

# FAST ALGORITHMS FOR EARTH MOVER'S DISTANCE BASED ON OPTIMAL TRANSPORT AND $L_1$ TYPE REGULARIZATION I

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ABSTRACT. We propose a new algorithm to approximate the Earth Mover's distance (EMD). Our main idea is motivated by the theory of optimal transport, in which EMD can be reformulated as a familiar homogeneous degree 1 regularized minimization. The new minimization problem is very similar to problems which have been solved in the fields of compressed sensing and image processing, where several fast methods are available. In this paper, we adopt a primal-dual algorithm designed there, which uses very simple updates at each iteration and is shown to converge very rapidly. Several numerical examples are provided.

## 1. INTRODUCTION

In this paper we propose a new algorithm to approximate the Earth Mover's distance (EMD), which is motivated by the theory of optimal transport and methods related to those used in compressed sensing and image processing.

We begin by reviewing some well known facts. EMD, which is also named the Monge problem or the Wasserstein metric, plays a central role in many applications, including image processing, computer vision and statistics e.t.c [8, 11, 15, 19]. The EMD is a particular metric defined on the probability space of a convex, compact set  $\Omega \subset \mathbb{R}^d$ . Given two probability densities  $\rho^0, \rho^1$  in a probability set  $\mathcal{P}(\Omega)$ , where

$$\mathcal{P}(\Omega) = \{\rho(x) \in L^1(\Omega) : \int_{\Omega} \rho(x)dx = 1, \quad \rho(x) \geq 0\} .$$

The EMD deals with following minimization problem

$$EMD(\rho^0, \rho^1) := \min_{\pi} \int_{\Omega \times \Omega} \|x - y\|_2 \pi(x, y) dx dy \quad (1)$$

with the constraint that the joint measure (also called the transport function)  $\pi(x, y)$  has  $\rho^0(x)$  and  $\rho^1(y)$  as marginals, i.e.

$$\int_{\Omega} \pi(x, y) dy = \rho^0(x) , \quad \int_{\Omega} \pi(x, y) dx = \rho^1(y) , \quad \pi(x, y) \geq 0 .$$

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*Key words and phrases.* Earth Mover's distance; Optimal transport; Compressed sensing; Primal-dual algorithm;  $L_1$  regularization.

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Here  $\|\cdot\|_2$  is the 2-norm in  $\mathbb{R}^d$ . Notice that (1) is an infinite dimensional linear programming problem. In particular, if  $\rho^0, \rho^1$  are delta measures centered at points  $x, y$ , EMD returns a Euclidean metric on  $\mathbb{R}^d$ , i.e.  $\|x - y\|_2$ .

In recent years, (1) has been well studied by the theory of optimal transport [1, 5, 17, 20, 21]. The theory (remarkably) points out that (1) is equivalent to a new minimization problem

$$EMD(\rho^0, \rho^1) = \inf_m \left\{ \int_{\Omega} \|m(x)\|_2 dx : \nabla \cdot m(x) + \rho^1(x) - \rho^0(x) = 0 \right\}, \quad (2)$$

where  $m : \Omega \rightarrow \mathbb{R}^d$  is a flux vector satisfying some boundary conditions, such as zero flux or periodic conditions. Here we use a zero flux condition, i.e.  $m(x) \cdot \nu(x) = 0$ , where  $\nu(x)$  is the unit normal vector for the boundary of  $\Omega$ .

The minimization (2) has an interesting fluid dynamics interpretation. It turns out that (2) can be viewed as the following optimal control problem

$$\inf_{\rho, m} \left\{ \int_0^1 \int_{\Omega} \|m(t, x)\|_2 dx dt : \frac{\partial \rho}{\partial t} + \nabla \cdot m = 0, \quad \rho(0) = \rho^0, \quad \rho(1) = \rho^1 \right\}, \quad (2b)$$

where the minimum is taken among all possible flux functions  $m(t, x)$ , such that the probability density function is moved continuously in time, from  $\rho^0$  to  $\rho^1$ . The problem (2b) has many minimizers. One of them is such that  $\rho(t, x) = t\rho_1 + (1 - t)\rho_0$ . Here the flux function  $m(t, x)$  does not depend on the time and  $\nabla \cdot m = -\frac{\partial \rho}{\partial t} = \rho^0 - \rho^1$ , so that the control problem becomes (2), see [3, 6, 20, 21] for details.

Moreover, the formulation (2) has two benefits numerically. First, the dimension in (2) is lower than the one in the original problem (1). Suppose we discretize  $\Omega$  by a grid with  $N$  nodes. Since the unknown variable  $\pi(x, y)$  in (1) is supported on  $\Omega \times \Omega$  and  $m(x)$  in (1) only depends on  $\Omega$ , the number of grid points used in (2) is  $N$  while the number needed in (1) is  $N^2$ . Second, (2) is an  $L_1$ -type minimization problem, which shares its structure with many problems in compressed sensing and image processing. It is possible to borrow a very fast and simple algorithm used there to solve EMD, see e.g. [7, 16, 22].

In this paper, we propose a new algorithm for EMD, leveraging the structure of the formulation (2). The algorithm mainly uses a finite volume method to discretize the domain  $\Omega$  and then applies the framework of primal-dual iterations designed in [4, 13]. Since (2) is a special homogenous degree one minimization with linear constraints, our algorithm inherits all the benefits of the primal-dual algorithm: First, we use a shrink operator at each step, which handles the sparsity easily, see e.g. [7]; Second, the algorithm converges rapidly and each step involves very simple formulae. Thus the complexity of the algorithm is very low and the program is very simple.

Many methods have been proposed for (1), see e.g. [10, 12, 14, 18]. As we discussed, they either solve (2) with more variables or apply linear programming techniques for an  $L_1$ -type minimization. In [19], the authors designed an algorithm for EMD using (2). However, it is totally different from the strategy we use. In [19], they use the Hodge decomposition and solve an inverse Laplacian to transfer (2) into an unconstrained problem, which is then solved by the alternating direction method of multipliers (ADMM). Every step in

our algorithm has simple explicit solutions, is much easier to implement and seems to converge rapidly.

The outline of this paper is as follows. In section 2, we propose a primal-dual algorithm for (2) on a uniform grid. We analyze the algorithm in section 3. Several numerical examples are presented in section 4.

## 2. ALGORITHM

The EMD problem, as presented in (2), has similar structure to many homogenous degree one regularized problems. In this section we will use a finite volume discretization to approximate (2). The discretized problem becomes an  $\ell_1$ -like optimization with linear constraints, which allows us to apply the hybrid primal-dual method designed in [4, 13].

We shall consider a uniform lattice graph  $G = (V, E)$  with spacing  $\Delta x$  to discretize the spatial domain, where  $V$  is the vertex set

$$V = \{1, 2, \dots, N\} ,$$

and  $E$  is the edge set. Here  $i = (i_1, \dots, i_d) \in V$  represents a point in  $\mathbb{R}^d$ .

We consider a discrete probability set supported on all vertices:

$$\mathcal{P}(G) = \{(p_i)_{i=1}^N \in \mathbb{R}^n \mid \sum_{i=1}^N p_i = 1, p_i \geq 0, i \in V\} ,$$

where  $p_i$  represents a probability at node  $i$ , i.e.  $p_i = \int_{C_i} \rho(x) dx$ ,  $C_i$  is a cube centered at  $i$  with length  $\Delta x$ . So  $\rho^0(x)$ ,  $\rho^1(x)$  is approximated by  $p^0 = (p_i^0)_{i=1}^N$  and  $p^1 = (p_i^1)_{i=1}^N$ .

We use two steps to consider the EMD on  $\mathcal{P}(G)$ . We first define a flux on a lattice. Denote a matrix  $m = (m_{i+\frac{1}{2}})_{i=1}^N \in \mathbb{R}^{N \times d}$ , where each component  $m_{i+\frac{1}{2}}$  is a row vector in  $\mathbb{R}^d$ , i.e.

$$m_{i+\frac{1}{2}} = (m_{i+\frac{1}{2}e_v})_{v=1}^d = \left( \int_{C_{i+\frac{1}{2}e_v}} m(x) dx \right)_{v=1}^d ,$$

where  $e_v = (0, \dots, \Delta x, \dots, 0)^T$ ,  $\Delta x$  is at the  $v$ -th column. In other words, if we denote  $i = (i_1, \dots, i_d) \in \mathbb{R}^d$  and  $m(x) = (m^1(x), \dots, m^d(x))$ , then

$$m_{i+\frac{1}{2}e_v} \approx m^v(i_1, \dots, i_{v-1}, i_v + \frac{1}{2}\Delta x, i_{v+1}, \dots, i_d)\Delta x^d .$$

We consider a zero flux condition. So if a point  $i + \frac{1}{2}e_v$  is outside the domain  $\Omega$ , we let  $m_{i+\frac{1}{2}e_v} = 0$ . Based on such a flux  $m$ , we define a discrete divergence operator  $\text{div}_G(m) := (\text{div}_G(m_{i+\frac{1}{2}}))_{i=1}^N$ , where

$$\text{div}_G(m_{i+\frac{1}{2}}) := \frac{1}{\Delta x} \sum_{v=1}^d (m_{i+\frac{1}{2}e_v} - m_{i-\frac{1}{2}e_v}) .$$

We next introduce the discrete cost functional

$$\|m\| := \sum_{i=1}^N \|m_{i+\frac{1}{2}}\|_2 = \sum_{i=1}^N \sqrt{\sum_{v=1}^d |m_{i+\frac{e_v}{2}}|^2} .$$

To summarize, (2) forms an optimization problem

$$\begin{aligned} & \underset{m}{\text{minimize}} && \|m\| \\ & \text{subject to} && \operatorname{div}_G(m) + p^1 - p^0 = 0 , \end{aligned}$$

which can be written explicitly as

$$\begin{aligned} & \underset{m}{\text{minimize}} && \sum_{i=1}^N \sqrt{\sum_{v=1}^d |m_{i+\frac{e_v}{2}}|^2} \\ & \text{subject to} && \frac{1}{\Delta x} \sum_{v=1}^d (m_{i+\frac{1}{2}e_v} - m_{i-\frac{1}{2}e_v}) + p_i^1 - p_i^0 = 0 , \quad i = 1, \dots, N, \quad v = 1, \dots, d . \end{aligned} \tag{3}$$

We can prove that the set of  $m$  satisfying (3) is not empty. We observe that (3) is an optimization problem, which is very similar to some problems in compressed sensing and image processing e.g. [7], whose cost functional is convex and whose constraints are linear. Thus we solve (3) by looking at its saddle point structure. Denote  $\Phi = (\Phi_i)_{i=1}^N$  as (3)'s Lagrange multiplier, thus we have

$$\min_m \max_{\Phi} L(m, \Phi) := \min_m \max_{\Phi} \|m\| + \Phi^T (\operatorname{div}_G(m) + p^1 - p^0) . \tag{4}$$

Saddle point problems, such as (4), are well studied by the first-order primal dual algorithm [4, 13]. The iteration steps are as follows:

$$\begin{cases} m^{k+1} = \arg \min_m \|m\| + \Phi^T \operatorname{div}_G(m) + \frac{\|m - m^k\|_2^2}{2\mu} ; \\ \Phi^{k+1} = \arg \max_{\Phi} \Phi^T \operatorname{div}_G(m^{k+1} + \theta(m^{k+1} - m^k)) - \frac{\|\Phi - \Phi^k\|_2^2}{2\tau} , \end{cases} \tag{5}$$

where  $\mu, \tau$  are two small step sizes,  $\theta \in [0, 1]$  is a given parameter,  $\|m - m^k\|_2^2 = \sum_{i=1}^N \sum_{v=1}^d (m_{i+\frac{1}{2}e_v} - m_{i+\frac{1}{2}e_v}^k)^2$  and  $\|\Phi - \Phi^k\|_2^2 = \sum_{i=1}^N (\Phi_i - \Phi_i^k)^2$ . These steps are alternating a gradient ascent in the dual variable  $\Phi$  and a gradient descent in the primal variable  $m$ .

It turns out that iteration (5) can be solved by simple explicit formulae. Since the unknown variable  $m, \Phi$  is component-wise separable in this problem, each of its components  $m_{i+\frac{1}{2}}, \Phi_i$  can be independently obtained by solving (5).

First, notice

$$\begin{aligned}
& \min_m \|m\| + \Phi^T \operatorname{div}_G(m) + \frac{\|m - m^k\|_2^2}{2\mu} \\
&= \min_m \sum_{i=1}^N \sum_{v=1}^d \sqrt{m_{i+\frac{1}{2}e_v}^2} + \frac{1}{\Delta x} \sum_{i=1}^N \sum_{v=1}^d \Phi_i(m_{i+\frac{1}{2}e_v} - m_{i-\frac{1}{2}e_v}) + \frac{\|m - m^k\|_2^2}{2\mu} \\
&= \sum_{i=1}^N \min_{m_{i+\frac{1}{2}}} \left( \|m_{i+\frac{1}{2}}\|_2 - \nabla_G \Phi_{i+\frac{1}{2}}^T m_{i+\frac{1}{2}} + \frac{1}{2\mu} \|m_{i+\frac{1}{2}} - m_{i+\frac{1}{2}}^k\|_2^2 \right),
\end{aligned}$$

where  $\nabla_G \Phi_{i+\frac{1}{2}} := \frac{1}{\Delta x} (\Phi_{i+e_v} - \Phi_i)_{v=1}^d$ . The first iteration in (5) has an explicit solution, which is:

$$m_{i+\frac{1}{2}}^{k+1} = \operatorname{shrink}_2(m_{i+\frac{1}{2}}^k + \mu \nabla_G \Phi_{i+\frac{1}{2}}, \mu),$$

where we define the  $\operatorname{shrink}_2$  operation

$$\operatorname{shrink}_2(y, \alpha) := \frac{y}{\|y\|_2} \max\{\|y\|_2 - \alpha, 0\}, \quad \text{where } y \in \mathbb{R}^d.$$

Second, consider

$$\begin{aligned}
& \max_{\Phi} \Phi^T \operatorname{div}_G(m^{k+1} + \theta(m^{k+1} - m^k)) - \frac{\|\Phi - \Phi^k\|_2^2}{2\tau} \\
&= \sum_{i=1}^N \max_{\Phi_i} \left\{ \Phi_i [\operatorname{div}_G(m_{i+\frac{1}{2}}^{k+1} + \theta(m_{i+\frac{1}{2}}^{k+1} - m_{i+\frac{1}{2}}^k)) + p_i^1 - p_i^0] - \frac{\|\Phi_i - \Phi_i^k\|_2^2}{2\tau} \right\}.
\end{aligned}$$

Thus the second iteration in (5) becomes

$$\Phi_i^{k+1} = \Phi_i^k + \tau \{ \operatorname{div}_G(m_{i+\frac{1}{2}}^{k+1} + \theta(m_{i+\frac{1}{2}}^{k+1} - m_{i+\frac{1}{2}}^k)) + p_i^1 - p_i^0 \}.$$

**2.1. Algorithm.** We are now ready to state our algorithms.

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### primal-dual 1-Wasserstein metric

**Input:** Discrete probabilities  $\rho^0, \rho^1$ ;

Initial guess of  $m^0$ , step size  $\mu, \tau, \theta \in [0, 1]$ .

**Output:** Optimal plan  $m$  and EMD value  $\|m\|$ .

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1. for  $k = 1, 2, \dots$       Iterates until convergence
  2.      $m_{i+\frac{1}{2}}^{k+1} = \operatorname{shrink}_2(m_{i+\frac{1}{2}}^k + \mu \nabla_G \Phi_{i+\frac{1}{2}}, \mu)$  ;
  3.      $\Phi_i^{k+1} = \Phi_i^k + \tau \{ \operatorname{div}_G(m_{i+\frac{1}{2}}^{k+1} + \theta(m_{i+\frac{1}{2}}^{k+1} - m_{i+\frac{1}{2}}^k)) + p_i^1 - p_i^0 \}$  ;
  4.     **end**
- 

*Remark 1.* The  $\operatorname{shrink}_2$  operator we use is a multidimensional one, which is a projection to a ball, see e.g. [7]. In our next paper, we will use the conventional  $\operatorname{shrink}_1$ , where the distance function is  $\ell_1$ .

In practice, we usually select  $\theta = 1$  in the algorithm. In next section, we shall prove under this selection, the algorithm converges to the minimizer of (3).

### 3. NUMERICAL ANALYSIS

In this section, we show several numerical properties of the algorithm (5). We prove that the primal-dual algorithm (5) converges to the minimizer of our discretized minimization (3). In addition, such a minimizer satisfies an eikonal equation on the grid.

**Theorem 1.** *Denote a linear operator  $K : \mathbb{R}^{N \times d} \rightarrow \mathbb{R}^N$ , such that*

$$Km = (\operatorname{div}_G(m_{i+\frac{1}{2}}))_{i=1}^N ,$$

*and a saddle point of  $L$  in (4) as  $(m^*, \Phi^*)$ . Choose  $\theta = 1$ ,  $\tau\mu\|K\|_\infty^2 < 1$ . Then  $m^k, \Phi^k$  in iteration (5) converges to  $m^*, \Phi^*$ . Moreover,  $\Phi^*$  satisfies the eikonal equation on graphs, i.e.*

$$\|\nabla_G \Phi_{i+\frac{1}{2}}^*\|_2 = 1 , \quad \text{if } i \in V \text{ with } \|m_{i+\frac{1}{2}}^*\|_2 > 0 . \quad (6)$$

*Proof.* First, we only need to show that saddle point problem  $L$  satisfies the condition of Theorem 1 in [4, 13]. We rewrite  $L$  as

$$L(m, \Phi) = G(m) + \Phi^T Km - F(\Phi) ,$$

where  $G(m) = \|m\|$ ,  $Km = (\operatorname{div}_G(m_{i+\frac{1}{2}}))_{i=1}^N$ , and  $F(\Phi) = \sum_{i=1}^N \Phi_i(p_i^0 - p_i^1)$ . It is easy to observe that  $G, F$  is a convex continuous function and  $K$  is a linear operator. From Theorem 1 in [4], we prove the convergence result.

Second, since  $\|m_{i+\frac{1}{2}}^*\|_2 > 0$ , we have

$$0 = \frac{\partial L}{\partial m_{i+\frac{1}{2}}} \Big|_{(m^*, \Phi^*)} = \frac{m_{i+\frac{1}{2}}^*}{\|m_{i+\frac{1}{2}}^*\|_2} - \nabla_G \Phi_{i+\frac{1}{2}}^* .$$

Thus

$$1 = \left\| \frac{m_{i+\frac{1}{2}}^*}{\|m_{i+\frac{1}{2}}^*\|_2} \right\|_2 = \|\nabla_G \Phi_{i+\frac{1}{2}}^*\|_2 .$$

We have proven  $\Phi^*$  satisfies (6). □

From above convergence result, we are ready to show that the computational complexity for this primal-dual algorithm is  $O(NM)$ , where  $M$  is the iteration number for a given error. It is known in [4], the algorithm converges with the rate of  $O(\frac{1}{M})$ . We also have the fact that each iteration has simple updates, which only need  $O(N)$  operations. So our method requires overall  $O(N) \times O(M)$  computations. In practice, we observe good performance, see the next section. This is because of the well known performance of shrink operations as observed in compressed sensing and image processing calculations.

We do not claim that our discrete approximation (3) converges as  $\Delta x \rightarrow 0$  to the solution of (2). However, if as  $\Delta x \rightarrow 0$  the family  $m$  stays uniformly bounded, then it is easy to show that  $m$  converges to a weak solution of (2).

## 4. EXAMPLES

In this section, we demonstrate several numerical results on a square  $[-2, 2] \times [-2, 2]$ . Our discretization used in the Figure 1-5 is a uniform  $40 \times 40$  lattice. The parameters are chosen as  $\mu = \tau = 0.025$ ,  $\theta = 1$ . The initial flux  $m$  is chosen as all zeros.

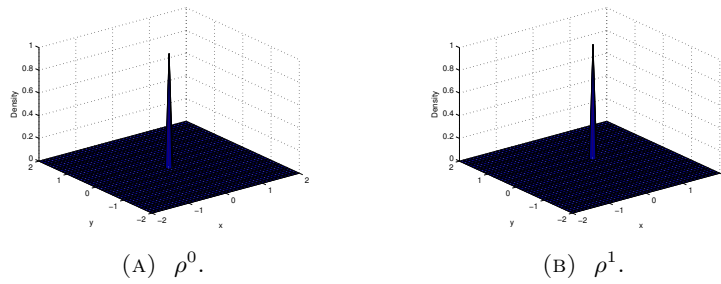


FIGURE 1. Here  $\rho^0$  and  $\rho^1$  are concentrated at  $(0, 0)$ ,  $(0.4, 0.4)$ , i.e.  $\rho^0 = \delta_{(0,0)}$ ,  $\rho^1 = \delta_{(0.4,0.4)}$ . The computed earth mover's metric is 0.6232

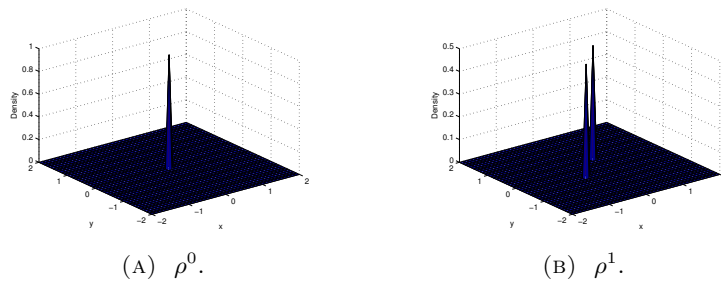


FIGURE 2. Here  $\rho^0$  is concentrated at  $(0, 0)$ ,  $\rho^1$  is concentrated at two positions,  $(0.4, 0.4)$  and  $(-0.4, -0.4)$ , i.e.  $\rho^0 = \delta_{(0,0)}$ ,  $\rho^1 = \frac{1}{2}(\delta_{(0.4,0.4)} + \delta_{(-0.4,-0.4)})$ . The computed earth mover's metric is 0.6232.

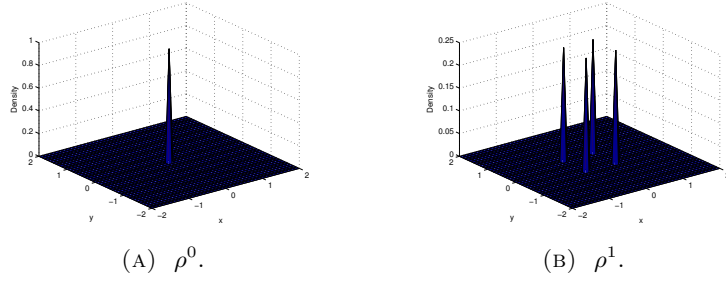


FIGURE 3. Here  $\rho^0$  is concentrated at  $(0,0)$ ,  $\rho^1$  is concentrated at four positions,  $(0.4, 0.4)$ ,  $(0.4, -0.4)$ ,  $(-0.4, 0.4)$ ,  $(-0.4, -0.4)$ , i.e.  $\rho^0 = \delta_{(0,0)}$ ,  $\rho^1 = \frac{1}{4}(\delta_{(0.4,0.4)} + \delta_{(0.4,-0.4)} + \delta_{(-0.4,0.4)} + \delta_{(-0.4,-0.4)})$ . The computed earth mover's metric is 0.5882.

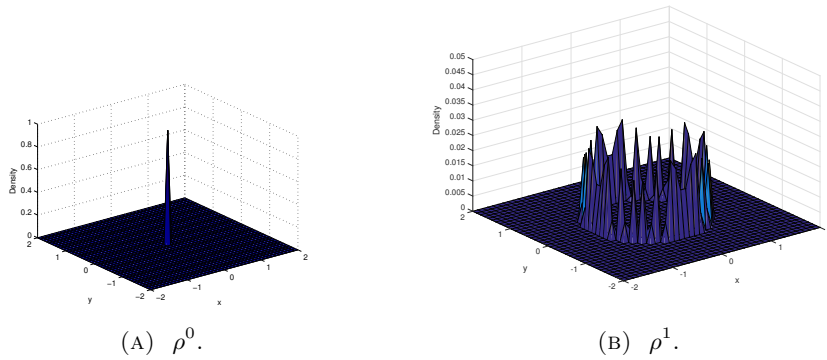


FIGURE 4. Here  $\rho^0$  is concentrated at  $(0,0)$ ,  $\rho^1$  is a measure supported on a circle, i.e.  $\rho^0 = \delta_{(0,0)}$ ,  $\rho^1 = \frac{1}{K}(e^{\frac{x^2}{\sigma} - \frac{x^4}{\sigma}})$ , where  $K$  is a normalized constants and  $\sigma = 10^{-3}$ . The computed earth mover's metric is 0.6943.



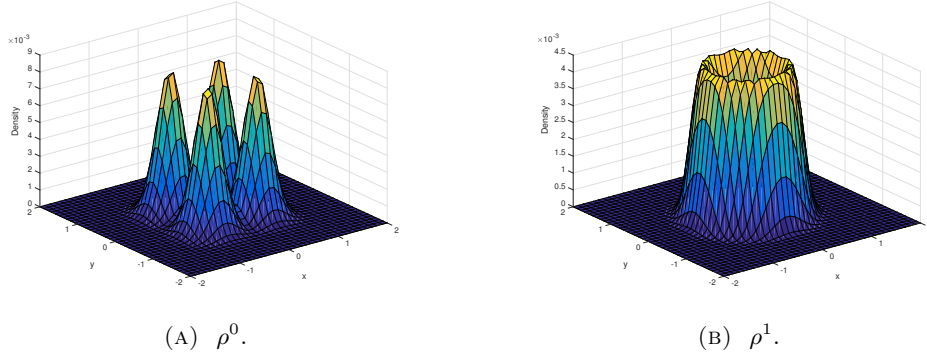


FIGURE 5. Here  $\rho^0 = \frac{1}{K_1} e^{-\frac{x^2 - |x_1| - |x_2|}{\sigma}}$ ,  $\rho^1 = \frac{1}{K_2} (e^{\frac{x^2}{2\sigma}} - \frac{x^4}{2\sigma})$ , where  $K_1, K_2$  are normalized constants,  $\sigma = 0.1$ . The computed earth mover's metric is 0.1236.

For the above computations, we use the stopping criteria

$$\sum_{i=1}^N |\operatorname{div}_G(m_{i+\frac{1}{2}}^k) + p_i^1 - p_i^0| \leq 0.01 .$$

Table 1 reports the time under different grids for computing EMD in Figure 1, 2, 3. The implementation is done in MATLAB 2016, on a 3.33GHZ Intel i5 processor with 4GB RAM.

**Example 1:**

Grids number (N)	Time (s)	Iteration	Relative Error
100	0.07	32	0.167
400	0.49	118	0.119
1600	6.19	560	0.080
6400	202.28	4884	0.052

**Example 2:**

Grids number (N)	Time (s)	Iteration	Relative Error
100	0.15	83	0.165
400	1.04	281	0.120
1600	18.58	1526	0.080
6400	378	9121	0.052

**Example 3:**

Grids number (N)	Time (s)	Iteration	Relative Error
100	0.14	115	0.052
400	1.16	299	0.042
1600	19.80	1633	0.032
6400	486	11804	0.022

TABLE 1. Time is in seconds, which is for experiments in Figure 1, 2, 3. Denote absolute error as the difference between computed value of EMD and its true value (Euclidean distance  $\frac{\sqrt{2}}{2}$ ). Then the relative error is computed by the ratio between the absolute error and true value.

In the above examples, we observe that the number of iteration is roughly  $O(N)$ . From the complexity estimation in previous section, we claim that the complexity of our algorithm is  $O(N^2)$ , which closely matches the result of computation time in table 1.

## 5. CONCLUSIONS

We have designed a fast algorithm which enables us to approximate the Earth Mover’s distance easily. The method is inspired by key ideas in optimal transport and homogeneous degree one regularizers. From optimal transport, the Earth Mover’s distance can be rewritten into a fluid dynamics setting, which forms an  $L_1$ -type minimization. Motivated by compressed sensing and image processing, we borrow a primal-dual algorithm, which is simple and converges rapidly.

Compared with existing methods, our algorithm has following advantages:

- First, it leverages the key observation of optimal transport, which transfers the problem into an  $L_1$ -type minimization. The new problem uses  $N$  variables, which is much less than the original  $N^2$  linear optimization;
- Second, it uses simple exact formulae at each iteration and is guaranteed to converge to the minimizer;

- Last, it not only gives the value of the distance, but also provides a map for initial and target distributions. In particular, it gives a solution of a discrete eikonal equation (6) on grid, which can be used to find geodesics.

Our results open a door to many new questions. On one hand, EMD provides a special  $L_1$ -type minimization. In order to solve it efficiently, we use particular convex homogeneous degree one techniques. which arise elsewhere. On the other hand, if we replace the Euclidean distance in (1) by a Manhattan distance,  $\|x - y\|_1 = \sum_{v=1}^d |x_v - y_v|$ , its new equivalent formulation (2) will be an  $\ell_1$  optimization, which is known to be solvable by a standard shrink operator very easily and converges even more rapidly. In the future, we will study such problems and use our algorithms for many practical applications.

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