Are Random Flutter Shutter Codes Good Enough?

Yohann Tendero*

Abstract

Flutter shutter (coded exposure) cameras permit to make invertible arbitrarily severe motion blurs, when the camera-scene motion is uniform. To achieve this prowess Agrawal, Raskar et al. proposed in [1, 2, 3, 25, 26, 27] to interrupt the photon flux during the exposure time. The photon flux is interrupted according to a sequence called *flutter shutter code.* In their founding papers, the inventors of the *flutter shutter*, proposed to use binary *flutter shutter codes.* The proposed *flutter shutter code* was obtained by randomly generating and testing a very large number of binary *flutter shutter codes.* The computational burden of such a random search is huge because of the number of possible codes, e.g., 2^{52} in [26]. However, a practical case can require a much longer code. Following this work, many other authors have used random binary *flutter shutter codes* to get invertible motion kernels. Others have proposed accelerated schemes attempting to find "good" binary *flutter shutter codes.* However, it is striking that, to the best of our knowledge, none of these papers prove that some sort of optimality can be reached by (binary or not) random *flutter shutter codes.* In this paper, we prove that random codes coming from independent and identically distributed random variables are not the adequate code family to optimize a *flutter shutter* in terms of MSE. Indeed, we prove that their expected MSE is asymptotically not better than a simple snapshot.

Keywords: Motion blur, Poisson noise, *snapshot*, *flutter shutter*, optimization, signal to noise ratio (SNR), mean square error (MSE), coded exposure.

1 Introduction

A digital camera count at each pixel sensor the number of photons emitted by the observed scene during an interval of time called exposure time. The photon count is a Poisson random variable. Its mean is the noiseless pixel value. The difference between the noiseless pixel value and the observed sensor photon count is called shot noise. A passive camera has no control over the landscape photon emission. Consequently, with a passive camera the only safe way to increase the signal to noise ratio (SNR) is to acquire more photons by increasing the exposure time. However, when the camera and the scene are in relative motion during the exposition process the resulting images are corrupted by motion blur. A motion blur kernel is not invertible as soon as its support exceeds two pixels. Obtaining a longer exposure time without the effects of motion blur is therefore one of the core problems of photography.

For uniform motion blurs a setup that solves this dilemma was proposed in [1, 2, 3, 25, 26, 27]. The authors propose to attach a *flutter shutter* to a camera. The shutter interrupts the photon flux on sub-intervals of the exposure time. As a byproduct of this fluttering shutter, the resulting motion blur kernel can be made invertible. Numerically a *flutter shutter* is described by a

*Department of Mathematics - University of California Los Angeles 520 Portola Plaza, Los Angeles, CA 90095, United States, tendero@math.ucla.edu. Y. Tendero is supported on the Grants No: 12-DARPA-1042, UC Lab 12-LR-236660, ONR N000141210838 and NSF DMS-1118971.

binary shutter sequence -or *flutter shutter code*- that gives the intervals where the photon flux is interrupted. If the *flutter shutter code* is well chosen, a *flutter shutter* can guarantee the invertibility of any uniform motion blur. Thus, for arbitrarily high velocities or equivalently for arbitrary long exposures one can invert the blur by a simple deconvolution (see figure 1). Therefore, many more photons can be sensed by the camera.

Figure 1: On the left panel: simulated observed (blurry and noisy) image using the Agrawal, Raskar et al. code [1]. The blur interval length is 52 pixels. Notice the stroboscopic effect of the *flutter shutter* apparatus. On the right panel: the reconstructed image obtained by a direct deconvolution. See [33] for a peer reviewed implementation and on line demonstration of a *flutter shutter* camera simulator.

Can a *flutter shutter* indefinitely increase the SNR by an increased exposure time? Tendero et al. [35] prove that the answer is no. More precisely, given a landscape moving in uniform translation at a known velocity v, the gain of a *flutter shutter* in terms of root mean square error (RMSE) is no more than 1.17 compared to an optimal *snapshot*. This bound is optimal in the sense that the 1.17 factor can be attained. However, this analysis does not cover the case of random *flutter shutter codes* with infinite support. Therefore, their analysis does not permit to bring the discussion on the MSE of random *flutter shutter* as we shall do here. In the literature on the *flutter shutter*, to the best of our knowledge, most *flutter shutter codes* are obtained by randomly generating and comparing the performance of a very large number of binary codes. Such random optimization is computationally very expensive, due to the number of possible codes, even for simple binary *flutter shutter codes*. To the best of our knowledge, no argument of optimality has been proved for random (binary or not) *flutter shutter codes*. This paper gives a mathematical formalism that permits to analyze random *flutter shutter codes*. Formulae give a lower bound of the expected MSE, for both variant of the *flutter shutter* namely the *analog* and *numerical flutter shutter*. It proves that *flutter shutter codes* coming from independent and identically distributed random variables are not optimal. Indeed we prove that, as soon as the relative camera-scene velocity is non zero, the expected MSEs are bounded from below by the MSE of a simple *snapshot*. As a byproduct, we can conclude that good *flutter shutter codes* in terms of MSE are *not* random *flutter shutter codes*.

1.1 Related work

There is a considerable literature on the problem of increasing the SNR of moving scenes. However, few papers seem to have addressed the optimization of a *flutter shutter* or claimed to have found optimal codes. We shall review them next. The question of the optimal exposure time using a conventional camera is investigated in [4]. This paper [4] considers the case of non invertible blurs with supports larger than two pixels that therefore requires a regularized deconvolution [9]. Following the *flutter shutter* literature [1, 2, 3, 25, 26, 27, 35, 40, 34, 33, 18, 17, 20, 21, 13, 8, 12] we shall not follow that path here. Only well-posed deconvolution strategies will be compared. Indeed, the main goal of the *flutter shutter* is to make uniform motion blur kernels invertible. Therefore, a regularized deconvolution scheme is not needed.

In [32] a multi-image framework that acquires a burst of sharp but noisy images was proposed. The obtained image is sharp and has an increased SNR. The burst denoising method [5] is the most convincing competitor to *flutter shutter* methods. However, it relies on registration algorithms that are glutton in memory and computational resources. By opposition, a *flutter shutter* camera acquires and records only one image.

In [10, 24, 31, 37, 38, 39] the authors use hybrid and/or complex camera systems. These systems raise other problems such as the computational costs and/or hardware issues. We shall not discuss further these issues as they are out of the scope of this paper.

The simplest setup was proposed by Agrawal, Raskar et al. [1, 2, 3, 25, 26, 27]. They propose to modulate the photon flux that enters the camera. Indeed, they proposed to open/close the camera shutter according to a pseudo random binary sequence. When the motion is uniform the resulting blur kernel becomes invertible. In other words, the Fourier transform of the *flutter shutter* kernel does not vanish on the support of the Fourier transform of the signal. Due to this fluttering shutter, the visual result of an image acquired by *flutter shutter* is close to a stroboscopic image (see figure 1). Nevertheless, a *fluter shutter* permits to get back a neat image after deconvolution.

In [15, 19, 23, 36] the authors use a dynamic lighting pattern instead of the shutter to recreate a *flutter shutter* effect. In [20] the *flutter shutter* apparatus is applied to iris images and in [40] to bar-codes. In [28] the authors investigate the question of denoising an image taken by a *flutter shutter* camera and suggest an user assisted estimation of the blur. Their conclusion is that the denoising should be applied before and after the deconvolution [28].

As we have just seen, most of these works actually propose more complex hardware setups than the original *flutter shutter*. However, the common denominator is to obtain a neat image while increasing the exposure time. The exact evaluation of the mean square error (MSE) is thus the core question of these camera designs.

The optimization of a *flutter shutter* is specifically investigated in [1, 2, 3, 13, 17, 26]. In [26, page 799] Raskar, Agrawal et al. "computed a near-optimal code by implementing a randomized linear search and considered approximately 3×10^6" binary codes of length 52 with 26 "1". Such a random search is mathematically equivalent to generating 52 realizations of independent and identically distributed (i.i.d.) Bernoulli random variables $\mathcal{B}(\frac{1}{2})$, and keeping the ones that have 26 "1". A code "that (i) maximizes the minimum of the magnitude of the DFT values and (ii) minimizes the variance of the DFT values" [26, page 799] is proposed. In [2], Agrawal, Raskar et al. experimentally investigate the possibility that for binary codes of length 52 the optimal number of "1" may be smaller than 26. Their conclusion is that "optimal codes for coded exposure need not be 50% on-off if signal-dependent noise is taken into account" [2, page 2567]. Recall that, due to the Poisson nature of photon emission, the signal dependency of the noise must be taken into account. In [3], Agrawal and Xu propose "design rules for a code to have good PSF estimation capability and outline two search criteria for finding the optimal code for a given length" [3, page 2066]. The search remains a random search among binary sequences. In [1], Agrawal and Raskar use the *flutter shutter* apparatus to "propose a method to selectively increase the resolution of a high frequency region on an object when the blur is small" [1, page 2]. The method still requires a *flutter shutter code* and the authors "analyze the relationship between the blur size and parameters of the code and show how to choose these parameters" [1, page 5]. A binary code found by random search is proposed. In [17], McCloskey proposes to adapt the code to the "object velocity, and propose a method for computing such velocity-dependent sequences" [17, page 1] (codes). The considered codes are binary and found by a random search. In a paper entitled "Designing the optimal shutter sequences for the flutter shutter imaging method" [13], Jelinek argues that " the number of possible sequences grows exponentially in both the subject's motion velocity and desired exposure value, with their majority being useless" [13, page 77010N-1] and propose an accelerated random scheme.

As a matter of fact, all the above mentioned papers use a random search to find "near optimal" or "optimal" binary *flutter shutter* codes. The search is not exhaustive. The exact evaluation and comparison of the MSE with the MSE of an optimal *snapshot* is not performed. Thus, the optimality claims are empty. Indeed, there are based on heuristics. Note that any finite (random or not) *flutter shutter code* can be analyzed with the theory developed in [35]. It is proved [35] that for a fixed velocity the gain cannot exceed 17%. The hope with random *flutter shutter* strategies would be that infinitely supported random codes (binary or not) could beat this bound, in expectation. Indeed, the above mentioned theory does not apply to this situation.

In this paper, the question that shall be answered is: Can random *flutter shutter codes* be optimal in terms of the expected MSE? If such codes are optimal, how do they compare to the optimal *snapshot* [35]? To the best of our knowledge, these questions are not considered in the literature on the *flutter shutter*. Theorem 3.2 and corollary 3.3 prove that, as soon as the relative camera-scene velocity is non zero, an *analog flutter shutter* camera equipped with a random code has an expected MSE bounded from below by the MSE of a *snapshot.* Theorem 4.1 and corollary 4.2 we prove that the same result holds for a *numerical flutter shutter*. These two results mean that a *flutter shutter* camera equipped with a random *flutter shutter code* cannot gain over a *snapshot* in terms of image quality (MSE). As a byproduct, a good *flutter shutter code* is therefore not a random code.

Outline of the paper

Section 2 briefly presents the mathematical requisites of the *flutter shutter* formalism. Section 3 develops the formalism needed to analyze random coded *analog flutter shutters.* Corollary 3.3 proves that for a fixed non zero velocity their MSE does improve over the MSE of a *snapshot.* The same result is proved to hold for the *numerical flutter shutter.* A glossary of notations is in annex C, page 19. (In the sequel latin numerals refer to the glossary of notations page 19.)

2 Mathematical requisites

This section gives the whole *flutter shutter* formalism as it was developed by Tendero et al. in [35]. This is needed to bring the discussion on random *flutter shutter codes.* The exposition is self-contained.

The relative camera-scene motion can be associated with a one dimensional box kernel. Its support increases linearly with the aperture time $\Delta t > 0$ and the velocity v of the relative camera scene motion. If the exposure time is too long and the blur support exceeds two pixels, then the blur is no more invertible. In that case, the restoration process is an ill-posed problem [4]. The *flutter shutter* [1, 2, 3, 27, 25] (coded exposure) permits to ensure an invertible motion kernel for arbitrarily severe, i.e., large, motion blur support. There are two different acquisition tools that implement a *flutter shutter* with a moving sensor (or scene). The *flutter shutter gain function* can be implemented as an optical (temporally changing) filter. This filter controls the percentage of incoming photons allowed to travel to the sensor. The filtering function is generally assumed to be piecewise constant [1, 2, 3, 27, 25] with a *flutter shutter code* $(\alpha_k)_{k=0}^{L-1}$, where L is the length of the code. This setup, which corresponds to the initial technology of the inventors, is called *analog flutter shutter*.

A more flexible set up, the *numerical flutter shutter*, is a mere temporal filter. In a nutshell, the camera takes a burst of L images. The k-th elementary image is assigned a numerical gain $\alpha_k \in \mathbb{R}$. The observed image is obtained as the weighted sum of elementary images with numerical weights $(\alpha_k)_{k=0}^{L-1}$. An image sensor can have a duty ratio of nearly 100% (the duty ratio is the ratio of light integration time over readout, storage, reset times - that is the percentage of useful time), see, e.g., [16, 22]. Consequently, there are sensors that can integrate photons without interruption. This means that the *numerical flutter shutter*, as it is described below, i.e., without "dead time" between two consecutive weights α_k, is doable from a technological point of view. Notice that in both cases we have a *flutter shutter code*, but the formulae for the resulting image are not the same, as illustrated in table 1.

Table 1: This table summarizes the main formulae on *numerical* and *analog flutter shutters*. The first column describes the structure of the *numerical flutter shutter*, the second describes the *analog flutter shutter*. Using the same code the MSE of a *numerical flutter shutter* is lower than the MSE of an *analog flutter shutter*. All codes usable with an *analog flutter shutter* are usable with a *numerical flutter shutter* while the converse is not true. See text.

Type of *flutter shutter*	*Numerical flutter shutter*	*Analog flutter shutter*
Flutter shutter gain function $\alpha(t)$	$\alpha(t) = \sum_{k=0}^{L-1} \alpha_k \mathbb{1}_{[k\Delta t,(k+1)\Delta t[}(t)$ (with $\alpha_k \in \mathbb{R}$ and $\Delta t > 0$)	$\alpha(t) = \sum_{k=0}^{L-1} \alpha_k \mathbb{1}_{[k\Delta t,(k+1)\Delta t[}(t)$ (with $\alpha_k \in [0,1]$ and $\Delta t > 0$)
Observed samples $obs(n)$	$obs(n) \sim \sum_{k=0}^{L-1} \alpha_k \mathcal{P}\left(\int_{k\Delta t}^{(k+1)\Delta t} u(n-vt)dt\right)$	$obs(n) \sim \mathcal{P}\left(\frac{1}{v}(\alpha\left(\frac{\cdot}{v}\right) * u)(n)\right)$
$\mathbb{E}\left(obs(n)\right)$ (observed)	$\left(\frac{1}{v}\alpha\left(\frac{\cdot}{v}\right) * u\right)(n)$	$\frac{1}{v}(\alpha\left(\frac{\cdot}{v}\right) * u)(n)$
$\mathrm{var}(obs(n))$ (observed)	$\left(\frac{1}{v}\alpha^2\left(\frac{\cdot}{v}\right) * u\right)(n)$	$\frac{1}{v}(\alpha\left(\frac{\cdot}{v}\right) * u)(n)$
Inverse filter $\hat{\gamma}(\xi)$	$\frac{\mathbb{1}_{[-\pi,\pi]}(\xi)}{\hat{\alpha}(\xi v)}$	$\frac{\mathbb{1}_{[-\pi,\pi]}(\xi)}{\hat{\alpha}(\xi v)}$
$\mathbb{E}(\hat{u}_{est}(\xi))$ (deconvolved)	$\hat{u}(\xi)\mathbb{1}_{[-\pi,\pi]}(\xi)$	$\hat{u}(\xi)\mathbb{1}_{[-\pi,\pi]}(\xi)$
$\mathrm{MSE}_{\mathrm{spectral}}(\xi)$ (deconvolved)	$\frac{\|\alpha\|_{L^2(\mathbb{R})}^2\|u\|_{L^1(\mathbb{R})}}{\|\hat{\alpha}(\xi v)\|^2}\mathbb{1}_{[-\pi,\pi]}(\xi)$ (1)	$\frac{\|\alpha\|_{L^1(\mathbb{R})}\|u\|_{L^1}}{\|\hat{\alpha}(\xi v)\|^2}\mathbb{1}_{[-\pi,\pi]}(\xi)d\xi$ (2)
MSE (deconvolved)	$\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\|\alpha\|_{L^2(\mathbb{R})}^2\|u\|_{L^1(\mathbb{R})}}{\|\hat{\alpha}(\xi v)\|^2}\mathbb{1}_{[-\pi,\pi]}(\xi)d\xi$ (3)	$\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\|\alpha\|_{L^1(\mathbb{R})}\|u\|_{L^1}}{\|\hat{\alpha}(\xi v)\|^2}\mathbb{1}_{[-\pi,\pi]}(\xi)d\xi$ (4)

The *flutter shutter* study is generally performed as if the image were a one-dimensional signal, recorded on a line in the direction of the camera-landscape motion. This is valid because the motion blur is one-dimensional and that the convolution and deconvolution model is applied on each line of the image. From the mathematical viewpoint, the *flutter shutter* reduces to the 1D convolution of a *flutter shutter gain function* α with the one dimensional observed stochastic landscape. The expected value at position x of this stochastic landscape will be denoted by $u(x)$. In all statements, this ideal (noiseless) landscape u is assumed to have finite energy: $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $[-\pi, \pi]$ band limited (thanks to the combined camera and sensor frequency cut-off). Therefore, u is well sampled at a unit rate.

The formalism of the *flutter shutter* is summarized in table 1. Its first row indicates the kind of implementable *flutter shutter gain function* α, depending on the *flutter shutter* type and with a discrete code. The second row of the table gives the exact formula of the observed samples where $X \sim Y$ denotes that the random variables X and Y have the same law. The notation $\mathcal{P}(\lambda)$ denotes a Poisson random variable with intensity λ, and v denotes the relative camera scene velocity (in pixel per second). For the *analog flutter shutter*, the observed digital image at a pixel n is a Poisson noise whose intensity is $\frac{1}{v}(\alpha\left(\frac{\cdot}{v}\right) * u)(n)$. This intensity is the convolution of the landscape u with the (rescaled) *flutter shutter gain function* α. (Here and elsewhere, $*$ denotes the classic continuous convolution on $\mathbb{R}$, see **(x)**). For the *numerical flutter shutter*, the formula is similar. However, it can only be stated with a discrete *flutter shutter code*. Indeed, we have a linear combination of weighted acquired images. Each one is a Poisson noise. In both cases the observed samples $obs(n)$ are obtained for $n \in \mathbb{Z}$.

The expected value of the observed image is the same for the *analog* and *numerical flutter shutters*. As explicit in the third row, it is nothing but the convolution of the landscape u with the *flutter shutter gain function* α. There is a significant difference in the fourth row. The variance of the observed value at a pixel n depends on the square of the *flutter shutter gain function* α for the *numerical flutter shutter*. The dependency is linear for the *analog flutter shutter*.

The fifth row which gives the inverse filter applied to the observed samples in order to deconvolve the observed image. (Hereinafter $\hat{f}$ denotes the classic continuous Fourier transform on $\mathbb{R}$ and $\check{f}$ the inverse Fourier transform on $\mathbb{R}$, see **(xv)**). This inverse filter is the inverse of the Fourier transform of the convolution kernel, $\frac{1}{v}(\alpha\left(\frac{\cdot}{v}\right) * \text{sinc})(n)$. The sinc function ensures that it is applied only on the $[-\pi, \pi]$ frequencies. Indeed, u is assumed to be $[-\pi, \pi]$ band-limited.

The sixth row gives the expected value of the deconvolved (restored) image. The inverse filter is designed to give back the observed landscape u in expectation.

The seventh row gives the variance [35] of the deconvolved (restored) (denoted $\text{MSE}_{\text{spectral}}(\xi)$) for any frequency $\xi \in \mathbb{R}$. We have $\text{MSE}_{\text{spectral}}(\xi) = 0$ for any ξ that satisfies $|\xi| > \pi$ because the inverse filter is only applied to $[-\pi, \pi]$ frequencies. This is valid because u is assumed to be band limited in this range.

The last row gives the MSE of the restored signal. Note that u intervenes in the above formulae as a mere multiplication factor by the constant $\|u\|_{L^1(\mathbb{R})}$. Thus, optimizing a *flutter shutter* amounts to find *flutter shutter gain functions* α that minimize equation (3) or (4), that are different. Recall that our goal is to evaluate the expected value of the MSE when the *flutter shutter code* is random.

Having stated the *flutter shutter* formalism we can move on extending the above formalism to cope with random *flutter shutter codes*. Consider the example of random *flutter shutter code* given by the Agrawal, Raskar et al. *flutter shutter code* [26, 27].

The example of the Agrawal, Raskar et al. code [26, 27] For any code coming from a random sequence $(\alpha_k)_{k=0}^{L-1}$ where the α_k are i.i.d. such that $\mathbb{E}(\alpha_k) = m$ and $\text{var}(\alpha_k) = \sigma^2$.

From table 1 we have $\alpha_L(x) = \sum_{k=0}^{L-1} \alpha_k \mathbb{1}_{[k\Delta t,(k+1)\Delta t[}(x)$ and therefore

$$\mathbb{E}\left(|\hat{\alpha}_L(\xi)|^2\right) = L\Delta t^2 \left(\sigma^2 \text{sinc}^2\left(\frac{\Delta t\xi}{2\pi}\right) + Lm^2 \text{sinc}^2\left(\frac{L\Delta t\xi}{2\pi}\right)\right). \quad (5)$$

(The calculation is in annex A.) Note that from equation (5) we can deduce the general "shape" of the modulus of the Fourier transform of a random *flutter shutter gain function*. As explicit in (5), it consists in the sum picked at $\xi = 0$ of the function $\mathbb{R} \ni \xi \mapsto Lm^2\text{sinc}^2\left(\frac{L\Delta t\xi}{2\pi}\right)$ and of the function $\mathbb{R} \ni \xi \mapsto \sigma^2\text{sinc}^2\left(\frac{\Delta t\xi}{2\pi}\right)$. The later provides the invertibility of the convolution kernel as soon as $\sigma^2 > 0$. An interesting example is the Agrawal, Raskar et al. code [26, page 5] and patent application [27] shown in figure 2. For this [26, page 5] code the authors chose Bernoulli random variables $\mathcal{B}(\frac{1}{2})$. Such a *flutter shutter code* is binary and half the α_k are equal to 1 as considered in [26]. The (modulus of) Fourier transform of this *flutter shutter code* [26, page 5] has, indeed, this generic shape as can be seen in figure 2. This means that the random optimization scheme of [26, 27] was a success: it is close to the expected value. In addition, $\mathbb{E}\left(|\hat{\alpha}_L(\xi)|^2\right)$ increases with the code length L. Therefore, from (4) one can hope that the MSE will decrease significantly when $L \to +\infty$. The question of optimality of binary random *flutter shutter codes* coming form Bernoulli random variables $\mathcal{B}(p)$ is specifically treated in remark page 13. Indeed, to the best of our knowledge, all random *flutter shutter codes* of the literature are binary.

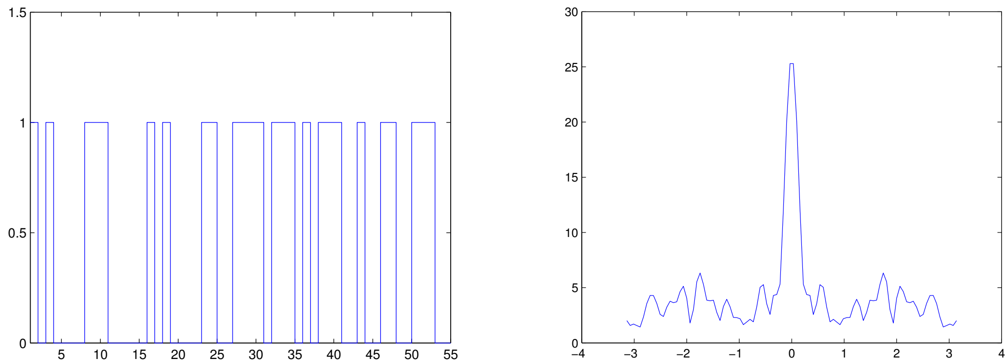

Figure 2: Agrawal, Raskar et al. code. Left: the binary *flutter shutter gain function* for the optimized Agrawal, Raskar et al. code. Right: The Fourier transform (modulus) of the Agrawal, Raskar et al. code found by an random research among binary sequences of length 52 [26, page 5] and patent application [27].

For any finite $L \in \mathbb{N}^+$, the random *flutter shutter gain function* α_L is in $L^1(\mathbb{R})$ and can be analyzed like any other *flutter shutter* with the formulas of table 1. However, when $L \to +\infty$ the *flutter shutter gain function* α_L is no more in $L^1(\mathbb{R})$. Therefore, the theory of [35] does not apply anymore. In addition, as we have just seen, most *flutter shutter* codes are binary and obtained as realizations of, e.g., Bernoulli random variables. Consequently, the theory of [35] does not permit to analyze the asymptotic case of most of the *flutter shutter codes* in the literature. Therefore, this paper proposes a theory that permit to bring the discussion on the MSE of any random *flutter shutter*.

3 The analog flutter shutter formalism for random codes

This section considers the case of random *flutter shutter gain function* coming from i.i.d. random *flutter shutter codes*. A close formula for the expectation of the MSE will be given. The theorem 3.2 and corollary 3.3 permit to answer the question of the gain of the *analog flutter shutter* in terms of MSE with respect to a *snapshot*.

As it is done in the literature on the *flutter shutter* [1, 2, 3, 13, 17, 26] we assume that α_k, $k \in \{0, \cdots, L-1\}$, are i.i.d. random variables with w.l.o.g (without loss of generality) $var(\alpha_k) = \sigma^2$. This assumption represents no limitation. Indeed, with *analog flutter shutters* one must have $\alpha_k \in [0,1]$ for $k \in \{0, \cdots, L-1\}$ for obvious physical reasons. This implies that the random variables α_k have finite second order moments i.e., $\text{var}(\alpha_k) < \infty$. Moreover, the random *flutter shutter gain function* defined by

$$\alpha_L(x) = \sum_{k=0}^{L-1} \alpha_k \mathbb{1}_{[k\Delta t,(k+1)\Delta t[}(x) \tag{6}$$

is in $L^1(\mathbb{R})$. In addition, we have

$$\|\alpha_L\|_{L^1(\mathbb{R})} = \Delta t \sum_{k=0}^{L-1} |\alpha_k| = \Delta t \sum_{k=0}^{L-1} \alpha_k. \tag{7}$$

since $\alpha_k \geqslant 0$ and (see, e.g., [35, equation 3.15])

$$|\hat{\alpha}_L(\xi)|^2 = \Delta t^2 \text{sinc}^2\left(\frac{\xi \Delta t}{2\pi}\right) |\sum_{k=0}^{L-1} \alpha_k e^{-ik\xi\Delta t}|^2. \tag{8}$$

When $L \to +\infty$, we have that $\alpha(\cdot) := \lim_{L\to\infty} \alpha_L(\cdot) = \sum_{k=0}^{+\infty} \alpha_k \mathbb{1}_{[k\Delta t,(k+1)\Delta t[}(\cdot)$ is locally integrable. Recall that for obvious feasibility reasons $\alpha_k \in [0,1]$. Therefore, any realization of α is bounded. Consequently, α enjoys a Fourier transform in the tempered distribution space $S'(\mathbb{R})$.

Let $\mathbb{E}(\alpha_k) = m$ so that $E(|\alpha_k|^2) = \text{var}(\alpha_k) + (\mathbb{E}(\alpha_k))^2 = \sigma^2 + m^2 =: \beta$. We assume w.l.o.g. that $\mathbb{E}(\alpha_k) = m > 0$. Indeed, if $m = 0$ since and $\alpha_k \geqslant 0$ we would have $\mathbb{E}(|\alpha_k|) = 0$ and $\alpha_k = 0$ almost surely. In this case, with probability 1 no photon would be acquired by the camera and no signal observed. We also assume w.l.o.g. that $\text{var}(\alpha_k) > 0$. The measurability of α_L and α are obvious. Indeed, both are elementary functions (see, e.g., [7, p. 12]). Recall the formula (2) that gives the spectral MSE (or variance) of the estimated landscape: for any frequency $\xi \in \mathbb{R}$ and any *flutter shutter gain function* $\alpha_L \in L^1(\mathbb{R})$ we have

$$\text{MSE}_{\text{spectral}(\alpha_L)}(\xi) := \frac{\|u\|_{L^1(\mathbb{R})}\|\alpha_L\|_{L^1(\mathbb{R})}}{|\hat{\alpha}_L(\xi v)|^2} \mathbb{1}_{[-\pi,\pi]}(\xi). \tag{9}$$

Indeed, for any realization of the random *flutter shutter gain function* α_L, we have $0 \leqslant \alpha_L \leqslant 1$, for obvious feasibility constraint. Thus, $\hat{\alpha}_L(0) = \|\alpha\|_{L^1(\mathbb{R})} < +\infty$ and (9) is therefore valid in $\bar{\mathbb{R}}^+$. The evaluation of

$$\mathbb{E}\left(\text{MSE}_{\text{spectral}(\alpha_L)}(\xi)\right) \quad = \mathbb{E}\left(\frac{\|u\|_{L^1(\mathbb{R})}\|\alpha_L\|_{L^1(\mathbb{R})}}{|\hat{\alpha}_L(\xi v)|^2} \mathbb{1}_{[-\pi,\pi]}(\xi)\right) \tag{10}$$

at the limit when $L \to +\infty$ requires some work. Indeed, equation (10) involves the quotient of two *dependent* random variables namely $\|\alpha_L\|_{L^1(\mathbb{R})}$ and $|\hat{\alpha}_L(\xi v)|^2$. (It is not possible to split the expectation and pass to the limit on L.) We successively consider the case $v = 0$ in section 3.1, and the case $v \neq 0$ in section 3.2.

3.1 The case $v = 0$

Consider first the case $v = 0$ and any realization of the random *flutter shutter gain function* α_L. Recall that since $\alpha_L \geqslant 0$ we have $\hat{\alpha}_L(0) = \|\alpha_L\|_{L^1(\mathbb{R})} < +\infty$ and (9) applies. Thus, for any $\xi \in \mathbb{R}$ and any $L \in \mathbb{N}^+$ we have

$$\begin{aligned}\mathrm{MSE}_{\mathrm{spectral}}(\xi) &= \frac{\|u\|_{L^1(\mathbb{R})}\|\alpha_L\|_{L^1(\mathbb{R})}}{|\hat{\alpha}_L(0)|^2}\mathbb{1}_{[-\pi,\pi]}(\xi) = \frac{\|u\|_{L^1(\mathbb{R})}}{\|\alpha_L\|_{L^1(\mathbb{R})}}\mathbb{1}_{[-\pi,\pi]}(\xi)\\ &= \frac{\|u\|_{L^1(\mathbb{R})}}{\Delta t\sum_{k=0}^{L-1}\alpha_k}\mathbb{1}_{[-\pi,\pi]}(\xi) \text{ (from (6))}\\ &= \frac{\frac{1}{L}\|u\|_{L^1(\mathbb{R})}}{\Delta t\frac{1}{L}\sum_{k=0}^{L-1}\alpha_k}\mathbb{1}_{[-\pi,\pi]}(\xi). \end{aligned} \tag{11}$$

Let $\xi \in [-\pi,\pi]$ and, from (11) let $X_L := \frac{1}{L}$ and $Y_L = \Delta t\frac{1}{L}\sum_{k=0}^{L-1}\alpha_k$. We have $X_L \xrightarrow{L\to+\infty} 0$ and $Y_L \xrightarrow{L\to+\infty} m\Delta t \in \mathbb{R}$ in probability by the (weak) law of large numbers. The constant $m\Delta t$ is positive. Therefore, from Slutsky's theorem (see, e.g., [29, page 19]) we deduce that when $L \to +\infty$ the quotient $\frac{X_L}{Y_L}$ tends to the deterministic constant 0 in distribution. Therefore, (see, e.g., [14, page 140]) $\frac{\frac{1}{L}\|u\|_{L^1(\mathbb{R})}}{\Delta t\frac{1}{L}\sum_{k=0}^{L-1}\alpha_k} = \frac{X_L}{Y_L} \xrightarrow{L\to+\infty} 0 \in \mathbb{R}$ in probability. Since the random sequence $\frac{\frac{1}{L}\|u\|_{L^1(\mathbb{R})}}{\Delta t\frac{1}{L}\sum_{k=0}^{L-1}\alpha_k} = \frac{\|u\|_{L^1(\mathbb{R})}}{\Delta t\sum_{k=0}^{L-1}\alpha_k}$ is monotone and nonnegative (see, e.g., [11, Ex5, page 228]) for any $\xi \in [-\pi,\pi]$ we have $\frac{\frac{1}{L}\|u\|_{L^1(\mathbb{R})}}{\Delta t\frac{1}{L}\sum_{k=0}^{L-1}\alpha_k} \xrightarrow{L\to+\infty} 0$ almost surely. Thus, for any $\xi \in \mathbb{R}$ the spectral MSE is almost surely zero. Therefore, the MSE is almost surely zero. This means that, if $v = 0$ then the MSE of random *flutter shutter codes* is equal to zero, with probability 1. Thus, random *flutter shutter codes* "work" when the relative camera-scene motion is zero. Indeed, *flutter shutter codes* behave like a mere accumulation. However, if $v = 0$ one does not need a *flutter shutter* because there is no convolution to make invertible. We now turn to the case $v \neq 0$.

3.2 The case $v \neq 0$

To prove that random *flutter shutter codes* are not an effective way to improve the MSE of an *analog flutter shutter* over a simple *snapshot* it suffices to provide an adequate lower bound of the expected MSE. This is our goal.

Let $\xi \in \mathbb{R}$ and consider the random variables

$$X_L(\xi) := \frac{1}{L}|\sum_{k=0}^{L-1}\alpha_k e^{-ik\xi}|^2 \text{ and } Y_L := \frac{1}{L}\sum_{k=0}^{L-1}\alpha_k. \tag{12}$$

Note that $X_L(\xi)$ is 2π periodic. Recall that the random variables α_k are i.i.d. and are such that $\mathbb{E}(\alpha_k) = m > 0$. We have

Proposition 3.1. ***(Convergence of the periodogram** $X_L(\xi)$**)***
Let $(\alpha_k)_{k\in\{0,\cdots,L-1\}}$ be i.i.d. random variables of finite variances $\sigma^2 > 0$. For any $\xi \in \mathbb{R}\backslash\{0\}$ the sequence of random variables $X_L(\xi) := \frac{1}{L}|\sum_{k=0}^{L-1}\alpha_k e^{-ik\xi}|^2$ converges in distribution to a random variable that we can therefore call $X(\xi)$. In addition, for any $\xi \in \mathbb{R}\backslash\{0\}$, $X(\xi)$ has $\mathbb{E}(X(\xi)) = \sigma^2$.

Proof. See annex B, page 17. □

We wish to provide a lower bound of the expected spectral MSE (10), that is $\mathbb{E}\left(\frac{Y_L}{X_L(\xi)}\right)$ and of the expected MSE at the limit when $L \to +\infty$. Note that (10) involves the quotient of two dependent random variables. Therefore, our work-plan is

Work-plan:

Step 1 Find a suitable convergence for $\frac{X_L(\xi)}{Y_L}$ when $L \to +\infty$. Find an adequate lower bound for $\liminf_{L\to+\infty} \mathbb{E}\left(\frac{Y_L}{X_L(\xi)}\right)$. This is treated in section 3.2.1.

Step 2 Find a suitable lower bound for $\mathbb{E}\left(\mathrm{MSE}_{\mathrm{spectral}}(\xi)\right)$ at the limit when $L \to +\infty$. This is treated in section 3.2.2.

Step 3 Computation of the MSE and comparison with the *snapshot*. This is done in section 3.2.3.

3.2.1 Step 1: Convergence of $\frac{X_L(\xi)}{Y_L}$, when $L \to +\infty$

This section tackles the Step 1 of our work-plan. We first prove that $\frac{X_L(\xi)}{Y_L}$ converges in distribution then exhibit a lower bound for $\liminf_{L\to+\infty} \mathbb{E}\left(\frac{Y_L}{X_L(\xi)}\right)$.

In order to prove that, for any $\xi \in [-\pi, \pi]$, the sequence $\left(\frac{X_L(\xi)}{Y_L}\right)_{L\in\mathbb{N}^+}$ converges, we shall use, once more, Slutsky's theorem. Therefore, we need to prove that $(X_L(\xi))_{L\in\mathbb{N}^+}$ converges in distribution and that $(Y_L)_{L\in\mathbb{N}^+}$ converges in probability to a non zero constant.

On the one hand, from its definition (12) and for any $\xi \in \mathbb{R}$, by the (weak) law of large number we have $Y_L \to m$ (a deterministic constant) in probability. Recall that, by w.l.o.g. assumption we have $m > 0$.

On the other hand, from its definition (12), the random variable $X_L(\xi)$ is the periodogram of the sequence $(\alpha_k)_{k=0}^{L-1}$. Since $\mathrm{var}(\alpha_k) = \sigma^2 < +\infty$ from proposition 3.1 we have that, for any $\xi \in \mathbb{R}\backslash\{0\}$ the sequence of random variables $(X_L(\xi))_{L\in\mathbb{N}^+}$ converges in distribution to a random variable that we can therefore call $X(\xi)$.

Hence, combining these two convergences, by Slutsky's theorem (see, e.g., [29, page 19]) we obtain that

$$\frac{X_L(\xi)}{Y_L} = \frac{\frac{1}{L}\left|\sum_{k=0}^{L-1} \alpha_k e^{-ik\xi}\right|^2}{\frac{1}{L}\|\alpha_L\|_{L^1(\mathbb{R})}} \xrightarrow{L\to+\infty} \frac{X(\xi)}{m} \text{ in distribution, for any } \xi \in \mathbb{R}\backslash\{0\}. \tag{13}$$

In addition, from proposition 3.1, we deduce that

$$\mathbb{E}\left(\frac{X(\xi)}{m}\right) = \frac{\sigma^2}{m} \text{ for any } \xi \in \mathbb{R}\backslash\{0\}. \tag{14}$$

Moreover, since for any $\xi \in \mathbb{R}\backslash\{0\}$ the sequence $\left(\frac{X_L(\xi)}{Y_L}\right)_{L\in\mathbb{N}^+}$ converges in distribution, for any f continuous bounded, we have (see, e.g., [14, page 138])

$$\mathbb{E}\left(f\left(\frac{X_L(\xi)}{Y_L}\right)\right) \xrightarrow{L\to+\infty} \mathbb{E}\left(f\left(\frac{X(\xi)}{m}\right)\right) \text{ for any } \xi \in \mathbb{R}\backslash\{0\}.$$

For any $\xi \in \mathbb{R}\backslash\{0\}$, the random sequence $\left(\frac{X_L(\xi)}{Y_L}\right)_{L\in\mathbb{N}^+}$ is non negative. Thus, in particular, for any $n \in \mathbb{N}^+$ and choosing the continuous bounded function defined by $f_n(x) := \frac{1}{|x|+\frac{1}{n}}$ we have that

$$\lim_{L\to+\infty} \mathbb{E}\left(\frac{1}{\frac{X_L(\xi)}{Y_L}+\frac{1}{n}}\right) = \mathbb{E}\left(\frac{1}{\frac{X(\xi)}{m}+\frac{1}{n}}\right) \text{ for any } \xi \in \mathbb{R}\backslash\{0\} \text{ and any } n \in \mathbb{N}^+.$$

In addition, by Jensen's inequality, for any $\xi \in \mathbb{R}\backslash\{0\}$, any $L \in \mathbb{N}^+$ and any $n \in \mathbb{N}^+$, we have $\mathbb{E}\left(\frac{1}{\frac{X(\xi)}{m}+\frac{1}{n}}\right) \geqslant \frac{1}{\mathbb{E}\left(\frac{X(\xi)}{m}+\frac{1}{n}\right)}$. From (14), for any $\xi \in \mathbb{R}\backslash\{0\}$, any $L \in \mathbb{N}^+$ and any $n \in \mathbb{N}^+$, we obtain $\mathbb{E}\left(\frac{X(\xi)}{m}+\frac{1}{n}\right) \leqslant \frac{\sigma^2}{m}+\frac{1}{n}$. Therefore, we deduce that $\mathbb{E}\left(\frac{1}{\frac{X(\xi)}{m}+\frac{1}{n}}\right) \geqslant \frac{1}{\frac{\sigma^2}{m}+\frac{1}{n}}$ for any $\xi \in \mathbb{R}\backslash\{0\}$ and any $n \in \mathbb{N}^+$. Therefore, we proved that

$$\lim_{L\to+\infty} \mathbb{E}\left(\frac{1}{\frac{X_L(\xi)}{Y_L}+\frac{1}{n}}\right) \geqslant \frac{1}{\frac{\sigma^2}{m}+\frac{1}{n}} \text{ for any } \xi \in \mathbb{R}\backslash\{0\} \text{ and any } n \in \mathbb{N}^+. \tag{15}$$

Recall that, for any $\xi \in \mathbb{R}\backslash\{0\}$, the random sequence $\left(\frac{X_L(\xi)}{Y_L}\right)_{L\in\mathbb{N}^+}$ is non negative. Thus, for and $L \in \mathbb{N}^+$ and any $\xi \in \mathbb{R}\backslash\{0\}$, we have

$$\frac{Y_L}{X_L(\xi)} = \frac{1}{\frac{X_L(\xi)}{Y_L}} > \frac{1}{\frac{X_L(\xi)}{Y_L}+\frac{1}{n}} \geqslant 0.$$

Hence,

$$\mathbb{E}\left(\frac{Y_L}{X_L(\xi)}\right) > \mathbb{E}\left(\frac{1}{\frac{X_L(\xi)}{Y_L}+\frac{1}{n}}\right) \geqslant 0 \text{ for any } \xi \in \mathbb{R}\backslash\{0\} \text{ and any } n \in \mathbb{N}^+. \tag{16}$$

Therefore, combining (15) and (16) by Fatou's Lemma (see, e.g., [7, Thm. 5.8 page 223]), we deduce that

$$\liminf_{L\to+\infty} \mathbb{E}\left(\frac{Y_L}{X_L(\xi)}\right) \geqslant \frac{1}{\frac{\sigma^2}{m}+\frac{1}{n}} \text{ for any } \xi \in \mathbb{R}\backslash\{0\} \text{ and any } n \in \mathbb{N}^+.$$

Thus, we deduce that

$$\liminf_{L\to+\infty} \mathbb{E}\left(\frac{Y_L}{X_L(\xi)}\right) \geqslant \frac{m}{\sigma^2} \text{ for any } \xi \in \mathbb{R}\backslash\{0\}. \tag{17}$$

At $\xi = 0$, from their definition (12), and any $L \in \mathbb{N}^+$ we have we have $\frac{Y_L}{X_L(0)} = \frac{\sum_{k=0}^{L-1}\alpha_k}{(\sum_{k=0}^{L-1}\alpha_k)^2} = \frac{\frac{1}{L}}{\frac{1}{L}\sum_{k=0}^{L-1}\alpha_k}$. Since $\frac{1}{L} \xrightarrow{L\to+\infty} 0$ and $\frac{1}{L}\sum_{k=0}^{L-1}\alpha_k \xrightarrow{L\to+\infty} m$ in probability, once again, by Slutsky's theorem we deduce that $\frac{Y_L}{X_L(0)} \xrightarrow{L\to+\infty} 0$ in distribution. This concludes the Step 1 of our work-plan.

3.2.2 Step 2: Lower bound of the spectral MSE

This section treats the step 2 of our work-plan. We provide an adequate lower bound for the expected MSE of any *analog flutter shutter* equipped with random codes.

For any $\xi \in \mathbb{R}$ and $L \in \mathbb{N}^+$ the expectation of the spectral MSE (defined in equation (2)) with respect to the random *flutter shutter gain function* α_L defined by (6) satisfies

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{MSE}_{\mathrm{spectral}(\alpha_L)}(\xi)\right) = \mathbb{E}\left(\frac{\|u\|_{L^1(\mathbb{R})}\|\alpha_L\|_{L^1(\mathbb{R})}}{|\hat{\alpha}_L(\xi v)|^2}\right) &(18)\\
&= \|u\|_{L^1(\mathbb{R})}\mathbb{E}\left(\frac{\Delta t\sum_{k=0}^{L-1}\alpha_k}{\Delta t^2\mathrm{sinc}^2(\frac{\xi v}{2\pi})|\sum_{k=0}^{L-1}\alpha_k e^{-ik\xi v\Delta t}|^2}\right)\mathbb{1}_{[-\pi,\pi]}(\xi) \text{ (from (7) and (8))}\\
&= \frac{\|u\|_{L^1(\mathbb{R})}}{\Delta t\mathrm{sinc}^2(\frac{\xi v\Delta t}{2\pi})}\mathbb{E}\left(\frac{\sum_{k=0}^{L-1}\alpha_k}{|\sum_{k=0}^{L-1}\alpha_k e^{-ik\xi v\Delta t}|^2}\right)\mathbb{1}_{[-\pi,\pi]}(\xi)\\
&= \frac{\|u\|_{L^1(\mathbb{R})}}{\Delta t\mathrm{sinc}^2(\frac{\xi v\Delta t}{2\pi})}\mathbb{E}\left(\frac{\frac{1}{L}\sum_{k=0}^{L-1}\alpha_k}{\frac{1}{L}|\sum_{k=0}^{L-1}\alpha_k e^{-ik\xi v\Delta t}|^2}\right)\mathbb{1}_{[-\pi,\pi]}(\xi)\\
&= \frac{\|u\|_{L^1(\mathbb{R})}}{\Delta t\mathrm{sinc}^2(\frac{\xi v\Delta t}{2\pi})}\mathbb{E}\left(\frac{Y_L}{X_L(\xi v\Delta t)}\right)\mathbb{1}_{[-\pi,\pi]}(\xi) \text{ (from (12))}. &(19)
\end{aligned}$$

Thus combining (17) and (18)-(19), for any $\xi \in \mathbb{R}\backslash\{0\}$, we have

$$\begin{aligned}
\liminf_{L\to+\infty}\mathbb{E}\left(\mathrm{MSE}_{\mathrm{spectral}(\alpha_L)}(\xi)\right) &= \frac{\|u\|_{L^1(\mathbb{R})}}{\Delta t\mathrm{sinc}^2(\frac{\xi v\Delta t}{2\pi})}\liminf_{L\to+\infty}\mathbb{E}\left(\frac{X_L(\xi v\Delta t)}{Y_L}\right)\mathbb{1}_{[-\pi,\pi]}(\xi)\\
&\geqslant \frac{\|u\|_{L^1(\mathbb{R})}}{\Delta t\mathrm{sinc}^2(\frac{\xi v\Delta t}{2\pi})}\frac{m}{\sigma^2}\mathbb{1}_{[-\pi,\pi]}(\xi). \qquad (20)
\end{aligned}$$

The equation (20) is valid for any $\xi \in \mathbb{R}\backslash\{0\}$. Therefore, (20) is valid on $\mathbb{R}$ except on the zero Lebesgue's measure set $\{0\}$, i.e., almost everywhere.

This concludes the step 2 of our work-plan. We shall now turn to Step 3, namely the evaluation of the expected MSE and comparison with the *snapshot*.

3.2.3 Step 3: Expected MSE and comparison with the *snapshot*

This section tackles the step 3 of our work-plan. Our first goal is to compute an adequate lower bound for the expected MSE of *analog flutter shutter* camera equipped with any random *flutter shutter code*. Our second goal is to compare this lower bound with the MSE of a *snapshot*.

For any $L \in \mathbb{N}^+$ and any realization of α_L, from its definition (4), the expectation of the MSE, with respect to α_L, satisfies

$$\mathbb{E}\left(\mathrm{MSE}(\alpha_L)\right) = \frac{1}{2\pi}\int_{\mathbb{R}}\mathbb{E}\left(\mathrm{MSE}_{\mathrm{spectral}(\alpha_L)}(\xi)\right)d\xi, \qquad (21)$$

where we used Fubini's theorem. Indeed, the spectral MSE is non negative. In addition, by Fatou's lemma, again, we obtain

$$\liminf_{L\to+\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbb{E}\left(\mathrm{MSE}_{\mathrm{spectral}}(\xi)\right) \geqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\liminf_{L\to+\infty}\mathbb{E}\left(\mathrm{MSE}_{\mathrm{spectral}}(\xi)\right)d\xi. \qquad (22)$$

Therefore, combining (20), (21) and (22) we deduce that

$$\liminf_{L\to+\infty}\mathbb{E}\left(\mathrm{MSE}(\alpha_L)\right) \geqslant \frac{m}{\sigma^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\|u\|_{L^1(\mathbb{R})}}{\Delta t\mathrm{sinc}^2(\frac{\xi v\Delta t}{2\pi})}d\xi.$$

Indeed, equation (20) is valid for almost every $\xi \in \mathbb{R}$. Therefore, we proved the following theorem.

Theorem 3.2. ***(A lower bound for the expected MSE of any random analog flutter shutters.)***
Let $u(x-vt)$ a scene moving at velocity $v \neq 0$. The expected MSE of any analog flutter shutter equipped with a random flutter shutter code coming from i.i.d. coefficients α_k satisfies

$$\liminf_{L\to+\infty} \mathbb{E}_{\alpha_L}(MSE(\alpha_L)) \geqslant \frac{m}{\sigma^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\|u\|_{L^1(\mathbb{R})}}{\Delta t sinc^2(\frac{\xi v \Delta t}{2\pi})} d\xi,$$

where $\mathbb{E}(\alpha_k) = m$ and $var(\alpha_k) = \sigma^2$.

Remark From (4), the MSE of a Δt-*snapshot* is equal to

$$\text{MSE}(\Delta t\text{-snapshot}) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\|u\|_{L^1(\mathbb{R})}}{\Delta t \text{sinc}^2(\frac{\xi v \Delta t}{2\pi})} d\xi. \tag{23}$$

Indeed, a *snapshot* is a *flutter shutter* with a *flutter shutter gain function* α of the form $\alpha = \mathbb{1}_{[0,\Delta t]}$. In addition, from theorem 3.2 we deduce that if the α_k comes from Bernoulli random variables $\mathcal{B}(p)$ we have $m = \mathbb{E}(\alpha_k) = p$ and $\sigma^2 = \text{var}(\alpha_k) = p(1-p)$. Therefore, we have

$$\liminf_{L\to+\infty} \mathbb{E}_{\alpha_L}(\text{MSE}(\alpha_L)) \geqslant \frac{1}{1-p}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\|u\|_{L^1(\mathbb{R})}}{\Delta t \text{sinc}^2(\frac{\xi v \Delta t}{2\pi})} d\xi, \tag{24}$$

for any $p > 0$. For example, a binary code with half of its coefficients "1" we have $p = \frac{1}{2}$. Therefore, as soon as $v \neq 0$, this random code multiplies the MSE by a factor 2 compared to a *snapshot*. This theoretical result coincides with the experimental result of [33, table 1, page 8]. In addition, a code with $p = \frac{1}{4}$ has a lower MSE than the initial heuristic guess that assumed that $p = \frac{1}{2}$ is the optimal choice. This remark agrees with the experimental investigation of [2]. Indeed, the authors state that "[...] for signal-dependent noise, smaller s can be used" [2, figure 4, page 2564] (s refers to the number of "1" of the *flutter shutter code*). Indeed, $p = \frac{1}{2}$ is not good choice. Furthermore, it is clear from (24) that binary codes are not the adequate code family for the *flutter shutter*: as soon as the velocity $v \neq 0$, they can only increase the MSE compared to a *snapshot*.

Corollary 3.3. ***(The expected MSE of any random analog flutter shutter is higher than the MSE of a snapshot.)***
Let $u(x-vt)$ a scene moving at velocity $v \neq 0$. The expected MSE of any analog flutter shutter equipped with any random flutter shutter code coming from i.i.d. code coefficients α_k satisfies

$$\liminf_{L\to+\infty} \mathbb{E}_{\alpha_L}(MSE(\alpha_L)) \geqslant MSE(\Delta t\text{-}snapshot),$$

where $MSE(\Delta t - snapshot)$ denotes the MSE of a snapshot with exposure time equal to Δt.

Proof. Recall that $\mathbb{E}(\alpha_k) = m$ and $\sigma^2 = var(\alpha_k) = \beta - m^2$. On the one hand we have, $\sigma^2 \leqslant \beta$ hence $\frac{1}{\sigma^2} \geqslant \frac{1}{\beta}$ and $\frac{m}{\sigma^2} \geqslant \frac{m}{\beta}$. On the other hand, for any random variable that takes values in $[0,1]$, we have, by Jensen's inequality, $m = \mathbb{E}(\alpha_k) \geqslant \mathbb{E}(|\alpha_k|^2) = \beta$. Thus, we deduce that $\frac{m}{\sigma^2} \geqslant \frac{m}{\beta} \geqslant 1$. The result follows from theorem (3.2) combined with (23). □

4 The *numerical flutter shutter* formalism for random codes

Similarly to the *analog flutter shutter* case treated in section 3 we assume that α_k are i.i.d. and bounded. The boundedness assumption represents no loss of generality. Indeed, the coefficient α_k of the *flutter shutter code* are digitals and therefore bounded. For any finite *flutter shutter code* the spectral MSE of the *numerical flutter shutter* is given by (1). Similarly to section 3 we set

$$X_L(\xi) := \frac{1}{L}|\sum_{k=0}^{L-1} \alpha_k e^{-ik\xi}|^2 \text{ and } Y_L := \frac{1}{L}\sum_{k=0}^{L-1} \alpha_k^2. \tag{25}$$

By the weak law of large numbers $Y_L \xrightarrow{L\to+\infty} \mathbb{E}(|\alpha_k|^2)$ in probability. The rest of the proof follows from the argumentation developed in section 3.2.1 (Step 1): for any $\xi \in \mathbb{R}\backslash\{0\}$ it holds

$$\liminf_{L\to+\infty} \mathbb{E}\left(\frac{Y_L}{X_L(\xi)}\right) \geqslant \frac{\beta}{\sigma^2}.$$

The step 2 and 3 are identical, provided we replace $\mathbb{E}(\alpha_k) = m$ by $\mathbb{E}(|\alpha_k|^2) = \beta$ everywhere. Thus, we have

Theorem 4.1. ***(A lower bound for the expected MSE of random numerical flutter shutters.)***
Let $u(x-vt)$ a scene moving at velocity $v \neq 0$. The expected MSE of any numerical flutter shutter equipped with a random code coming from i.i.d. code coefficients α_k of finite variances satisfies

$$\liminf_{L\to+\infty} \mathbb{E}_{\alpha_L}(MSE(\alpha_L)) \geqslant \frac{\beta}{\sigma^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\|u\|_{L^1(\mathbb{R})}}{\Delta t sinc^2(\frac{\xi v \Delta t}{2\pi})} d\xi,$$

where $var(\alpha_k) = \sigma^2 := \beta - \mathbb{E}(\alpha_k)^2$.

Corollary 4.2. ***(The expected MSE of the numerical flutter shutter is higher than the MSE of a snapshot.)***
Let $u(x-vt)$ a scene moving at velocity $v \neq 0$. The expected MSE of any analog flutter shutter equipped with a random code coming from i.i.d. code coefficients α_k satisfies

$$\liminf_{L\to+\infty} \mathbb{E}_{\alpha_L}(MSE(\alpha_L)) \geqslant MSE(\Delta t\text{-}snapshot),$$

where $MSE(\Delta t - snapshot)$ is the MSE of a snapshot with exposure time equal to Δt.

Proof. Since $\text{var}(\alpha_k) = \sigma^2 = \beta - m^2$ we have $\sigma^2 \leqslant \beta$. Therefore, $1 \leqslant \frac{\beta}{\sigma^2}$ which concludes the proof by the same argumentations as in the proof of corollary 3.3. □

5 Conclusion

The goal of the *flutter shutter* is to gain in terms of RMSE compared to a classic *snapshot.* The hope with random *flutter shutter codes* is that they could provide a substantial improvement in terms of RMSE compared to the 1.17 bound proved in [35]. Indeed, the assumptions of [35] did not permit to treat general random *flutter shutter codes.* This paper has considered random *flutter shutter codes*, coming from independent and identically distributed bounded random variables. This set up fits to the random *flutter shutter codes* proposed in most *flutter shutter* random optimization papers. Two simples formulae proved in the present paper have given a lower bound of the expected MSE of the final restored image, for both the *analog* and the *numerical flutter shutter*. For a fixed non zero velocity $v \neq 0$, it has been proven that these codes (binary or not) do not beat a standard *snapshot* in terms of MSE. In other words, as soon as $v \neq 0$ a *flutter shutter* camera equipped with a random *flutter shutter code* cannot gain over a simple *snapshot* in terms of MSE. This result is in direct contradiction with most of the *flutter shutter* literature, that has adopted random *flutter shutter codes*. As a byproduct, we have proved that a good *flutter shutter code* is not a random code.

Acknowledgement

Yohann Tendero is supported on the Grants No: 12-DARPA-1042, UC Lab 12-LR-236660, ONR N000141210838 and NSF DMS-1118971.

A Computation of $\mathbb{E}\left(|\hat{\alpha}_L(\xi)|^2\right)$

We have

$$\begin{aligned}
\mathbb{E}\left(|\hat{\alpha}_L(\xi)|^2\right) &= \mathbb{E}\left(|\int_{-\infty}^{\infty} \alpha_L(x)e^{-ix\xi}dx|^2\right) = \mathbb{E}\left(\int_{-\infty}^{\infty} \alpha_L(x)e^{-ix\xi}dx \int_{-\infty}^{\infty} \alpha_L(y)e^{iy\xi}dy\right) \\
&= \mathbb{E}\left(\int_{-\infty}^{\infty} \sum_{k=0}^{L-1} \alpha_k \mathbb{1}_{[k\Delta t,(k+1)\Delta t[}(x)e^{-ix\xi}dx \int_{-\infty}^{\infty} \sum_{l=0}^{L-1} \alpha_l \mathbb{1}_{[l\Delta t,(l+1)\Delta t[}(y)e^{iy\xi}dy\right) \\
&= \mathbb{E}\left(\sum_{k=0}^{L-1} \alpha_k \int_{k\Delta t}^{(k+1)\Delta t} e^{-ix\xi}dx \sum_{l=0}^{L-1} \alpha_l \int_{l\Delta t}^{(l+1)\Delta t} e^{iy\xi}dy\right) \\
&= \mathbb{E}\left(\sum_{k=0}^{L-1}\sum_{l=0}^{L-1} \alpha_k\alpha_l \int_{k\Delta t}^{(k+1)\Delta t} e^{-ix\xi}dx \int_{l\Delta t}^{(l+1)\Delta t} e^{iy\xi}dy\right) \\
&= \sum_{k=0}^{L-1}\sum_{l=0}^{L-1} \mathbb{E}(\alpha_k\alpha_l) \int_{k\Delta t}^{(k+1)\Delta t} e^{-ix\xi}dx \int_{l\Delta t}^{(l+1)\Delta t} e^{iy\xi}dy \\
&= \sum_{k=0}^{L-1} \mathbb{E}(\alpha_k\alpha_k) \int_{k\Delta t}^{(k+1)\Delta t} e^{-ix\xi}dx \int_{k\Delta t}^{(k+1)\Delta t} e^{iy\xi}dy \\
+ &\sum_{k=0}^{L-1}\sum_{\substack{l=0\\ l\neq k}}^{L-1} \mathbb{E}(\alpha_k)\mathbb{E}(\alpha_l) \int_{k\Delta t}^{(k+1)\Delta t} e^{-ix\xi}dx \int_{l\Delta t}^{(l+1)\Delta t} e^{iy\xi}dy \\
&= \sum_{k=0}^{L-1} \mathbb{E}(\alpha_k\alpha_k) \int_{k\Delta t}^{(k+1)\Delta t} e^{-ix\xi}dx \int_{k\Delta t}^{(k+1)\Delta t} e^{iy\xi}dy \\
+ &\sum_{k=0}^{L-1}\sum_{l=0}^{L-1} \mathbb{E}(\alpha_k)\mathbb{E}(\alpha_l) \int_{k\Delta t}^{(k+1)\Delta t} e^{-ix\xi}dx \int_{l\Delta t}^{(l+1)\Delta t} e^{iy\xi}dy \\
- &\sum_{k=0}^{L-1} \mathbb{E}(\alpha_k)\mathbb{E}(\alpha_k) \int_{k\Delta t}^{(k+1)\Delta t} e^{-ix\xi}dx \int_{k\Delta t}^{(k+1)\Delta t} e^{iy\xi} \\
&= \sum_{k=0}^{L-1} \beta \int_{k\Delta t}^{(k+1)\Delta t} e^{-ix\xi}dx \int_{k\Delta t}^{(k+1)\Delta t} e^{iy\xi}dy \qquad (26) \\
+ &\sum_{k=0}^{L-1}\sum_{l=0}^{L-1} m^2 \int_{k\Delta t}^{(k+1)\Delta t} e^{-ix\xi}dx \int_{l\Delta t}^{(l+1)\Delta t} e^{iy\xi}dy \qquad (27) \\
- &\sum_{k=0}^{L-1} m^2 \int_{k\Delta t}^{(k+1)\Delta t} e^{-ix\xi}dx \int_{k\Delta t}^{(k+1)\Delta t} e^{iy\xi}dy. \qquad (28)
\end{aligned}$$

Recall that by assumption the α_k, $k \in \{0, \cdots, L-1\}$, are independent. Thus, $\mathbb{E}(\alpha_k\alpha_l) = \mathbb{E}(\alpha_k)\mathbb{E}(\alpha_l) = m^2$ if $k \neq l$ and $\mathbb{E}(|\alpha_k|^2) = \beta$ otherwise. We shall treat separately the terms given by equations (26), (27) and (28). For the term in equation (26) we have

$$\sum_{k=0}^{L-1}\beta\int_{k\Delta t}^{(k+1)\Delta t}e^{-ix\xi}dx\int_{k\Delta t}^{(k+1)\Delta t}e^{iy\xi}dy=\sum_{k=0}^{L-1}\beta\int_0^{\Delta}te^{-i(x+k\Delta t)\xi}dx\int_0^{\Delta}te^{+i(y+k\Delta t)\xi}dy$$
$$=L\beta\int_0^{\Delta}te^{-ix\xi}dx\int_0^{\Delta}te^{+iy\xi}dy=L\beta\int_{-\frac{\Delta t}{2}}^{\frac{\Delta t}{2}}e^{-i(x+\frac{\Delta t}{2})\xi}dx\int_{-\frac{\Delta t}{2}}^{\frac{\Delta t}{2}}e^{+i(y+\frac{\Delta t}{2})\xi}dy$$
$$=L\beta\int_{-\frac{\Delta t}{2}}^{\frac{\Delta t}{2}}e^{-ix\xi}dx\int_{-\frac{\Delta t}{2}}^{\frac{\Delta t}{2}}e^{+iy\xi}dy=L\beta\left(2\frac{\sin(\frac{\xi\Delta t}{2})}{\xi}\right)^2=L\beta\left(\frac{\sin(\frac{\xi\Delta t}{2})}{\frac{\xi}{2}}\right)^2$$
$$=L\beta\left(\Delta t\frac{\sin(\frac{\xi\Delta t}{2})}{\frac{\xi\Delta t}{2}}\right)^2=L(\beta\Delta t)^2\mathrm{sinc}^2\left(\frac{\Delta t\xi}{2\pi}\right). \tag{29}$$

For the term given by equation (27) we have

$$\sum_{k=0}^{L-1}\sum_{l=0}^{L-1}m^2\int_{k\Delta t}^{(k+1)\Delta t}e^{-ix\xi}dx\int_{l\Delta t}^{(l+1)\Delta t}e^{iy\xi}dy=m^2\sum_{k=0}^{L-1}\int_{k\Delta t}^{(k+1)\Delta t}e^{-ix\xi}dx\sum_{l=0}^{L-1}\int_{l\Delta t}^{(l+1)\Delta t}e^{iy\xi}dy \tag{30}$$
$$=m^2\int_0^{L\Delta t}e^{-ix\xi}dx\int_0^{L\Delta t}e^{+iy\xi}dy=m^2\int_0^{L\Delta t}e^{-ix\xi}dx\int_0^{-L\Delta t}-e^{-iy\xi}dy$$
$$=m^2\int_0^{L\Delta t}e^{-ix\xi}dx\int_{-L\Delta t}^{0}e^{-iy\xi}dy=m^2\int_{-\frac{L\Delta t}{2}}^{-\frac{L\Delta t}{2}}e^{-ix\xi}dx\int_{-\frac{L\Delta t}{2}}^{\frac{L\Delta t}{2}}e^{-iy\xi}dy$$
$$=m^2\left(2\frac{\sin(\frac{L\Delta t\xi}{2})}{\xi}\right)^2=m^2\left(\frac{\sin(\frac{L\Delta t\xi}{2})}{\frac{\xi}{2}}\right)^2=m^2\left(L\Delta t\frac{\sin(\frac{L\Delta t\xi}{2})}{\frac{L\Delta t\xi}{2}}\right)^2$$
$$=(mL\Delta t)^2\mathrm{sinc}^2\left(\frac{L\Delta t\xi}{2\pi}\right). \tag{31}$$

For the term of equation (28) we obtain

$$\textstyle\sum_{k=0}^{L-1}m^2\int_{k\Delta t}^{(k+1)\Delta t}e^{-ix\xi}dx\int_{k\Delta t}^{(k+1)\Delta t}e^{iy\xi}dy=L(m\Delta t)^2\mathrm{sinc}^2\left(\frac{\Delta t\xi}{2\pi}\right). \tag{32}$$

Combining (26) with (29), (27) with (30)-(31) and (28) with (32) entails

$$\begin{aligned}\mathbb{E}\left(|\hat{\alpha}_L(\xi)|^2\right) &= L(\beta\Delta t)^2\mathrm{sinc}^2\left(\frac{\Delta t\xi}{2\pi}\right)+(mL\Delta t)^2\mathrm{sinc}^2\left(\frac{L\Delta t\xi}{2\pi}\right)-L(m\Delta t)^2\mathrm{sinc}^2\left(\frac{\Delta t\xi}{2\pi}\right)\\ &= L\Delta t^2\mathrm{sinc}^2\left(\frac{\Delta t\xi}{2\pi}\right)(\beta-m^2)+(mL\Delta t)^2\mathrm{sinc}^2\left(\frac{L\Delta t\xi}{2\pi}\right)\\ &= L\Delta t^2\left(\sigma^2\mathrm{sinc}^2\left(\frac{\Delta t\xi}{2\pi}\right)+Lm^2\mathrm{sinc}^2\left(\frac{L\Delta t\xi}{2\pi}\right)\right),\end{aligned}$$

and we get equation (5).

B Proof of proposition 3.1

We have

$$X_L(\xi):=\frac{1}{L}|\sum_{k=0}^{L-1}\alpha_k e^{-ik\xi}|^2=\frac{1}{L}|\sum_{k=0}^{L-1}(\alpha_k-m)e^{-ik\xi}+\sum_{k=0}^{L-1}me^{-ik\xi}|^2. \tag{33}$$

Set $\tilde{\alpha}_k := \alpha_k - m$ so that $\mathbb{E}(\tilde{\alpha}_k) = 0$ and $\mathrm{var}(\tilde{\alpha}_k) = \mathrm{var}(\alpha_k) = \sigma^2$. From (33) we have

$$X_L(\xi) \quad = \frac{1}{L}|\sum_{k=0}^{L-1} \tilde{\alpha}_k e^{-ik\xi}|^2 = \underbrace{\frac{1}{L}|\sum_{k=0}^{L-1} \tilde{\alpha}_k e^{-ik\xi}|^2}_{\tilde{X}_L(\xi)} + \underbrace{\frac{m^2}{L}|\sum_{k=0}^{L-1} e^{-ik\xi}|^2}_{\phi_L(\xi)} \tag{34}$$

$$+ \underbrace{\frac{m}{L}\left(\sum_{k=0}^{L-1} e^{-ik\xi}\right)\left(\sum_{k=0}^{L-1} \tilde{\alpha}_k e^{+ik\xi}\right)}_{Z_L(\xi)} + \underbrace{\frac{m}{L}\left(\sum_{k=0}^{L-1} e^{+ik\xi}\right)\left(\sum_{k=0}^{L-1} \tilde{\alpha}_k e^{-ik\xi}\right)}_{\overline{Z_L(\xi)}}. \tag{35}$$

Our goal is to find suitable convergences for each term in equations (34) and (35).

We have that $\tilde{X}_L(\xi)$ is the periodogram (see definition [6, equation 12.37, page 147]) of the zero mean sequence $(\tilde{\alpha}_k)_{k=0}^{L-1}$. Since the α_k are i.d.d. we have that the $\tilde{\alpha}_k$ are i.i.d. Therefore, the $\tilde{\alpha}_k$ can be seen as the realization of a linear random process of the form $\alpha_k = \eta_k$, where η is a zero mean white noise process with finite variance σ^2. Therefore, the autocorrelation of η satisfies $\mathbb{E}(\eta_{t_1}\eta_{t_2}) = \sigma^2\delta_0(t_2 - t_1)$, where $\delta_0(\cdot)$ denotes a Dirac-mass at zero, implying that η is wide sense stationary. Consequently, by Wiener-Khinchin's theorem the power spectrum density of η is $P_\eta(\xi) = \sigma^2 > 0$. Thus, for any $\xi \in [-\pi, \pi]$ the sequence of random variables $(\tilde{X}_L(\xi))_{L\in\mathbb{N}^+}$ converges in distribution to a random variable that we can therefore call $\tilde{X}(\xi)$ (see, e.g., [6, Thm. 12.10, page 150]).

Set $\phi_L(\xi) := \frac{m^2}{L}|\sum_{k=0}^{L-1} e^{-ik\xi}|^2$. We have that ϕ_L is up to a $\frac{2\pi}{m^2}$ multiplicative factor the Fejér kernel (see e.g. [30, page 443]). Therefore, for any $\xi \in \mathbb{R}\backslash\{0\}$ $\lim_{L\to+\infty} \phi_L(\xi) = 0$.

Set $Z_L(\xi) := \frac{m}{L}\left(\sum_{k=0}^{L-1} e^{-ik\xi}\right)\left(\sum_{k=0}^{L-1} \tilde{\alpha}_k e^{+ik\xi}\right)$. For any $\xi \in \mathbb{R}$ we have

$$\mathbb{E}\left(Z_L(\xi)\right) = \frac{m}{L}\left(\sum_{k=0}^{L-1} e^{-ik\xi}\right)\left(\sum_{k=0}^{L-1} \mathbb{E}\left(\tilde{\alpha}_k\right) e^{+ik\xi}\right) = 0. \tag{36}$$

Indeed, by definition the $\tilde{\alpha}_k$ satisfy $\mathbb{E}(\tilde{\alpha}_k) = 0$. In addition, from (36) and the definition of $Z_L(\xi)$, for any $\xi \in \mathbb{R}$ we have $\mathrm{var}(Z_L(\xi)) = \mathbb{E}(|Z_L(\xi)|^2)$. Therefore, we deduce that for any $\xi \in \mathbb{R}$ and any $L \in \mathbb{N}^+$

$$\mathrm{var}\left(Z_L(\xi)\right) \quad = \frac{m^2}{L}|\sum_{k=0}^{L-1} e^{-ik\xi}|^2 \mathbb{E}\left(\frac{1}{L}|\sum_{k=0}^{L-1} \tilde{\alpha}_k e^{+ik\xi}|^2\right) = \phi_L(\xi)\mathbb{E}\left(\tilde{X}_L(\xi)\right),$$

where the last equality is justified by the definitions of $\phi_L(\xi)$ and $\tilde{X}_L(\xi)$. Since for any $\xi \in \backslash\{0\}$ $\lim_{L\to+\infty} \phi_L(\xi) = 0$ and $\lim_{L\to+\infty} \mathbb{E}\left(\tilde{X}_L(\xi)\right) = P_\eta(\xi) = \sigma^2$ (see, e.g., [6, equation 12.40, page 147]), we obtain

$$\lim_{L\to+\infty} \mathrm{var}(Z_L(\xi)) = 0 \text{ for any } \xi \in \mathbb{R}\backslash\{0\}. \tag{37}$$

Therefore, from Bienaymé-Tchebychev's inequality and (36) for any $\varepsilon > 0$, any $L \in \mathbb{N}^+$ and any $\xi \in \backslash\{0\}$ we have $\mathbb{P}\left(|Z_L(\xi)| \geqslant \varepsilon\right) \leqslant \frac{\mathrm{var}(Z_L(\xi))}{\varepsilon^2}$. Thus, by passing to the limit $L \to +\infty$ from (37) we have that

$$Z_L(\xi) \xrightarrow{L\to+\infty} 0 \text{ in probability, for any } \xi \in \backslash\{0\}. \tag{38}$$

Similarly $\overline{Z_L(\xi)} \xrightarrow{L\to+\infty} 0$ in probability, for any $\xi \in \mathbb{R}\backslash\{0\}$.

Consequently, from (34)-(35), by Slutsky's, again, theorem we deduce that, for any $\xi \in [-\pi, \pi]\backslash\{0\}$ the sequence of random variables $X_L(\xi) \xrightarrow{L\to+\infty} \tilde{X}(\xi)$ in distribution. Moreover, from (33) we have that $X_L(\xi)$ is 2π periodic. Therefore, for any $\xi \in \mathbb{R}\backslash\{0\}$ the sequence of random variables $X_L(\xi) \xrightarrow{L\to+\infty} \tilde{X}(\xi) := X(\xi)$ in distribution. In addition, for any $\xi \in \mathbb{R}\backslash\{0\}$ the random variable $\tilde{X}(\xi)$ has $\mathbb{E}(\tilde{X}(\xi)) = \sigma^2$ (see, e.g., [6, Thm. 12.10, page 150]). This proves the proposition 3.1.

C Main notations and formulae

(i) $t \geqslant 0$ time variable

(ii) Δt length of a time interval (exposure time)

(iii) $x \in \mathbb{R}$ spatial variable

(iv) $\bar{z}$, for $z \in \mathbb{C}$ denotes the conjugate of a complex number

(v) $X \sim Y$ means that the random variables X and Y have the same law

(vi) $\mathbb{P}(A)$ probability of an event A

(vii) $\mathbb{E}(X)$ expected value of a random variable X

(viii) $\text{var}(X)$ variance of a random variable X

(ix) $\mathcal{P}(\lambda)$ Poisson random variable with intensity $\lambda > 0$. Thus, if $X \sim \mathcal{P}(\lambda)$ we have $\mathbb{P}(X = k) = \frac{\exp(-\lambda)\lambda^k}{k!}$

(x) $f * g$ convolution of two functions $(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$

(xi) $obs(n)$, $n \in \mathbb{Z}$ observation of the landscape at pixel n

(xii) v relative velocity between the landscape and the camera (unit: pixels per second)

(xiii) $\alpha_L(x) = \sum_{k=0}^{L-1} \alpha_k \mathbb{1}_{[k\Delta t,(k+1)\Delta t[}(x)$ *flutter shutter gain function*

(xiv) $\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f(x)|dx$, $\|f\|_{L^2(\mathbb{R})} = \sqrt{\int_{\mathbb{R}} |f(x)|^2 dx}$

(xv) Let $f, g \in L^1(\mathbb{R})$ or $L^2(\mathbb{R})$, then $\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$ and $\mathcal{F}^{-1}(\mathcal{F}(f))(x) := \widetilde{\mathcal{F}(f)}(x) = f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}(f)(\xi)e^{ix\xi}d\xi$. Moreover $\mathcal{F}(f*g)(\xi) = \mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)$ and (Plancherel)

$$\int_{\mathbb{R}} |f(x)|^2 dx = \|f\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\int_{\mathbb{R}} |\mathcal{F}(f)|^2(\xi)d\xi = \frac{1}{2\pi}\|\mathcal{F}(f)\|^2_{L^2(\mathbb{R})}$$

(xvi) u ideal (noiseless) observable landscape. Assumption: $[-\pi, \pi]$ band-limited, and $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

(xvii) $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} = \frac{1}{2\pi}\mathcal{F}(\mathbb{1}_{[-\pi,\pi]})(x) = \mathcal{F}^{-1}(\mathbb{1}_{[-\pi,\pi]})(x)$

(xviii) $\mathbb{1}_{[a,b]}$ indicator function of an interval $[a, b]$

(xix) $(\alpha_k)_k$ *flutter shutter code*. The code coefficients α_k are assumed to be independent and identically distributed random variables

(xx) $m = \mathbb{E}(\alpha_k)$

(xxi) $\text{var}(\alpha_k) = \sigma^2 := \beta - m^2$

(xxii) $\mathbb{E}(|\alpha_k|^2) = \beta$

References

[1] A. Agrawal and R. Raskar. Resolving objects at higher resolution from a single motion-blurred image. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, pages 1–8, 2007.

[2] A. Agrawal and R. Raskar. Optimal single image capture for motion deblurring. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, pages 2560–2567, 2009.

[3] A. Agrawal and Y. Xu. Coded exposure deblurring: Optimized codes for PSF estimation and invertibility. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, pages 2066–2073, 2009.

[4] G. Boracchi and A. Foi. Uniform Motion Blur in Poissonian Noise: Blur/Noise Tradeoff. *IEEE Trans. on Image Processing*, 20(2):592–598, 2011.

[5] T. Buades, Y. Lou, J.-M. Morel, and Z. Tang. A note on multi-image denoising. In *Proceedings of the IEEE International Workshop on Local and Non-Local Approximation in Image Processing (LNLA)*, pages 1–15, 2009.

[6] T. Chonavel and J. Ormrod. *Statistical Signal Processing: Modelling and Estimation.* Advanced Textbooks in Control and Signal Processing. Springer London, 2002.

[7] E. Çınlar. *Probability and Stochastics.* Graduate texts in mathematics. Springer, 2011.

[8] O. Cossairt, M. Gupta, and S.K. Nayar. When Does Computational Imaging Improve Performance? *IEEE Trans. on Image Processing*, 2012.

[9] A. Foi, S. Alenius, M. Trimeche, V. Katkovnik, and K. Egiazarian. A spatially adaptive Poissonian image deblurring. In *Proceedings of the IEEE International Conference on Image Processing (ICIP)*, volume 1, pages I–925, 2005.

[10] M.E. Gehm, R. John, D.J. Brady, R.M. Willett, and T.J. Schulz. Single-shot compressive spectral imaging with a dual-disperser architecture. *Optics Express*, 15(21):14013–14027, 2007.

[11] C.R. Heathcote. *Probability: Elements of the Mathematical Theory.* Dover Publications, 2012.

[12] J. Jelinek. Designing the optimal shutter sequences for the flutter shutter imaging method. In *Proceedings of the Society of Photo-Optical Instrumentation Engineers (SPIE)*, volume 7701, page 18, 2010.

[13] Jan Jelinek. Designing the optimal shutter sequences for the flutter shutter imaging method. In *SPIE Defense, Security, and Sensing*, pages 77010N–77010N. International Society for Optics and Photonics, 2010.

[14] Alan F. Karr. *Probability.* Springer, New York, 1993.

[15] S.J. Koppal, S. Yamazaki, and S.G. Narasimhan. Exploiting DLP Illumination Dithering for Reconstruction and Photography of High-Speed Scenes. *International Journal of Computer Vision (IJCV)*, pages 1–20, 2011.

[16] H. Kozuka. Image sensor, October 29 2002. US Patent 6,473,538.

[17] S. McCloskey. Velocity-dependent shutter sequences for motion deblurring. In *Proceedings of the Springer-Verlag European Conference on Computer Vision (ECCV)*, pages 309–322, 2010.

[18] S. Mccloskey. Acquisition system for obtaining sharp barcode images despite motion, February 16 2011. EP Patent 2,284,764.

[19] S. McCloskey. Temporally coded flash illumination for motion deblurring. In *Proceedings of the IEEE International Conference on Computer Vision (ICCV)*, pages 683–690, 2011.

[20] S. McCloskey, W. Au, and J. Jelinek. Iris capture from moving subjects using a fluttering shutter. In *Proceedings of the IEEE International Conference on Biometrics: Theory Applications and Systems (BTAS)*, pages 1–6, 2010.

[21] S. McCloskey, J. Jelinek, and K.W. Au. Method and system for determining shutter fluttering sequence, April 9 2009. US Patent 12/421,296.

[22] K. Nakamura, H. Ohzu, and I. Ueno. Image sensor in which reading and resetting are simultaneously performed, November 16 1993. US Patent 5,262,870.

[23] S.G. Narasimhan, S.J. Koppal, and S. Yamazaki. Temporal Dithering of Illumination for Fast Active Vision. In *Proceedings of the Springer-Verlag European Conference on Computer Vision (ECCV)*, pages 830–844, 2008.

[24] S.K. Nayar and M. Ben-Ezra. Motion-based motion deblurring. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 26(6):689–698, 2004.

[25] R. Raskar. Method and apparatus for deblurring images, July 13 2010. US Patent 7,756,407.

[26] R. Raskar, A. Agrawal, and J. Tumblin. Coded exposure photography: motion deblurring using fluttered shutter. *ACM Transactions on Graphics (TOG)*, 25(3):795–804, 2006.

[27] R. Raskar, J. Tumblin, and A. Agrawal. Method for deblurring images using optimized temporal coding patterns, August 25 2009. US Patent 7,580,620.

[28] A. Sarker and L.G.C. Hamey. Improved Reconstruction of Flutter Shutter Images for Motion Blur Reduction. In *Proceedings of the (IEEE) International Conference on Digital Image Computing: Techniques and Applications (DICTA)*, pages 417–422, 2010.

[29] R.J. Serfling. *Approximation Theorems of Mathematical Statistics*. Wiley Series in Probability and Statistics. Wiley, 2009.

[30] A.N. Shiriaev. *Probability*. Graduate texts in mathematics. Springer-Verlag, 1984.

[31] Y.W. Tai, H. Du, M.S. Brown, and S. Lin. Correction of spatially varying image and video motion blur using a hybrid camera. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 32(6):1012–1028, 2010.

[32] J. Telleen, A. Sullivan, J. Yee, O. Wang, P. Gunawardane, I. Collins, and J. Davis. Synthetic shutter speed imaging. In *Computer Graphics Forum*, volume 26, pages 591–598. Wiley Online Library, 2007.

[33] Y. Tendero. The Flutter Shutter Camera Simulator. *Image Processing On Line*, 2012, 2012.

[34] Y. Tendero and J.-M Morel. An optimal blind temporal motion blur deconvolution filter. *IEEE Trans. on Signal Processing letters*, 20(5):523–526, 2013.

[35] Y. Tendero, J.-M. Morel, and B. Rougé. The Flutter Shutter Paradox. *SIAM Journal on Imaging Sciences*, 6(2):813–847, 2013.

[36] R. Tsai. Pulsed control of camera flash, June 14 2011. US Patent 7,962,031.

[37] A. Veeraraghavan, D. Reddy, and R. Raskar. Coded Strobing Photography: Compressive Sensing of High Speed Periodic Events. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 33(4):671–686, 2011.

[38] B. Wilburn, N. Joshi, V. Vaish, E.V. Talvala, E. Antunez, A. Barth, A. Adams, M. Horowitz, and M. Levoy. High performance imaging using large camera arrays. *ACM Transactions on Graphics (TOG)*, 24(3):765–776, 2005.

[39] S. Xu, H. Liang, D. Tu, and G. Li. A deblurring technique for large scale motion blur images using a hybrid camera. In *Proceedings of the IEEE International Congress on Image and Signal Processing (ICISP)*, volume 2, pages 806–810, 2010.

[40] W. Xu and S. McCloskey. 2D Barcode localization and motion deblurring using a flutter shutter camera. In *Proceedings of the IEEE Workshop on Applications of Computer Vision (WACV)*, pages 159–165, 2011.