SPECTRAL RESULTS FOR PERTURBED VARIATIONAL EIGENVALUE PROBLEMS AND THEIR APPLICATIONS TO COMPRESSED PDES*

ÖMER FARUK TEKİN†, KE YIN†, AND FARZIN BAREKAT†

Abstract. We consider the solutions to a modification of the Courant's minimax characterization of the Dirichlet eigenfunctions of second order linear symmetric elliptic operators in a bounded domain Ω in $\mathbb{R}^d$. In particular, we perturb the objective functional by an arbitrary bounded penalty term. Without perturbation, it is well-known that Courant minimax principle yields the eigenfunctions, which form an orthonormal basis for $L^2(\Omega)$. We prove that the solutions of the perturbed problem still form an orthonormal basis in the case of $d = 1$, and $d = 2$, provided that the perturbation is sufficiently small in the latter case. As an application, we prove completeness results for compressed plane waves and compressed modes, which are the solutions to analogous variational problems with perturbations being an L^1-regularization term. The completeness theory for these functions sets a foundation for finding a computationally efficient basis for the representation of the solution of differential equations.

Key words. variational eigenvalue problems, spectral methods, L^1-regularization

1. Introduction.

For a second-order symmetric elliptic operator L on a bounded domain $\Omega \subset \mathbb{R}^d$, consider the following Dirichlet eigenvalue problem

$$\begin{aligned} Lu &= \lambda u \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

The spectral theorem for the second-order elliptic operators asserts that the Dirichet eigenfunctions of L forms an orthonormal basis for $L^2(\Omega)$. Furthermore, the eigenvalues and eigenfunctions can be characterized via a hierarchical variational procedure involving the minimization of the functional $B[u,u] = \langle Lu, u\rangle_{L^2}$, known as Courant variational method (see e.g. [6, 8]).

In this paper, we consider the Courant variational problem under a non-negative perturbation, and prove for it an analogue of the spectral theorem. We treat this problem in a general Hilbert space setting, where we consider an arbitrary self-adjoint operator T whose inverse is compact. Namely, we rather minimize the functional

$$\mathcal{J}(u) := P(u) + \langle Tu, u\rangle,$$

in the domain of definition for T, where P is a non-negative penalty term. We verify that the spectral theorem still holds, as long as the eigenvalues of the original problem grow sufficiently fast. In particular, the growth condition holds when T is the the weak realization of an elliptic operator on a bounded domain Ω lying inside $\mathbb{R}$, or $\mathbb{R}^2$.

As an application, we consider the spatially localized ("sparse") modes introduced in [9, 10]. In [5, 7], it was proven that perturbing the variational quantity of certain elliptic and parabolic PDEs would result in compact support. In [9, 10], the authors consider variational problems corresponding to Schrödinger's equation, or Laplace's equation, whose functional is modified by an L^1 regularization term. Namely, their

*This research was supported by NSF DMS 0914561 and 1118971. Research of Barekat was also supported by an NSERC PGS-D award.

†Mathematics Department, University of California at Los Angeles, Los Angeles, CA 90095-1555 USA(omerftekin@math.ucla.edu, kyin@math.ucla.edu, fbarekat@math.ucla.edu.)

construction involves the minimization of the functional

$$\mathcal{J}(u) := \frac{1}{\mu}||u||_1 + \langle Lu, u\rangle_{L^2},$$

where $L = -\frac{1}{2}\Delta$, or $L = -\frac{1}{2}\Delta + V(x)$. By minimizing the above functional subject to the orthogonality constraint, so-called "compressed modes" are obtained by the authors in [9]. In a different formulation, where shift-orthogonality is imposed in addition to orthogonality, the minimizers are called the "compressed plane waves" [10]. Numerical experiments suggest that the regularization parameter μ for the L^1 term is used to balance between the consistency with the original problem and the sparsity of the solution. The consistency and the sparsity of compressed modes as μ varies have been verified in [1, 2, 3].

Our results yield a further consistency results for compressed modes and compressed plane waves, as we conclude that the analogous spectral properties are preserved under perturbation. Such "completeness" conjectures are proposed in [9, 10]. In [9], the authors predicted that the compressed modes approximate the true eigenstates with high precision, whereas in [10], they conjectured that the compressed plane waves form a basis in $L^2(\Omega)$, where Ω is some bounded rectangular domain in $\mathbb{R}^d$. We will provide an affirmative answer to these conjectures in dimensions one and two, and propose Conjecture 1, which is phrased in a general Hilbert space setting. We note that this conjecture implies the above mentioned conjectures.

This paper is organized as follows. In Section 2, we study the theory of completeness for the solutions of perturbed Courant-Fisher variational problem for arbitrary linear operators defined on a Hilbert space. We prove the completeness in Theorems 2.8 and 2.9 under linear and super-linear growth conditions on the eigenvalues of the original operator. Without the completeness, we still have Theorem 2.10, which yields an estimation of the deviation from the true solutions. Lemma 2.6 is the essential tool in the proof of these results. In the Sections 3, and 4, we apply the theory developed in Section 2 to the second-order linear symmetric elliptic operators.

2. Perturbed variational problems associated to linear operators. The proof of the spectral theorem for elliptic operators relies on the fact that the "inverse" of the elliptic differential operator is compact, hence the following spectral theorem for symmetric compact operators holds:

THEOREM 2.1 (Spectral Theorem for Compact Operators). *Let $\mathcal{H}$ be a Hilbert space, and $K : \mathcal{H} \to \mathcal{H}$ be a symmetric compact operator. Then,*

1. *K has real eigenvalues ν_k, and $\nu_k \to 0$ as $k \to \infty$,*
2. *The (normalized) eigenvectors $\{\phi_k\}_{k=1}^{\infty}$, with $K\phi_k = \nu_k\phi_k$, form a complete orthonormal system in $\mathcal{H}$.*

As a consequence of this version of the spectral theorem, the inverse T, of a positive compact operator K satisfies the following spectral theorem:

THEOREM 2.2 (Spectral Theorem for Inverse-Compact Operators). *Let $\mathcal{H}$ be a Hilbert space, and $K : \mathcal{H} \to \mathcal{H}$ be a bijective symmetric compact operator. Then, $T = K^{-1}$ satisfies the following properties:*

1. *T has real eigenvalues λ_k, with $\{\lambda_k\}_{k=1}^{\infty}$ in increasing order, and $\lambda_k \to +\infty$ as $k \to \infty$,*
2. *The (normalized) eigenvectors $\{\phi_k\}_{k=1}^{\infty}$, with $T\phi_k = \lambda_k\phi_k$, form a complete orthonormal system.*

REMARK 2.3. *Notice that the operator T defined in Theorem 2.2 is unbounded, hence T must have a domain of definition, $\mathcal{D}(T)$, for which it is self-adjoint. We* *consider the following natural choice of the domain of definition,*

$$\mathcal{D}(T) = \{\alpha = \sum_{n\in\mathbb{N}} \hat{\alpha}_n \phi_n \in \mathcal{H} | \sum_{n\in\mathbb{N}} \lambda_n \hat{\alpha}_n \phi_n \in \mathcal{H}, \text{ or equivalently, } \sum_{n\in\mathbb{N}} \lambda_n^2 |\hat{\alpha}_n|^2 < \infty\},$$

in which there will be no ambiguity of the definition of T.

We will now work with the operators T that can be represented as the inverse of some bijective symmetric compact operator. The eigenvalues and eigenvectors of T can be characterized via the following Courant-Fisher variational formulae (see e.g. [6]):

$$\begin{aligned}
\lambda_1 &= \min_{\substack{u\in\mathcal{D}(T)\\ ||u||=1}} \langle Tu, u\rangle, \\
\phi_1 &= \operatorname*{argmin}_{\substack{u\in\mathcal{D}(T)\\ ||u||=1}} \langle Tu, u\rangle, \\
\lambda_k &= \min_{\substack{u\in\mathcal{D}(T)\\ u\in\{\phi_1,\ldots,\phi_{k-1}\}^\perp\\ ||u||=1}} \langle Tu, u\rangle, \\
\phi_k &= \operatorname*{argmin}_{\substack{u\in\mathcal{D}(T)\\ u\in\{\phi_1,\ldots,\phi_{k-1}\}^\perp\\ ||u||=1}} \langle Tu, u\rangle.
\end{aligned}$$

We consider a similar variational problem, where we perturb the objective functional $\langle Tu, u\rangle$. Strictly speaking, we define

$$J[u] = \langle Tu, u\rangle + P(u),$$

where $P : \mathcal{H} \to \mathbb{R}$ is a non-negative penalty term. We view the term $J[u]$ as the "energy" of the element u, and run a progressive energy-minimization procedure as in the Courant-Fisher formulae. In other words, we define

$$\begin{aligned}
\zeta_1 &= \operatorname*{argmin}_{\substack{u\in\mathcal{D}(T)\\ ||u||=1}} J[u], \\
\zeta_k &= \operatorname*{argmin}_{\substack{u\in\mathcal{D}(T)\\ u\in\{\zeta_1,\ldots,\zeta_{k-1}\}^\perp\\ ||u||=1}} J[u].
\end{aligned} \tag{2.1}$$

In case of a non-uniqueness in the minimization above, we define ζ_k to be one of the solutions to the corresponding minimization problem. To ensure the existence of ζ_k's, we impose that P is bounded and lower semi-continuous with respect to the norm-convergence, in the sense that

$$\begin{aligned}
P(u) &\leq C||u||, \\
||u_n - u|| \to 0 \implies P(u) &\leq \liminf P(u_n).
\end{aligned} \tag{2.2}$$

The smallest constant C satisfying the boundedness of P is the functional norm of P, and is denoted by $||P||$.

We require that the eigenvectors of T, i.e. $\{\phi_n\}_{n\in\mathbb{N}}$, form a complete orthonormal system in $\mathcal{H}$. We conjecture that as long as the perturbation satisfies the existence criteria (2.2), such a spectral result still holds:

Conjecture 1. *The set $\{\zeta_n\}_{n\in\mathbb{N}}$ obtained via the variational procedure (2.1) forms a complete orthonormal system in $\mathcal{H}$.*

This section mainly focuses on verifying this conjecture under certain growth assumptions on the eigenvalues λ_n. In order to verify this conjecture, one needs to show that

$$\phi_k \in \overline{\operatorname{span}\{\zeta_n\}_{n\in\mathbb{N}}} \quad \forall k \in \mathbb{N}, \tag{2.3}$$

where $\operatorname{span} E$ denotes the space consisting of the finite linear combinations of the elements in E. The following Hilbert theoretic result quantifies the relation (2.3):

Lemma 2.4. *Let $\{e_n\}_{n\in\mathbb{N}}$ be a maximal orthonormal system in a Hilbert space $\mathcal{H}$. Let $\{f_n\}_{n\in\mathbb{N}}$ be some orthonormal system in $\mathcal{H}$. Assume that each f_n has the expansion*

$$f_n = \sum_{k\in\mathbb{N}} a_{n,k} e_k, \; a_{n,k} \in \mathbb{C}.$$

Then, for each $k \in \mathbb{N}$,

$$d(e_k, \operatorname{span}\{f_n\})^2 = 1 - \sum_{n\in\mathbb{N}} |a_{n,k}|^2,$$

where $d(e, M)$ denotes the distance between some $e \in \mathcal{H}$, and some linear subspace M of $\mathcal{H}$.

Proof: Let w be the projection of e_k onto $\operatorname{span}\{f_n\}_{n\in\mathbb{N}}$. Then, since $\{f_n\}_{n\in\mathbb{N}}$ is an orthonormal system, w is given by

$$\begin{aligned} w &= \sum_{n\in\mathbb{N}} \langle e_k, f_n\rangle f_n \\ &= \sum_{n\in\mathbb{N}} \langle e_k, \sum_j a_{n,j} e_j\rangle f_n \\ &= \sum_{n\in\mathbb{N}} \overline{a_{n,k}} f_n. \end{aligned}$$

Hence, we can compute the size of w:

$$||w||^2 = \sum_{n\in\mathbb{N}} |a_{n,k}|^2. \tag{2.4}$$

Note by the property of the projection that $e_k - w \perp w$, therefore, by the Pythagorean identity, we have

$$||e_k||^2 = ||e_k - w||^2 + ||w||^2. \tag{2.5}$$

Notice also that w, being the projection of e_k onto $\operatorname{span}\{f_n\}_{n\in\mathbb{N}}$, is the closest point to e_k inside $\operatorname{span}\{f_n\}_{n\in\mathbb{N}}$, so that

$$d(e_k, \operatorname{span}\{f_n\})^2 = ||e_k - w||^2. \tag{2.6}$$

Combining (2.4)-(2.6), we get

$$d(e_k, \operatorname{span}\{f_n\})^2 = ||e_k||^2 - ||w||^2 = 1 - \sum_{n\in\mathbb{N}} |a_{n,k}|^2,$$

as desired. □

Corollary 2.5. *Let $\{e_n\}_{n\in\mathbb{N}}$, $\{f_n\}_{n\in\mathbb{N}}$, $a_{n,k}$ be as in Lemma 2.4. Then, for each $k \in \mathbb{N}$,*

$$\sum_{n\in\mathbb{N}} |a_{n,k}|^2 \leq 1, \tag{2.7}$$

and

$$e_k \in \overline{\operatorname{span}\{f_n\}} \iff \sum_{n\in\mathbb{N}} |a_{n,k}|^2 = 1.$$

The following lemma yields an estimate for the elements that are lying inside the orthogonal complement of any arbitrary orthonormal system, in terms of the deviation of their functional values from the sum of the eigenvalues corresponding to the true eigenstates. We denote the deviation of the first N elements by $F(N)$.

Lemma 2.6. *Let ϕ_n be the eigenvectors of the operator T, with the corresponding eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ being in increasing order. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal system satisfying*

$$\sum_{n=1}^{N} \langle Te_n, e_n\rangle \leq F(N) + \sum_{n=1}^{N} \lambda_n \quad \forall N \in \mathbb{N}. \tag{2.8}$$

Suppose that there exists $f \in \{e_n\}_{n\in\mathbb{N}}^{\perp}$, $||f|| = 1$, with the expansion

$$f = \sum_{n\in\mathbb{N}} f_n \phi_n, \ f_n \in \mathbb{C}.$$

Then, we have

$$\sum_{n=1}^{N} |f_n|^2 (\lambda_{N+1} - \lambda_n) \leq F(N), \quad \forall N \in \mathbb{N}. \tag{2.9}$$

Proof: Let the $\{a_{n,k}\}_{n,k\in\mathbb{N}}$ denote the coefficients when e_n expanded in the basis $\{\phi_k\}_{k\in\mathbb{N}}$, i.e.

$$e_n = \sum_{k\in\mathbb{N}} a_{n,k}\phi_k.$$

Applying the result (2.7) of Lemma 2.4 to the orthonormal systems $\{e_n\}_{n\in\mathbb{N}} \cup \{f\}$, and $\{\phi_n\}_{n\in\mathbb{N}}$, for each $k \in \mathbb{N}$, we get

$$\sum_{n\in\mathbb{N}} |a_{n,k}|^2 \leq 1 - |f_k|^2. \tag{2.10}$$

By the bilinearity of the inner product,

$$\langle Te_n, e_n\rangle = \sum_{k=1}^{\infty} |a_{n,k}|^2 \lambda_k.$$

Hence,

$$\sum_{n=1}^{N}\langle Te_n, e_n\rangle = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{N}|a_{n,k}|^2\right)\lambda_k. \tag{2.11}$$

Since e_n's have norm 1, in the expression (2.11), the coefficients of λ_k summed over k equals N. Given that the λ_k's are in the increasing order, the expression (2.11) is minimized when the coefficients of λ_k are maximized for small k. Having the constraint (2.10), we get

$$\sum_{n=1}^{N}\langle Te_n, e_n\rangle = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{N}|a_{n,k}|^2\right)\lambda_k \geq \sum_{k=1}^{N}\left(1-|f_k|^2\right)\lambda_k + \lambda_{N+1}\sum_{k=1}^{N}|f_k|^2.$$

Combining this, with the inequality (2.8) we get

$$\sum_{n=1}^{N}\lambda_n + F(N) \geq \sum_{n=1}^{N}\langle Te_n, e_n\rangle \geq \sum_{n=1}^{N}\left(1-|f_n|^2\right)\lambda_n + \lambda_{N+1}\sum_{n=1}^{N}|f_n|^2,$$

which implies

$$\sum_{n=1}^{N}|f_n|^2(\lambda_{N+1}-\lambda_n) \leq F(N),$$

as desired. □

Lemma 2.6 will be essential for proving the completeness result. The estimate (2.9) provides us an understanding of the elements lying inside the orthogonal complement in terms of $F(N)$, and the eigenvalues of T. If the estimate (2.9) is incompatible with the growth of eigenvalues, then we deduce that the orthogonal complement is empty, and hence the orthonormal system is maximal.

The next lemma provides an estimate for the deviation of the functional values of ζ_n, from the eigenvalues corresponding to the true eigenstates, so that Lemma 2.6 is applicable.

Lemma 2.7. *Let $\{\zeta_n\}_{n\in\mathbb{N}}$ be the solutions to the variational procedure (2.1). Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the eigenvalues of T, in increasing order. Then,*

$$J[\zeta_n] \leq \lambda_n + ||P||.$$

In particular, we have

$$\sum_{n=1}^{N}\langle T\zeta_n, \zeta_n\rangle \leq \sum_{n=1}^{N}J[\zeta_n] \leq \sum_{n=1}^{N}\lambda_n + ||P||N. \tag{2.12}$$

Proof: Let $\{a_{n,k}\}_{n,k\in\mathbb{N}}$ denote the coefficients when ζ_n expanded in the basis $\{\phi_k\}_{k\in\mathbb{N}}$:

$$\zeta_n = \sum_{k\in\mathbb{N}} a_{n,k}\phi_k.$$

Let $n\in\mathbb{N}$ be fixed. For integers j with $1 \leq j \leq n-1$, define

$$\eta_j = \sum_{k=1}^{n} a_{j,k}\phi_k.$$

Clearly, $\{\eta_1, \eta_2, \ldots, \eta_{n-1}\} \subset \text{span}\{\phi_1, \phi_2, \ldots, \phi_n\}$. Furthermore,

$$\dim \text{span}\{\phi_1, \phi_2, \ldots, \phi_n\} = n,$$

i.e. the space $\text{span}\{\phi_1, \phi_2, \ldots, \phi_n\}$ has dimension larger than the cardinality of $\{\eta_1, \eta_2, \ldots, \eta_{n-1}\}$, so that we can find $\eta \in \text{span}\{\phi_1, \phi_2, \ldots, \phi_n\}$, $\eta \neq 0$, such that

$$\eta \perp \eta_j, \quad \forall j : 1 \leq j \leq n-1.$$

Now, since η_j's and η lie inside $\text{span}\{\phi_1, \phi_2, \ldots, \phi_n\}$, we get

$$\langle \eta, \zeta_j \rangle = \langle \eta, \eta_j \rangle = 0, \quad \forall j : 1 \leq j \leq n-1,$$

so that

$$\eta \perp \zeta_j, \quad \forall j : 1 \leq j \leq n-1.$$

By rescaling, we may assume $||\eta|| = 1$, so that η lies precisely in the class of functions where we look for a minimizer to determine ζ_n. The function η is a sub-solution to the variational problem (2.1) at the n^{th} step, hence

$$J[\zeta_n] \leq J[\eta] = \langle T\eta, \eta \rangle + P(\eta). \tag{2.13}$$

Note first by the boundedness of P that

$$P(\eta) \leq ||P||\, ||\eta|| = ||P||. \tag{2.14}$$

Suppose η has the expansion

$$\eta = \sum_{k=1}^{n} b_k \phi_k.$$

Since, $||\eta|| = 1$, we have $\sum_{k=1}^{n} |b_k|^2 = 1$. Furthermore, by the bilinearity of the inner product,

$$\langle T\eta, \eta \rangle = \sum_{k=1}^{n} |b_k|^2 \lambda_k.$$

We also have that λ_k's are in increasing order, so that

$$\langle T\eta, \eta \rangle = \sum_{k=1}^{n} |b_k|^2 \lambda_k \leq \lambda_n \sum_{k=1}^{n} |b_k|^2 = \lambda_n. \tag{2.15}$$

Combining (2.13)-(2.15), we obtain

$$J[\zeta_n] \leq \lambda_n + ||P||. \tag{2.16}$$

Summing up the inequality (2.16) for $n = 1, 2, \ldots, N$, and combining with the non-negativity of P, we verify (2.12). □

The following theorem provides the completeness of the orthonormal system $\{\zeta_n\}_{n \in \mathbb{N}}$, provided the eigenvalues satisfy the super-linear growth:

THEOREM 2.8. *Suppose the eigenvalues of T satisfy*

$$\lim_{n \to \infty} \frac{\lambda_n}{n} = \infty. \tag{2.17}$$

Then, $\{\zeta_n\}_{n\in\mathbb{N}}$, which is defined by the variational procedure (2.1), forms a complete orthonormal system in $\mathcal{H}$.

Proof: Lemma 2.7 implies

$$\sum_{n=1}^{N} \langle T\zeta_n, \zeta_n\rangle \le \sum_{n=1}^{N} \lambda_n + ||P||N,$$

so that Lemma 2.6 is applicable to the orthonormal system $\{\zeta_n\}_{n\in\mathbb{N}}$ with the function $F(N) = ||P||N$. That is, assuming the existence of an $f \in \{\zeta_n\}^{\perp}_{n\in\mathbb{N}}$, $||f|| = 1$ with the expansion

$$f = \sum f_n \phi_n, \ f_n \in \mathbb{C},$$

we obtain the estimate

$$\sum_{n=1}^{N} |f_n|^2(\lambda_{N+1} - \lambda_n) \le ||P||N, \quad \forall N \in \mathbb{N}. \tag{2.18}$$

This last inequality implies (assuming $f_n \neq 0$)

$$\lambda_{N+1} - \lambda_n \le \frac{||P||N}{|f_n|^2}, \quad \forall N \in \mathbb{N},$$

which yields a contradiction by violating the growth condition (2.17) on λ_{N+1}, as we pass to limit as $N \to \infty$. Therefore, there is no non-zero function $f \in \{\zeta_n\}^{\perp}_{n\in\mathbb{N}}$, so that the orthonormal system $\{\zeta_n\}_{n\in\mathbb{N}}$ is complete, as desired. □

By trading the magnitude of the penalty term with the growth of λ_n, we can generalize the Theorem 2.8 so that it holds under a weaker growth condition on λ_n:

Theorem 2.9. *Suppose that the eigenvalues λ_n grows linearly in the sense that they satisfy*

$$\lambda_n = Mn + o(n), \ \text{as } n \to \infty \tag{2.19}$$

for some constant M. Suppose also that the penalty term P satisfies the following bound

$$||P|| < M. \tag{2.20}$$

Then, $\{\zeta_n\}_{n\in\mathbb{N}}$ forms a complete orthonormal system in $\mathcal{H}$.

Proof: Proceeding similarly as in the proof of Theorem 2.8, we get the inequality (2.18), namely that if f is a function with unit norm, lying in the orthogonal complement of $\{\zeta_n\}_{n\in\mathbb{N}}$, then we have

$$\sum_{n=1}^{N} |f_n|^2(\lambda_{N+1} - \lambda_n) \le ||P||N, \quad \forall N \in \mathbb{N}. \tag{2.21}$$

Now, for $m < N$, by the monotonicity of λ_k, we can lower-bound the LHS of (2.21) by

$$(\lambda_{N+1} - \lambda_m) \sum_{n=1}^{m} |f_n|^2,$$

so that

$$\frac{\lambda_{N+1}-\lambda_m}{N}\sum_{n=1}^{m}|f_n|^2 \le ||P||,\ \forall N\in\mathbb{N},\ \forall m: 0<m<N. \tag{2.22}$$

Taking limit in (2.22) as $N\to\infty$, with the aid of the growth condition (2.19), we obtain

$$M\sum_{n=1}^{m}|f_n|^2 \le ||P||,\ \forall m\in\mathbb{N}.$$

Now, taking limit as $m\to\infty$, and keeping in mind that $||f||=1$, we obtain

$$M \le ||P||,$$

contradicting (2.20). □

The following theorem establishes how close the functions ζ_n approximate the subspaces generated by the first few eigenvectors of the operator T:

THEOREM 2.10. *Let V_m be the subspace generated by the functions $\{\zeta_1,\zeta_2,\ldots,\zeta_m\}$. Then, for any $n\le m$, we have*

$$\sum_{k=1}^{n} d(\phi_k,V_m)^2 \le \frac{m||P||}{\lambda_{m+1}-\lambda_n},$$

provided $\lambda_{m+1}\ne\lambda_n$.

Proof: Recall from Lemma 2.7 that

$$\sum_{j=1}^{m}\langle T\zeta_j,\zeta_j\rangle \le \sum_{j=1}^{m} J[\zeta_j] \le m||P|| + \sum_{j=1}^{m}\lambda_j. \tag{2.23}$$

Recall by the bilinearity of the inner product that

$$\langle T\zeta_j,\zeta_j\rangle = \sum_{k=1}^{\infty}|a_{j,k}|^2\lambda_k. \tag{2.24}$$

Combining (2.24) and (2.23), yields that

$$\sum_{j=1}^{m}\sum_{k=1}^{\infty}|a_{j,k}|^2\lambda_k \le m||P|| + \sum_{j=1}^{m}\lambda_j. \tag{2.25}$$

Rearranging (2.25), we obtain

$$\sum_{k=m+1}^{\infty}\sum_{j=1}^{m}|a_{j,k}|^2\lambda_k - \sum_{k=1}^{m}\left(1-\sum_{j=1}^{m}|a_{j,k}|^2\right)\lambda_k \le m||P||.$$

Lower-bounding the terms λ_k with $k>m$, by λ_{m+1} in the last expression, we obtain

$$\lambda_{m+1}\sum_{k=m+1}^{\infty}\sum_{j=1}^{m}|a_{j,k}|^2 - \sum_{k=1}^{m}\left(1-\sum_{j=1}^{m}|a_{j,k}|^2\right)\lambda_k \le m||P||. \tag{2.26}$$

Since $\sum_{k=1}^{\infty} |a_{j,k}|^2 = 1$ for $j = 1, 2, \ldots, m$, we conclude that

$$\sum_{k=m+1}^{\infty} \sum_{j=1}^{m} |a_{j,k}|^2 = \sum_{k=1}^{m} \left(1 - \sum_{j=1}^{m} |a_{j,k}|^2 \right).$$

Substituting this into the inequality (2.26), and rearranging further we obtain

$$\sum_{k=1}^{m} \left(1 - \sum_{j=1}^{m} |a_{j,k}|^2 \right) (\lambda_{m+1} - \lambda_k) \leq m||P||. \tag{2.27}$$

Notice by Lemma 2.4 that the coefficient in front of $\lambda_{m+1} - \lambda_k$ in (2.27) is equal to $d(\phi_k, V_m)^2$, so that (2.27) becomes

$$\sum_{k=1}^{m} d(\phi_k, V_m)^2 (\lambda_{m+1} - \lambda_k) \leq m||P||.$$

Exploiting the monotonicity of λ_k once more, we obtain

$$\sum_{k=1}^{n} d(\phi_k, V_m)^2 (\lambda_{m+1} - \lambda_n) \leq \sum_{k=1}^{m} d(\phi_k, V_m)^2 (\lambda_{m+1} - \lambda_k) \leq m||P||,$$

implying

$$\sum_{k=1}^{n} d(\phi_k, V_m)^2 \leq \frac{m||P||}{\lambda_{m+1} - \lambda_n},$$

as desired. □

3. Perturbed variational problems associated to elliptic operators. The results of Section 2 can now be applied to second-order linear symmetric elliptic operators. Let L be a second-order linear symmetric elliptic operator defined on a bounded domain Ω in $\mathbb{R}^d$. For simplicity we will consider elliptic operators with principal parts $-\Delta$, i.e.

$$Lu = -\Delta u + \vec{b} \cdot \nabla u + cu,$$

where $\vec{b} : \Omega \to \mathbb{R}^d, c : \Omega \to \mathbb{R}$, are bounded measurable functions.
As we noted in the beginning of Section 2, according to the spectral theorem for second-order linear symmetric elliptic operators, L satisfies the following properties:

1. L has real (Dirichlet) eigenvalues, λ_k, with $\{\lambda_k\}_{k=1}^{\infty}$ in increasing order, and $\lambda_k \to +\infty$ as $k \to \infty$,
2. The (normalized) eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$, with $L\phi_k = \lambda_k \phi_k$, form a complete orthonormal system in $L^2(\Omega)$.

Furthermore, the Courant-Fisher principle applies to L, so that the eigenvalues and eigenfunctions of L is found by the following variational formulae:

$$\begin{aligned}
\lambda_1 &= \min_{\substack{u\in H_0^1(\Omega)\\ ||u||_2=1}} B[u,u],\\
\phi_1 &= \operatorname{argmin}_{\substack{u\in H_0^1(\Omega)\\ ||u||_2=1}} B[u,u],\\
\lambda_k &= \min_{\substack{u\in H_0^1(\Omega)\\ u\in\{\phi_1,\dots,\phi_{k-1}\}^\perp\\ ||u||_2=1}} B[u,u],\\
\phi_k &= \operatorname{argmin}_{\substack{u\in H_0^1(\Omega)\\ u\in\{\phi_1,\dots,\phi_{k-1}\}^\perp\\ ||u||_2=1}} B[u,u],
\end{aligned}$$

where

$$B[u,v] = \langle Lu, v\rangle = \int_\Omega \nabla u\cdot\nabla v + (\vec{b}\cdot\nabla u)v + cuv\, d\mathbf{x}$$

is the bilinear form associated to L.
We proceed similarly as in Section 2, where we perturb the functional $B[u,u]$. This time, we restrict ourselves to the penalty terms given by a constant multiple of the L^1 norm. Namely, we consider the energy functional

$$J_\mu[u] = B[u,u] + \frac{1}{\mu}||u||_{L^1},$$

and analogously define the functions $\{\psi_k\}_{k\in\mathbb{N}}$ via

$$\begin{aligned}
\psi_1 &= \operatorname{argmin}_{\substack{u\in H_0^1(\Omega)\\ ||u||_2=1}} J_\mu[u],\\
\psi_k &= \operatorname{argmin}_{\substack{u\in H_0^1(\Omega)\\ u\in\{\psi_1,\dots,\psi_{k-1}\}^\perp\\ ||u||_2=1}} J_\mu[u].
\end{aligned} \tag{3.1}$$

We call these functions $\{\psi_k\}_{k\in\mathbb{N}}$ "compressed modes of type two" by analogy to the "compressed modes" defined in [9]. The following lemma establishes the existence of $\{\psi_k\}_{k\in\mathbb{N}}$ by verifying the existence criteria (2.2) for the L^1 penalty term in the definition of J_μ:

Lemma 3.1. *Let $P : L^2(\Omega) \to \mathbb{R}$ be defined by $P(u) = \frac{1}{\mu}||u||_{L^1}$. Then, P satisfies the criteria (2.2). Furthermore,*

$$||P|| = \frac{|\Omega|^{\frac{1}{2}}}{\mu}.$$

Proof: By Cauchy-Schwarz inequality,

$$\frac{1}{\mu}||u||_{L^1} \le \frac{1}{\mu}||u||_{L^2}||\mathbb{1}_\Omega||_{L^2} = \frac{|\Omega|^{\frac{1}{2}}}{\mu}||u||_{L^2}, \tag{3.2}$$

with equality when u is a non-zero constant function. Therefore, P is bounded with functional norm $\frac{|\Omega|^{\frac{1}{2}}}{\mu}$.

As for the lower semi-continuity, consider a sequence $u_n \in L^2(\Omega)$ converging to some $u \in L^2(\Omega)$ in L^2. The inequality 3.2 implies that u_n converges to u also in L^1, so that $P(u) = \lim P(u_n)$, as desired. □

Now, the Theorems 2.8, 2.9, 2.10, can be derived for $\{\psi_k\}_{k\in\mathbb{N}}$. Theorems 2.8, and 2.9 holds true when the eigenvalues grow super-linearly, and linearly, respectively. Weyl's law yields the exact asymptotic behavior of the eigenvalues of a second-order linear symmetric elliptic operator:

Theorem 3.2 (Weyl's Law (see e.g. [8])). *Let L be a second-order linear elliptic operator on a bounded domain $\Omega \subset \mathbb{R}^d$, of the form*

$$Lu = -\Delta u + \vec{b} \cdot \nabla u + cu,$$

where $\vec{b} : \Omega \to \mathbb{R}^d, c : \Omega \to \mathbb{R}$, are bounded measurable functions. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the eigenvalues of L, in increasing order. Then,

$$\lambda_n = \frac{(2\pi)^d}{\omega_d|\Omega|} n^{2/d} + o(n^{2/d}), \text{ as } n \to \infty,$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

Therefore, we can deduce from the Weyl's law that super-linear and linear growth conditions on eigenvalues holds true precisely in dimensions 1, and 2, so that we have the following corollaries as direct consequences of the Theorems 2.8, 2.9, 2.10:

Corollary 3.3 (Corollary to Theorem 2.8). *Let L be a second-order linear symmetric elliptic operator defined on a bounded domain $\Omega \subset \mathbb{R}$. Then, $\{\psi_n\}_{n\in\mathbb{N}}$, which is defined by the variational procedure (3.1), forms a complete orthonormal system in $L^2(\Omega)$.*

Corollary 3.4 (Corollary to Theorem 2.9). *Let L be a second-order linear symmetric elliptic operator defined on a bounded domain $\Omega \subset \mathbb{R}^2$. Suppose also that the penalty parameter μ satisfies the following bound*

$$\mu > \frac{1}{4\pi|\Omega|^{\frac{3}{2}}}.$$

Then, $\{\psi_n\}_{n\in\mathbb{N}}$, which is defined by the variational procedure (3.1), forms a complete orthonormal system in $L^2(\Omega)$.

Corollary 3.5 (Corollary to Theorem 2.10). *Let $\{\phi_k\}_{k\in\mathbb{N}}$ be the (Dirichlet) eigenfunctions of a second-order linear symmetric elliptic operator L, defined on a bounded domain $\Omega \subset \mathbb{R}$. Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the associated eigenvalues. Let $\{\psi_n\}_{n\in\mathbb{N}}$ be the functions defined by the variational procedure (3.1), and V_m be the subspace generated by the functions $\{\psi_1, \psi_2, \ldots, \psi_m\}$. Then, for any $n \le m$, we have*

$$\sum_{k=1}^{n} d(\phi_k, V_m)^2 \le \frac{m|\Omega|^{\frac{1}{2}}}{\mu(\lambda_{m+1} - \lambda_n)},$$

provided $\lambda_{m+1} \neq \lambda_n$.

The following theorem establishes that the elements that lie in the orthogonal complement of $\{\psi_n\}_{n\in\mathbb{N}}$ cannot lie inside the space $H_0^1(\Omega)$. In other words, the orthogonal complement consists of highly irregular functions:

Theorem 3.6. *Let $\{\psi_n\}_{n\in\mathbb{N}}$ be the solutions to the variational procedure (2.1). Then,*

$$\{\psi_n\}^\perp \cap H_0^1(\Omega) = \{0\}.$$

Proof: Assume to the contrary that there is a non-zero $f \in \{\psi_n\}^\perp \cap H_0^1(\Omega)$. We may normalize f such that $||f||_2 = 1$. Since $f \in \{\psi_1, \ldots, \psi_{n-1}\}^\perp$ for all n, f is in the class of functions where we look for a minimizer to obtain ψ_n, hence is a subsolution to the variational problem (2.1) at n^{th} step. As ψ_n is the actual solution to the corresponding minimization problem, we have

$$J_\mu[\psi_n] \leq J_\mu[f]. \tag{3.3}$$

We now prove that

$$\lim_{n\to\infty} J_\mu[\psi_n] = \infty, \tag{3.4}$$

which together with (3.3) implies

$$J_\mu[f] = \infty. \tag{3.5}$$

Recall that

$$\sum_{n=1}^{N} B[\psi_n, \psi_n] = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{N} |a_{n,k}|^2 \right) \lambda_k, \tag{3.6}$$

where the coefficients in front of λ_k have magnitude less than or equal to 1 for each k, and their sum over k is N. Since λ_k's are in increasing order, the expression (3.6) is minimized when the coefficients of λ_k are maximized for small k. Therefore,

$$\sum_{n=1}^{N} B[\psi_n, \psi_n] \geq \sum_{k=1}^{N} \lambda_k,$$

and since P is non-negative

$$\sum_{n=1}^{N} J_\mu[\psi_n] = \sum_{n=1}^{N} B[\psi_n, \psi_n] + P(\psi_n) \geq \sum_{n=1}^{N} B[\psi_n, \psi_n] \geq \sum_{k=1}^{N} \lambda_k. \tag{3.7}$$

We know that $\lim_{n\to\infty} \lambda_n = \infty$, and both λ_n's and $J_\mu[\psi_n]$'s are in increasing order. Therefore, the inequality (3.7) can hold only if (3.4) holds. Hence, we verify (3.5), i.e.

$$J_\mu[f] = B[f, f] + P(f) = \infty. \tag{3.8}$$

Now, since P is bounded, the expression (3.8) yields

$$B[f, f] = \infty. \tag{3.9}$$

On the other hand, as B is the bilinear form associated to a second order elliptic operator, it is bounded in the sense that

$$|B[u, v]| \leq C||u||_{H^1(\Omega)}||v||_{H^1(\Omega)}. \tag{3.10}$$

Combining (3.9), and (3.10) applied to $u = v = f$, we get

$$||f||_{H^1(\Omega)} = \infty,$$

i.e. $f \notin H^1(\Omega)$, contradicting the assumption that $f \in \{\psi_n\}^\perp \cap H_0^1(\Omega)$. □

4. Applications. We now establish the analogues of the Theorems 2.8-2.10 for the compressed modes and the compressed plane waves. We first provide the precise definitions of CM and CPW as introduced in [9, 10], and establish their connections to the theory we developed in Section 2, and then proceed with the verification of the analogous theorems.

4.a. Compressed Modes. Compressed modes are defined via the following minimization procedure

$$\Psi^{(m)} = \{\psi_1^{(m)}, \dots, \psi_m^{(m)}\} = \underset{h_1,h_2,\dots,h_m}{\operatorname{argmin}} \sum_{i=1}^{m} J_\mu[h_i] \quad \text{s.t.} \quad \langle h_j, h_k\rangle = \delta_{jk}, \tag{4.1}$$

where

$$J_\mu[u] = \frac{1}{\mu}||u||_{L^1} + \langle u, \left(-\frac{1}{2}\Delta + V\right) u\rangle = \frac{1}{\mu}||u||_{L^1} + \frac{1}{2}||\nabla u||_{L^2} + \int_\Omega V u^2 d\mathbf{x}, \tag{4.2}$$

where V is a bounded measurable real-valued function defined on Ω. Here, the quantity $\langle u, \left(-\frac{1}{2}\Delta + V\right) u\rangle$ corresponds to the bilinear form associated to the elliptic operator $-\frac{1}{2}\Delta + V$. We denote the eigenvalues and eigenfunctions of $-\frac{1}{2}\Delta + V$ by λ_n and ϕ_n, with λ_n's being in increasing order, as usual.
Notice here that CMs are defined as the minimizers of an energy sum under orthogonality constraints, rather than as compressed modes of type two, which are solutions to an iterative minimization procedure. However, compressed modes of type two $\{\psi_1, \psi_2, \dots, \psi_m\}$, defined by the variational procedure (3.1), being an orthonormal sequence, is a subsolution to the minimization problem (4.1), so that

$$\sum_{i=1}^{m} J_\mu[\psi_i^{(m)}] \le \sum_{i=1}^{m} J_\mu[\psi_i].$$

Combining this with the estimate (2.12), we obtain

$$\sum_{i=1}^{m} J_\mu[\psi_i^{(m)}] \le m\frac{|\Omega|^{\frac{1}{2}}}{\mu} + \sum_{j=1}^{m} \lambda_j. \tag{4.3}$$

The proof of Theorem 2.10 relies essentially on the estimation (2.23), and the orthonormality of the sequence $\{\psi_1, \dots, \psi_m\}$. We still have the orthonormality, and the estimation (4.3) analogous to (2.23). Hence, the following corollary holds.

Corollary 4.1 (Corollary to Theorem 2.10). *Let V_m be the subspace generated by the compressed modes $\Psi = \{\psi_1^{(m)}, \psi_2^{(m)}, \dots, \psi_m^{(m)}\}$. Then, for any $n \le m$, we have*

$$\sum_{k=1}^{n} d(\phi_k, V_m)^2 \le \frac{m|\Omega|^{\frac{1}{2}}}{\mu(\lambda_{m+1} - \lambda_n)},$$

provided $\lambda_{m+1} \neq \lambda_n$.

From Corollary 4.1, we deduce the following approximation result:

Corollary 4.2. *Let $\{\phi_k\}_{k\in\mathbb{N}}$ be the (Dirichlet) eigenfunctions of a second-order linear symmetric elliptic operator L, defined on a bounded domain $\Omega \subset \mathbb{R}$. Given any fixed parameter μ, the first m compressed modes up to a linear transformation, denoted by $\{\xi_1^{(m)}, \dots, \xi_m^{(m)}\}$, satisfies*

$$\lim_{m\to\infty} \|\phi_k - \xi_k^{(m)}\|_2 = 0, \ \forall k \in \mathbb{N}.$$

Proof: Let $\xi_k^{(m)}$ denote the projection of $\phi_k^{(m)}$ onto the linear subspace spanned by $\{\psi_1^{(m)}, \ldots, \psi_m^{(m)}\}$, which we denote by V_m. Then, clearly, $\xi_k^{(m)}$ is a linear combination of $\{\psi_1^{(m)}, \ldots, \psi_m^{(m)}\}$. Furthermore, as a property of the projection, we have

$$d(\phi_k, V_m) = ||\phi_k^{(m)} - \xi_k^{(m)}||_2,$$

so that Corollary 4.1 implies

$$\sum_{k=1}^{n} ||\phi_k - \xi_k^{(m)}||_2^2 \leq \frac{m|\Omega|^{\frac{1}{2}}}{\mu(\lambda_{m+1} - \lambda_n)}. \tag{4.4}$$

As Ω is a bounded domain inside $\mathbb{R}$, By Weyl's law, we know that λ_m grows quadratically in m. Hence, passing to limit in (4.4) as $m \to \infty$, we conclude that the summands in the LHS of (4.4) decays to zero, i.e.

$$\lim_{m\to\infty} \|\phi_k - \xi_k^{(m)}\|_2 = 0, \tag{4.5}$$

as desired. □

Corollary 4.2 can be viewed as a completeness result, since (4.5) yields that any eigenfunction ϕ_k is well approximated by its projection $\xi_k^{(m)}$ onto V_m. In a sense, V_m's trace the full space, as $m \to \infty$.

4.b. Compressed Plane Waves. The construction of compressed plane waves is closely related to that of compressed modes, where both involve minimizing a certain functional. The difference is that compressed plane waves have multi-resolution capabilities, which is achieved by adding the shift-orthogonality constraints. Let $\mathbf{w} = (w_1, \ldots, w_d) \in \mathbb{R}_+^d$ be a basis of a d-dimensional lattice and let Ω be a rectangular box with

$$\Omega = [0, n_1 w_1] \times, \ldots, \times [0, n_d w_d], \ \ (n_1, \ldots, n_d) \in \mathbb{N}^d.$$

Define the lattice

$$\Gamma_{\mathbf{w}} = \{\mathbf{jw} := (j_1 w_1, \ldots, j_d w_d) | 0 \leq j_1 < n_1, \ldots, 0 \leq j_d < n_d\}.$$

The first n basic compressed plane waves (BCPWs,) $\{\psi^k\}_{k=1}^n$, are defined via

$$\psi^1 = \underset{\psi}{\operatorname{argmin}}\ J_\mu[\psi] \text{ s.t. } \langle \psi(\mathbf{x}), \psi(\mathbf{x} - \mathbf{jw})\rangle = \delta_{\mathbf{j},\mathbf{0}}\ \forall \mathbf{jw} \in \Gamma_{\mathbf{w}};$$

$$\psi^k = \underset{\psi}{\operatorname{argmin}}\ J_\mu[\psi] \text{ s.t. } \begin{cases} \langle \psi(\mathbf{x}), \psi(\mathbf{x} - \mathbf{jw})\rangle = \delta_{\mathbf{j},\mathbf{0}}\ \forall \mathbf{jw} \in \Gamma_{\mathbf{w}} \\ \langle \psi(\mathbf{x}), \psi^i(\mathbf{x} - \mathbf{jw})\rangle = 0 \quad \forall i : 0 < i < k, \end{cases}$$

where the functional J_μ is defined by

$$J_\mu[u] = \frac{1}{\mu}||u||_{L^1} + \langle u, -\frac{1}{2}\Delta u\rangle = \frac{1}{\mu}||u||_{L^1} + \frac{1}{2}||\nabla u||_{L^2}.$$

Notice that this functional is a special case of the functional (4.2), with $V \equiv 0$.
The translations of the BCPWs on the lattice $\Gamma_{\mathbf{w}}$ produce all CPWs. Unlike compressed modes that are solved in a single minimization problem, the compressed plane waves are constructed hierarchically. This is similar to the shift-orthogonality

wavelets, but a distinction of CPWs is that it is adapted to the Laplace's operator. Existence of CPW's essentially follows from the observation that shift orthogonality (i.e the constraints in the definition of BCPWs) is preserved under L^2-limits, so that any minimizing sequence has a subsequential limit, which still satisfies the shift orthogonality properties.
The following theorem (see for example [4]) characterizes any orthonormal sequence of shift orthogonal functions:

THEOREM 4.3. *Let $\Omega \subset \mathbb{R}^d$, and the lattice $\Gamma_{\mathbf{w}}$ be defined as above. Let $\{\xi^k\}_{k=1}^{\infty}$ be an orthonormal sequence of shift orthogonal functions. Then, the (Hilbert) space $\mathcal{H} = L^2(\Omega)$ can be written as a direct sum*

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_N,$$

where each $\mathcal{H}_k$ is the Hilbert space spanned by some eigenfunctions for the Laplace's equation in the rectangular box Ω, with the property that if ξ^k has the decomposition

$$\sqrt{N}\xi^k = e_1^k + e_2^k + \ldots + e_N^k, \quad e_j^k \in \mathcal{H}_j,$$

then $E = \{e_j^k | j = 1, 2, \ldots, N; k \in \mathbb{N}\}$ forms an orthonormal system in $\mathcal{H}$. Furthermore, $N = n_1 n_2 \cdots n_d = |\Gamma_{\mathbf{w}}|$.

A detailed discussion of Theorem 4.3, with a characterization of the Hilbert spaces $\mathcal{H}_k$ is given in the appendix.

REMARK 4.4. *For a fixed k, both $\{\xi_{\mathbf{j}}^k | \mathbf{jw} \in \Gamma_{\mathbf{w}}\}$, and $\{e_j^k | j = 1, 2, \ldots, N\}$ form an orthonormal system, and have the same cardinality, hence their linear span agree. Therefore,*

$$\text{span}\{\xi_{\mathbf{j}}^k | \mathbf{jw} \in \Gamma_{\mathbf{w}}; k = 1, \ldots, M\} = \text{span}\{e_j^k | j = 1, \ldots, N; k = 1, \ldots, M\}$$

for any $M \in \mathbb{N} \cup \{\infty\}$.

With this remark and Theorem 4.3 in mind, instead of working with the CPW's $\{\psi_{\mathbf{j}}^k\}$, it is natural to switch to $\{e_j^k\}$ for completeness results. Let's define

$$J_\infty[u] = \frac{1}{2}||\nabla u||_{L^2(\Omega)}^2.$$

Then, we can write

$$J_\mu[u] = J_\infty[u] + \frac{1}{\mu}||u||_{L^1}.$$

As the Hilbert spaces $\mathcal{H}_j$ are the span of some eigenfunctions of the Laplacian, the functional J_∞ satisfies the following linearity property:

$$J_\infty[e_1 + e_2 + \ldots + e_N] = J_\infty[e_1] + J_\infty[e_2] + \ldots J_\infty[e_N], \quad e_j \in \mathcal{H}_j,\ j = 1, 2, \ldots, N.$$

Now, the minimization procedure for $\{e_j^k\}$ becomes

$$\{e_1^1, e_2^1, \ldots, e_N^1\} = \underset{\substack{f_j \in \mathcal{H}_j \\ ||f_j||_2 = 1}}{\text{argmin}} J_\infty[f_1] + \ldots J_\infty[f_N] + \frac{1}{\mu}||f_1 + \ldots + f_N||_1,$$

$$\{e_1^k, e_2^k, \ldots, e_N^k\} = \underset{\substack{f_j \in \{e_j^1, \ldots, e_j^{k-1}\}^\perp \cap \mathcal{H}_j \\ ||f_j||_2 = 1}}{\text{argmin}} J_\infty[f_1] + \ldots J_\infty[f_N] + \frac{1}{\mu}||f_1 + \ldots + f_N||_1.$$

This is analogous to the variational procedure (2.1), except that at each step in the minimization, we obtain multiple functions. Nevertheless, we might view one particular e_j^k, say e_1^k for simplicity, as the solution to the following minimization problem over $\mathcal{H}_1$:

$$e_1^k = \operatorname*{argmin}_{\substack{f\in\{e_1^1,\ldots,e_1^{k-1}\}^\perp\cap\mathcal{H}_1\\ ||f||_2=1}} J_\infty[f] + J_\infty[e_2^k] + \ldots J_\infty[e_N^k] + \frac{1}{\mu}||f + e_2^k + \ldots + e_N^k||_1.$$

We still have the boundedness of the penalty term $P(f) = \frac{1}{\mu}||f + e_2^k + \ldots + e_N^k||_1$, as

$$\begin{aligned}\frac{1}{\mu}||f + e_2^k + \ldots + e_N^k||_1 &\le \frac{1}{\mu}\left(||f||_1 + ||e_2^k||_1 \ldots + ||e_N^k||_1\right)\\ \text{(Cauchy-Schwarz)} \ &\le \frac{|\Omega|^{\frac{1}{2}}}{\mu}\left(||f||_2 + ||e_2^k||_2 \ldots + ||e_N^k||_2\right)\\ &= \frac{N|\Omega|^{\frac{1}{2}}}{\mu},\end{aligned}$$

i.e.

$$||P|| \le \frac{N|\Omega|^{\frac{1}{2}}}{\mu}. \tag{4.6}$$

Therefore, $\{e_j^k\}_{k\in\mathbb{N}}$ could be viewed as the solutions to an analogue of the variational procedure (2.1), in the Hilbert space $\mathcal{H}_j$, with the linear functional being the restriction of $-\Delta$ on $\mathcal{H}_j$.
Let's enumerate the eigenfunctions forming each Hilbert space $\mathcal{H}_j$ as follows

$$\begin{aligned}\mathcal{H}_j &= \operatorname{span}\{\phi_j^k | k = 1, 2, \ldots\},\\ -\frac{1}{2}\Delta\phi_j^k &= \lambda_j^k\phi_j^k\\ \lambda_j^1 &\le \lambda_j^2 \le \ldots.\end{aligned} \tag{4.7}$$

The following theorem establishes an analogue of the Weyl's Law:

Theorem 4.5. *Let λ_j^k be defined as in (4.7), then*

$$\lambda_j^k = \frac{N(2\pi)^d}{2\omega_d|\Omega|}k^{2/d} + o(k^{2/d}),\ \text{as } k\to\infty,$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

A discussion of Theorem 4.5 can be found in the appendix.

We verified that the functions $\{e_j^k\}$ are obtained via a variational procedure analogous to (2.1). We also noted in Remark 4.4 that the spans of $\{e_j^k\}$, and CPWs agree. Therefore, the theory developed in Section 2 applies to CPWs, so that we obtain the following corollaries as direct consequences of the Theorems 2.8, 2.9, and 2.10:

Corollary 4.6 (Corollary to Theorem 2.8). *Let Ω be a bounded interval in $\mathbb{R}$. Then, for any lattice $\Gamma_{\mathbf{w}}$, and parameter μ, the set of compressed plane waves $\{\psi_{\mathbf{j}}^k\}$ defined on Ω forms a complete orthonormal system in $L^2(\Omega)$.*

Proof: Notice that $\{e_j^k\}_{k\in\mathbb{N}} \subset \mathcal{H}_j$ are obtained as the solutions to a variational problem in $\mathcal{H}_j$, analogous to the variational procedure (2.1). Furthermore, since Ω lies inside $\mathbb{R}$, by Theorem 4.5, the corresponding eigenvalues grow super-linearly. Therefore, Theorem 2.8 applies so that $\{e_j^k\}_{k\in\mathbb{N}}$ forms a complete orthonormal system in $\mathcal{H}_j$, for each $j = 1, 2, \ldots, N$. Finally, by Remark 4.4, $\{\psi_{\mathbf{j}}^k\}$ is a complete orthonormal system in $\mathcal{H}$. ☐

Corollary 4.7 (Corollary to Theorem 2.9). *Let Ω be a rectangular domain inside $\mathbb{R}^2$. Then, for any lattice $\Gamma_{\mathbf{w}}$, and parameter μ satisfying*

$$\mu > \frac{|\Omega|^{\frac{3}{2}}}{2\pi}, \tag{4.8}$$

the set of compressed plane waves $\{\psi_{\mathbf{j}}^k\}$ defined on Ω forms a complete orthonormal system in $L^2(\Omega)$.

Proof: Since Ω lies inside $\mathbb{R}^2$, notice by Theorem 4.5 that the corresponding eigenvalues grow linearly, with the linearity constant $\frac{2\pi N}{|\Omega|}$. The proof proceeds analogous to the proof of Corollary 4.6, except that we rather apply Theorem 2.9. The bound for the penalty term is provided in (4.6), so that Theorem 2.9 holds precisely when the bound (4.8) is satisfied. ☐

Corollary 4.8 (Corollary to Theorem 2.10). *Let V_j^m be the subspace generated by the functions $\{e_j^1, e_j^2, \ldots, e_j^m\}$. Then, for any $n \leq m$, we have*

$$\sum_{k=1}^{n} d(\phi_j^k, V_j^m)^2 \leq \frac{mN|\Omega|^{\frac{1}{2}}}{\mu(\lambda_j^{m+1} - \lambda_j^n)},$$

provided $\lambda_j^{m+1} \neq \lambda_j^n$. Defining V^m via

$$V^m = \mathrm{span}\{\psi_{\mathbf{j}}^k | \mathbf{j} \in \mathbb{Z}^d, k = 1, 2, \ldots, m\},$$

we further have

$$\sum_{k\leq m, j\leq N} d(\phi_j^k, V^m)^2 = \frac{mN|\Omega|^{\frac{1}{2}}}{\mu} \sum_{j\leq N} \frac{1}{\lambda_j^{m+1} - \lambda_j^n}.$$

5. Appendix. We will provide an explicit characterization of the Hilbert spaces $\mathcal{H}_j$ in Theorem 4.3. The eigenfunctions of the Laplace's operator in a rectangular domain $\Omega = [0, n_1 w_1] \times, \ldots, \times [0, n_d w_d]$ is given by

$$\phi_{m_1,\ldots,m_d}(\mathbf{x}) = e^{2\pi i\left(\frac{m_1 x_1}{n_1 w_1} + \frac{m_2 x_2}{n_2 w_2} + \ldots + \frac{m_d x_d}{n_d w_d}\right)}.$$

where $(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$. Hence, if we form the lattice

$$\Pi_{\mathbf{w}} = \{\left(\frac{m_1}{n_1 w_1}, \frac{m_2}{n_2 w_2}, \ldots, \frac{m_d}{n_d w_d}\right) | (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d\},$$

then each of the eigenfunctions of the Laplace's operator in the domain Ω can be represented as

$$\phi_\upsilon(\mathbf{x}) = e^{2\pi i \upsilon \cdot \mathbf{x}},\ \upsilon \in \Pi_{\mathbf{w}},$$

with the corresponding eigenvalue $\lambda_\upsilon = 4\pi^2 |\upsilon|^2$. Now, we define

$$\Lambda_{\mathbf{w}} = \{\left(\frac{m_1}{n_1 w_1}, \frac{m_2}{n_2 w_2}, \ldots, \frac{m_d}{n_d w_d}\right) | 0 \leq m_1 < n_1, \ldots, 0 \leq m_d < n_d\}.$$

Each $\rho \in \Lambda_{\mathbf{w}}$ has a natural periodic extension in $\Pi_{\mathbf{w}}$ with respect to $\Gamma_{\mathbf{w}}$. For each $\rho \in \Lambda_{\mathbf{w}}$, we denote such extension by Σ_ρ. Now, the family of Hilbert spaces $\mathcal{H}_j$ in Theorem 4.3 consists of the Hilbert spaces

$$\mathcal{H}_\rho = \operatorname{span}\{\phi_\upsilon | \upsilon \in \Sigma_\rho\}.$$

Since $\Lambda_{\mathbf{w}}$, and $\Gamma_{\mathbf{w}}$ each has cardinality $n_1 n_2 \ldots n_d$; N, the cardinality of the family of Hilbert spaces $\{\mathcal{H}_j\}_{j=1}^N$, satisfies $N = n_1 n_2 \ldots n_d = |\Gamma_{\mathbf{w}}|$, as asserted in Theorem 4.3.

We have already observed that the eigenvalue corresponding to ϕ_υ is $\lambda_\upsilon = 4\pi^2|\upsilon|^2$. Weyl's law in the rectangular domain case can be viewed as the growth of the size of the distance between lattice points and the origin. Therefore, with all these lattice characterization of the eigenfunctions, it is not hard to see that the growth of the eigenvalues corresponding to the eigenfunctions in each of the Hilbert spaces $\mathcal{H}_j$ are given precisely as in Theorem 4.5.

As an illustration, let's consider $\Omega = [0,2] \times [0,3] \subset \mathbb{R}^2$, and $\mathbf{w} = (1,1)$. Then, $\Gamma_{\mathbf{w}}$ becomes

$$\Gamma_{\mathbf{w}} = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}.$$

The eigenfunctions for the Laplace's equation in Ω are given by

$$\phi_{m,n}(x,y) = e^{2\pi i(\frac{mx}{2}+\frac{ny}{3})},$$

so that

$$\Pi_{\mathbf{w}} = \{\left(\frac{m}{2},\frac{n}{3}\right)|m,n \in \mathbb{Z}\}.$$

Now, the finite lattice $\Lambda_{\mathbf{w}}$ becomes

$$\Lambda_{\mathbf{w}} = \{(0,0), \left(0,\frac{1}{3}\right), \left(0,\frac{2}{3}\right), \left(\frac{1}{2},0\right), \left(\frac{1}{2},\frac{1}{3}\right), \left(\frac{1}{2},\frac{2}{3}\right)\}$$

Finally, the decomposition given in Theorem 4.3 becomes

$$L^2(\Omega) = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}._5 \oplus \mathcal{H}_6,$$

where

$$\mathcal{H}_1 = \operatorname{span}\{\phi_{2k,3l}\}_{k,l\in\mathbb{Z}}, \quad \mathcal{H}_2 = \operatorname{span}\{\phi_{2k,3l+1}\}_{k,l\in\mathbb{Z}}, \quad \mathcal{H}_3 = \operatorname{span}\{\phi_{2k,3l+2}\}_{k,l\in\mathbb{Z}},$$
$$\mathcal{H}_4 = \operatorname{span}\{\phi_{2k+1,3l}\}_{k,l\in\mathbb{Z}}, \quad \mathcal{H}_5 = \operatorname{span}\{\phi_{2k+1,3l+1}\}_{k,l\in\mathbb{Z}}, \quad \mathcal{H}_6 = \operatorname{span}\{\phi_{2k+1,3l+2}\}_{k,l\in\mathbb{Z}}.$$

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