

# PHASELESS STABILITY FOR HELMHOLTZ

ALDEN WATERS

ABSTRACT. We prove a Hölder conditional stability estimate for identifying attenuation coefficients from phaseless boundary value measurements

## 1. INTRODUCTION

Interest in phaseless measurements has seen an increasing rise in popularity in the theory of signal processing. When measuring a wave from its source, many times it is only possible to gain information about the modulus of the signal. Algorithms found in [7] and [6] center on decomposing a wave into a sequence of Fourier modes. In this article, we are interested in phaseless measurements of solutions to the Helmholtz equation. The goal of this article is slightly different. We will show having an idea of which partial differential equation the wave comes from is enough to give complete reconstruction of the coefficients in the partial differential equation.

As a practical application, in multi-wave tomography usually some type of wave is sent to the portion of the body being imaged. In electromagnetic or optical radiation tomography the wave interaction with the tissues of the patient are measured [3]. Naturally one cannot measure inside the patient, so some initial boundary value problem must be considered. One such mathematical model of the emitted ultrasound waves is the following Helmholtz equation with a high-frequency source term. The equation is

$$(1.1) \quad Lu = \Delta u + (i\lambda n^2(x) + \lambda^2)u = h(x, \lambda) \quad x \in \mathbb{R}^d$$

The scalar  $\lambda$  is large and  $n^2(x)$  is the attenuation coefficient. The source  $h(x, \lambda)$ , which emits the waves, we also assume to depend on  $\lambda$  and be compactly supported. Given a smooth, strictly convex, bounded domain  $\Omega$  we assume that on the boundary of  $\Omega$ ,  $\partial\Omega$  that  $n^2(x) \equiv 1$ .

The measurements we consider are then of the form

$$(1.2) \quad \{|u|, x \in \partial\Omega\}$$

with the collection of  $h(x, \lambda)$  varying over all  $x \in \partial\Omega$ .

In this paper, we derive a stability result for the attenuation coefficients of the Helmholtz equation. Stable reconstructions have been made from Robin conditions for lower order terms than considered here [4]. In the related case, for the acoustic wave equation for Dirichlet boundary conditions in [9] [13] and Robin conditions in [5] and [19] the potential can also be recovered. However stability estimates from phaseless measurements have not been previously given. In [11], [10], uniqueness results in dimension 3 for lower order terms than the ones considered are derived from phaseless measurements. These papers are predicated on analyticity arguments, which require data in a small neighborhood of the source. The question of phaseless stability from internal measurements for Schrödinger was also examined in [2]. These measurements are in contrast to the boundary data we require. We are not able to prove uniqueness results unless  $\lambda \rightarrow \infty$ . The inverse problem of recovering the source for wave equations from Dirichlet boundary conditions is also examined in [18], [16], [8], [15], and [17].

## 2. CONSTRUCTION OF SOLUTIONS

We are looking for an asymptotic model to (1.1) of the form

$$(2.1) \quad U(x) = a(x) \exp(i\lambda\psi(x))$$

which we show in the high frequency limit solves the equation (1.1) up to suitable error terms. We use a Gaussian beam Ansatz which involves the construction of a phase function  $\psi(x)$  and an amplitude  $a$  for the first order beams as

$$\begin{aligned} \psi(x) &= S(s) + (x - x(s)) \cdot p(s) + \frac{1}{2}(x - x(s)) \cdot M(s)(x - x(s)) \\ a(x) &= a_0(s) + \mathcal{O}(|x - x(s)|) \end{aligned}$$

where  $x(s)$  is a curve which describes a Hamiltonian flow. For the construction of the Ansatz, we follow the work of [12] quite closely. We chose the operator  $L$  for the Ansatz construction here instead of

$$(2.2) \quad \tilde{L} = \Delta + \lambda^2 n^2(x) + i\lambda n^2(x)$$

which corresponds to the acoustic wave equation. These operators have corresponding Hamiltonian flows  $H = p^2 - 1$  and  $\tilde{H} = p^2 - n^2$ . The main

problem of isolating the X-ray transform of the attenuation coefficients  $n^2(x)$  is control of the nul-bicharacteristics. The ordinary differential equations which govern the rath path  $\{(s, x(s)) : 0 \leq s \leq T\}$  along which solutions are concentrated are

$$(2.3) \quad \frac{d^2 x(s)}{ds^2} = -\nabla_x n(x(s))$$

which is associated to  $\tilde{L}$  and

$$(2.4) \quad \frac{d^2 x(s)}{ds^2} = 0$$

which is associated to  $L$ . The second equation gives that the ray paths in  $\mathbb{R}^d$  coincide while the first one does not. The problem of recovering  $n^2(x)$  for the operator  $\tilde{L}$  is more difficult and we leave it as an open question.

### 3. CONSTRUCTION OF SOLUTIONS

We will use the ansatz,

$$(3.1) \quad U(x) = \sum_{j=0}^l \exp(i\lambda\psi(x))a_j(x)\lambda^{-j}$$

to build asymptotic solutions to (1.1) in the high frequency limit, and then give estimates on the difference between these approximate solutions and the true solutions. We will follow [12] for the Gaussian beam ansatz in constructing the phase  $a$  and amplitude  $\psi(x)$ . Every geometrics optics solution concentrates on an open set around the ray path  $\{(s, x(s)) : 0 \leq s \leq T\}$ . The flow for the ray path of  $H$  is defined by the set of ODEs

$$(3.2) \quad \dot{x} = 2p \quad \dot{p} = 0.$$

We assume that the obstacle  $\Omega$  is a smooth compact subdomain of  $\mathbb{R}^d$  and that  $n^2(x) \equiv 1$  on  $\partial\Omega$ . We have that

$$(3.3) \quad LU = \sum_{j=0}^l \exp(i\lambda\psi(x))c_j(x)\lambda^{-j}.$$

The coefficients  $c_j$  are defined recursively as follows

$$\begin{aligned} c_{-2} &= (1 - |\nabla_x \psi|^2) a_0(x) = E(x) a_0 \\ c_{-1} &= i n^2(x) a_0 + \nabla_x \cdot (a_0 \nabla_x \psi) + \nabla_x a_0 \cdot \nabla_x \psi + E a_1(x) \\ c_j &= i n^2(x) a_{j+1} + \nabla_x \cdot (a_{j+1} \nabla_x \psi) + \nabla_x a_{j+1} \cdot \nabla_x \psi + E(x) a_{j+1} + \delta_x a_j \\ j &= 0, 1, \dots, l \end{aligned}$$

If we are Taylor expanding the coefficients  $a_j(x)$  around the central ray  $x(s)$  then we arrive at the following set of ordinary differential equations

$$(3.4) \quad \dot{S} = 2 \quad \dot{a}_0 = -n^2(x(s)) a_0$$

If we have chosen  $c_{-2}$  and  $c_{-1}$  to vanish on the ray to third and first order respectively. For the stability estimates we prove the ODE which defines  $a_0$  explicitly is the most important for identifying the attenuation coefficient  $n^2(x)$ . This leads to the following set of differential equations

$$(3.5) \quad \dot{S} = 2 \quad \dot{M} = -2M^2 \quad \dot{a}_0 = -\text{tr}(M(s)) a_0 - n^2(x(s)) a_0.$$

We see that

$$(3.6) \quad a_0(s) = \exp \left( \int_0^s -n^2(x(t)) - \text{tr} M(t) dt \right).$$

The phase  $\psi$  needs to verify the conditions

$$(3.7) \quad \psi(x(s)) = S(s), \quad \nabla \psi(x(s)) = p(s) \quad D^s \psi(x(s)) = M(s)$$

which can be done if we set

$$\begin{aligned} \psi(x) &= S(s) + (x - x(s)) \cdot p(s) + \frac{1}{2} (x - x(s)) \cdot M(s) (x - x(s)) \\ a(x(s)) &= a_0(s). \end{aligned}$$

We use the initial data  $S(0) = 0$  and  $M(0)$  such that

$$(3.8) \quad M(0) = M(0)^T \quad M(0) \dot{x}(0) = \dot{p}(0) \quad \Im M(0) \text{ positive definite on } \dot{x}(0)^\perp$$

cf. Section 2 of [12].

We now define a cutoff function  $\phi_\lambda(x) \in C^\infty(\mathbb{R}^n)$  for  $\lambda > 0$  such that

$$(3.9) \quad \phi_\lambda(x) = \begin{cases} 0 & \text{if } x \in \{x : |x - x(s)| > 2\lambda^{-\frac{1}{2a}} \mid 0 \leq s \leq T\} \\ 1 & \text{if } x \in \{x : |x - x(s)| < \lambda^{-\frac{1}{2a}} \mid 0 \leq s \leq T\} = A_\lambda \end{cases}$$

One can arrange so that there is a constant  $C$  such that

$$(3.10) \quad |\nabla_x^m \phi_\lambda| \leq C\lambda^{-m}$$

We drop the subscript  $\lambda$  for the rest of this paper. It follows that a zeroth order Gaussian beam looks like

$$(3.11) \quad U^0(x) = (a_0(s) + \mathcal{O}(|x - x(s)|)) \exp(i\lambda\psi(x))\phi(x) + \mathcal{O}(\lambda^{-1}).$$

We also take the following Corollary 5 from [21].

**Corollary 1.** *Let  $\psi(x)$  correspond to a zeroth order beam. Let  $\sim$  denote the equivalence relation bounded above and below by, then we have*

$$\exp(-2\lambda\Im\psi(x)) \sim \exp(-\lambda C|x - x(s)|^2).$$

where  $C$  is independent of  $\lambda$ . As a consequence if we let  $B$  denote the set

$$B = \{x : |x - x(s)| > \lambda^{-(\frac{1}{2}-\sigma)}, \quad 0 \leq t \leq T\} \quad \sigma > 0, \sigma \in \mathbb{R}$$

then since  $2\Im\psi(x) \sim |x - x(s)|^2$ ,  $\exp(-2\lambda\Im\psi(x))$  is exponentially decreasing in  $\lambda$  for all  $x \in B$ .

*Proof.* We need only observe that  $M(s)$  is a bounded and positive definite matrix. From the form of the phase functions constructed the desired result follows.  $\square$

#### 4. INTRODUCTION OF THE SOURCE TERMS

We introduce source functions. This is the same argument as in Section 2.1 of [12] and is repeated for completeness. We let  $\rho$  be a function such that  $|\nabla\rho| = 1$  on the set  $\{x : \rho(x) = 0\}$  and let the hypersurface  $\{x : \rho(x) = 0\}$  be denoted as  $\Sigma$ . Let  $x_0$  be a point in  $\Sigma$  and we let  $(x(s), p(s))$  be the solution path- in other words the nul-bicharacteristics with  $(x(0), p(0)) = (x_0, n(x_0)\nabla\rho(x_0))$ . The hypersurface  $\Sigma$  is given by  $s = \sigma(y)$  with  $\sigma(0) = 0$  and  $\nabla\sigma(0) = 0$  where  $x = (s, y)$  and  $y = (y_1, \dots, y_{d-1})$  is transversal. We let the optics Ansatz  $U(x)$  have initial data  $(x(0), p(0))$ , and be defined in in this tubular neighborhood. We let  $U^+$  be the restriction of  $U$  to the surface  $\{x : \rho(x) \geq 0\}$ . Because we need to have a source term which is a multiple of  $\delta(\rho)$  we also need a second ‘outgoing’ solution  $U^-$  which is defined on  $\{x : \rho(x) \leq 0\}$  which is then equal to  $U^+$  on the hypersurface

$\Sigma$ . For example, we can write the ingoing and outgoing optics solutions as

$$(4.1) \quad U^+ = A^+(x, \lambda) \exp(i\lambda\psi^+(x)) \quad U^- = A^-(x, \lambda) \exp(i\lambda\psi^-(x))$$

where we have set  $\psi^+ = \psi^-$  and  $A^+ = A^-$  on  $\Sigma$ , then requirement that their Taylor series coincide on the boundary is equivalent to setting  $\psi^+(\sigma(y)) = \psi^-(\sigma(y))$  and differentiating with respect to  $y$  and evaluating at  $y = 0$ . We extend  $U^+$  to be 0 on  $\{x : \rho(x) < 0\}$  and  $U^-$  to be 0 on  $\{x : \rho(x) > 0\}$ . We define our geometric optics Ansatz solution  $U$  to be  $U = U^+ + U^-$ . We set  $A = A^+ = A^-$  on  $\Sigma$ . In order to add the source term we notice that

$$\begin{aligned} LU &= i\lambda \left( \left( \frac{\partial\psi^+}{\partial\nu} - \frac{\partial\psi^-}{\partial\nu} \right) A(x, \lambda) + \frac{\partial A^+}{\partial\nu}(x, \lambda) - \frac{\partial A^-}{\partial\nu}(x, \lambda) \right) \exp(i\lambda\phi^+) \delta(\rho) \\ &+ f_{gb} = \\ &g_0\delta(\rho) + f_{gb} \end{aligned}$$

where  $\nu(x) = \nabla\rho(x)$  is the unit normal to  $\Sigma$ . We consider the singular part of  $L_n u_{gb}$  that is  $g_0\delta(\rho)$  to be the same source term and  $f_{gb}$  the error then we obtain

$$(4.2) \quad f = \exp(i\lambda\psi^+) \sum_{j=-2}^l c_j^+(x) \lambda^{-j} + \exp(i\lambda\psi^-) \sum_{j=-2}^l c_j^-(x) \lambda^{-j}$$

whenever  $c_j^+$  are extended to be zero if  $\rho(x) < 0$  and  $c_j^-$  are extended to be zero when  $\rho(x) > 0$ . We know by construction that  $c_{-2}^\pm = \mathcal{O}(|x - x(s)|^3)$  and  $\mathcal{O}(|x - x(s)|)$  respectively.

## 5. ERROR ESTIMATES

We have that the  $c_j(x)^\pm$  are bounded and

$$(5.1) \quad c_j^\pm(x) = \sum_{|\beta|=l-2j-2} d_{\beta,j}^\pm (x - x(s))^\beta \quad j = -2, \dots, l-1$$

where the  $d_{\beta,j}$  are bounded by Taylor's theorem. We obtain

$$(5.2) \quad |c_j^\pm(x)| \leq |C_j| |x - x(s)|^{l-2j-2} \phi(x).$$

with the  $C_j$  uniformly bounded independent of  $\lambda$ . The choice of  $M(s)$  a matrix with positive definite imaginary part results in the following

$$(5.3) \quad \text{Im}\psi^\pm(x) \geq C|x - x(s)|^2$$

for some positive constant  $C$ . Exactly as in [12] we have

$$(5.4) \quad b^p \exp(-ab^2) \leq C_p a^{-p/2} \exp(-ab^2/2) \quad C_p = \left(\frac{p}{e}\right)^{\frac{p}{2}}$$

Setting  $p = N - 2j - 2$ ,  $a = kc$  and  $b = |x - x(s)|$  with  $x \in A_\lambda$  we obtain

$$(5.5) \quad |f_{gb}(x)| \leq \exp(-\lambda \Im \psi^\pm(x)) \sum_{j=-2}^l |c_j(x)| \lambda^{-j}$$

If we differentiate  $c_j^\pm(x)$  and  $\exp(i\lambda\psi(x))$  then we have for the  $N^{\text{th}}$  order beams

$$\begin{aligned} & \|f_{gb}(x)\|_{H^m(|x|<R)}^2 \leq \\ & C \lambda^{-N+2+2m} \int_{x \in A_\lambda} \exp(-2\lambda C|x - x(s)|^2) dx \leq \lambda^{-N+2+(1-d)/2+2m} \end{aligned}$$

## 6. EXTENSION OF THE ANSATZ

The extension of the Gaussian beam ansatz to  $\mathbb{R}^d$  will allow us the necessary means to make observations further away from the source and still identify source terms. We briefly review the results of [12]. We set

$$(6.1) \quad \tilde{f}_u = LU - g_0 \delta(\rho)$$

We would like this function to be supported in  $|x| < 6R$  and be  $\mathcal{O}(\lambda^{-1})$ . We let  $G_\lambda(x)$  be the Green's function for the Helmholtz operator  $L$ . In order to extend our approximate solution  $U$  we introduce a smooth cutoff  $\chi_a(x)$  such that

$$\begin{aligned} \eta_a(x) &= 1 & |x| < (a-1)R \\ \eta_a(x) &= 0 & |x| > aR. \end{aligned}$$

Now we set

$$(6.2) \quad \tilde{U}(x) = \chi_3(x)U(x) + \int G_\lambda(x-y)\chi_5(y)L[(1-\chi_3(y))U(y)] dy.$$

In [12], they prove

$$(6.3) \quad \left\| U - \tilde{U} \right\|_{H^m(\mathbb{R}^d)} = \mathcal{O}(\lambda^{-n}).$$

Therefore the size of  $\tilde{U}$  and the extension depends on the size of  $f_{gb}$ . Furthermore, using ideas in Vainberg [20], they also prove

$$(6.4) \quad \|U - u\|_{H^m(|x| < R)} \leq C\lambda^{-1} \|f_{gb}\|_{H^m(|x| < R)}.$$

The triangle inequality allows us to conclude

$$(6.5) \quad \left\| u - \tilde{U} \right\|_{H^m(\mathbb{R}^d)} \leq C\lambda^{-1} \|f_{gb}\|_{H^m(|x| < R)}$$

## 7. OBSERVABILITY INEQUALITIES

We use the notation in [1]. We consider again the problem

$$(7.1) \quad (-\Delta - \lambda^2 - i\lambda n^2(x))u^{x_0, \omega_0} = h^{x_0, \omega_0}(x, \lambda)$$

where we let  $x_0$  denote the position of the center of the plane wave source. Sources are indexed by  $x_0$  and  $\omega_0$ . Let

$$(7.2) \quad \partial\mathcal{S}\Omega^+ = \{(x_0, \omega_0) : x_0 \in \partial\Omega, \langle \nu, \omega_0 \rangle > 0\}$$

where  $\nu$  denotes the outward unit normal to the boundary.

**Theorem 1.** *Let  $N \geq (1 - d)/2 + 4$ , then there exists a constant  $C_1$  which depends on  $\text{diam}(\Omega)$  the  $C^0(\Omega)$  norm of  $n_i^2(x)$ ,  $i = 1, 2$  and a constant  $C_2$  which depends on the  $\text{diam}(\Omega)$  and the  $C^{N+1}(\Omega)$  norm of  $n_i^2(x)$ ,  $i = 1, 2$  such that if  $u_1^{x_0, \omega_0}$  and  $u_2^{x_0, \omega_0}$  solve the radiation problem with attenuation coefficients  $n_1^2$  and  $n_2^2$  then it follows that if*

$$(7.3) \quad \lambda^{-1} < \epsilon_0 \quad \delta = \sup_{\partial\mathcal{S}\Omega^+} \| |u_1^{x_0, \omega_0}| - |u_2^{x_0, \omega_0}| \| < \epsilon_0$$

then this implies

$$(7.4) \quad \| |n_2^2 - n_1^2| \|_{H^{-1/2}(\Omega)} \leq C_1 \left( \frac{C_2}{\lambda^{\beta'}} + \delta \right)$$

for some  $\beta' \in (0, (2d)^{-1})$ .

The uniqueness corollary follows immediately.

**Corollary 2.** *Let  $\delta$  be defined in Theorem 1 above. Let  $\delta = 0$  and  $\lambda \rightarrow \infty$  then this implies  $n_1^2 = n_2^2$ .*

We consider our globally defined complex optics solutions  $\tilde{U}_1$  and  $\tilde{U}_2$  which were constructed previously. Dropping the superscripts  $x_0, \omega_0$  where it is understood, from our approximation we know the main term of interest is

$$(7.5) \quad \| |u_1| - |u_2| \| = \left| |\tilde{U}_1| - |\tilde{U}_2| \right| + \mathcal{O}(\lambda^{-\beta'})$$



for  $x \in \partial\Omega$ . This estimate is a result of making our approximate solution with  $N$  sufficiently large so we can conclude from (6.5) and (6.1)

$$(7.6) \quad \left\| \tilde{U}_1 - u_1 \right\|_{H^1(\mathbb{R}^n)} \leq \frac{C}{\lambda}$$

Here we let constant  $C$  be a generic constant depending on  $C^{N+1}(\Omega)$  norm of  $n_1^2(x)$  by construction. We use the fact  $\tilde{U}_1$ , and  $u_1$  are bounded in  $C^{N+1}(\Omega)$  norm to conclude

$$(7.7) \quad \left\| \tilde{U}_1 - u_1 \right\|_{C^0(\mathbb{R}^n)} \leq \frac{C}{\lambda^{\beta'}}$$

for some  $\beta' \in (0, 1)$ . We then use the estimate on the first order terms

$$(7.8) \quad \|U_1 - a_0 \exp(i\lambda\psi)\|_{C^0(\mathbb{R}^n)} \leq \frac{C}{\lambda}$$

Combining the estimates (7.8) and (7.12), gives the estimate (7.5). Now we need to combine the estimates to recover the X-ray transform. We start with the following Lemma

**Lemma 1.** *Let  $A(x)$  and  $B(x)$  be positive  $C^0(\mathbb{R})$  functions and  $\epsilon \in (0, 1)$  such that*

$$(7.9) \quad \|\exp(-A(x)) - \exp(-B(x))\|_{C^0(\mathbb{R})} < \epsilon$$

*It follows that there is a constant  $C$  depending on the  $C^0(\mathbb{R})$  norm of  $A$  and  $B$  such that*

$$(7.10) \quad \|A(x) - B(x)\|_{C^0(\mathbb{R})} < C\epsilon$$

*Proof.* By the mean value theorem, there exists an  $r_*$  between  $B(x)$  and  $A(x)$  for each fixed  $x$  such that

$$(7.11) \quad |(\exp(-A) - \exp(-B))| = \left| \left( - \int_B^A \exp(-r) dr \right) \right| = |((B - A) \exp(-r_*))|$$

taking the supremum over  $x$ , and then applying (7.9) yields the desired result.  $\square$

We would like to use the zeroth order coefficients to a reconstruction of the attenuation coefficient. We know using (3.11) that

$$(7.12) \quad \left| a(x) \exp(i\lambda\psi) - a_0(0) \exp \left( - \int_0^s (n_1^2(x(t))) dt - \int_0^s \text{tr}M(t) dt \right) \exp(\lambda\underline{\psi(x)})\phi(x) \right| = \mathcal{O}(\lambda^{-1/2d})$$

by choice of the cutoff function in (3.9). Examining (7.5), we are interested in the left hand side which using (7.12) can be approximated by

$$(7.13) \quad |a_0(0) \left( \exp \left( - \int_0^s (n_1^2(x(t))) dt \right) - \exp \left( - \int_0^s (n_2^2(x(t))) dt \right) \right) \times \exp \left( - \int_0^s \text{tr}M(t) dt \right) \exp(\lambda\underline{\psi(x)})\phi(x)|$$

where

$$(7.14) \quad \underline{\psi(x)} = (-(x - x(s))\Im M(s) \cdot (x - x(s))).$$

Because  $\Im M(s)$  is a positive definite matrix by Corollary 1 we have

$$(7.15) \quad \sup_{x \in \Omega} \left| \exp(\lambda\underline{\psi(x)})\phi(x) \right| = 1$$

since we know  $x(s)$  reaches the boundary, and is contained in  $A_\lambda$ . Because the rest of the coefficients of (7.13) are independent of  $x$ , we obtain that the supremum of (7.13) over  $x \in \partial\Omega$  is equal

$$(7.16) \quad C(s) \left| a_0(0) \left( \exp \left( - \int_0^s (n_1^2(x(t))) dt \right) - \exp \left( - \int_0^s (n_2^2(x(t))) dt \right) \right) \right|$$

where

$$(7.17) \quad C(s) = \exp \left( - \int_0^s \text{tr}M(t) dt \right)$$

We apply Lemma 1 which gives

$$(7.18) \quad \|I(n_1^2 - n_2^2)\|_{C^0(\partial\mathcal{S}\Omega^+)} \leq C_1 \left( \delta + \frac{C_2}{\lambda^{\beta'}} \right)$$

where  $C_1$  denotes a generic constant depending on the  $C^0(\Omega)$  norm of  $n_1^2, n_2^2$  and  $C_2$  depends on the  $C^{N+1}(\Omega)$  norm respectively. Now we set  $n_1^2 - n_2^2 = \tilde{n}^2$ , which we recall has compact support. Because we have assumed that  $\Omega$  is strictly convex, it follows from (7.18) that

$$(7.19) \quad \sup_{x \in \Omega, \theta \in \mathbb{S}^{d-1}} \left| \int_{-\infty}^{\infty} \tilde{n}^2(x + s\theta) ds \right| \leq C_1 \left( \delta + \frac{C_2}{\lambda^{\beta'}} \right)$$

For  $x \in \mathbb{R}^d$  and  $\theta \in \mathbb{S}^{n-1}$  we set

$$(7.20) \quad Xf(x, \theta) = \int_{-\infty}^{\infty} \tilde{n}^2(x + s\theta) ds.$$

Because  $\tilde{n}^2$  is compactly supported we can conclude

$$(7.21) \quad \left| \int_{\mathbb{S}^{d-1}} \int_{x \in \theta^\perp} |X\tilde{n}^2(x, \theta)|^2 dx d\theta \right|^{1/2} \leq C_1 \left( \delta + \frac{C_2}{\lambda^{\beta'}} \right)$$

As per [14], we define the following norm for all  $\alpha_0 \in \mathbb{R}$

$$(7.22) \quad \|g\|_{H^{\alpha_0}(T)} = \int_{\mathbb{S}^{d-1}} \int_{\omega_0^\perp} (1 + |\eta|^2)^{\frac{\alpha_0}{2}} |\hat{g}(\theta, \eta)|^2 d\eta d\theta.$$

where the Fourier transform is defined by

$$(7.23) \quad \hat{f}(\eta) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^d} \exp(-i\eta \cdot x) f(x) dx$$

The monograph [14] gives a stability estimate for functions in terms of the transform  $Xf$ .

**Theorem 2.** [14]/[Theorem 2.18] *For every  $\alpha$  there are positive constants  $c(\alpha, d, \Omega)$  and  $C(\alpha, d, \Omega)$  such that for  $f \in C_0^\infty(\mathbb{R}^d)$  we have that*

$$(7.24) \quad c(\beta, d) \|f\|_{H_0^\alpha(\Omega)} \leq \|Xf\|_{H^{\alpha+1/2}(T)} \leq C(\alpha, d) \|f\|_{H_0^\alpha(\Omega)}$$

We can now give a reconstruction theorem. We recall the Fourier transform is an isometry on  $L^2$ . To finish the proof of Theorem 1 we use (7.21) along with the stability estimate above with  $\alpha = -1/2$ . We notice that  $\tilde{n}^2 \in C_0^\infty(\Omega)$  by choice of  $n_1^2 = n_2^2 \equiv 1$  on  $\partial\Omega$ .

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ALDEN.WATERS@ENS.FR, CNRS, ECOLE NORMALE SUPERIEURE, DEPARTMENT OF MATHEMATICS AND APPLICATIONS, 45 RUE D’ULM, PARIS 75005