Primal-dual method for continuous max-flow approaches

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ABSTRACT: We review the continuous max-flow approaches for the variational image segmentation models with piecewise constant representations. The review is conducted by exploring the primal-dual relationships between the continuous min-cut and max-flow problems. In addition, we introduce the parameter free primal-dual method for solving those max-flow problems. Empirical results show that the primal-dual method is competitive to the augmented Lagrangian method.

1 INTRODUCTION

Multi-phase image segmentation (or multi-labelling) is an important problem in image processing and has a wide range of applications in related areas such as computer vision (Paragios, Chen, & Faugeras 2005), stereo (Kolmogorov & Zabih 2002, Kolmogorov & Zabih 2004a) and 3D reconstruction (Vogiatzis, Esteban, Torr, & Cipolla 2007).The basic task in image segmentation is to optimally assign different labels to different pixels of an image with respect to some energy functional.

In the spatially discrete setting, the image is usually modelled as a graph, and the solution of the multi-phase image labelling problem can be found by computing the min-cut or max-flow solutions of the graph, see (Kolmogorov & Boykov 2005, Kolmogorov & Zabih 2004b, Boykov, Veksler, & Zabih 2001, Boykov & Kolmogorov 2003) and references therein. In the spatially continuous setting, the variational approach has been widely studied, where the problem is formulated as the minimization problem of a continuous energy functional. Compared with the graph-based method in the discrete setting, the variational method has several advantages: 1) it can avoid the metrication errors thanks to the crucial rotation invariance property; 2) a wide range of reliable numerical schemes are available, and these schemes can be easily implemented and accelerated; 3) it requires less memory in computation; 4) it is easy to use GPU and parallel processors.

In this paper, we focus on the variational approach for the image segmentation problem. Let Ω denote the domain of an input image. Then the task is to find a partition $\{\Omega_i\}_{i=1}^n$ of Ω which minimises the following energy functional

$$\min_{\{\Omega_i\}_{i=1}^n} \sum_{i=1}^n \int_{\Omega_i} f_i(x)dx + \alpha R\left(\{\Omega_i\}_{i=1}^n\right)$$

(1)

$$\text{s.t. } \cup_{i=1}^n \Omega_i = \Omega \text{ and } \Omega_k \cap \Omega_l = \emptyset \text{ for } k \neq l,$$

where $R(\cdot)$ is a regularisation term. Concrete representations for $\{\Omega_i\}_{i=1}^n$ are needed so that we can design numerical algorithms to solve (1). Over the last 30 years, many representations have been proposed, for example the level set method (Osher & Sethian 1988, Chan & Vese 2001). In this work, we

are interested in the piecewise constant representations where each region is represented by a unique binary or integer value (Lie, Lysaker, & Tai 2005, Lie, Lysaker, & Tai 2006b, Lie, Lysaker, & Tai 2006a). With the piecewise constant representations, convex relaxations can often be constructed for (1) and so efficient and robust numerical algorithms can be developed based on those relaxations. So far there are three typical piecewise constant representations for $\{\Omega_i\}_{i=1}^n$ and each representation will result in a particular form of (1).

Binary value representation: For each partition Ω_i, we can define an indicator function

$$u_i(x) = \begin{cases} 1 & x \in \Omega_i \\ 0 & x \notin \Omega_i, \end{cases} \quad i = 1, \cdots, n.$$

With these indicator functions, model (1) with the Potts regulariser $R(\{\Omega_i\}_{i=1}^n) = \sum_{i=1}^n |\partial\Omega_i|$ can be rewritten as[1]

$$\min_{u_i \in \{0,1\}, u \in S} \sum_{i=1}^n \int_\Omega u_i f_i dx + \alpha \sum_{i=1}^n \int_\Omega |\nabla u_i|\, dx \quad (2)$$

where

$$S = \left\{ (u_1, \cdots, u_n): \ \sum_{i=1}^n u_i = 1, \ u_i \geq 0 \right\}.$$

Such a representation was used in (Lellmann, Kappes, Yuan, Becker, & Schnörr 2009, Zach, Gallup, Frahm, & Niethammer 2008). In (Bae, Yuan, & Tai 2011), global minimization property for this model was proven. In (Yuan, Bae, Tai, & Boykov 2010), this model was interpreted as a min-cut problem with a corresponding max-flow model.

Integer value representation: Let $u : \Omega \to \{1, \cdots, n\}$ be a labelling function such that $u(x) = i$ if $x \in \Omega_i$, $i = 1, \cdots, n$. Let $\psi_i(u)$ be the corresponding indicator functions for Ω_i which is related to u. Then model (1) with the regulariser $R(\{\Omega_i\}_{i=1}^n) = \sum_{i=1}^n |\partial\Omega_i|$ reduces to

$$\min_{u \in \{1,2\cdots n\}} \sum_{i=1}^n \int_\Omega f_i \psi_i(u) dx + \alpha \sum_{i=1}^n \int_\Omega |\nabla\psi_i(u)| dx,$$

$$(3)$$

This model was first introduced in (Lie, Lysaker, & Tai 2005, Lie, Lysaker, & Tai 2006b). Augmented Lagrangian was used in (Lie, Lysaker, & Tai 2005, Lie, Lysaker, & Tai 2006b) to solve it. In (Bae & Tai 2009), discrete graph cut was used to solve this model which is very much related to the graph based method

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[1]In the sequel, we will omit the notation x when there is no confusion from the context.

in (Ishikawa 2003, Darbon & Sigelle 2006). The interpretation of this model as a min-cut problem and its corresponding max-flow model was given in (Bae, Yuan, Tai, & Boykov 2010, Bae, Yuan, Tai, & Boykov 2014a). This problem can also be transformed into a binary minization problem. If we replace the regulariser by $R(\{\Omega_i\}_{i=1}^n) = \int_\Omega |\nabla u|\, dx$ and introduce

$$\lambda_i(x) = \begin{cases} 1 & \text{if } u(x) > i \\ 0 & \text{if } u(x) \leq i \end{cases}, \ i = 1, \cdots, n-1$$

and $\lambda_0(x) = 1$, $\lambda_n(x) = 0$. Then model (1) can also be written as a binary minimization problem (Liu, Tai, Leung, & Huang 2014)

$$\min_{\lambda_i \in \{0,1\}, \lambda \in B} \sum_{i=1}^n \int_\Omega (\lambda_{i-1} - \lambda_i) f_i dx + \alpha \sum_{i=1}^{n-1} \int_\Omega |\nabla\lambda_i| dx,$$

$$(4)$$

where

$$B = \{(\lambda_0, \cdots, \lambda_n): \ 1 = \lambda_0 \geq \cdots \geq \lambda_n = 0\}.$$

After solving (4), the labelling function u can be recovered by $u = \sum_{i=1}^n (\lambda_{i-1} - \lambda_i) \cdot i$.

Product of binary values representation: This representation was was first given in (Lie, Lysaker, & Tai 2005, Lie, Lysaker, & Tai 2006a) extending the multiphase level set framework of Vese-Chan (Vese & Chan 2002). In the case of two-phase labelling, its convex relaxation and global minimization was proposed in (Chan, Esedoglu, & Nikolova 2006). For simplicity, assume $n = 2^m$. Define $w_0(s) = 1 - s$ and $w_1(s) = s$ and introduce m binary function $\phi_1, \cdots, \phi_m : \Omega \to \{0, 1\}$ such that $x \in \Omega_i$ if and only if $\phi_1 \cdots \phi_m$ is a binary representation of the integer i. With these notations, model (1) with the regulariser $R(\{\Omega_i\}_{i=1}^n) = \sum_{i=1}^m |\nabla\phi_i|\, dx$ can be formulated as

$$\min_{\phi_i \in \{0,1\}} \int_\Omega \sum_{i=1}^n \prod_{k=1}^m w_{a_k^i}(\phi_k) f_i dx + \alpha \sum_{k=1}^m \int_\Omega |\nabla\phi_k|\, dx,$$

$$(5)$$

where $a_1^i \cdots a_m^i$ is a binary representation of i, denoted by $i = [a_1^i \cdots a_m^i]$. The interpretation of this model as a min-cut problem and its corresponding max-flow model was given in (Bae, Lellmann, & Tai 2013, Bae & Tai 2015, Liu, Tai, Leung, & Huang 2014).

2 CONTINUOUS MAX-FLOW APPROACHES

Three piecewise constant representations and the corresponding formulations for labelling problems were

introduced in the last section. All the three formulations are non-convex because either the constraints are non-convex or both the objective functional and the constraints are non-convex. A number of approaches have appeared in the literature recently showing that these models can be interpreted as continuous min-cut problems. There exists a corresponding continuous max-flow model for each of these three min-cut problems. Exploring the connection between these continuous min-cut and max-flow problems, convex global minimization method could be derived for these non-convex problems.

Since the pioneer work of Chan et al. (Chan, Esedoglu, & Nikolova 2006), one of the important research topics in image segmentation is to find good convex relaxations for the non-convex minimization problems arising from the variational methods and then design numerical algorithms for the corresponding convex relaxations. In a series of research papers (Yuan, Bae, Tai, & Boykov 2014, Bae & Tai 2015, Bae, Yuan, Tai, & Boykov 2014b, Bae, Lellmann, & Tai 2013, Yuan, Bae, Tai, & Boykov 2010), the authors have developed a set of computationally efficient algorithms for the models in the last section. The essential idea is to interpret these models as continuous min-cut problems and each of them has a corresponding continuous max-flow problem. This section reviews the continuous max-flow approaches from the primal-dual perspective. However, we use a different derivation to get these continuous max-flow models.

2.1 *Two-phase image segmentation*

We first consider the Chan-Vese model (Chan, Esedoglu, & Nikolova 2006) for the two phase segmentation problem

$$\min_{u \in \{0,1\}} \int_{\Omega} (1-u)C_s dx + \int_{\Omega} uC_t dx + \alpha \int_{\Omega} |\nabla u|\, dx.$$
(6)

Model (6) can be viewed as a special case of either the formulation (2) or (5). Since (6) is a non-convex optimization problem, it is not tractable to find its global solution. Thus the authors propose to relax the binary valued function u to $0 \leq u \leq 1$ and solve the following convex problem

$$\min_{u \in [0,1]} \int_{\Omega} (1-u)C_s dx + \int_{\Omega} uC_t dx + \alpha \int_{\Omega} |\nabla u|\, dx.$$
(7)

Moreover, it is proved in (Chan, Esedoglu, & Nikolova 2006) that the solution to (6) can be obtained by thresholding the solution to (7). However,

the numerical algorithms for (7) usually suffer from the non-smoothness of the TV term. We can form a dual problem of (7) as follows:

$$\min_{u \in [0,1]} \int_{\Omega} (1-u)C_s dx + \int_{\Omega} uC_t dx + \alpha \int_{\Omega} |\nabla u|\, dx$$

$$= \min_{u \in [0,1]} \max_{\substack{p_s \leq C_s \\ p_t \leq C_t \\ |q| \leq \alpha}} \int_{\Omega} (1-u)p_s dx + \int_{\Omega} u p_t dx + \int_{\Omega} u \operatorname{div} q dx$$

$$= \min_{u \in \mathbb{R}} \max_{\substack{p_s \leq C_s \\ p_t \leq C_t \\ |q| \leq \alpha}} \int_{\Omega} (1-u)p_s dx + \int_{\Omega} u p_t dx + \int_{\Omega} u \operatorname{div} q dx$$

$$= \max_{\substack{p_s \leq C_s \\ p_t \leq C_t \\ |q| \leq \alpha}} \min_{u \in \mathbb{R}} \int_{\Omega} (1-u)p_s dx + \int_{\Omega} u p_t dx + \int_{\Omega} u \operatorname{div} q dx$$

$$= \max_{\substack{p_s \leq C_s \\ p_t \leq C_t \\ |q| \leq \alpha}} \min_{u \in \mathbb{R}} \int_{\Omega} p_s dx + \int_{\Omega} u \left(\operatorname{div} q - p_s + p_t \right) dx$$

$$= \max \int_{\Omega} p_s dx \ \text{ s. t. } \ \begin{cases} p_s \leq C_s, \ p_t \leq C_t, \ |q| \leq \alpha \\ \operatorname{div} q - p_s + p_t = 0, \end{cases}$$
(8)

where in the first equality, we use the fact (Giusti 1977)

$$\int_{\Omega} \alpha \, |\nabla u|\, dx = \max_{|q| \leq \alpha} \int_{\Omega} u \operatorname{div} q dx$$

in the distributional sense; and the third equality follows from the minmax theorem (Ekeland & Teman 1999, Chapter 6, Proposition 2.4). Instead of regarding (7) as a convex relaxation, our view point is that (6) is a *min-cut problem* over a continuous domain and the last maximization problem in (8) is a *max-flow problem*. Problem (7) is the *dual problem* of (8). The pair of min-max problems (7) and (8) are continuous analogue of the min-cut and max-flow problems in the graph based methods (Yuan, Bae, & Tai 2010, Yuan, Bae, Tai, & Boykov 2014). In the discrete case, it is well-known that the min-cut problem is equivalent to the max-flow problem. In the continuous setting, the following three problem are equivalent, c.f. (Yuan, Bae, Tai, & Boykov 2014, Prop. 3.1), and thus gives another way to explain the convex relaxation model (7):

Min-cut problem (6) $\Leftrightarrow$ Max-flow problem (8)

$\Leftrightarrow$ Dual of max-flow problem (7)

From the above derivation, we can see that the so-called cut for (7) is the Lagrangian multiplier for the

flow conservation constraint $\operatorname{div} q - p_s + p_t = 0$ of the max-flow problem.

2.2 *Multi-phase image segmentation*

We will derive the max-flow models for the multi-phase image formulations (2), (4) and (5) one by one and show that the equivalence between the min-cut and max-flow problems can also be extended to multi-phase cases.

2.2.1 *Binary value representation*

Following the convex relaxation technique for the two-phase Chan-Vese model, a natural convex relaxation for (2) is

$$\min_{u \in S} \ \sum_{i=1}^{n} \int_{\Omega} u_i f_i dx + \alpha \sum_{i=1}^{n} \int_{\Omega} |\nabla u_i| \, dx. \qquad (9)$$

We can form a dual problem of (9) by introducing another n variables $p_i, i = 1, \cdots, n$ which corresponds to the data fidelity term and an auxiliary variable p_s:

$$\min_{(u_1, \cdots, u_n) \in S} \ \sum_{i=1}^{n} \int_{\Omega} u_i f_i dx + \alpha \sum_{i=1}^{n} \int_{\Omega} |\nabla u_i| \, dx$$

$$= \min_{u_i \geq 0} \max_{p_s} \ \int_{\Omega} \left(1 - \sum_{i=1}^{n} u_i \right) p_s dx + \sum_{i=1}^{n} \int_{\Omega} u_i f_i dx$$

$$+ \alpha \sum_{i=1}^{n} \int_{\Omega} |\nabla u_i| \, dx$$

$$= \min_{\substack{u_i \in \mathbb{R} \\ p_i \leq f_i \\ |q_i| \leq \alpha}} \max_{p_s} \ \int_{\Omega} \left(1 - \sum_{i=1}^{n} u_i \right) p_s dx + \sum_{i=1}^{n} \int_{\Omega} u_i f_i dx$$

$$+ \sum_{i=1}^{n} \int_{\Omega} u_i \operatorname{div} q_i dx$$

$$= \max_{\substack{p_s \\ p_i \leq f_i \\ |q_i| \leq \alpha}} \min_{u_i \in \mathbb{R}} \ \int_{\Omega} \left(1 - \sum_{i=1}^{n} u_i \right) p_s dx + \sum_{i=1}^{n} \int_{\Omega} u_i f_i dx$$

$$+ \sum_{i=1}^{n} \int_{\Omega} u_i \operatorname{div} q_i dx$$

$$= \max_{\substack{p_s \\ p_i \leq f_i \\ |q_i| \leq \alpha}} \min_{u_i \in \mathbb{R}} \ \int_{\Omega} p_s dx + \sum_{i=1}^{n} \int_{\Omega} u_i \left(\operatorname{div} q_i - p_s + p_i \right) dx$$

$$= \max \int_{\Omega} p_s dx \ \text{ s. t. } \begin{cases} p_i \leq f_i, |q_i| \leq \alpha \\ \operatorname{div} q_i - p_s + p_i = 0, \end{cases} \qquad (10)$$

where the third equality follows from the minmax theorem. The maximization problem (10) corresponds to a type of max-flow model in the spatially continuous domain (Yuan, Bae, Tai, & Boykov 2010).

2.2.2 *Integer value representation*

Relaxing $\lambda_i \in \{0, 1\}$ to $\lambda_i \in [0, 1]$ gives the convex relaxation of (11)

$$\min_{\lambda \in B} \sum_{i=1}^{n} \int_{\Omega} (\lambda_{i-1} - \lambda_i) f_i dx + \alpha \sum_{i=1}^{n-1} \int_{\Omega} |\nabla \lambda_i| dx. \quad (11)$$

By fixing $\lambda_0 = 1$ and $\lambda_n = 0$, the dual problem of (11) can be computed as follows:

$$\min_{\lambda \in B} \sum_{i=1}^{n} \int_{\Omega} (\lambda_{i-1} - \lambda_i) f_i dx + \alpha \sum_{i=1}^{n-1} \int_{\Omega} |\nabla \lambda_i| dx$$

$$= \min_{\lambda_i \in \mathbb{R}} \max_{\substack{p_i \leq f_i \\ |q_i| \leq \alpha}} \sum_{i=1}^{n} \int_{\Omega} (\lambda_{i-1} - \lambda_i) p_i dx + \sum_{i=1}^{n-1} \int_{\Omega} \lambda_i \operatorname{div} q_i dx$$

$$= \max_{\substack{p_i \leq f_i \\ |q_i| \leq \alpha}} \min_{\lambda_i \in \mathbb{R}} \sum_{i=1}^{n} \int_{\Omega} (\lambda_{i-1} - \lambda_i) p_i dx + \sum_{i=1}^{n-1} \int_{\Omega} \lambda_i \operatorname{div} q_i dx$$

$$= \max_{\substack{p_i \leq f_i \\ |q_i| \leq \alpha}} \min_{\lambda_i \in \mathbb{R}} \int_{\Omega} p_1 dx + \sum_{i=1}^{n-1} \lambda_i \left(\operatorname{div} q_i - p_i + p_{i+1} \right) dx$$

$$= \max \int_{\Omega} p_1 dx \ \text{ s. t. } \begin{cases} p_i \leq f_i, |q_i| \leq \alpha \\ \operatorname{div} q_i - p_i + p_{i+1} = 0, \end{cases} \qquad (12)$$

where in the first equality $1 \leq i \leq n-1$ for λ_i and q_i, whereas $1 \leq i \leq n$ for p_i; again the second equality follows from the minmax theorem. The maximisation problem (12) also corresponds to a continuous max-flow model (Bae, Yuan, Tai, & Boykov 2014b).

2.2.3 *Product of binary values representation*

The energy functional in (5) is also non-convex in addition to the non-convexity of the constraints. In order to find a convex relaxation of (5), we can compute the pixel-wise convex envelop of the data term, which can be obtained by computing the bi-conjugate of the functional twice (Bae, Lellmann, & Tai 2013). The bi-conjugate of $g(u) := \sum_{i=1}^{n} \prod_{k=1}^{m} w_{a_k^i}(u_k) f_i$, $u_k \in \{0, 1\}$ $(k = 1, \cdots, m)$ is

$$g^*(p) = \max_{u_k \in \{0,1\}} \left\{ \sum_{k=1}^{m} p_k u_k - \sum_{i=1}^{n} \prod_{k=1}^{m} w_{a_k^i}(u_k) f_i \right\}.$$

Consequently the bi-conjugate of g^* is

$$g^{**}(\phi) = \max_{p \in \mathbb{R}^m} \left\{ \sum_{k=1}^{m} \phi_k p_k - g^*(p) \right\}$$

$$= \max_{p \in \mathbb{R}^m} \left\{ \sum_{k=1}^m \phi_k p_k \right.$$

$$\left. + \min_{u_k \in \{0,1\}} \left(\sum_{i=1}^n \prod_{k=1}^m w_{a_k^i} (u_k) f_i - \sum_{k=1}^m p_k u_k \right) \right\}$$

$$= \max_{p \in \mathbb{R}^m} \sum_{k=1}^m \{ \phi_k p_k + p_0 \}$$

$$\text{s. t. } p_0 \le - \sum_{k=1}^m u_k^i p_k + f_i, \text{ for any } i = [u_1^i \cdots u_m^i],$$

where the constraint in the last equality follows from the fact

$$\sum_{i=1}^n \prod_{k=1}^m w_{a_k^i} \left(u_k^i \right) f_i = f_i \text{ if } i = [u_1^i \cdots u_m^i].$$

Thus the problem (5) with the regularisation term can be relaxed to

$$\min_\phi \max_p \int_\Omega p_0 dx + \int_\Omega \sum_{k=1}^m \phi_k p_k dx + \alpha \int_\Omega \sum_{k=1}^m |\nabla \phi_k| dx$$

$$= \min_\phi \max_{p, |q_i| \le \alpha} \int_\Omega p_0 dx + \int_\Omega \sum_{k=1}^m \phi_k p_k dx$$

$$+ \int_\Omega \sum_{k=1}^m \phi_k \operatorname{div} q_k dx$$

$$= \max_{p, |q_i| \le \alpha} \min_\phi \int_\Omega p_0 dx + \int_\Omega \sum_{k=1}^m \phi_k (\operatorname{div} q_k + p_k) dx$$

$$= \max \int_\Omega p_0 dx \text{ s. t. } \begin{cases} p_0 \le -\sum_{k=1}^m u_k^i p_k + f_i \\ |q_k| \le \alpha \\ \operatorname{div} q_k + p_k = 0. \end{cases}$$

$$(13)$$

Moreover, if the data term of (5) is sub-modular, (13) is a tight convex relaxation for (5). In particular, the data term is sub-modular when $n = 4$ (Bae & Tai 2015).

3 GENERAL ALGORITHMIC FRAMEWORK

3.1 *Augmented Lagrangian method*

The maximisation problems (8), (10), (12) and (13) share the following general form

$$\max_p F(p) \text{ s. t. } \begin{cases} p \in C_p \\ q \in C_q, \\ Lq + Dp = 0, \end{cases} \quad (14)$$

where

- $F(p)$ is a (linear) functional of p,
- L and D are linear mappings,
- C_p and C_q are convex sets.

The augmented Lagrangian formulation of (14) is

$$\mathcal{L}(p, q, \gamma) = F(p) + \langle \gamma, Lq + Dp \rangle - \frac{c}{2} \|Lq + Dp\|^2.$$

$$(15)$$

Then with a triple of intial points (p^0, q^0, γ^0), we can solve (14) by the augmented Lagrangian method as follows (Alg. 1):

- $p^{l+1} = \arg\max_{p \in C_p} \mathcal{L}(p, q^l, \gamma^l)$,
- $q^{l+1} = \arg\max_{q \in C_q} \mathcal{L}(p^{l+1}, q, \gamma^l)$,
- $\gamma^{l+1} = \gamma^l - c(Lq + Dp)$.

When there are no explicit solutions to the first two subproblems, they can be solved approximately by several projected ascent iterations (Yuan, Bae, Tai, & Boykov 2014, Bae & Tai 2015, Bae, Yuan, Tai, & Boykov 2014b, Bae, Lellmann, & Tai 2013, Yuan, Bae, Tai, & Boykov 2010). The penalization parameter c is chosen by trial-and-error in numerical tests.

3.2 *Primal-dual method*

In this section, we will provide another method for solving the continuous max-flow problems in Sec. 2. As it can be observed from the derivation process, each continuous max-flow approach is associated with a min-max problem of the form

$$\min_{\gamma \in C_\gamma} \max_{\substack{p \in C_p \\ q \in C_q}} F(p, q, \gamma), \quad (16)$$

where $F(p, q, \gamma)$ is (linear) functional of p, q and γ. Then following the general set-up of the primal-dual method in (Chambolle & Pock 2011, Esser, Zhang, & Chan 2010), we can solve (16) as follows (Alg. 2):

- $p^{l+1} = p^l + \tau_l^p \frac{\partial F(p, q^l, \gamma^l)}{\partial p}$,
- $q^{l+1} = q^l + \tau_l^q \frac{\partial F(p^{l+1}, q, \gamma^l)}{\partial q}$,
- $\gamma^{l+1} = \gamma^l - \beta_l \frac{\partial F(p^{l+1}, q^{l+1}, \gamma)}{\partial \gamma}$,

where τ_l^p, τ_l^q and β_l are adaptive stepsizes. The primal-dual method is parameter free compared with the augmented Lagrangian method in Sec. 3.1 as there is no proximal regularizer in Alg. 2. Moreover, we can choose the stepsizes automatically to accelerate the algorithm, for example see Tab. 1. In general, τ_l^i $(i = p, q)$ and β_l should be inverse to each other, which are proportional to l and $1/l$ respectively.

4 NUMERICAL EXPERIMENTS

This section compares the aforementioned two algorithms on the image segmentation problems. The tests are conducted on a Mac Pro laptop with Intel Core i5 CPUs @ 2.4 GHz and executed from Matlab. For the discretized version of the gradient, divergence and TV operators, we refer the readers to (Zhu 2008, Ch. 1.3).

In our tests, the data term has the form $f_i = |u - c_i|^\beta$, where u is the input image and c_i is the average intensity of the region i which is assumed to be fixed. Apparently, f_i is convex when $\beta \geq 1$ and non-convex when $\beta < 1$. Here β is set to 1.

In order to compare the convergence rate of the two tested algorithms, we first (numerically) compute the optimal primal energy, denoted by E^*, by running either of the algorithms 10000 iterations. Then we evaluate the progress of the algorithm by computing the relative error of the primal energy defined as

$$e^l = \frac{|E^l - E^*|}{E^*}, \tag{17}$$

where E^l is the primal energy in the lth iteration.

We test the primal-dual and the augmented Lagrangian methods on the standard cameraman image (see the left of Fig. 1) for the two-phase image segmentation problem and on the standard noisy 3 color image (see the right of Fig. 1) for the multi-phase image segmentation problem. Due to the limited space, we only conduct the tests for the variational model with the integer value representation in Sec. 2.2.2. In Alg. 1, c is selected to be 0.3, and the adaptive stepsize selections in Alg. 2 are listed in Tab. 1.

Table 1: Stepsize selections for the primal-dual method (Alg. 2)

	τ_l^p	τ_l^q	β_l
Cameraman	$0.1 * l$	$0.04 * l - 0.02$	$\frac{1}{0.08 * l + 0.2}$
3 color	$0.005 * l + 0.03$	$0.005 * l$	$\frac{1}{0.02 * l}$

The segmentation results are plotted in Figs. 2. By comparing the plots before and after thresholding, we can see that both the primal-dual and the augmented Lagrangian methods can achieve very good performance since they all converge to a result which takes the values 0 and 1 almost everywhere. The computational results are presented in Tab. 2 which includes the number of iterations and computational time needed for the algorithm to converge to certain accuracy. Table 2 clearly shows that the primal-dual method with the adaptive stepsize selections can be faster than the augmented Lagrangian method.

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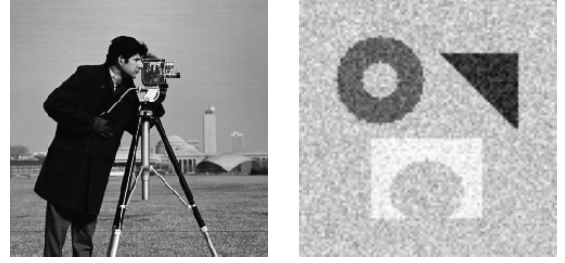
Figure 1: Input images for image segmentation. Left: cameraman for two-phase image segmentation; Right: 3 color for multiphase image segmentation.

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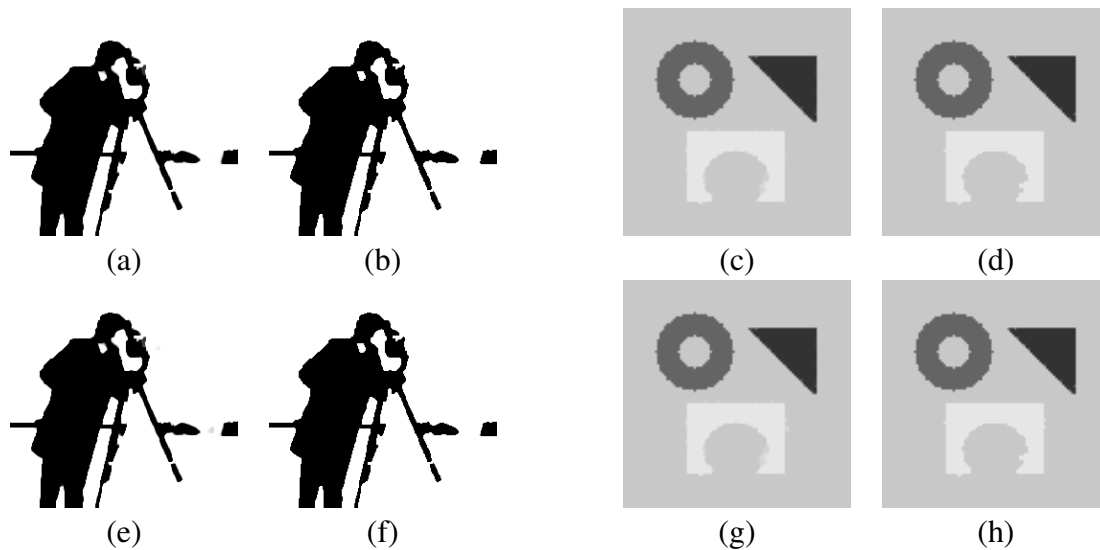
Figure 2: The first row: rescaled labelling function returned by the primal-dual method (Alg. 2) before ((a) and (c))and after ((b) and (d)) thresholding; The second row: rescaled labelling function returned by the augmented Lagrangian method (Alg. 1) before ((e) and (g)) and after ((f) and (h)) thresholding. The thresholding value is 0.5.

Table 2: Computational results of the image segmentation for the two-phase (cameraman) and multi-phase (3 colour) image segmentations by the primal-dual and the augmented Lagrangian methods. The algorithm terminates if e^l defined in (17) is less than ε.

| | cameraman | | | | 3 color | | | |
| | $\varepsilon = 1e-3$ | | $\varepsilon = 1e-5$ | | $\varepsilon = 1e-3$ | | $\varepsilon = 1e-5$ | |
	iter	time (s)	iter	time (s)	iter	time (s)	iter	time (s)
Primal-dual (Alg. 2)	32	0.15	118	0.51	25	0.11	65	0.25
Augmented Lagrangian (Alg. 1)	32	0.16	149	0.67	40	0.18	215	0.89

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