

The physics of multilayer networks

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The study of networks plays a crucial role in investigating the structure, dynamics, and function of a wide variety of complex systems in myriad disciplines. Despite the success of traditional network analysis, standard networks provide a limited representation of these systems, which often includes different types of relationships (i.e., “multiplexity”) among their constituent components and/or multiple interacting subsystems. Such structural complexity has a significant effect on both dynamics and function. Throwing away or aggregating available structural information can generate misleading results and provide a major obstacle towards attempts to understand the system under analysis. The recent “multilayer” approach for modeling networked systems explicitly allows the incorporation of multiplexity and other features of realistic networked systems. On one hand, it allows one to couple different structural relationships by encoding them in a convenient mathematical object. On

the other hand, it also allows one to couple different dynamical processes on top of such interconnected structures. The resulting framework plays a crucial role in helping to achieve a thorough, accurate understanding of complex systems. The study of multilayer networks has also revealed new physical phenomena that remained hidden when using the traditional network representation of graphs. Here we survey progress towards a deeper understanding of dynamical processes on multilayer networks, and we highlight some of the physical phenomena that emerge from multilayer structure and dynamics.

Introduction

Networks provide a powerful representation of interaction patterns in complex systems^{1–3}. The structure of social relations among individuals, interactions between proteins, food webs, and many other situations can be represented using networks. Until recently, the vast majority of studies in network science have focused networks that consist of a single type of entity, with different entities connected to each other via a single type of connection. Such networks are now called single-layer (or monolayer) networks. The idea of incorporating additional information — such as multiple types of interactions, subsystems, and time-dependence — has long been pointed out in various fields, such as sociology^{4,5} and engineering⁶, but an effective unified framework for the mathematical treatment of such multidimensional structures, which are usually called multilayer networks, has been developed only recently^{7,8}.

Multilayer networks can be used to model many complex systems. For example, relationships between humans include different kinds of interactions — such as relationships between family members, friends, and coworkers — that constitute different layers of a social system. Different layers of connectivity also arise naturally in natural and human-made systems in transportation⁹, ecology¹⁰, neuroscience¹¹, and numerous other areas. The potential of multilayer networks for representing complex systems more accurately than was previously possible has led to an explosion

of work on the physics of multilayer networks.

The main question posed a few years ago concerned the implications of multilayer structures for the dynamics of complex systems, and several seminal papers about interdependent networks — a special type of multilayer network — revealed that such structures can change the qualitative behaviors in a significant way^{12,13}. These findings thus posed the challenge for how to account for multiple layers of connectivity in a consistent mathematical way. An explosion of recent papers has developed the field of multilayer networks into its modern form, and there is now a suitable mathematical framework¹⁴ and novel structural descriptors^{15–19} for studying these systems. Many studies have also started to highlight the importance of analyzing multilayer networks, instead of relying on their monolayer counterparts^{20,21}, to gain new insights about empirical systems.

Nowadays, we know that the study of multilayer networks is fundamental for enhancing our understanding of dynamical processes on real networked systems, such as flow (or its congestion) in transportation network^{22,23}, cascading failures in interdependent infrastructures¹², and information and disease spreading in social networks^{24–27}. For instance, when two spreading processes are coupled in a multilayer network, the onset of one disease-spreading process depends on the onset of the other one, yielding a curve of critical points in the phase diagram of the parameters that govern a system’s spreading dynamics²⁵. Such a curve reveals the existence of two distinct regimes, such that the criticality of the two dynamics may or may not be interdependent. Similarly, cooperative behavior can be enhanced by multilayer structures, providing a novel way for cooperation to survive in structured populations²⁸. For additional examples, see various reviews and surveys^{7,8,13,24,29–31} on multilayer networks and specific topics within them.

A multilayer framework allows a natural representation of coupled structures and coupled dynamical processes. Here, after a brief overview on the representation of multilayer networks, we will focus on dynamical processes in which multilayer analysis has revealed new physical be-

havior. Specifically, we will discuss two cases: (i) a single dynamical process, such as continuous and discrete diffusion, running on the top of a multilayer structure; and (ii) different dynamical processes, each one running on the top of each layer and coupled/mixed by a multilayer structure.

Structural representation of multilayer networks

One can represent a monolayer network mathematically by using an adjacency matrix, which is a 2nd-order tensor. This tensor encodes information about (possibly directed and/or weighted) relationships among entities in a network. Because multilayer networks include multiple dimensions of connectivity, called aspects, that have to be considered simultaneously, their structure is much richer than that of ordinary networks. Possible aspects include different types of interactions or communication channels, different subsystems, different spatial locations, different points in time, and more. One uses higher-order tensors to encode the connectivity of multilayer networks as (multi)linear-algebraic objects^{7,14}. Multilayer networks include three types of edges: intra-layer edges (connecting nodes within the same layer), inter-layer edges between replica nodes (i.e., copies of the same entity) in different layers, and inter-layer edges between nodes that represent distinct entities.

Let N be the number of nodes in a multilayer network, and let L be the number of layers. The components $m_{i\alpha}^{j\beta}$ of a 4th-order tensor $\mathbf{M}$ encode the relationship between any node i in layer α and any node j in layer β in the system ($i, j \in \{1, 2, \dots, N\}$ and $\alpha, \beta \in \{1, 2, \dots, L\}$).

We use δ_i^j and δ_α^β , respectively, to indicate the Kronecker delta function for indices corresponding to nodes and layers. We separate the contributions of node–node relationships within

and across layers of the multilayer network as

$$\begin{aligned}
m_{i\alpha}^{j\beta} &= \underbrace{m_{i\alpha}^{j\beta} \delta_{\alpha}^{\beta} \delta_i^j + m_{i\alpha}^{j\beta} \delta_{\alpha}^{\beta} (1 - \delta_i^j)}_{\text{intra-layer relationships}} + \underbrace{m_{i\alpha}^{j\beta} (1 - \delta_{\alpha}^{\beta}) \delta_i^j + m_{i\alpha}^{j\beta} (1 - \delta_{\alpha}^{\beta}) (1 - \delta_i^j)}_{\text{inter-layer relationships}} \\
&= \underbrace{m_{i\alpha}^{i\alpha}}_{\text{self-relationships}} + \underbrace{m_{i\alpha}^{j\alpha}}_{\text{endogenous}} + \underbrace{m_{i\alpha}^{j\beta}}_{\text{exogenous}} + \underbrace{m_{i\alpha}^{i\beta}}_{\text{intertwining}} \\
&= \mathbb{S}_{i\alpha}(\mathbf{M}) + \mathbb{N}_{i\alpha}^j(\mathbf{M}) + \mathbb{X}_{i\alpha}^{j\beta}(\mathbf{M}) + \mathbb{I}_{i\alpha}^{\beta}(\mathbf{M}). \tag{1}
\end{aligned}$$

We call Eq. (1) the “structural ($\mathbb{S}\mathbb{N}\mathbb{X}\mathbb{I}$) decomposition” of the multilayer adjacency tensor $\mathbf{M}$. Different types of multilayer networks arise from contributions of different $\mathbb{S}\mathbb{N}\mathbb{X}\mathbb{I}$ components in the tensorial representation of a network. In Fig. 1, we illustrate some common types of multilayer networks.

Once the connectivity of the nodes and layers are encoded in a tensor, one can define novel measures to characterize the multilayer structure. However, this is a delicate process, as naively generalizing existing concepts from monolayer networks can lead to qualitatively incorrect or non-sensical results⁷. Studies of structural properties of multilayer networks include descriptors to identify the most central nodes according to a specific notion of importance^{14–18} and quantify triadic relations such as clustering and transitivity^{14, 17, 19}. Significant advances have been achieved to reduce the structural complexity of a multilayer system³², to unveil mesoscale structures (e.g., communities of densely-connected nodes)^{33–35}, and to quantify intra-layer and inter-layer correlations^{36–38} in empirical networked systems.

The structural properties depend crucially on how layers are coupled together to form a multilayer structure. Inter-layer edges provide the coupling and help encode structural and dynamical features of a system, and their presence (or absence) produces fascinating structural and dynamical effects. For example, in multimodal transportation systems, in which layers represent different transportation modes, the weight of inter-layer connections might encode an economic or tempo-

ral cost to switching between two modes³⁹. In multilayer social networks, inter-layer connections allow models to tune, in a natural way, an individual's self-reinforcement in opinion dynamics⁴⁰. Depending on the relative importance of intra-layer and inter-layer connections, a multilayer network can act either as a system of independent entities, in which layers are structurally decoupled, or as a single-layer system, in which layers are indiscernible in practice. In some multilayer networks, one can even derive a sharp transition between these two regimes^{41,42}.

Single and mixed dynamics on multilayer networks

There are two different categories of dynamical processes on multilayer networks: (i) a single dynamical process on top of the coupled structure of a multilayer network (see Fig. 2a); and (ii) two or more dynamical processes are defined on each layer separately and are coupled together by the presence of inter-layer connections between nodes, mixing their effects (see Fig. 2b).

As in the case of structure, it is possible to introduce a unifying framework, in terms of a dynamical $\mathbb{S}\mathbb{N}\mathbb{X}$ decomposition, to describe dynamics on multilayer networks (this is in the same spirit of coupled-cell networks⁴³, but we are explicitly separating the structural and dynamical effects.). Let $x_{i\alpha}^{[\ell]}$ (where $\ell \in \{1, 2, \dots, C\}$) denote the ℓ th component of a C -dimensional vector $\mathbf{x}_{i\alpha}$ that represents the state of node i in layer α . The most general (and possibly nonlinear) dynamics

governing the evolution of each state is given by the systems of equations

$$\begin{aligned}
\dot{\mathbf{x}}_{i\alpha}(t) &= F_{i\alpha}(\mathbf{X}(t)) = \sum_{\beta=1}^L \sum_{j=1}^N f_{i\alpha}^{j\beta}(\mathbf{X}(t)) \\
&= \underbrace{\sum_{\beta=1}^L \sum_{j=1}^N f_{i\alpha}^{j\beta}(\mathbf{X}(t)) \delta_{\alpha}^{\beta} \delta_i^j + \sum_{\beta=1}^L \sum_{j=1}^N f_{i\alpha}^{j\beta}(\mathbf{X}(t)) \delta_{\alpha}^{\beta} (1 - \delta_i^j)}_{\text{intra-layer dynamics}} \\
&\quad + \underbrace{\sum_{\beta=1}^L \sum_{j=1}^N f_{i\alpha}^{j\beta}(\mathbf{X}(t)) (1 - \delta_{\alpha}^{\beta}) \delta_i^j + \sum_{\beta=1}^L \sum_{j=1}^N f_{i\alpha}^{j\beta}(\mathbf{X}(t)) (1 - \delta_{\alpha}^{\beta}) (1 - \delta_i^j)}_{\text{inter-layer dynamics}} \\
&= \underbrace{f_{i\alpha}^{i\alpha}(\mathbf{X}(t))}_{\text{self-interaction}} + \underbrace{\sum_{j \neq i} f_{i\alpha}^{j\alpha}(\mathbf{X}(t))}_{\text{endogenous interaction}} + \underbrace{\sum_{\beta \neq \alpha} \sum_{j \neq i} f_{i\alpha}^{j\beta}(\mathbf{X}(t))}_{\text{exogenous interaction}} + \underbrace{\sum_{\beta \neq \alpha} f_{i\alpha}^{i\beta}(\mathbf{X}(t))}_{\text{intertwining}} \\
&= \mathbb{S}_{i\alpha}(\mathbf{X}(t)) + \mathbb{N}_{i\alpha}(\mathbf{X}(t)) + \mathbb{X}_{i\alpha}(\mathbf{X}(t)) + \mathbb{I}_{i\alpha}(\mathbf{X}(t)), \tag{2}
\end{aligned}$$

where $\mathbf{X}(t) \equiv (\mathbf{x}_{11}, \mathbf{x}_{21}, \dots, \mathbf{x}_{N1}, \mathbf{x}_{12}, \mathbf{x}_{22}, \dots, \mathbf{x}_{N2}, \dots, \mathbf{x}_{1L}, \mathbf{x}_{2L}, \dots, \mathbf{x}_{NL})$.

Similar to the structural decomposition in Eq. (1), we have decoupled the different contributions of intra-layer and inter-layer dynamics, allowing us to classify different dynamical processes in terms of the corresponding dynamical $\mathbb{S}\mathbb{N}\mathbb{X}\mathbb{I}$ components. In the next two sections, we review a few emblematic examples in which the observed physical behavior is a direct consequence of multilayer interactions.

Single dynamics. In this section, we analyze physical phenomena that arise from a single dynamical process on top of a multilayer structure. The behavior of such a process depends both on intra-layer structure (i.e., the usual considerations in networks) and on inter-layer structure (i.e., the presence and strength of interactions between nodes on different layers).

One of the simplest types of dynamics is a diffusion process (either continuous or discrete). The physics of diffusion, which has been analyzed thoroughly in multiplex networks^{44,45} (i.e., it

networks of $\mathbb{SNI}$ type), reveals an intriguing and unexpected phenomenon: diffusion can be faster in a multiplex network than in any of the layers considered independently⁴⁴.

One can understand diffusion in multiplex networks in terms of the spectral properties of a (combinatorial) Laplacian tensor, obtained from the adjacency tensor of the multilayer network, that governs the diffusive dynamics. One first “flattens”⁴⁶ — without loss of information, provided one keeps the layer labels — the Laplacian tensor¹⁴ into a special lower-order tensor called “supra-Laplacian matrix”⁴⁴. The supra-Laplacian matrix has a block-diagonal structure, where diagonal blocks encode the associated Laplacian matrices corresponding to each layer separately and off-diagonal blocks encode inter-layer connections.

The time scale of diffusion is controlled by the the smallest positive eigenvalue Λ_2 of the supra-Laplacian matrix. In Fig. 3, we show a representative result that evinces the existence of two distinct regimes in multiplex networks as a function of the inter-layer coupling. The regimes illustrate how multilayer structure can influence the outcome of a physical process. For small values of the inter-layer coupling, the multilayer structure slows down the diffusion; for large values, the diffusion speed converges to the mean diffusion speed of the superposition of layers. In many cases, the diffusion in the superposition is faster than that in any of the separate layers. The transition between the two regimes is a structural transition⁴¹, a characteristic of multilayer networks that can arise also in other contexts^{47,48}.

The above phenomenology can also occur in discrete processes. Perhaps the most canonical examples of discrete dynamics are random walks, which are used to model Markovian dynamics on monolayer networks and which have yielded numerous insights over the last several decades^{49,50}. In a random walk, a discretized form of diffusion, a walker jumps between nodes through available connections. In a multilayer network, the available connections include layer switching via an inter-layer edge, a transition that has no counterpart in monolayer networks and which enriches

random-walk dynamics^{33,39,42}. An important physical insight of the interplay between multilayer structure and the dynamics of random walkers is “navigability”³⁹, which we take to be the mean fraction of nodes that are visited by a random walker in a finite time, which (similar to the case of continuous diffusion) can be larger than the navigability of an aggregated network of layers. In terms of navigability, multilayer networks are more resilient to uniformly random failures than their individual layers, and such resilience arises directly from the interplay between the multilayer structure and the dynamical process.

Another physical phenomenon that arises in multilayer networks is related to congestion, which arises from a balance between flow over network structures and the capacity of such structures to support flow. Congestion in networks was analyzed many years ago in the physics literature^{51–53}, but it has only recently been studied in multilayer networks^{23,54}, which can be used to model multimodal transportation systems. It is now known that the multilayer structure of a multiplex network can induce congestion even when a system would remain decongested in each layer independently²³. This seemingly surprising effect is explained by the multilayer structure: the intertwining favors shortest paths in one of the layers, causing a flow overload across those paths and thereby congesting them.

Mixed dynamics. Mixed dynamical processes are a second archetypical family of dynamics in which multilayer structure plays a crucial role. Thus far, the best studied examples are mixed spreading processes, which are crucial for understanding phenomena such as the spreading dynamics of two concurrent diseases in two-layer multiplex networks^{26,30,55–57} and spread of disease coupled with the spread of information or behavior^{24,25,27,58,59}. We illustrate two basic effects: (i) the two spreading processes can enhance each other (e.g., one disease facilitates infection by the other²⁶), and (ii) one process can inhibit the spread of the other (e.g., a disease can inhibit infection by another disease²⁶ or the spreading of awareness about a disease can inhibit the spread of the

disease²⁵). Interacting spreading processes also exhibit other fascinating dynamics, and multilayer networks provide a natural means to explore them ²⁴.

The corresponding phenomenology is characterized by the existence of a curve of critical points that separate the endemic and non-endemic phases. This curve exhibits a crossover between two different regimes: (i) a regime in which the critical properties of one spreading process are independent of the other, and (ii) a regime in which the critical properties of one spreading process do depend on those of the other. The point at which this crossover occurs is called a “metacritical” point.

In Fig. 4, we show (left) a phase diagram of the incidence in one layer of two reciprocally enhanced disease spreading processes; and (right) a phase diagram of the incidence in one layer of one inhibiting disease spreading process acting on the other. The metacritical point delineates the transition between independence (dashed line) and dependence (solid curve) of the critical properties of the two processes.

Conclusions and outlook

In most natural and engineered systems, entities interact with each other in complicated patterns that include multiple types of relationships and/or multiple subsystems, change in time, and incorporate other complications. The theory of multilayer networks seeks to take such features into account to improve our understanding of such complex systems.

In the last few years, there have been intense efforts to generalize traditional network theory by developing and validating a framework to study multilayer systems in a comprehensive fashion. The implications of multilayer network structure and dynamics are now being explored in fields as diverse as neuroscience^{21,60}, transportation^{9,61}, ecology¹⁰, granular materials⁶², evolutionary game

theory³¹, and many others. Despite considerable progress in the last few years^{7,8}, much remains to be done to obtain a deep understanding of the new physics of multilayer network structure and multilayer network dynamics (both dynamics of and dynamics on such networks). In seeking such a deep understanding, it is crucial to underscore the inextricable interdependence of the structure and dynamics of networks.

Recent efforts have revealed fundamental new physics in multilayer networks. The richer types of spreading and random-walk dynamics can lead to enhanced navigability, induced congestion, and the emergence of new critical properties. Such new phenomena also have a major impact on practical goals such as coarse-graining networks to examine mesoscale features and evaluating the importance of nodes — two goals that date back to the beginning of investigations of networks^{1,3,63}. For multilayer networks to achieve their vast potential, there remain crucial problems to address. For example, it is much easier to measure edge weights reliably for intra-layer edges than for inter-layer edges. Moreover, inter-layer edges not only play a different role from intra-layer ones, but they also play different roles in different applications, and the research community is only scratching the surface of the implications of their presence and the new phenomena to which they lead. The correlation structure across different layers and the cost of moving across layers affects both structural and dynamical properties. For example, which structures slow down the spread of information and diseases, and which ones speed it up? Which structures promote robustness, and which ones hinder it? Which structures promote synchronous dynamics and which ones impede it? The answers to such questions can be different for different types of dynamical processes (and for different variants of the same type of process), and it is crucial to further explore the new physical phenomena emerging from multilayer networks to answer these questions.

Contributions

All of the authors wrote the paper and contributed equally to the production of the manuscript.

Competing financial interests

The authors declare no competing financial interests.

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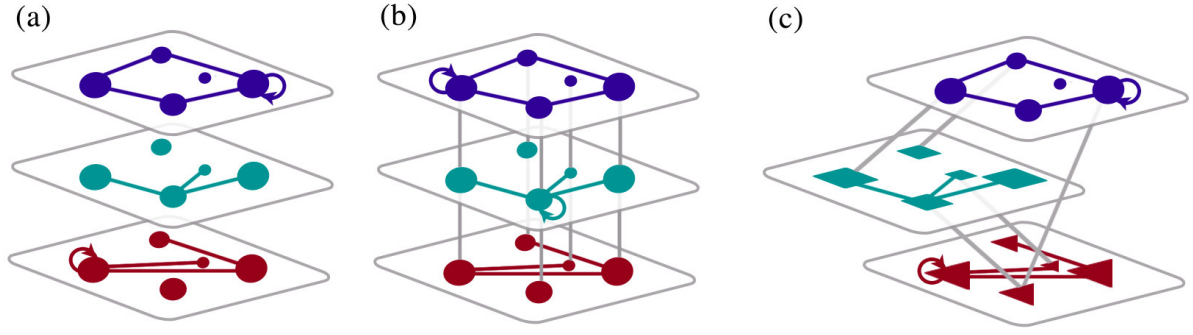


Figure 1: **Multilayer networks.** (a) An edge-colored multigraph, in which nodes can be connected by different types (i.e., colors) of interactions; structural intertwining and exogenous terms are absent and the others are present, so this is an $\mathbb{S}\mathbb{N}$ multilayer network. (b) A multiplex network, which consists of an edge-colored multigraph along with inter-layer edges that connect entities with their replicas on other layers; there are no structural exogenous terms but all other terms are present, so this is an $\mathbb{S}\mathbb{N}\mathbb{I}$ multilayer network. (c) An interdependent network, in which each layer contains nodes of a different type (circles, squares, and triangles) and includes inter-layer edges to nodes in other layers; in this case, all terms are present except for the structural intertwining term, so this is an $\mathbb{S}\mathbb{N}\mathbb{X}$ multilayer network.

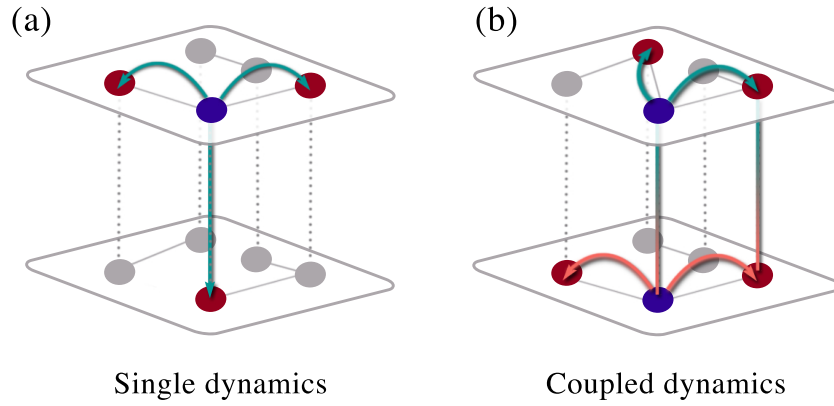


Figure 2: **Dynamical processes on multilayer networks.** (a) Schematic of a single type of dynamical process running on all layers of a multiplex network. (Arcs of the same color represent the same dynamical process.) (b) Schematic of two dynamical processes, each of which is running on a different layer, that are coupled by the interconnected structure of a multilayer network.

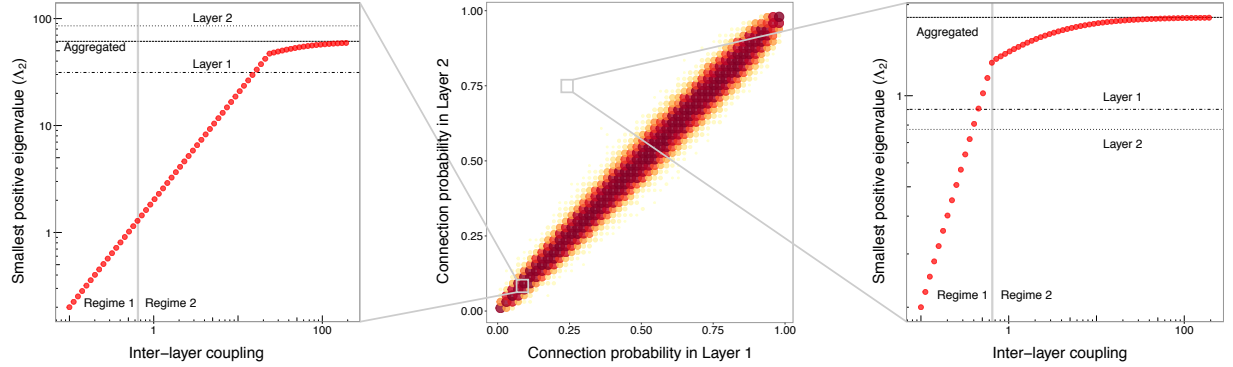


Figure 3: Single dynamics on multilayer networks. The speed of Laplacian diffusion dynamics of $\mathbb{SN}$ type is characterized by the second smallest eigenvalue Λ_2 of the corresponding combinatorial Laplacian tensor. We consider a pair of coupled Erős–Rényi (ER) networks in which we independently vary the probability to connect two nodes within the same layer between 0 and 1. The condition⁴⁴ to observe faster diffusion in the multilayer network than diffusion in each layer separately is $\Lambda_2^{\text{multiplex}} \geq \max\{\Lambda_2^{\text{layer } 1}, \Lambda_2^{\text{layer } 2}\}$. In the central panel, we see that the condition is satisfied when the two layers have similar edge-connection probabilities. In the side panels, we show the behavior of Λ_2 as a function of the inter-layer coupling in the two different regions of behavior.

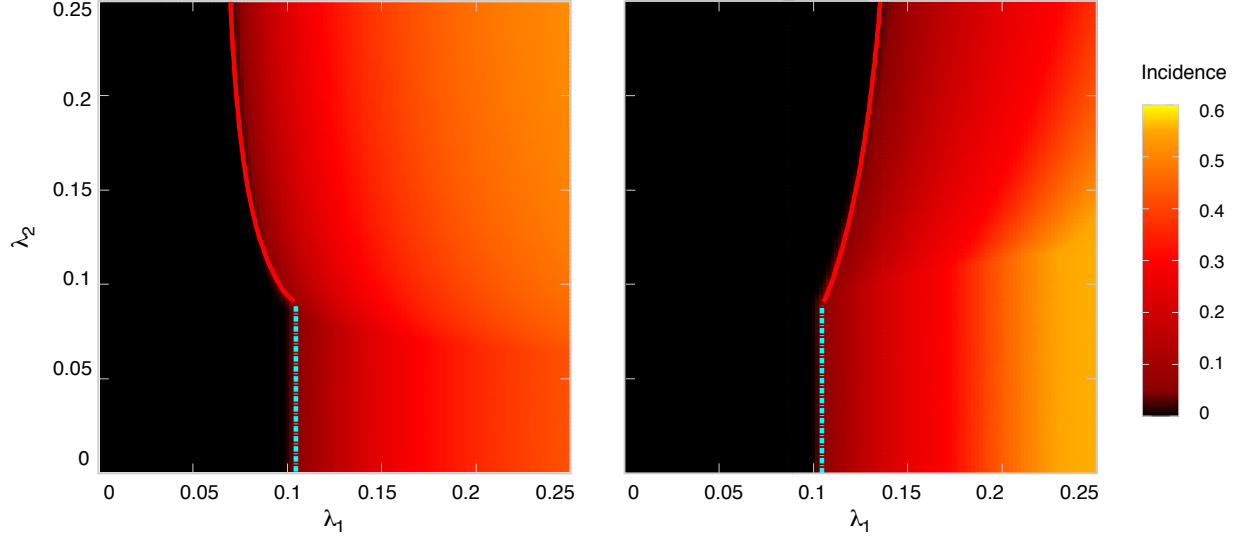


Figure 4: **Mixed dynamics on multilayer networks.** Two (left) reciprocally-enhanced and (right) reciprocally-inhibited disease-spreading processes of susceptible–infected–susceptible (SIS) type. We computed these diagrams for multiplex networks formed by two layers of 5000-node Erdős–Rényi graphs of 5000 with mean intra-layer degree $\langle k \rangle = 7$. The colors in the figure represent the prevalence levels of the diseases at a steady state of Monte-Carlo simulations. Note the emergence of a curve of critical points (at a “metacritical point”) in which the spreading in one layer depends on the spreading on the other.