Minimization of Transformed L_1 Penalty: Theory, Difference of Convex Function Algorithm, and Robust Application in Compressed Sensing

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Abstract

We study the minimization problem of a non-convex sparsity promoting penalty function, the transformed l_1 (TL1), and its application in compressed sensing (CS). The TL1 penalty interpolates l_0 and l_1 norms through a nonnegative parameter $a \in (0, +\infty)$, similar to l_p with $p \in (0, 1]$. TL1 is known in the statistics literature to enjoy three desired properties: unbiasedness, sparsity and Lipschitz continuity. We first consider the constrained minimization problem and prove the uniqueness of global minimizer and its equivalence to l_0 norm minimization if the sensing matrix A satisfies a restricted isometry property (RIP) and if $a > a^*$, where a^* depends only on A. This result contains the well-known equivalence of l_1 norm and l_0 norm, in the limit $a \to +\infty$. The solution is stable under noisy measurement. For general sensing matrix A, we show that the support set of a local minimizer corresponds to linearly independent columns of A, and recall sufficient conditions for a critical point to be a local minimum. Next, we present difference of convex algorithms for TL1 (DCATL1) in computing TL1-regularized constrained and unconstrained problems in CS. The DCATL1 algorithm involves outer and inner loops of iterations, one time matrix inversion, repeated shrinkage operations and matrix-vector multiplications. For the unconstrained problem, we prove convergence of DCALT1 to a stationary point satisfying the first order optimality condition. Finally in numerical experiments, we identify the optimal value $a = 1$, and compare DCATL1 with other CS algorithms on three classes of sensing matrices: Gaussian random matrices, over-sampled discrete cosine transform matrices (ODCT), and uniformly distributed M-sphere

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matrices (whose columes are uniformly distributed unit vectors on the M-dimensional sphere). Among existing algorithms, the iterated reweighted least squares method based on $L_{1/2}$ norm is the best in sparse recovery for Gaussian matrices, and the DCA algorithm based on $L_1 - L_2$ penalty is the best for ODCT matrices (the most coherent among the three classes). We find that for all three classes of sensing matrices, the performance of DCATL1 algorithm (initiated with L_1 minimization) always ranks near the top (if not the top), and is the *most robust choice* insensitive to RIP (incoherence) of the underlying CS problems.

Index Terms

Transformed l_1 penalty, sparse signal recovery theory, difference of convex function algorithm, convergence analysis, coherent random matrices, compressed sensing, robust recovery.

I. INTRODUCTION

Compressed sensing [4], [8] has generated enormous interest and research activities in mathematics, statistics, signal processing, imaging and information sciences, among numerous other areas. One of the basic problems is to reconstruct a sparse signal under a few linear measurements (linear constraints) far less than the dimension of the ambient space of the signal. Consider a sparse signal $\beta \in \Re^N$, an $M \times N$ sensing matrix A and an observation $y \in \Re^M$, $M \ll N$, such that:

$$y = A\beta + \epsilon,$$

where ϵ is an N-dimensional observation error. If β is sparse enough, it can be reconstructed exactly in the noisefree case and in stable manner in the noisy case provided that the sensing matrix A satisfies certain incoherence or the restricted isometry property (RIP) [4], [8].

The direct approach is l_0 optimization. The constrained formulation is:

$$\min_{\beta \in \Re^N} \|\beta\|_0, \quad s.t. \quad y = A\beta, \tag{1.1}$$

and the unconstrained l_0 regularized version is:

$$\min_{\beta \in \Re^N} \{\|y - A\beta\|_2^2 + \lambda \|\beta\|_0\} \tag{1.2}$$

for some positive parameter λ. Since minimizing L_0 norm is NP-hard [20], many viable alternatives are available. Greedy methods (matching pursuit [19], othogonal matching pursuits (OMP) [27], and regularized OMP (ROMP) [21]) work well if the dimension N is not too large. For the unconstrained problem (1.2), the penalty decomposition method [17] replaces the term $\lambda\|\beta\|_0$ by $\rho_k\|\beta - y\|_2^2 + \lambda\|y\|_0$, and

minimizes over (β, y) for a diverging sequence ρ_k. The variable y allows the iterative hard thresholding procedure.

The relaxation methods replace l_0 norm by a continuous sparsity promoting penalty functions $p(\cdot)$. The minimization takes the form:

$$min \quad P(\beta), \quad s.t. \quad y = A\beta. \tag{1.3}$$

for the constrained problem and

$$\min_{\beta \in \Re_N} \{\|y - A\beta\|_2^2 + \lambda P(\beta)\} \tag{1.4}$$

for the unconstrained problem. Convex relaxation uniquely selects $P(\cdot)$ as the l_1 norm. The resulting problems are known as basis pursuit (LASSO in the over-determined regime [26]). The l_1 algorithms include l_1-magic [4], Bregman and split Bregman methods [34], [14] and yall1 [32]. Theoretically, Candés and Tao introduced RIP condition and used it to establish the equivalent and unique global solution to l_0 minimization via l_1 relxation among other stable recovery results [2], [4], [1].

There are many choices of P for non-convex relaxation. One is the l_p norm ($p \in (0, 1)$) with l_0 equivalence under RIP [6]. The $l_{1/2}$ norm is representative of this class of functions, with the reweighted least squares and half-thresholding algorithms for computation [12], [30], [29]. Near the RIP regime, $l_{1/2}$ penalty tends to have higher success rate of sparse reconstruction than l_1, however, it is not as good as l_1 if the sensing matrix is far away from RIP [15], [33] as we shall see later as well. In the highly non-RIP (coherent) regime, it is recently found that the difference of L_1 and L_2 norm minimization gives the best sparse recovery results [33], [15]. It is therefore of both theoretical and practical interest to find a non-convex penalty that is consistently better than l_1 and always ranks among the top in sparse recovery whether the sensing matrix is RIP or non-RIP (as an all-around champion).

In the statistics literature of vairable selection, Fan and Li [11] advocated for classes of penalty functions with three desired properties: unbiasedness, sparsity and continuity. To help identify such a penalty function denoted by ρ, Fan and Lv [18] proposed the following condition for characterizing unbiasedness and sparsity promoting properties.

Condition 1. *The penalty function $\rho(.)$ satisfies:*

C1.1: *$\rho(t)$ is increasing and concave in $t \in [0, \infty)$;*

C1.2: *$\rho'(t)$ is continuous with $\rho'(0+) \in (0, \infty)$;*

C1.3: *if $\rho(t)$ depends on a positive parameter λ, then $\rho'(t; \lambda)$ is increasing in $\lambda \in (0, \infty)$ and $\rho'(0+)$ is independent of λ.*

It follows that $\rho'(t)$ is positive and decreasing, and $\rho'(0+)$ is the upper bound of $\rho'(t)$. It is shown in [11] that penalties satisfying Condition 1 and $\lim_{t\to\infty}\rho^{'}(t) = 0$ enjoy both unbiasedness and sparsity. Though continuity does not generally hold for this class of penalty functions, a special one parameter family of functions, the so called transformed l_1 functions (TL1) $\rho_a(|t|)$, where $\rho_a(t) = \dfrac{(a+1)t}{a+t}$, $a \in (0, +\infty)$, satisfies all three desired properties [11]. We shall study the minimization of TL1 functions for CS problems, in terms of theory, algorithms and computation. We shall show via numerical results that a difference of convex algorithm of TL1 (DCATL1) is the all around champion among the existing representative CS algorithms based on three classes of coherent random sensing matrices. Same as $L_{1/2}$ regularization [30], [31], there also exists thresholding algorithm for TL1, which is being studied in the companion paper [16].

The rest of the paper is organized as follows. In section 2, we study the elementary inequalities of TL1. In section 3, a RIP condition is given for finding the unique global minimizer of the constrained TL1 model, which is also proven to be stable under noisy measurement. The local minimizers of both the constrained and unconstrained models share the property that they extract independent columns from the sensing matrix A. In section 4, we present two DC algorithms for TL1 optimization (DCATL1), and establish the relevant convergence theory. In section 5, we compare the performance of DCATL1 with l_1, reweighted least squares, $l_1 - l_2$ algorithms in constrained and unconstrained test problems for three classes of matrices: the Gaussian, the oversampled discrete cosine transform (DCT), and the uniformly distributed M-sphere matrices of varying degrees of incoherence. Numerical experiments indicate that our algorithm — DCATL1 is most robust and consistently top ranked while maintaining high sparse recovery rates across all sensing matrices. Concluding remarks are in section 6.

II. Transformed l_1 (TL1) and Preliminaries

The transformed l_1 functions (TL1) are $\rho_a(|t|)$ [18] with:

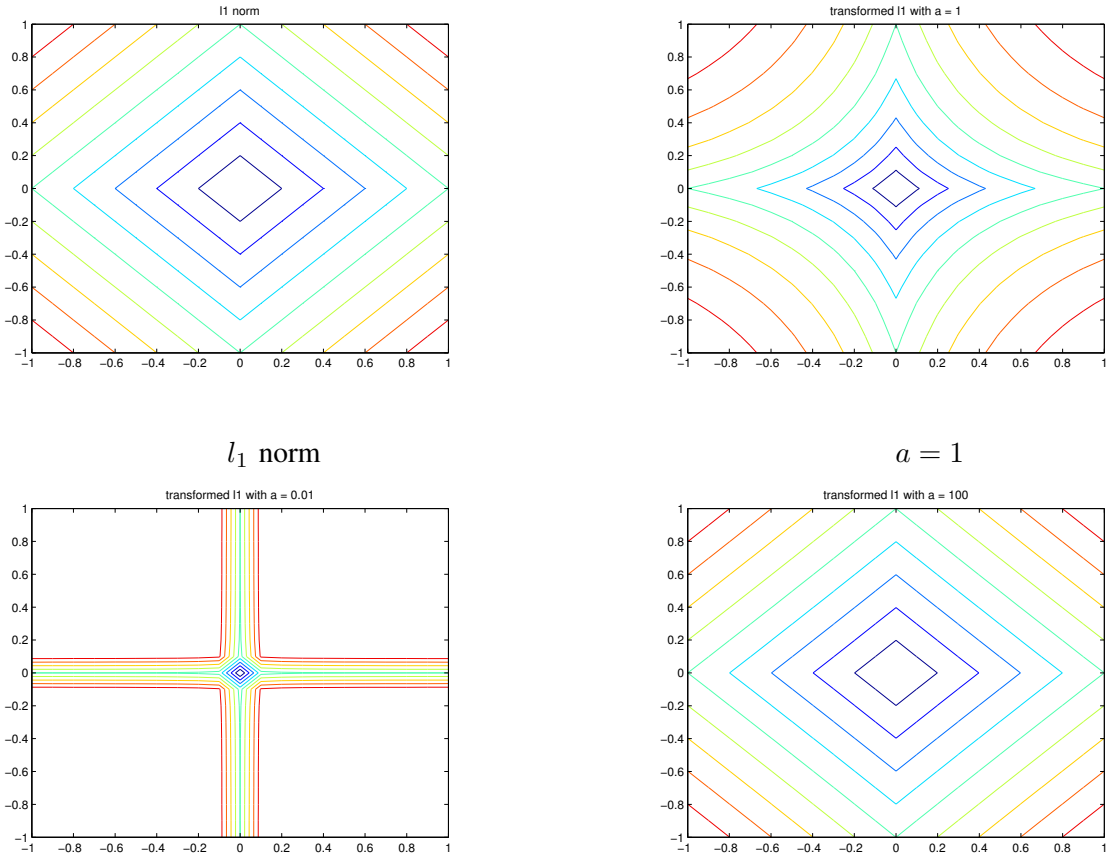
$$\rho_a(t) = \frac{(a+1)t}{a+t}, \;\; t \geq 0, \tag{2.1}$$

where the parameter $a \in (0, +\infty)$. It interpolates the l_0 and l_1 norms as

$$\lim_{a\to 0^+} \rho_a(|t|) = \chi_{\{t\neq 0\}}$$

and

$$\lim_{a\to\infty} \rho_a(|t|) = |t|.$$



l_1 norm $\qquad$ $a = 1$

$a = 0.01$ $\qquad$ $a = 100$

Fig. 1: Level lines of two sparsity promoting measures. Compared with l_1, the level line of TL1 is closer to the axes or those of l_0 when parameter a is small ($a = 0.01$). When a becomes larger ($a = 100$), the level lines converge to those of l_1.

In Fig. (1), we draw level lines of l_1 and TL1. With the adjustment of parameter a, the TL1 can approximate both l_1 and l_0 well. The TL1 function is Lipschitz continuous and satisfies Condition 1, thus enjoying the unbiasedness, sparsity and continuity properties [18].

Let us define:

$$P_a(x) = \sum_{i=1,\ldots N} \rho_a(|x_i|), \tag{2.2}$$

and focus on the constrained TL1 minimization model:

$$\min_{x\in\Re^N} f(x) = \min_{x\in\Re^N} P_a(x) \;\; s.t. \quad Ax = y, \tag{2.3}$$

and the unconstrained TL1-regularized model:

$$\min_{x\in\Re^N} f(x) = \min_{x\in\Re^N} \frac{1}{2}\|Ax-y\|_2^2 + \lambda P_a(x). \tag{2.4}$$

First, we prove elementary inequalities of ρ_a for later use.

Lemma II.1. *For $a \geq 0$, any x_i and x_j in $\Re$, the following inequalities hold:*

$$\rho_a(|x_i+x_j|) \leq \rho_a(|x_i|+|x_j|) \leq \rho_a(|x_i|)+\rho_a(|x_j|) \leq 2\rho_a(\frac{|x_i+x_j|}{2}) \tag{2.5}$$

Proof: Let us prove these inequalities one by one, starting from the left.

1.) According to Condition 1, we know that $\rho_a(|t|)$ is increasing in the variable $|t|$. By triangle inequality $|x_i+x_j| \leq |x_i|+|x_j|$, we have:

$$\rho_a(|x_i+x_j|) \leq \rho_a(|x_i|+|x_j|)$$

2.) Since $\rho_a(|t|) = \dfrac{(a+1)|t|}{a+|t|}$,

$$\begin{aligned}
\rho_a(|x_i|)+\rho_a(|x_j|) &= \frac{(a+1)|x_i|}{a+|x_i|} + \frac{(a+1)|x_j|}{a+|x_j|} \\
&= \frac{a(a+1)(|x_i|+|x_j|+2|x_ix_j|/a)}{a(a+|x_i|+|x_j|+|x_ix_j|/a)} \\
&\geq \frac{(a+1)(|x_i|+|x_j|+|x_ix_j|/a)}{(a+|x_i|+|x_j|+|x_ix_j|/a)} \\
&= \rho_a(|x_i|+|x_j|+|x_ix_j|/a) \\
&\geq \rho_a(|x_i|+|x_j|)
\end{aligned}$$

3.) By concavity of the function ρ_a ,

$$\frac{\rho_a(|x_i|)+\rho_a(|x_j|)}{2} \leq \rho_a\left(\frac{|x_i|+|x_j|}{2}\right).$$

■

Remark II.1. *It follows from Lemma II.1 that the triangular inequality holds for the function $\rho(x) \equiv \rho_a(|x|)$: $\rho(x_i+x_j) = \rho_a(|x_i+x_j|) \leq \rho_a(|x_i|)+\rho_a(|x_j|) = \rho(x_i)+\rho(x_j)$.*
Also we have: $\rho(x) \geq 0$, and $\rho(x) = 0 \Leftrightarrow x = 0$. Our penalty function ρ acts almost like a norm. However, it lacks absolute scalability, or $\rho(cx) \neq |c|\rho(x)$ in general. The next lemma further analyses this inequality.

Lemma II.2.

$$\rho_a(|cx|) = \begin{cases} \leq |c|\rho_a(|x|) & \textit{if } |c| > 1; \\ \geq |c|\rho_a(|x|) & \textit{if } |c| \leq 1. \end{cases} \tag{2.6}$$

Proof:

$$\begin{aligned} \rho_a(|cx|) &= \frac{(a+1)|c||x|}{a+|c||x|} \\ &= |c|\rho_a(|x|)\ \frac{a+|x|}{a+|cx|} \end{aligned}$$

So if $|c| \leq 1$, the factor $\frac{a+|x|}{a+|cx|} \geq 1$. Then $\rho_a(|cx|) \geq |c|\rho_a(|x|)$. Similarly when $|c| > 1$, we have $\rho_a(|cx|) \leq |c|\rho_a(|x|)$. ■

III. Theory of TL1 Minimization

A. RIP Condition for Constrained Model

For the constrained TL1 model (2.3), we present a theory on sparse recovery based on RIP [2].

Suppose β^0 is a sparsest solution for l_0 minimization s.t. $A\beta^0 = y$, while another vector β is defined as

$$\beta = arg \min_{\beta \in \Re_N} \{P_a(\beta)|\ \ A\beta = y\}. \tag{3.1}$$

We addressed the question whether the two vectors β and β^0 are equal to each other. That is to say, under what condition we can recover the sparsest solution β^0 via solving the relaxation problem (2.3).

For an $M \times N$ matrix A and set $T \subset \{1, ..., N\}$, let A_T be the matrix consisting of the column a_j of A for $j \in T$. Similarly for vector x, x_T is a sub-vector, consisting of components indexed from the set T.

Definition III.1. *(Restricted Isometry Constant)*

*For each number s, define the **s-restricted isometry constant** of matrix A as the smallest number δ_s, such that for all subset T with $|T| \leq s$ and all $\beta \in \Re_{|T|}$, the inequality*

$$(1 - \delta_s)\|\beta\|_2^2 \leq \|A_T\beta\|_2^2 \leq (1 + \delta_s)\|\beta\|_2^2,$$

holds.

For a fixed y, the under-determined linear system has infinitely many solutions. Let x be one solution of $Ax = y$. It does not need to be the l_0 or ρ_a minimizer. If $P_a(x) > 1$, we scale y by the positive scalar C as:

$$y_C = \frac{y}{C}; \quad x_C = \frac{x}{C}. \tag{3.2}$$

Now x_C is a solution to the modified problem: $Ax_C = y_C$. When C becomes larger, the number $P_a(x_C)$ is smaller and tends to 0 in the limit $C \to \infty$. Thus, we can find a constant $C \geq 1$, such that $P_a(x_C) \leq 1$. That is to say, for scaled vector x_C, we always have: $P_a(x_C) \leq 1$.

Since the penalty $\rho_a(t)$ is increasing in positive variable t, we have the inequality:

$$\begin{aligned} P_a(x_C) &\leq |T|\rho_a(|x_C|_\infty) \\ &= |T|\rho_a(\tfrac{|x|_\infty}{C}) \\ &= \frac{|T|(a+1)|x|_\infty}{aC+|x|_\infty}, \end{aligned}$$

where $|T|$ is the cardinality of the support set of vector x. For $P_a(x_C) \leq 1$, it suffices to impose:

$$\frac{|T|(a+1)|x|_\infty}{aC+|x|_\infty} \leq 1,$$

or:

$$C \geq \frac{|x|_\infty}{a}\left(a|T| + |T| - 1\right), \tag{3.3}$$

where x is one of solutions for underdetermined system $Ax = y$.

Let β^0 be the l_0 minimizer for the constrained l_0 optimization problem (1.1) with support set T. Due to the scale-invariance of l_0, β_C^0 (defined similarly as above) is a global l_0 minimizer for the modified problem:

$$\min \quad \|\beta\|_0, \quad s.t. \quad y_C = A\beta. \tag{3.4}$$

with the same support set T. Then for the modified ρ_a optimization:

$$\min \quad P_a(\beta), \quad s.t. \quad y_C = A\beta, \tag{3.5}$$

we have the following:

Theorem III.1. *(Exact Sparse Recovery) For a given sensing matrix A, if there is a number $R > |T|$, such that*

$$\delta_R + \frac{R}{|T|}\delta_{R+|T|} < \frac{R}{|T|} - 1, \tag{3.6}$$

then there exists $a^* > 0$, depending only on matrix A, such that for any $a > a^*$, the minimizer β_C for (3.5) is unique and equal to the minimizer β_C^0 in (3.4) for any C satisfying (3.3).

Proof:

The proof generally follows the lines of arguments in [2] and [6], while using special properties of the penalty function ρ_a.

For simplicity, we denote β_C by β and β_C^0 by β^0.

Define the function:

$$f(a) = \frac{a^2}{(a+1)^2}\frac{R}{|T|}(1-\delta_{R+|T|}) - 1 - \delta_R$$

It is continuous and increasing in the parameter a. Note that at $a = 0$, $f(0) = -1 - \delta_M < 0$, and as $a \to \infty$, $f(a) \to \frac{R}{|T|}(1-\delta_{R+|T|}) - 1 - \delta_R > 0$ by (3.6). There exists a constant a^*, such that $f(a^*) = 0$. The number a^* depends on the RIP of matrix A only, and so it is independent of the scalar C.

For $a > a^*$:

$$\delta_R + \frac{a^2}{(a+1)^2}\frac{R}{|T|}\delta_{R+|T|} < \frac{a^2}{(a+1)^2}\frac{R}{|T|} - 1. \tag{3.7}$$

Let $e = \beta - \beta^0$, and we want to prove that the vector $e = 0$. It is clear that, $e_{T^c} = \beta_{T^c}$, since T is the support set of β^0. By the triangular inequality of ρ_a, we have:

$$P_a(\beta^0) - P_a(e_T) = P_a(\beta^0) - P_a(-e_T) \le P_a(\beta_T).$$

Then

$$\begin{aligned} P_a(\beta^0) - P_a(e_T) + P_a(e_{T^c}) &\le P_a(\beta_T) + P_a(\beta_{T^c}) \\ &= P_a(\beta) \\ &\le P_a(\beta^0) \end{aligned}$$

It follows that:

$$P_a(\beta_{T^c}) = P_a(e_{T^c}) \le P_a(e_T). \tag{3.8}$$

Now let us arrange the components at T^c in the order of decreasing magnitude of $|e|$ and partition into L parts: $T^c = T_1 \cup T_2 \cup ... \cup T_L$, where each T_j has R elements (except possibly T_L with less). Also denote $T = T_0$ and $T_{01} = T \cup T_1$. Since $Ae = A(\beta - \beta^0) = 0$, it follows that

$$\begin{aligned} 0 &= \|Ae\|_2 \\ &= \|A_{T_{01}}e_{T_{01}} + \sum_{j=2}^{L} A_{T_j}e_{T_j}\|_2 \\ &\ge \|A_{T_{01}}e_{T_{01}}\|_2 - \sum_{j=2}^{L}\|A_{T_j}e_{T_j}\|_2 \\ &\ge \sqrt{1-\delta_{|T|+R}}\|e_{T_{01}}\|_2 - \sqrt{1+\delta_R}\sum_{j=2}^{L}\|e_{T_j}\|_2 \end{aligned} \tag{3.9}$$

At the next step, we derive two inequalities between the l_2 norm and function P_a, in order to use the inequality (3.8). Since

$$\begin{aligned} \rho_a(|t|) = \frac{(a+1)|t|}{a+|t|} &\le (\frac{a+1}{a})|t| \\ &= (1+\frac{1}{a})|t| \end{aligned}$$

we have:

$$\begin{aligned} P_a(e_{T_0}) &= \sum_{i\in T_0} \rho_a(|e_i|) \\ &\leq (1+\tfrac{1}{a})\|e_{T_0}\|_1 \\ &\leq (1+\tfrac{1}{a})\sqrt{|T|}\ \ \|e_{T_0}\|_2 \\ &\leq (1+\tfrac{1}{a})\sqrt{|T|}\ \ \|e_{T_{01}}\|_2. \end{aligned} \tag{3.10}$$

Now we estimate the l_2 norm of e_{T_j} from above in terms of P_a. It follows from β being the minimizer of the problem (3.5) and the definition of x_C (3.2) that

$$P_a(\beta_{T^c}) \leq P_a(\beta) \leq P_a(x_C) \leq 1,$$

where x_C is defined in (3.2) : a scaled vector from any solution of underdetermined system $Ax = b$.

For each $i \in T^c$, $\rho_a(\beta_i) \leq P_a(\beta_{T^c}) \leq 1$. Also since

$$\begin{aligned} & \frac{(a+1)|\beta_i|}{a+|\beta_i|} \leq 1 \\ \Leftrightarrow\ & (a+1)|\beta_i| \leq a+|\beta_i| \\ \Leftrightarrow\ & |\beta_i| \leq 1 \end{aligned} \tag{3.11}$$

we have

$$|e_i| = |\beta_i| \leq \frac{(a+1)|\beta_i|}{a+|\beta_i|} = \rho_a(|\beta_i|) \quad \text{for every } i \in T^c.$$

Using the property that $\rho_a(t)$ is increasing for non-negative variable $t \geq 0$, and that $|e_i| \leq |e_k|$ for each $i \in T_j$ and $k \in T_{j-1}$, $j = 2, 3, ..., L$, we have

$$\begin{aligned} & |e_i| \leq \rho_a(|e_i|) \leq P_a(e_{T_{j-1}})/R \\ \Rightarrow\ & \|e_{T_j}\|_2^2 \leq \frac{P_a(e_{T_{j-1}})^2}{R} \\ \Rightarrow\ & \|e_{T_j}\|_2 \leq \frac{P_a(e_{T_{j-1}})}{R^{1/2}} \\ \Rightarrow\ & \sum_{j=2}^{L} \|e_{T_j}\|_2 \leq \sum_{j=1}^{L} \frac{P_a(e_{T_j})}{R^{1/2}} \end{aligned} \tag{3.12}$$

Finally, plug (3.10) and (3.12) into inequality (3.9) to get:

$$\begin{aligned} 0 &\geq \sqrt{1-\delta_{|T|+R}}\frac{a}{(a+1)|T|^{1/2}}P_a(e_T) - \sqrt{1+\delta_R}\frac{1}{R^{1/2}}P_a(e_T) \\ &\geq \frac{P_a(e_T)}{R^{1/2}}\left(\sqrt{1-\delta_{R+|T|}}\frac{a}{a+1}\sqrt{\frac{R}{|T|}} - \sqrt{1+\delta_R}\right) \end{aligned} \tag{3.13}$$

By (3.7), the factor $\sqrt{1-\delta_{R+|T|}}\frac{a}{a+1}\sqrt{\frac{R}{|T|}} - \sqrt{1+\delta_R}$ is strictly positive, hence $P_a(e_T) = 0$, and $e_T = 0$. Also by inequality (3.8), $e_{T^c} = 0$. We have proved that $\beta_C = \beta_C^0$. The equivalence of (3.5) and (3.4) holds. ■

Remark III.1. *Theorem III.1 contains a sufficient condition for β to be the unique global minimizer of l_0 optimization problem (1.1). On the other hand, with a choice of $R = 3|T|$, our condition (3.6) becomes:*

$$\delta_{3|T|} + 3\delta_{4|T|} < 2 \tag{3.14}$$

which is exactly the condition (1.6) of Theorem 1.1 in [2]. This is consistent with the fact that when parameter a goes to $+\infty$, our penalty function ρ_a recovers the l_1 norm.

Next, we prove that TL1 recovery is stable under noisy measurements, i.e.,

$$\min \quad P_a(\beta), \quad s.t. \quad \|y_C - A\beta\|_2 \leq \tau. \tag{3.15}$$

Theorem III.2. *(Stable Recovery Theory) Under the same RIP condition and a^* in theorem III.1, for $a \geq a^*$, the solution β_C^n for optimization (3.15) satisfies*

$$\|\beta_C^n - \beta_C^0\|_2 \leq D\tau,$$

for some constant D depending only on the RIP condition.

Proof: Set $n = A\beta - y_C$. In the proof, we use three related notations listed below for clarity:

- $\beta_C^n \Rightarrow$ optimal solution for the noisy constrained problem (3.15);
- $\beta_C \Rightarrow$ optimal solution for the noiseless constrained problem (3.5);
- $\beta_C^0 \Rightarrow$ optimal solution for the l_0 problem (3.4).

Let T be the support set of β_C^0, i.e., $T = supp(\beta_C^0)$, and vector $e = \beta_C^n - \beta_C^0$. Following the proof of theorem III.1, we obtain:

$$\sum_{j=2}^{L} \|e_{T_j}\|_2 \leq \sum_{j=1}^{L} \frac{P_a(e_{T_j})}{R^{1/2}} = \frac{P_a(e_{T^c})}{R^{1/2}}$$

and

$$\|e_{T_{01}}\|_2 \geq \frac{a}{(a+1)\sqrt{|T|}} P_a(e_T).$$

Further, due to the inequality $P_a(\beta_{T^c}^n) = P_a(e_{T^c}) \leq P_a(e_T)$ and inequality (3.9), we get

$$\|Ae\|_2 \geq \frac{P_a(e_T)}{R^{1/2}} C_\delta,$$

where $C_\delta = \sqrt{1-\delta_{R+|T|}} \dfrac{a}{a+1}\sqrt{\dfrac{R}{|T|}} - \sqrt{1+\delta_R}$.

By the initial assumption on the size of observation noise, we have

$$\|Ae\|_2 = \|A\beta_C^n - A\beta_C^0\|_2 = \|n\|_2 \leq \tau, \tag{3.16}$$

so we have: $P_a(e_T) \leq \dfrac{\tau R_{1/2}}{C_\delta}$.

On the other hand, we know that $P_a(\beta_C) \leq 1$ and β_C is in the feasible set of the noisy problem (3.15). Thus we have the inequality: $P_a(\beta_C^n) \leq P_a(\beta_C) \leq 1$. By (3.11), $\beta_{C,i}^n \leq 1$ for each i. So, we have

$$|\beta_{C,i}^n| \leq \rho_a(|\beta_{C,i}^n|). \tag{3.17}$$

It follows that

$$\begin{aligned}
\|e\|_2 &\leq \|e_T\|_2 + \|e_{T^c}\|_2 = \|e_T\|_2 + \|\beta_{C,T^c}^n\|_2 \\
&\leq \frac{\|A_T e_T\|_2}{\sqrt{1-\delta_T}} + \|\beta_{C,T^c}^n\|_1 \\
&\leq \frac{\|A_T e_T\|_2}{\sqrt{1-\delta_T}} + P_a(\beta_{C,T^c}^n) = \frac{\|A_T e_T\|_2}{\sqrt{1-\delta_T}} + P_a(e_{T^c}) \\
&\leq \frac{\tau}{\sqrt{1-\delta_R}} + P_a(e_T) \leq D\tau.
\end{aligned}$$

where constant number D depends on δ_R and $\delta_{R+|T|}$. The second inequality uses the definition of RIP, while the first inequality in the last row comes from (3.16). ■

B. Sparsity of Local Minimizer

We study properties of local minimizers of both the constrained problem (2.3) and the unconstrained model (2.4). As in l_p and l_{1-2} minimization [33], [15], a local minimizer of TL1 minimization extracts linearly independent columns from the sensing matrix A, with no requirement for A to satisfy RIP. Reversely, we state additional conditions on A for a stationary point to be a local minimizer besides the linear independence of the corresponding column vectors.

Theorem III.3. *(Local minimizer of constrained model)*

Suppose x^* is a local minimizer of the constrained problem (2.3) and $T^* = supp(x^*)$, then A_{T^*} is of full column rank, i.e. columns of A_{T^*} are linearly independent.

Proof:

Here we argue by contradiction. Suppose that the column vectors of A_{T^*} are not linearly independent, then there exists non-zero vector $v \in ker(A)$, such that $supp(v) \subseteq T^*$. For any neighbourhood of x^*, $N(x^*, r)$, we can scale v so that:

$$\|v\|_2 \leq \min\{r; \quad |x_i^*|, \ i \in T^*.\} \tag{3.18}$$

Next we define:

$$\xi_1 = x^* + v;$$
$$\xi_2 = x^* - v,$$

so both ξ_1 and ξ_2, $\in \mathscr{N}(x^*, r)$, and $x^* = \frac{1}{2}(\xi_1 + \xi_2)$. On the other hand, from $supp(v) \subseteq T^*$, we have that $supp(\xi_1), supp(\xi_2) \subseteq T^*$. Moreover, due to the inequality (3.18), vectors x^*, x_1, and x_2 are located in the same orthant, i.e. $sign(x_i^*) = sign(\xi_{1,i}) = sign(\xi_{2,i})$, for any index i. It means that $\frac{1}{2}|\xi_1| + \frac{1}{2}|\xi_2| = \frac{1}{2}|\xi_1 + \xi_2|$. Since the penalty function $P_a(t)$ is strictly concave for non-negative variable t,

$$\begin{aligned} \tfrac{1}{2}P_a(\xi_1) + \tfrac{1}{2}P_a(\xi_2) &= \tfrac{1}{2}P_a(|\xi_1|) + \tfrac{1}{2}P_a(|\xi_2|) \\ &< P_a(\tfrac{1}{2}|\xi_1| + \tfrac{1}{2}|\xi_2|) = P_a(\tfrac{1}{2}|\xi_1 + \xi_2|) = P_a(x^*). \end{aligned}$$

So for any fixed r, we can find two vectors ξ_1 and ξ_2 in the neighbourhood $\mathscr{N}(x^*, r)$, such that $\min\{P_a(\xi_1), P_a(\xi_2)\} \leq \frac{1}{2}P_a(\xi_1) + \frac{1}{2}P_a(\xi_2) < P_a(x^*)$. Both vectors are in the feasible set of the constrained problem (2.3), in contradiction with the assumption that x^* is a local minimizer. ■

The same property also holds for local minimizers of the unconstrained problem (2.4), because a local minimizer of unconstrained problem is also a local minimizer for a constrained optimization [33] and [2]. We skip the details and state the result below.

Theorem III.4. *(Local minimizer of unconstrained model)*

Suppose x^* is a local minimizer of the unconstrained problem (2.4) and $T^* = supp(x^*)$, then columns of A_{T^*} are linearly independent.

From the two theorems above, we conclude the following facts:

Corollary III.1. (a) *For any local minimizer of (2.3) or (2.4), e.g. x^*, the sparsity of x^* is at most rank(A);*

(b) *The number of local minimizers is finite, both for problem (2.3) and (2.4).*

In [18], the authors studied sufficient conditions of a strict local minimizer for minimizing any penalty functions satisfying Condition 1. Here we specialize and simplify it for our concave TL1 function ρ_a.

For a convex function $h(\cdot)$, the sub-differential $\partial h(x)$ is the closed convex set:

$$\partial h(x) := \{y \in \Re^N : h(z) \geq h(z_0) + \langle z - z_0, y\rangle, \quad \forall z \in \Re^N\}, \tag{3.19}$$

which generalizes the derivative in the sense that h is differentiable at x if and only if $\partial h(x)$ is a singleton or $\{\nabla h(x)\}$.

The TL1 penalty function $p_a(\cdot)$ can be written as a difference of two convex functions:

$$\begin{aligned}\rho_a(t) &= \frac{(a+1)t}{a+t} \\ &= \frac{(a+1)t}{a} - \left(\frac{(a+1)t}{a} - \frac{(a+1)t}{a+t} \right) \\ &= \frac{(a+1)t}{a} - \frac{(a+1)t^2}{a(a+t)}.\end{aligned} \tag{3.20}$$

Thus the general derivative of function $P_a(\cdot)$, as a combination of two convex derivatives. Denote: $\partial P_a(x)$: $\partial P_a(x) = \prod_{i=1}^{N} I_i^a \subset \Re_N$, where

$$I_i^a = \begin{cases} sgn(x_i)\frac{(a+1)}{a+|x_i|} - \frac{(a+1)x_i}{(a+|x_i|)^2}, \text{ if } \ i \in supp(x), \\ [-1,1]\frac{(a+1)}{a+|x_i|} - \frac{(a+1)x_i}{(a+|x_i|)^2} = [-\frac{(a+1)}{a+|x_i|} - \frac{(a+1)x_i}{(a+|x_i|)^2}, \frac{(a+1)}{a+|x_i|} - \frac{(a+1)x_i}{(a+|x_i|)^2}], \text{otherwise.} \end{cases} \tag{3.21}$$

Also $\partial\|x\|_1 = \prod_{i=1}^{N} I_i$, where I_i is defined as:

$$I_i = \begin{cases} sgn(x_i), & \text{if } \ i \in supp(x), \\ [-1,1], & \text{otherwise.} \end{cases} \tag{3.22}$$

Thus we can rewrite $\partial P_a(x)$: $\partial P_a(x) = \prod_{i=1}^{N} \left(I_i \frac{(a+1)}{a+|x_i|} - \frac{(a+1)x_i}{(a+|x_i|)^2} \right) = \partial\|x\|_1 . * \partial P_a(|x|)$, where ". * " is the element-wise product of two vectors.

Definition III.2. *(Maximum concavity and local concavity of the penalty function)*

For a penalty function ρ, we define its maximum concavity as:

$$\kappa(\rho) = \sup_{t_1,t_2 \in (0,\infty), t_1<t_2} -\frac{\rho^{'}(t_2) - \rho^{'}(t_1)}{t_2 - t_1} \tag{3.23}$$

and its local concavity of ρ at a point $b = (b_1, b_2, ..., b_R)^t \in \Re^R$ with $\|b\|_0 = R$ as:

$$\kappa(\rho; b) = \lim_{\epsilon \to 0+} \max_{1 \le j \le R} \sup_{t_1,t_2 \in (|b_j|-\epsilon, |b_j|+\epsilon), t_1<t_2} -\frac{\rho^{'}(t_2) - \rho^{'}(t_1)}{t_2 - t_1} \tag{3.24}$$

Let us recabibitemll a set of sufficient conditions on the (strict) local minimizer of (2.4):

Condition 2. *For vector $\beta \in \Re^N$, $\lambda > 0$, and $T = supp(\beta)$:*

C2.1: *Matrix $Q = A_T^t A_T$ is non-singular, i.e. matrix A_T is column independent;*

C2.2: *for vector $z = \frac{1}{\lambda} A^t(y - A\beta)$, $\|z_{T^c}\|_\infty < \rho_a^{'}(0+) = \frac{a+1}{a}$;*

C2.3: *vector β_T satisfies the stationary point equation: $\beta_T = Q^{-1}A_T^t y - \lambda Q^{-1} w_T$, where $w_T \in \partial P_a(\beta)$;*

C2.4: $\lambda_{min}(Q) \geq \lambda\,\kappa(\rho_a;\beta_T)$, *where* $\lambda_{min}(\cdot)$ *denotes the smallest eigenvalues of a given symmetric matrix.*

Under Condition 2, Lv and Fan [18] showed the following:

Theorem III.5. *If a vector* $\beta \in \Re^N$ *satisfies all four requirements in Condition 2, then* β *is a local minimizer of problem (2.4).*
Furthermore, if the inequality of (C2.4) is strict, then the vector β *is a strict local minimizer.*

Proof: Here we gave a simplified proof of the theorem to illustrate each of the conditions (C2.1)-(C2.4). The objective function is:

$$\ell(x) = 2^{-1}\|A\,x - y\|_2^2 + \lambda P_a(x) = 2^{-1}\|Ax - y\|_2^2 + \lambda\sum_{j=1}^{N}\rho_a(x_j).$$

Let us define a subspace of $\Re^N$ as: $S = \{\beta \in \Re^N | \beta_{T^c} = 0\}$.

First, by Condition 1, (C2.4), and the definition of $\kappa(\rho_a;\beta_T)$, the objective function $\ell(\cdot)$ is convex in $\mathscr{N}(\beta, r_0) \cap S$, where r_0 is a positive number (the radius). By equation (C2.3), β_T is a local minimizer of $\ell(\cdot)$ in S.

Next, we show that the sparse vector β is indeed a local minimizer of $\ell(\cdot)$ in $\Re^N$. Because of inequality (C2.2), there exists $\delta \in (0,\infty)$, a positive number $r_1 < \delta$, such that $\rho_a^{'}(\delta) \in (0, \rho_a^{'}(0+)]$, and for any vector $x \in \mathscr{N}(\beta, r_1)$,

$$\|w_{T^c}\|_\infty < \rho_a^{'}(\delta),$$

where $w = \lambda^{-1}A^t(y - A\beta)$. We can further shrink r_1 if necessary so that $r_1 < r_0$, $\mathscr{N}(\beta, r_1) \subseteq \mathscr{N}(\beta, r_0)$. By the mean value theorem, $\forall \beta_1 \in \mathscr{N}(\beta, r_1)$,

$$\ell(\beta_1) = \ell(\beta_2) + \bigtriangledown^t\ell(\beta_0)(\beta_1 - \beta_2),$$

where β_2 is the projection of β_1 onto S and β_0 lies on the line segment joining β_1 and β_2. Since,

$$(\beta_1 - \beta_2)_T = 0, \quad \beta_0 \in \mathscr{N}(\beta, r_1) \;\; and \;\; sign(\beta_{0,T^c}) = sign(\beta_{1,T^c}),$$

we have the following inequality:

$$\begin{aligned}
\ell(\beta_1) - \ell(\beta_2) &= \partial\ell(\beta_0)_{T^c} * \beta_{1,T^c} \\
&= [A_{T^c}^t A\beta_0 - A_{T^c}^t y + \lambda P_a^{'}(\beta_{0,T^c})]^t\beta_{1,T^c} \\
&= -\lambda[\lambda^{-1}A_{T^c}^t(y - A\beta_0)]^t\beta_{1,T^c} + \lambda\left(\partial\|\beta_{0,T^c}\|_1 .* P_a^{'}(|\beta_{0,T^c}|)\right) * \beta_{1,T^c} \\
&> -\lambda\rho_a^{'}(\delta)\|\beta_{1,T^c}\|_1 + \lambda P_a^{'}(|\beta_{0,T^c}|) * |\beta_{1,T^c}| \\
&\geq -\lambda\rho_a^{'}(\delta)\|\beta_{1,T^c}\|_1 + \lambda\rho_a^{'}(\delta)\|\beta_{1,T^c}\|_1 = 0,
\end{aligned}$$

where we also used the property of generalized derivative $\partial\|.\|_1$, and $*$ stands for vector cross product.

So for any $\beta_1 \in \mathscr{N}(\beta, r_1)$, $\ell(\beta_1) > \ell(\beta_2)$. Since β_2 is a projection on S and it belongs to the ball $\mathscr{N}(\beta, r_1) \subseteq \mathscr{N}(\beta, r_0)$,

$$\ell(\beta_1) > \ell(\beta_2) \geq \ell(\beta).$$

The (C2.4) is only used in the first part of the proof. If we has the strict inequality $\lambda_{min}(Q) > \lambda\kappa(\rho_a; \beta_T)$, then β_T is a strict local minimizer in S, as the function $\ell(\cdot)$ is strictly convex in the intersection $\mathscr{N}(\beta, r_0) \cap S$. Further, the same proof shows that β is a strict local minimizer in $\Re^N$.

■

IV. DC Algorithm for Transformed l_1 Penalty

A. DC Programming

Generally speaking, a DC program is an optimization problem of the form:

$$\alpha = \inf\{f(x) = g(x) - h(x) : x \in \Re^d\} \qquad (P_{dc})$$

where g, h are lower semi-continuous proper convex functions on $\Re^d$, [25], [22]

The difference of convex function algorithm (DCA) approximates h, at the current point x^l of iteration, by its affine minorization defined by

$$h_l(x) = h(x^l) + \langle x - x^l, y^l \rangle, \quad y^l \in \partial h(x^l)$$

to produce a convex program in the form:

$$\inf\{g(x) - h_l(x) : x \in \Re^d\} \Leftrightarrow \inf\{g(x) - \langle x, y^l \rangle : x \in \Re^d\}$$

where the optimal solution is denoted as x^{l+1}.

B. Algorithm for Unconstrained Model — DCATL1

Consider the following unconstrained optimization problem (2.4):

$$\min_{x \in \Re^N} f(x) = \min_{x \in \Re^N} \frac{1}{2}\|Ax - y\|_2^2 + \lambda P_a(x)$$

where

$$P_a(x) = \sum_{i=1,\ldots N} \rho_a(|x_i|).$$

The DCA for this problem is:

$$\left\langle \begin{array}{rcl} f(x) & = & g(x) - h(x) \\ g(x) & = & \frac{1}{2}\|Ax-y\|_2^2 + c\|x\|_2^2 + \lambda\frac{(a+1)}{a}\|x\|_1 \\ h(x) & = & \lambda\frac{(a+1)}{a}\|x\|_1 - \lambda P_a(x) + c\|x\|_2^2 \end{array} \right. \tag{4.1}$$

Define:

$$v^n = \lambda\frac{a+1}{a}sign(x^n) - \lambda sign(x^n)\frac{(a+1)}{a+|x^n|} + \lambda\frac{(a+1)x^n}{(a+|x^n|)^2} + 2cx^n, \tag{4.2}$$

then $v^n \in \partial h(x^n)$.

Algorithm 1: DCA for unconstrained transformed $l1$ penalty minimization

Define: $\epsilon_{outer} > 0, \quad \epsilon_{inner} > 0$

Initialize: $x^0 = 0, n = 0$

while $|x^{n+1} - x^n| > \epsilon_{outer}$ **do**

$v^n = \lambda\frac{a+1}{a}sign(x^n) - \lambda sign(x^n)\frac{(a+1)}{a+|x^n|} + \lambda\frac{(a+1)x^n}{(a+|x^n|)^2} + 2cx^n$

$x^{n+1} = arg\min_{x\in\Re^N}\{\frac{1}{2}\|Ax-y\|_2^2 + c\|x\|^2 + \lambda\frac{(a+1)}{a}\|x\|_1 - \langle x, v^n\rangle\}$

then $n+1 \to n$

end while

At each step, we need to solve a strongly convex l_1-regularized sub-problem, which is:

$$\begin{array}{rcl} x^{n+1} & = & arg\min_{x\in\Re^N}\{\frac{1}{2}\|Ax-y\|_2^2 + c\|x\|^2 + \lambda\frac{(a+1)}{a}\|x\|_1 - \langle x, v^n\rangle\} \\ & = & arg\min_{x\in\Re^N}\{\frac{1}{2}x^t(A^tA+2cI)x - \langle x, v^n + A^ty\rangle \ + \lambda\frac{(a+1)}{a}\|x\|_1\} \end{array} \tag{4.3}$$

We now employ the Alternating Direction Method of Multipliers (ADMM). The sub-problem is recast as:

$$\begin{array}{rl} x^{n+1} = arg\min_{x\in\Re^N}\{ & \frac{1}{2}x^t(A^tA+2cI)x - \langle x, v^n + A^ty\rangle \ + \lambda\frac{(a+1)}{a}\|z\|_1 \ \} \\ & s.t. \quad x - z = 0. \end{array} \tag{4.4}$$

Define the augmented Lagrangian function as:

$$L(x,z,u) = \frac{1}{2}x^t(A^tA+2cI)x - \langle x, v^n + A^ty\rangle \ + \lambda\frac{(a+1)}{a}\|z\|_1 + \frac{\delta}{2}\|x-z\|_2^2 + u^t(x-z),$$

where u is the Lagrange multiplier, and $\delta > 0$ is a penalty parameter. The ADMM consists of three iterations:

$$\left\{ \begin{array}{rcl} x^{n+1} & = & arg\min_x \quad L(x, z^n, u^n); \\ z^{n+1} & = & arg\min_z \quad L(x^{n+1}, z, u^n); \\ u^{n+1} & = & u^n + x^{n+1} - z^{n+1}. \end{array} \right.$$

The first two steps have closed-form solutions and are described in Algorithm 2, where $shink(.,.)$ is a soft-thresholding operator given by:

$$shrink(x,r)_i = sign(x_i)\max\{|x_i| - r, 0\}.$$

Algorithm 2: ADMM for subproblem (4.3)

Initial guess: x^0, z^0 and u^0

while not converged **do**

$x^{n+1} := (A^tA + 2cI + \delta I)^{-1}(A^ty - v + \delta z^n - u^n)$

$z^{n+1} := shrink(\ \ x^{n+1} + u^n, \frac{a+1}{a\delta}\lambda\ \ )$

$u^{n+1} := u^n + x^{n+1} - z^{n+1}$

then $n \to n+1$

end while

C. Convergence Theory for Unconstrained DCATL1

We present a convergence theory for the Algorithm 1 (DCATL1). We prove that the sequence $\{f(x^n)\}$ is decreasing and convergent, while the sequence $\{x^n\}$ is bounded under some requirement on λ. Its sub-limit vector x^* is a stationary point satisfying the first order optimality condition. Our proof is based on the convergent theory of DCA for $l_1 - l_2$ penalty function [33] besides the general results [23] [24].

Definition IV.1. *(Modulus of strong convexity) For a convex function $f(x)$, the modulus of strong convexity of f on $\Re^N$, denoted as $m(f)$, is defined by*

$$m(f) := sup\{\rho > 0 : f - \frac{\rho}{2}\|.\|_2^2 \text{ is convex on } \Re^N\}.$$

Let us recall a useful inequality from [24] concerning the sequence $f(x^n)$.

Lemma IV.1. *Suppose that $f(x) = g(x) - h(x)$ is a D.C. decomposition, and the sequence $\{x^n\}$ is generated by (4.3), then*

$$f(x^n) - f(x^{n+1}) \geq \frac{\rho(g) + \rho(h)}{2}\|x^{n+1} - x^n\|_2^2.$$

Here is the convergence theory for our unconstrained Algorithm 1 — DCATL1. The objective function is : $f(x) = \frac{1}{2}\|Ax - y\|_2^2 + \lambda P_a(x)$.

Theorem IV.1. *The sequences $\{x^n\}$ and $\{f(x^n)\}$ in Algorithm 1 satisfy: (I)*

1) *Sequence $\{f(x^n)\}$ is decreasing and convergent.*
2) *$\|x^{n+1}-x^n\|_2 \to 0$ as $n \to \infty$. If $\lambda > \dfrac{\|y\|_2^2}{a+1}$, $\{x^n\}_{n=1}^{\infty}$ is bounded.*
3) *Any subsequential limit vector x^* of $\{x^n\}$ satisfies the first order optimality condition:*

$$0 \in A^T(Ax^*-y) + \lambda \partial P_a(x^*), \tag{4.5}$$

implying that x^* is a stationary point of (2.4).

Proof:

1) By the definition of $g(x)$ and $h(x)$ in equation (4.1), it is easy to see that:

$$\begin{aligned} \rho(g) &\geq 2c; \\ \rho(h) &\geq 2c. \end{aligned}$$

By Lemma IV.1, we have:

$$\begin{aligned} f(x^n)-f(x^{n+1}) &\geq \frac{\rho(g)+\rho(h)}{2}\|x^{n+1}-x^n\|_2^2 \\ &\geq 2c\|x^{n+1}-x^n\|_2^2. \end{aligned}$$

So the sequence $\{f(x^n)\}$ is decreasing and non-negative, thus convergent.

2) It follows from the convergence of $\{f(x^n)\}$ that:

$$\|x^{n+1}-x^n\|_2^2 \leq \frac{f(x^n)-f(x^{n+1})}{2c} \to 0, \quad as\ n \to \infty.$$

If $y = 0$, since the initial vector $x^0 = 0$, and the sequence $\{f(x^n)\}$ is decreasing, we have $f(x^n) = 0$, $\forall n \geq 1$. So $x^n = 0$, and the boundedness holds.

Consider non-zero vector y. Then

$$f(x^n) = \frac{1}{2}\|Ax^n-y\|_2^2 + \lambda P_a(x^n) \leq f(x^0) = \|y\|_2^2,$$

So $\lambda P_a(x^n) \leq \|y\|_2^2$, implying $\lambda \rho_a(\|x^n\|_\infty) \leq \|y\|_2^2$, or:

$$\frac{\lambda(a+1)||x^n\|_\infty}{a+\|x^n\|_\infty} \leq \|y\|_2^2.$$

So if $\lambda > \dfrac{\|y\|_2^2}{a+1}$, then

$$|x^n|_\infty \leq \frac{a\|y\|_2^2}{\lambda(a+1)-\|y\|_2^2},$$

or the sequence $\{x^n\}_{n=1}^{\infty}$ is bounded.

3) Let $\{x^{n_k}\}$ be a subsequence of $\{x^n\}$ which converges to x^*. So the optimality condition at the n_k-th step of Algorithm 1 is expressed as:

$$\begin{aligned} 0 \in \quad & A^T(Ax^{n_k} - y) + 2c(x^{n_k} - x^{n_k-1}) + \lambda(\tfrac{a+1}{a})\partial\|x^{n_k}\|_1 \\ & -\lambda(\tfrac{a+1}{a})\partial\|x^{n_k-1}\|_1 + \lambda\partial P_a(x^{n_k-1}). \end{aligned} \tag{4.6}$$

Since $\|x^{n+1} - x^n\|_2 \to 0$ as $n \to \infty$ and x^{n_k} converges to x^*, as shown in Proposition 3.1 of [33], we have that for sufficiently large index n_k,

$$\begin{aligned} &\partial\|x^{n_k}\|_1 \subseteq \partial\|x^*\|_1 \\ &\partial\|x^{n_k-1}\|_1 \subseteq \partial\|x^*\|_1 \\ &\partial P_a(x^{n_k-1}) \subseteq \partial P_a(x^*). \end{aligned}$$

Letting $n_k \to \infty$ in (4.6), we have $0 \in A^T(Ax^* - y) + \lambda\partial P_a(x^*)$.

■

Remark IV.1. *The above theorem says that the sub-sequence limit x^* is a stationary point for (2.4). Let $T^* = supp(x^*)$, there exists vector $w \in \partial P_a(x^*)$, s.t.*

$$\begin{aligned} 0 &= A^t(Ax^* - y) + \lambda w \\ \Rightarrow \quad 0 &= A^t_{T^*}(A_{T^*}x^*_{T^*} - y) + \lambda w_{T^*} \\ \Rightarrow \quad 0 &= Qx^*_{T^*} - A^t_{T^*}y + \lambda w_{T^*} \\ \Rightarrow \quad x^*_{T^*} &= Q^{-1}A^t_{T^*}y - \lambda Q^{-1}w_{T^*}. \end{aligned} \tag{4.7}$$

So (C2.3) is automatically satisfied by x^*. If (C2.1)-(C2.3) are also satisfied, the limit point x^* is a local minimizer of (2.4).

D. Algorithm for Constrained Model

We also use DCA to solve the constrained problem:

$$\begin{aligned} &\min_{x\in\Re^N} P_a(x) \;\; s.t. \quad Ax = y \\ &\Leftrightarrow \\ &\min_{x\in\Re^N} \frac{a+1}{a}\|x\|_1 - \{\frac{a+1}{a}\|x\|_1 - P_a(x)\} \;\; s.t. \quad Ax = y \end{aligned}$$

Choose vector $z \in \frac{a+1}{a}\partial\|x\|_1 - \partial P_a(x)$, then the convex sub-problem is:

$$\min_{x\in\Re^N} \frac{a+1}{a}\|x\|_1 - \langle z, x\rangle \;\; s.t. \quad Ax = y \tag{4.8}$$

To solve (4.8), we introduce two Lagrange multipliers u, v and define an augmented Lagrangian:

$$L_\delta(x, w, u, v) = \frac{a+1}{a}\|w\|_1 - z^t x + u^t(x - w) + v^t(Ax - y) + \frac{\delta}{2}\|x - w\|^2 + \frac{\delta}{2}\|Ax - y\|^2,$$

where $\delta > 0$. ADMM finds a saddle point (x^*, w^*, u^*, v^*), such that:

$$L_\delta(x^*, w^*, u, v) \leq L_\delta(x^*, w^*, u^*, v^*) \leq L_\delta(x, w, u^*, v^*) \quad \forall x, w, u, v$$

by alternately minimizing L_δ with respect to x, minimizing with respect to y and updating the dual variables u and v. The saddle point x^* will be a solution to (4.8). The overall algorithm for solving the constrained TL1 is described in Algorithm (3). The explicit expressions for z come from (3.21) and (3.22).

Algorithm 3: DCA method for constrained TL1 minimization

Define $\epsilon_{outer} > 0$, $\epsilon_{inner} > 0$ and initialize $x^0 = 0$

while $\|x^n - x^{n+1}\| \geq \epsilon_{outer}$ **do**

$z = (z_1, ...z_N)$, where $z_i = \frac{a+1}{a} sign(x_i) - \frac{(a+1)sign(x_i)}{a+|x_i|} + \frac{(a+1)x_i}{(a+|x_i|)^2}$.

$x^1_{in} = x^n$, $j = 1$, $w^1 = x^1_{in}$, $v^1 = 0$ and $u^1 = 0$.

while $\|x^j_{in} - x^{j+1}\| \geq \epsilon_{inner}$ **do**

$x^{j+1}_{in} := (A^tA + I)^{-1}(w^j + A^t y + \frac{z - u^j - A^t v^j}{\delta})$

$w^j = shrink(\ \ x^{j+1}_{in} + \frac{u^j}{\delta}, \frac{a+1}{a\delta}\ \ )$

$u^{j+1} := u^j + \delta(x^{j+1} - w^j)$

$v^{j+1} := v^j + \delta(Ax^{j+1} - y)$

end while

$n = n + 1$

$x^n = x^j_{in}$

end while

V. Numerical Results

In this section, we use three classes of randomly generated matrices to illustrate the effectiveness of our Algorithms: DCATL1 (difference convex algorithm for transformed l_1 penalty) and its constrained version. We compare them separately with several state-of-the-art solvers on recovering sparse vectors:

- unconstrained algorithms: reweighted $l_{1/2}$ [12], yall1 (an alternating direction l_1 algorithm)[32] and DCA l_{1-2} algorithm [33] [15];
- constrained algorithms: Bregman algorithm [34], yall1, and $Lp - RLS$ [7].

All our tests were performed on a $Lenovo$ desktop with 16 GB of RAM and Intel Core processor $i7-4770$ with CPU at $3.40GHz \times 8$ under 64-bit Ubuntu system.

The three classes of random matrices are:

1) Gaussian matrix.
2) Over-sampled DCT with factor F.
3) Uniformly distributed M-sphere matrix.

We did not use prior information of the true sparsity level. Also, for all the tests, the computation is initialized with zero vectors. In fact, the DCATL1 does not guarantee a global minimum in general, due to nonconvexity of the problem. Indeed we observe that DCATL1 with random starts often gets stuck at local minima especially when the matrix A is ill-conditioned (e.g. A has a large condition number or is highly coherent). In the numerical experiments, by setting $x_0 = 0$, we find that DCATL1 usually produces a global minimizer. The intuition behind our choice is that by using zero vector as initial guess, the first step of our algorithm reduces to solving an unconstrained l_1 problem. So basically we are minimizing TL1 on top of l_1, which possibly explains why minimization of TL1 initialized by $x0 = 0$ always outperforms l_1.

Choice of Parameter: a

In DCATL1, parameter a is also very important. When a tends to zero, the penalty function approaches the l_0 norm. If a goes to ∞, objective function will be more convex and act like the l_1 optimization. So choosing a better 'a' will improve the effectiveness and success rate for our algorithm.

We tested DCATL1 on recovering sparse vectors with different parameter a, varying among $\{0.1\ \ 0.3\ \ 1\ \ 2\ \ 10\}$. In this test, A is a 64×256 random matrix generated by normal Gaussian distribution. The true vector x^* is also a randomly generated sparse vector with sparsity k from the set $\{8\ \ 10\ \ 12\ ...\ \ 32\}$. Here the parameter λ was set to be 10^{-5} for all tests. Although the best λ should be dependent on a in general, we considered the noiseless case, and $\lambda = 10^{-5}$ is small enough to approximately enforce $Ax = Ax^*$. For each a, we sampled 100 times with different A and x^*. The recovered vector x_r is regarded as successful if the relative error: $\frac{\|x_r - x^*\|_2}{\|x^*\|_2} \leq 10^{-3}$.

Fig. (2) shows the success rate using DCATL1 over 100 independent trials for various parameter a and sparsity k. From the figure, we see that DCATL1 with $a = 1$ is the best among all tested values. Also numerical results for $a = 0.3$ and $a = 2$ (near 1), are better than those with 0.1 and 10. This is because the objective function is more non-convex at a smaller 'a' and thus more difficult to solve. On

the other hand, the iterations more likely stop at a local l_1 minima far from l_0 solution if a is too large. Thus for all the following tests, we set the parameter $a = 1$.

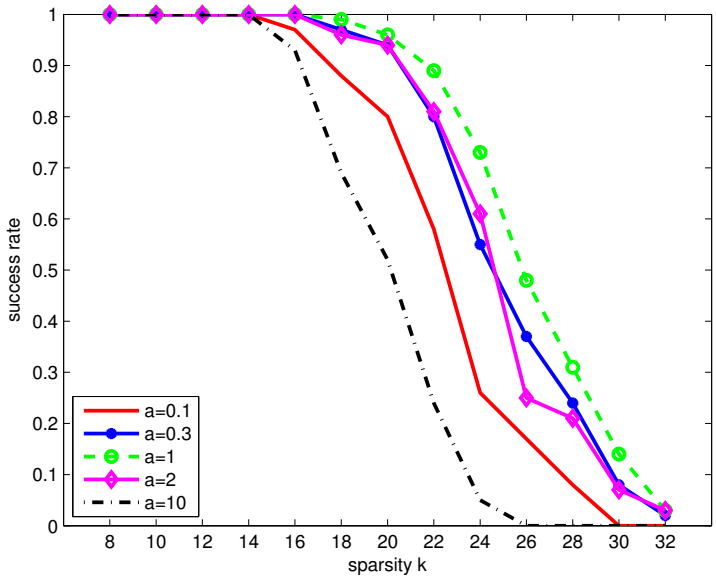


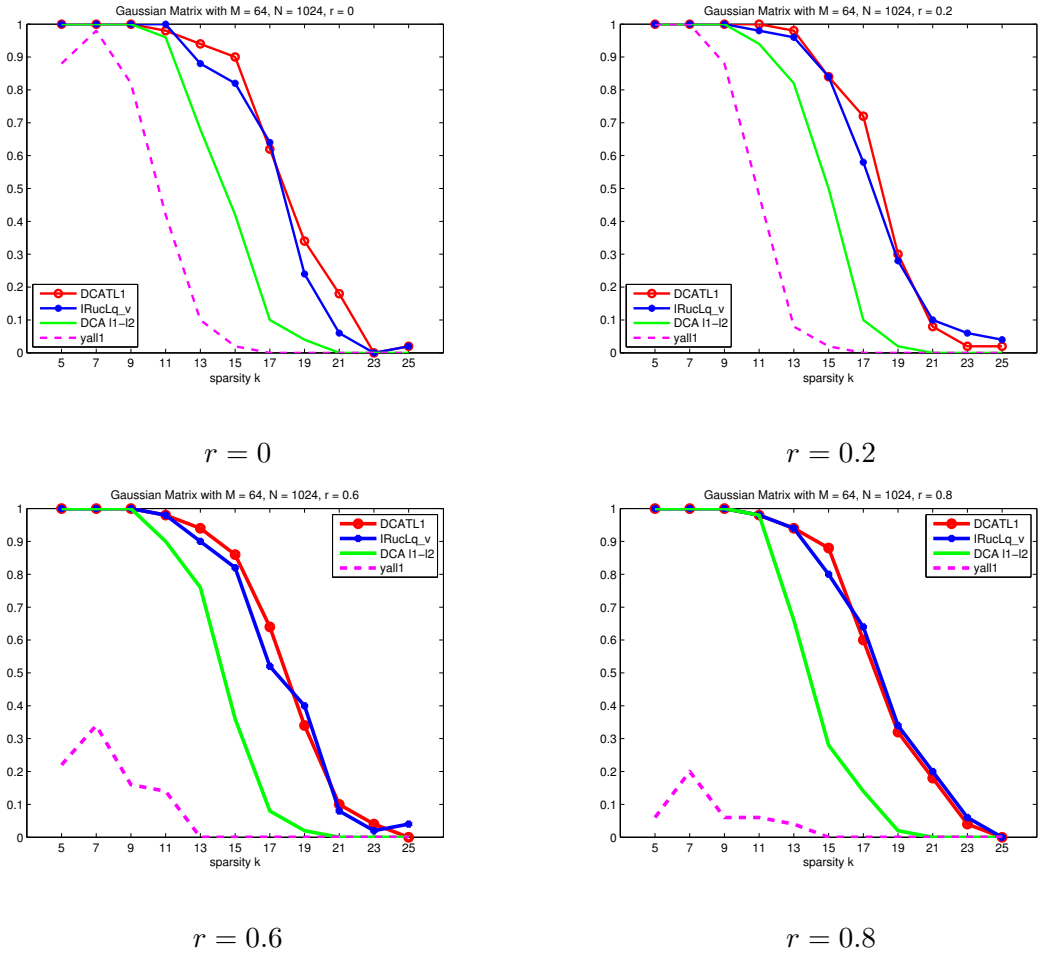
Fig. 2: Numerical tests on parameter a with $M = 64$, $N = 256$ by the unconstrained DCATL1 method.

A. *Numerical Experiment for Unconstrained Algorithm*

For each unconstrained algorithms experiment, an scheme is considered to be successful in this trial, if the relative error of the numerical result x_r from the ground truth x is less than 0.001, or $\frac{\|x_r - x\|}{\|x\|} < 0.001$. We did 50 trials to compute average success rates in all three classes of matrices. For all these numerical experiments using unconstrained algorithms, parameters of DCATL1 are fixed as: $a = 1$; $C = 10^{-9}$ (In application, the value of C is set to be very small); $\delta = 10^{-5}$.

1) ***Gaussian matrix****:* We use $\mathcal{N}(0, \Sigma)$, the multi-variable normal distribution to generate Gaussian matrix A. Here covariance matrix is $\Sigma = \{(1 - r) * \chi_{(i=j)} + r\}_{i,j}$, where the value of '$r$' varies from 0 to 0.8. In theory, the larger the r is, the more difficult it is to recovery true sparse vector. For matrix A, the row number and column number are set to be $M = 64$ and $N = 1024$. The sparsity k varies among $\{5 \quad 7 \quad 9 \ ... \quad 25\}$.

We compare four algorithms in terms of success rate. Denote x_r as a reconstructed solution by a certain algorithm. We consider one algorithm to be successful, if the relative error of x_r to the truth solution x



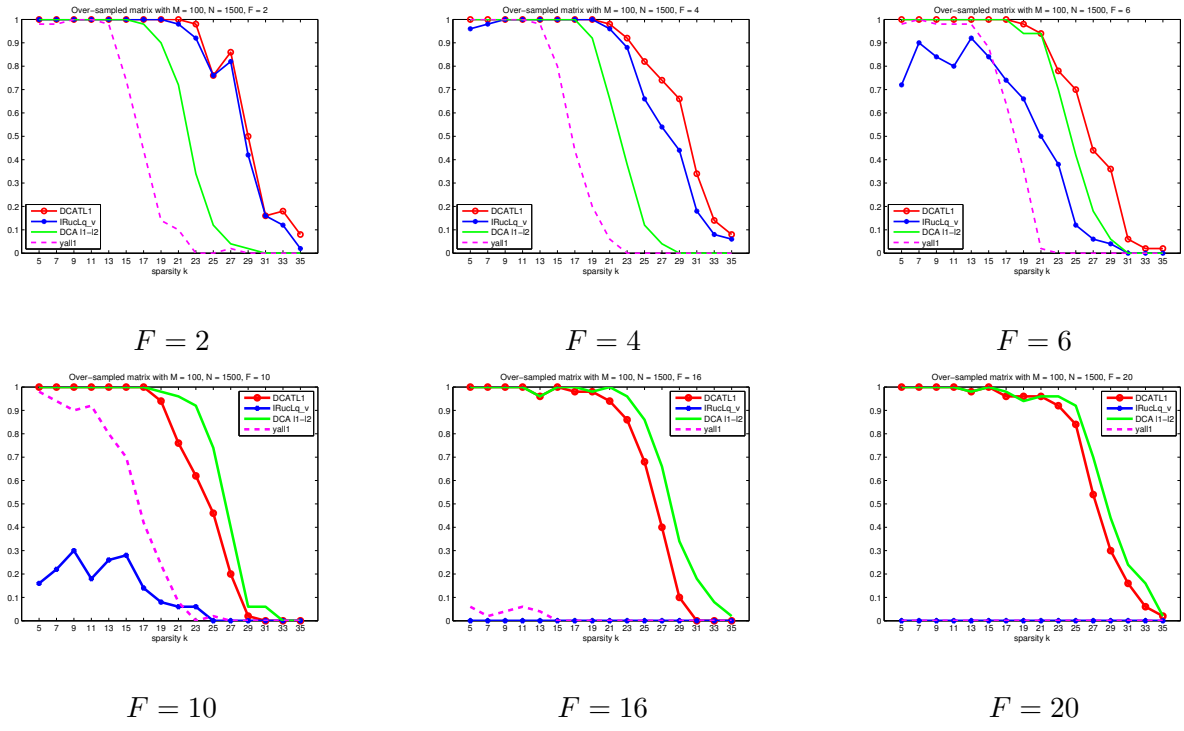
$r = 0$ $r = 0.2$

$r = 0.6$ $r = 0.8$

Fig. 3: Numerical tests for unconstrained algorithms under Gaussian generated matrices: M = 64 , N = 1024 with different coherence r.

is less that 0.001, $i.e.$, $\frac{\|x_r - x\|}{\|x\|} < 10^{-3}$. In order to improve success rates for all compared algorithms, we set tolerance parameter to be smaller or maximum cycle number to be higher inside each algorithm. As a result, it takes a long time to run one realization using all algorithms separately.

The success rate of each algorithm is plotted in Figure 3 with parameter r from the set: $\{0 \ \ 0.2 \ \ 0.6 \ \ 0.8\}$. For all cases, DCATL1 and reweighted $l_{1/2}$ algorithms (IRucLq-v) performed almost the same and both were much better than the other two, while the l_1 algorithm (yall1) has the lowest success rate.



$F = 2$ $F = 4$ $F = 6$

$F = 10$ $F = 16$ $F = 20$

Fig. 4: Numerical test for unconstrained algorithms under over-sampled DCT matrices: $M = 100$, $N = 1500$ with different F, and peaks of solutions separated by $2RL = 2F$.

*2) **Over-sampled DCT**:* The over-sampled DCT matrices A [13] [15] are:

$$A = [a_1, ..., a_N] \in \Re^{M \times N}$$
$$where \quad a_j = \frac{1}{\sqrt{M}} cos(\frac{2\pi\omega(j-1)}{F}), \quad j = 1, ..., N, \tag{5.1}$$

and ω is a random vector, drawn uniformly from $(0, 1)^M$

Such matrices appear as the real part of the complex discrete Fourier matrices in spectral estimation [13] An important property is their high coherence: for a 100×1000 matrix with $F = 10$, the coherence is 0.9981, while the coherence of the same size matrix with $F = 20$, is typically 0.9999.

The sparse recovery under such matrices is possible only if the non-zero elements of solution x are sufficiently separated. This phenomenon is characterized as $minimum\ separation$ in [5], and this minimum length is referred as the Rayleigh length (RL). The value of RL for matrix A is equal to the factor F. It is closely related to the coherence in the sense that larger F corresponds to larger coherence of a matrix. We find empirically that at least 2RL is necessary to ensure optimal sparse recovery with spikes further apart for more coherent matrices.

TABLE I: The success rates (%) of DCATL1 for different combination of sparsity and minimum separation lengths.

sparsity	5	8	11	14	17	20
1RL	100	100	95	70	22	0
2RL	100	100	98	74	19	5
3RL	100	100	97	71	19	3
4RL	100	100	100	71	20	1
5RL	100	100	96	70	28	1

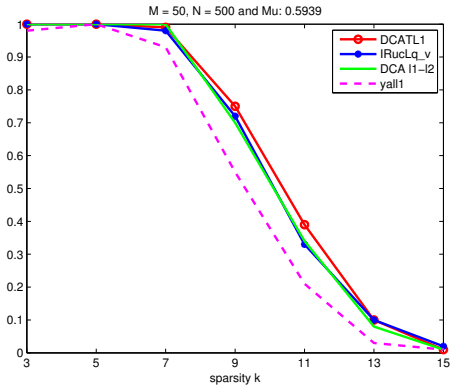
Under the assumption of sparse signal with 2RL separated spikes, we compare those four algorithms in terms of success rate. Denote x_r as a reconstructed solution by a certain algorithm. We consider one algorithm successful, if the relative error of x_r to the truth solution x is less that 0.001, $i.e.$, $\frac{\|x_r - x\|}{\|x\|} < 0.001$. The success rate is averaged over 50 random realizations.

Fig. 4 shows success rates for those algorithms with increasing factor F from 2 to 20. The sensing matrix is of size 100×1500. It is interesting to see that along with the increasing of value F, DCA of $l_1 - l_2$ algorithm performs better and better, especially after $F \geq 10$, and it has the highest success rate among all. Meanwhile, reweighted $l_{1/2}$ is better for low coherent matrices. When $F \geq 10$, it is almost impossible for it to recover sparse solution for the high coherent matrix. Our DCATL1, however, is more robust and consistently performed near the top, sometimes even the best. So it is a valuable choice for solving sparse optimization problems where coherence of sensing matrix is unknown.

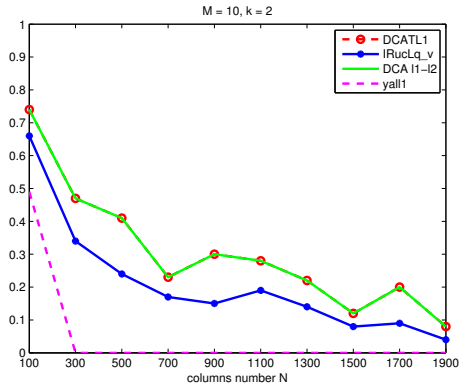
We further look at the success rates of DCATL1 with different combinations of sparsity and separation lengths for the over-sampled DCT matrix A. The rates are recorded in Table I, which shows that when the separation is above with the minimum length, the sparsity relative to M plays more important role in determining the success rates of recovery.

3) ***Uniformly distributed*** M***-sphere matrix****:* The column vectors of the $M \times N$ matrix A are sampled from uniform distribution on the surface of unit M-hypersphere. We use standard normal distribution $\mathcal{N}(0, I_M)$ to generate a random matrix B, then normalize each column vector of matrix B to unit l_2 norm. The density function of multi-variate normal random variables is:

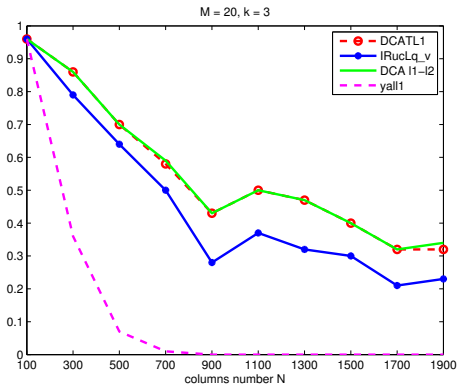
$$f_{\mathbf{x}}(x_1, \ldots, x_M) = \frac{1}{(2\pi)^{M/2}} \exp\left(- \sum_{i=1,2,..M} \frac{x_i^2}{2}, \right)$$



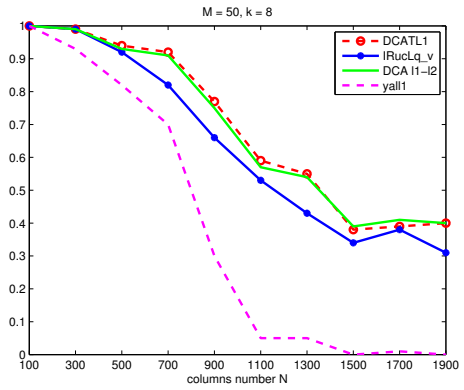
$(M, N) = (50, 500)$, mean coherence $\mu = 0.5939$.



$M = 10, k = 2$



$M = 20, k = 3$



$M = 50, k = 8$

Fig. 5: Comparison of success rates of four unconstrained algorithms for M-sphere random matrices. In the top left plot, $(M, N) = (50, 500)$, sparsity k varies from 3 to 15. In the other three plots, (M, k) are fixed, and N is varied.

and the column vectors of matrix A are independent. After normalization, distribution of column vectors in matrix A is uniform on the M-sphere surface.

In our numerical tests, we vary three parameters: column number N, row number M, and sparsity k. We fixed two parameters, and changed the value of the remaining one. The results are based on 100 trials. In the top-left plot of Fig. 5, we chose $M = 50$ and $N = 500$, the sparsity k varied from 3 to 13. We see that the success rate curves for reweighted $l_{1/2}$, DCA $l_1 - l_2$ and DCATL1 are almost the same, with a little higher value for TL1. In the other three plots, we fixed parameters: M and k, with row

number N changing from 100 to 1900. The mean values of the coherence μ of A (maximum absolute value of pairwise column vector inner product) for different combinations of M and N, are shown in Table II. It is interesting to see that for all three cases, curves for DCA l_1-l_2 and DCATL1 are almost identical, and better than the other two (reweighted $l_{1/2}$ and l_1). The DCA algorithm for both $l_1 - l_2$ and TL1 may have helped too.

TABLE II: Mean coherence of M-sphere random matrices for different values of (M, N) over 100 samples.

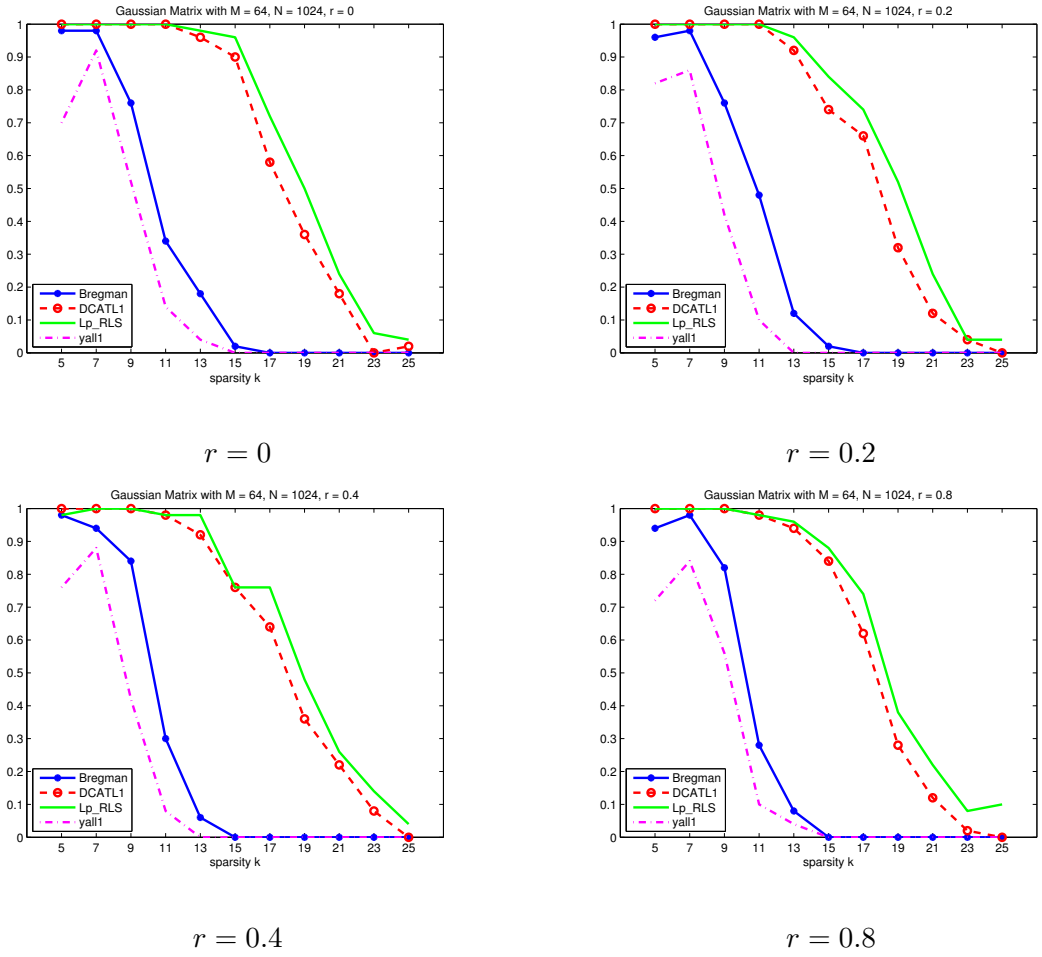
Column number N:	100	300	500	700	900
Mu: $M = 10$	0.9065	0.9431	0.9537	0.9605	0.9651
Mu: $M = 20$	0.7418	0.8024	0.8198	0.8363	0.8416
Mu: $M = 50$	0.5105	0.5680	0.5943	0.6031	0.6119
Column number N:	1100	1300	1500	1700	1900
Mu: $M = 10$	0.9674	0.9685	0.9708	0.9742	0.9733
Mu: $M = 20$	0.8526	0.8573	0.8643	0.8634	0.8663
Mu: $M = 50$	0.6209	0.6309	0.6364	0.6378	0.6454

B. Numerical Experiment for Constrained Algorithm

For constrained algorithms, we performed similar numerical experiments. An algorithm is considered to be successful in one sample experiment if the relative error of the numerical result x_r from the ground truth x is less than 0.001, or $\frac{\|x_r - x\|}{\|x\|} < 0.001$. We did 50 trials to compute average success rates for all the numerical experiments same as for the unconstrained algorithms. For all these numerical experiments using constrained algorithms, parameters of constrained DCATL1 are fixed as: $a = 1; \quad \delta = 10$.

1) ***Gaussian Random Matrices****:* We fix parameters $(M, N) = (64, 1024)$, while covariance parameter r is varied from 0 to 0.8. The reweighted $l_{1/2}$ and two l_1 algorithms (Bregman and yall1) are chose for Comparison.

In Fig. (6), we see that reweighted $l_{1/2}$ algorithm: Lp-RLS is the best among the four algorthms with DCATL1 trailing not much behind.



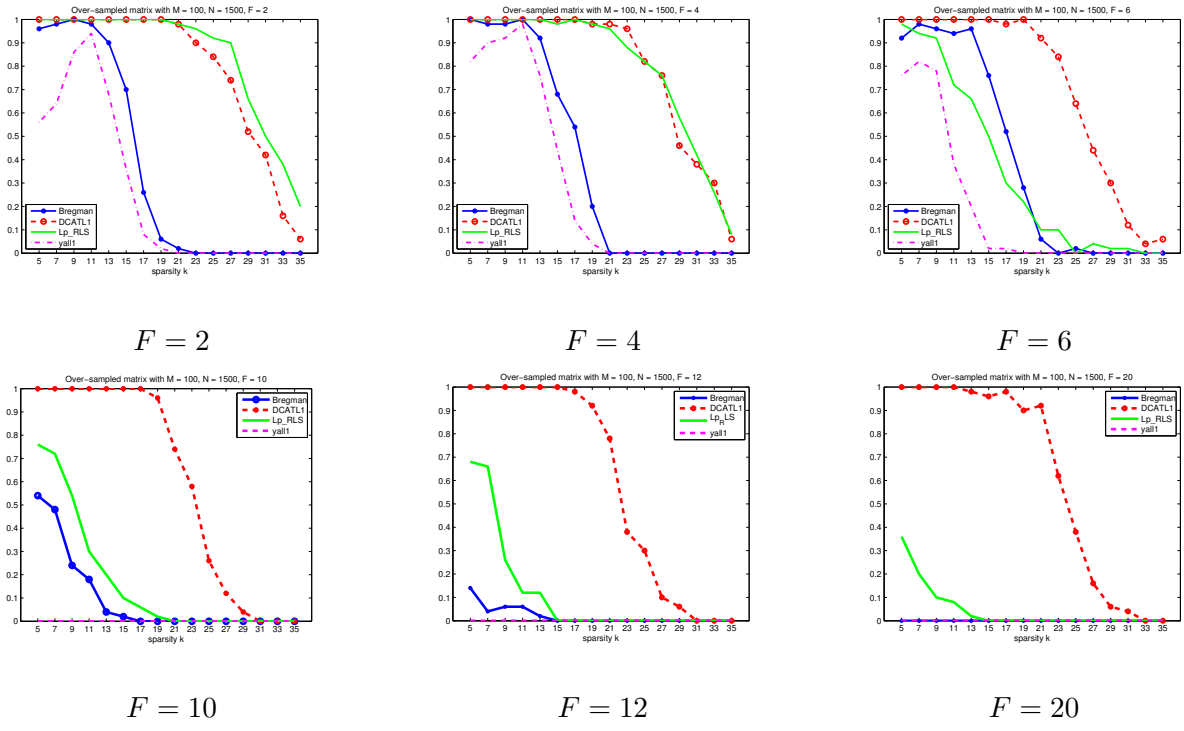
$r = 0$ $r = 0.2$

$r = 0.4$ $r = 0.8$

Fig. 6: Comparison of constrained algorithms for 64×1024 Gaussian random matrices with different coherence parameter r. The data points are averaged over 50 trials.

*2) **Over-sampled DCT**:* In this experiment, we choose parameter: $(M, N) = (100, 1500)$, while the value for F changes from 2 to 20. So the coherence of there matrices has a wider range and almost reaches 1 at the high end.

In Fig. (7), when F is small, say $F = 2, 4$, algorithm Lp-RLS still performs the best, similar to the case of Gaussian matrices. However, with increasing F, the success rates for Lp-RLS declines quickly, worse than the Bregman l_1 algorithm at $F = 6, 10$. The performance for DCATL1 is very stable and maintains a high level consistently even at the very high end of coherence ($F = 20$).



$F = 2$ $F = 4$ $F = 6$

$F = 10$ $F = 12$ $F = 20$

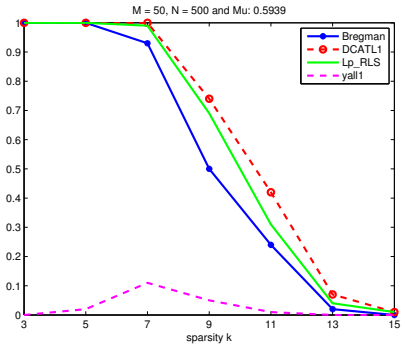
Fig. 7: Comparison of success rates of constrained algorithms for the over-sampled DCT random matrices: $(M, N) = (100, 1500)$ with different F values, peak separation by $2RL = 2F$.

*3) **Uniformly Distributed M-sphere Random Matrices**:* We conducted two types of experiments for the M-sphere random matrices. In one, we fixed (M, N) and vary sparsity k. In the other, we fixed (M, k), and varied N. The results are shown in Figure 8. DCATL1 is consistently at the top.

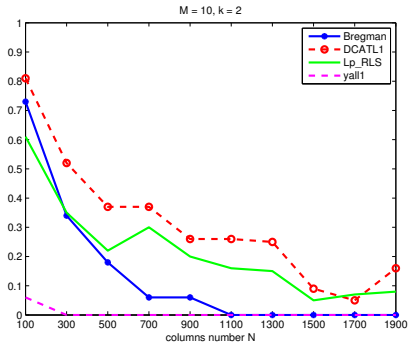
VI. Concluding Remarks

We have studied compressed sensing problem with the transformed l_1 penalty function for both the unconstrained and constrained models. We established a theory for the uniqueness and l_0 equivalence of global minimizer of the unconstrained model under RIP and analyzed properties of local minimizers. We presented two DC algorithms along with a convergence theory.

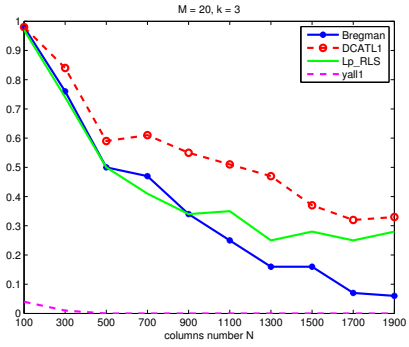
In numerical experiments, we observed that for incoherent Gaussian matrices, DCATl1 is on par with the best method reweighted $l_{1/2}$ $(Lp - RLS)$ in the unconstrained (constrained) model. For highly coherent over-sampled DCT matrices, DCATl1 is comparable to the best method DCA $l1 - l2$ algorithm. For random matrices of varied degree of coherence we tested (Gaussian, over-sampled DCT, uniform



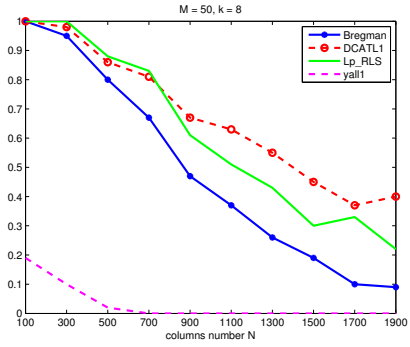
Varying sparsity k at $(M, N) = (50, 500)$.



Varying N at $(M, k) = (10, 2)$.



Varying N at $(M, k) = (20, 3)$.



Varying N at $(M, k) = (50, 8)$.

Fig. 8: Comparison of success rates of algorithms for M-sphere random matrices.

M-sphere), the DCATL1 algorithm is the most robust for constrained and unconstrained models alike.

In future work, we plan to develop TL1 algorithms for imaging processing applications such as deconvolution and deblurring.

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REFERENCES

[1] E. Candès, T. Tao, *Decoding by linear programming*, IEEE Trans. Info. Theory, 51(12):4203-4215, 2005.

[2] E. Candès, M. Rudelson, T. Tao, R. Vershynin, *Error correction via linear programming*, in 46th Annual IEEE Symposium on Foundations of Computer Science, pp. 668-681, 2005.

[3] E. Candès, J. Romberg, T. Tao, *Robust uncertainty principles: Exact signal reconstruction from highly incomplete Fourier information*, IEEE Trans. Info. Theory, 52(2), 489-509, 2006.

[4] E. Candès, J. Romberg, T. Tao, *Stable signal recovery from incomplete and inaccurate measurements*, Comm. Pure Applied Mathematics, 59(8):1207-1223, 2006.

[5] E. Candès, C. Fernandez-Granda, *Super-resolution from noisy data*, Journal of Fourier Analysis and Applications, 19(6):1229-1254, 2013.

[6] R. Chartrand, *Nonconvex compressed sensing and error correction*, ICASSP 2007, vol. 3, p. III 889.

[7] R. Chartrand, W. Yin, *Iteratively reweighted algorithms for compressive sensing*, ICASSP 2008, pp. 3869-3872.

[8] D. Donoho, *Compressed sensing*, IEEE Trans. Info. Theory, 52(4), 1289-1306, 2006.

[9] D. Donoho, M. Elad, *Optimally sparse representation in general (nonorthogonal) dictionaries via l_1 minimization*, Proc. Nat. Acad. Scien. USA, vol. 100, pp. 2197-2202, Mar. 2003.

[10] E. Esser, Y. Lou and J. Xin, *A Method for Finding Structured Sparse Solutions to Non-negative Least Squares Problems with Applications*, SIAM J. Imaging Sciences, 6(2013), pp. 2010-2046.

[11] J. Fan, and R. Li, *Variable selection via nonconcave penalized likelihood and its oracle properties,* Journal of the American Statistical Association, 96(456):1348-1360, 2001.

[12] M-J Lai, Y. Xu, and W. Yin, *Improved Iteratively Reweighted Least Squares for Unconstrained Smoothed L_q Minimization,* SIAM Journal on Numerical Analysis, 51(2):927-957, 2013.

[13] A. Fannjiang, W. Liao, *Coherence Pattern-Guided Compressive Sensing with Unresolved Grids*, SIAM J. Imaging Sciences, Vol. 5, No. 1, pp. 179–202, 2012.

[14] T. Goldstein and S. Osher, *The Split Bregman Method for l_1-regularized Problems*, SIAM Journal on Imaging Sciences, 2(1):323-343, 2009.

[15] Y. Lou, P. Yin, Q. He, and J. Xin, *Computing Sparse Representation in a Highly Coherent Dictionary Based on Difference of L1 and L2,* CAM Report 14-02, UCLA, 2014; J. Sci. Computing, to appear.

[16] S. Zhang and J. Xin, *Minimization of Transformed L_1 Penalty: A Thresholding Representation Theory and Fast Algorithms*, in preparation.

[17] Z. Lu and Y. Zhang, *Sparse approximation via penalty decomposition methods*, SIAM J. Optimization, 23(4):2448-2478, 2013.

[18] J. Lv, and Y. Fan, *A unified approach to model selection and sparse recovery using regularized least squares,* Annals of Statistics, 37(6A), pp. 3498-3528, September 2009.

[19] S. Mallat and Z. Zhang, *Matching pursuits with time-frequency dictionaries*, IEEE Trans. Signal Processing, 41(12):3397-3415, 1993.

[20] B. Natarajan, *Sparse approximate solutions to linear systems*, SIAM Journal on Computing, 24(2):227-234, 1995.

[21] D. Needell and R. Vershynin, *Signal recovery from incomplete and inaccurate measurements via regularized orthogonal matching pursuit,* IEEE Journal of Selected Topics in Signal Processing, 4(2):310-316, 2010.

[22] C.S. Ong, L.T.H. An, *Learning sparse classifiers with difference of convex functions algorithms*, Optimization Methods and Software, 28(4):830-854, 2013.

[23] P.D. Tao and L.T.H. An, *Convex analysis approach to d.c. programming: Theory, algorithms and applications*, Acta Mathematica Vietnamica, vol. 22, no. 1, pp. 289-355, 1997.

[24] P.D. Tao and L.T.H. An, *A DC optimization algorithm for solving the trust-region subproblem*, SIAM Journal on Optimization, 8(2), pp. 476–505, 1998.

[25] H.A.L. Thi, B.T.A. Thi, and H.M. Le, *Sparse signal recovery by difference of convex functions algorithms*, in Intelligent Information and Database Systems, pp. 387-397. Springer, 2013.

[26] R. Tibshirani, *Regression shrinkage and selection via the lasso*, J. Royal. Statist. Soc, 58(1):267-288, 1996.

[27] J. Tropp and A. Gilbert, *Signal recovery from partial information via orthogonal matching pursuit*, IEEE Trans. Inform. Theory, 53(12):4655-4666, 2007

[28] J. Zeng, S. Lin, Y. Wang, and Z. Xu, $L_{1/2}$ regularization: Convergence of iterative half thresholding algorithm, Signal Processing, IEEE Transactions on, 62(9):2317-2329, 2014.

[29] F. Xu and S. Wang, *A hybrid simulated annealing thresholding algorithm for compressed sensing*, Signal Processing, 93:1577-1585, 2013.

[30] Z. Xu, X. Chang, F. Xu, and H. Zhang, $L_{1/2}$ regularization: A thresholding representation theory and a fast solver, *Neural Networks and Learning Systems, IEEE Transactions on*, 23(7):1013-1027, 2012.

[31] W. Cao, J. Sun, and Z. Xu, Fast image deconvolution using closed-form thresholding formulas of regularization, *Journal of Visual Communication and Image Representation*, 24(1):31-41, 2013.

[32] J. Yang and Y. Zhang, *Alternating direction algorithms for l_1 problems in compressive sensing*, SIAM Journal on Scientific Computing, 33(1):250-278, 2011.

[33] P. Yin, Y. Lou, Q. He, and J. Xin, *Minimization of L1 - L2 for compressed sensing*, CAM Report 14-01, UCLA, 2014.

[34] W. Yin, S. Osher, D. Goldfarb, and J. Darbon, *Bregman iterative algorithms for l_1-minimization with applications to compressed sensing*, SIAM Journal on Imaging Sciences, 1(1):143-168, 2008.