mathfrak{u}}_{\mathrm{est,num}}(x)-\tilde{u}(x)|^2dx = \|\alpha\|^2_{L^2(\mathbb{R})}\|\gamma\|^2_{L^2(\mathbb{R})}\left(\frac{\sigma_r^2}{\Delta t}+\mu+\sigma_o^2\right)$$

and theorem 2 page 13 is proved.

E Main notations and formulae

(i) t time variable
(ii) Δt length of a time interval (exposure time)
(iii) $x \in \mathbb{R}$ spatial variable
(iv) $X \sim Y$ means that the random variables X and Y have the same law
(v) $\mathbb{P}(A)$ probability of an event A
(vi) $\mathbb{E}(X)$ expected value of a random variable X
(vii) $\text{var}(X)$ variance of a random variable X
(viii) $f * g$ convolution of two $L^2(\mathbb{R})$ functions $(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$ (the validity is discussed as soon as both functions are not in $L^2(\mathbb{R})$)
(ix) $l(t,x) > 0 \ \forall \ x \in \mathbb{R}^+ \times \mathbb{R}$ continuous landscape before passing through the optical system
(x) $\mathcal{P}(\lambda)$ Poisson random variable with intensity $\lambda > 0$
(xi) g *point-spread-function* of the optical system. Assumption $g \in S(\mathbb{R})$ ($S(\mathbb{R})$ denotes the Schwartz class on $\mathbb{R}$)
(xii) $\sigma_o^2 < +\infty$ variance of the thermal noise
(xiii) $\sigma_r^2 < +\infty$ variance of the additive (sensor readout, quantization) noise
(xiv) $\mu < +\infty$ average mean of $\tilde{u}$ (see **(xv)**)
(xv) $\tilde{u} = \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} * g * l$ ideal observable landscape just before sampling. Assumption: $\tilde{u}$ $[-\pi,\pi]$ band-limited, $\mu := \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \tilde{u}(x)dx$ and $u := (\tilde{u}-\mu) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$
(xvi) $obs(n)$, $n \in \mathbb{Z}$ observation of the landscape at pixel n
(xvii) $\tilde{u}_{\text{est,ana}}$ estimated landscape for the *analog flutter shutter* from the observed samples $obs(n)$ where $n \in \mathbb{Z}$. It is defined to have $\mathbb{E}(\tilde{u}_{\text{est,ana}}(x)) = \tilde{u}(x)$ for any $x \in \mathbb{R}$.
(xviii) $\tilde{u}_{\text{est,num}}$ estimated landscape for the *numerical flutter shutter* from the observed samples $obs(n)$ where $n \in \mathbb{Z}$. It is defined to have $\mathbb{E}(\tilde{u}_{\text{est,ana}}(x)) = \tilde{u}(x)$ for any $x \in \mathbb{R}$.
(xix) v relative velocity between the landscape and the camera (unit: pixels per second)
(xx) $\alpha(t)$ *flutter shutter gain function.*
(xxi) $\mathbb{1}_{[a,b]}$ indicator function of an interval $[a,b]$
(xxii) $\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f(x)|dx$, $\|f\|_{L^2(\mathbb{R})} = \sqrt{\int_{\mathbb{R}} |f(x)|^2 dx}$
(xxiii) Let $f, g \in L^1(\mathbb{R})$ or $L^2(\mathbb{R})$, then

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx;$$

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) := \widetilde{\mathcal{F}(f)}(x) = \check{f}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}(f)(\xi)e^{ix\xi}d\xi.$$

Moreover $\mathcal{F}(f*g)(\xi) = \mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)$ and (Plancherel)

$$\int_{\mathbb{R}} |f(x)|^2 dx = \|f\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\int_{\mathbb{R}} |\mathcal{F}(f)|^2(\xi)d\xi = \frac{1}{2\pi}\|\mathcal{F}(f)\|^2_{L^2(\mathbb{R})}. \quad (55)$$

(xxiv) $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} = \frac{1}{2\pi}\mathcal{F}(\mathbb{1}_{[-\pi,\pi]})(x) = \mathcal{F}^{-1}(\mathbb{1}_{[-\pi,\pi]})(x)$
(xxv) (Poisson summation formula) Let $\varphi \in L^1(\mathbb{R})$ be $[-\pi,\pi]$ band-limited.

$$\sum_n \varphi(n) = \sum_k \hat{\varphi}(2k\pi). \quad (56)$$

Since φ is $[-\pi,\pi]$ band-limited $\hat{\varphi}(\xi) = 0 \ \forall \xi \in \mathbb{R}$ such that $|\xi| > \pi$, then from Poisson's formula **(xxv)** equation (56) we have

$$\sum_n \varphi(n) = \hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x)dx. \quad (57)$$

This applies to any shift of φ, so we also have

$$\sum_n \varphi(x+n) = \int_{\mathbb{R}} \varphi(x)dx. \quad (58)$$

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