TRUNCATED L_{1-2} MODELS FOR SPARSE RECOVERY AND RANK MINIMIZATION[*]

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Abstract. We study a truncated difference of l_1 and l_2 norms as a nonconvex metric for recovering sparse vectors and low-rank matrices from linear measurements. By discarding large magnitudes in penalization, the proposed metric, denoted as truncated l_{1-2}, achieves a nearly unbiased approximation of the vector sparsity and matrix rank. We establish exact and stable recovery conditions of truncated l_{1-2} minimization under the restricted isometry property (RIP) framework. Computationally, we develop a difference of convex functions algorithm to efficiently solve truncated l_{1-2} minimization with guaranteed convergence. Experiments show that the proposed method is on par with the state-of-the-art solvers for sparse vector recovery and matrix completion.

Key words. compressed sensing, difference of convex functions algorithm, l_{1-2} minimization, matrix completion, rank minimization, sparse recovery.

AMS subject classifications. 15A83, 65K10, 90C26, 94A12

1. Introduction. We address two related problems: sparse recovery and rank minimization. Sparse recovery refers to recovering a sparse vector from a set of linear measurements, which is one of the most fundamental issues in compressed sensing (CS) [10, 18]. Mathematically, it can be expressed as

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{x}\|_0 \quad \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{y}, \tag{1.1}$$

where $\|\mathbf{x}\|_0$ counts the number of nonzero entries of $\mathbf{x}$, $\mathbf{A} \in \mathbb{R}^{m \times n}$ is called a sensing matrix, and $\mathbf{y} \in \mathbb{R}^m$ is a measurement vector. On the other hand, rank minimization aims at recovering a low-rank matrix from its linear observations, which arises in many areas of science and technology, including system identification [34], collaborative prediction [47], and low-dimensional embedding [33]. This problem is formulated as follows:

$$\min_{\mathbf{X} \in \mathbb{R}^{m \times n}} \text{rank}(\mathbf{X}) \quad \text{subject to} \quad \mathcal{A}(\mathbf{X}) = \mathbf{b}, \tag{1.2}$$

where $\mathcal{A} : \mathbb{R}^{m \times n} \to \mathbb{R}^p$ is a linear map and $\mathbf{b} \in \mathbb{R}^p$ denotes a measurement vector. A special case of (1.2) is to recover a low-rank matrix from a small number of its entries, referred to as matrix completion, where $\mathcal{A} := \mathcal{P}_\Omega$ samples entries from a two-dimensional index set Ω. Problems (1.1) and (1.2) are closely related, considering that the rank of a matrix is equal to the l_0 metric of its singular vectors.

Computationally, both (1.1) and (1.2) are NP-hard. Recent breakthroughs in CS [10, 18] and low-rank matrix recovery [46, 8] suggest that under certain conditions, solutions of (1.1) and (1.2) can be obtained by solving their convex surrogates, where the l_0 metric is replaced by the l_1 norm [49, 14] and the matrix rank is replaced by the

nuclear norm (sum of the singular values) [22]. Over the years, various research efforts have been devoted to solving l_1 minimization efficiently, e.g., alternating direction method of multipliers (ADMM) [4], split Bregman [25], fixed-point continuation (FPC) [26], and fast iterative shrinkage-thresholding algorithm (FISTA) [1]. Many of them have been extended to nuclear norm minimization, e.g., singular value thresholding (SVT) [5], FPC with approximate SVD (FPCA) [39], and accelerated proximal gradient algorithm [50].

Minimizing the l_1 norm is a convex problem, thus computationally tractable. However, it may sometimes yield suboptimal performance due to the biased approximation to l_0 in the sense that l_1 is dominated by entries with large magnitudes, unlike l_0 in which all nonzero entries have equal contributions. In this regard, nonconvex metrics give closer approximations to l_0, thus promoting sparsity better than l_1. Some popular nonconvex metrics include l_q quasi-norm with $0 < q < 1$ and its variants [13, 12, 16, 54], capped-l_1 [58], smoothly clipped absolute deviation (SCAD) [20], and minimax concave penalty (MCP) [57]. Some of them have been extended to rank minimization such as l_q quasi-norm on singular values [42, 32] and matrix MCP [31].

Two closely related metrics to this work are truncated l_1 [51, 29] and difference of l_1 and l_2 norms [19, 35, 55], denoted as $l_{t,1}$ and l_{1-2}, respectively. The $l_{t,1}$ metric discards large magnitudes in penalization, thus achieving a better approximation to l_0 than l_1. This idea was incorporated into the iterative support detection (ISD) method [51] of minimizing $\sum_{i \notin T} |x_i|$, where T is a fixed set containing the indices of large magnitude entries of the previous reconstruction. Recently, Lu et al. [38] proposed to truncate $\mathbf{x}$'s t largest magnitudes $\{|x|_{[i]}\}_{i=1}^{t}$, instead of a fixed set in ISD, and formalized a partial regularization framework $\sum_{i=t+1}^{n} \phi(|x|_{[i]})$ for a class of sparse promoting metrics ϕ. The idea of truncated penalization has been extended to rank minimization [29] and matrix decomposition of low-rank and sparse components [43].

The l_{1-2} metric, on the other hand, is defined as $\|\mathbf{x}\|_{1-2} := \|\mathbf{x}\|_1 - \|\mathbf{x}\|_2$. It was first introduced in [19] as a sparsity penalty for nonnegative least squares problems and was later applied to sparse recovery in [35, 55] with superior performance over other existing metrics when the matrix $\mathbf{A}$ in (1.1) is highly coherent. In addition, l_{1-2} has shown advantages in various applications such as image restoration [37], phase retrieval [56], and point source super-resolution [36]. One advantage of l_{1-2} over l_1 is its unbiased characterization of one-sparse vectors, since $\|\mathbf{x}\|_{1-2} = 0$ if and only if $\|\mathbf{x}\|_0 \leq 1$; see [55] for more details. However, l_{1-2} becomes biased and behaves like l_1 as the number of leading entries (in magnitude) increases. This effect can be seen from the fact that l_{1-2} tends to l_1 through vector cloning, i.e., for any $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and defining $\mathbf{x}^{(i)} := [\mathbf{x}^T \ldots \mathbf{x}^T]^T \in \mathbb{R}^{in}$, one has that $\lim_{i \to \infty} \|\mathbf{x}^{(i)}\|_{1-2} / \|\mathbf{x}^{(i)}\|_1 = 1$.

Motivated by this observation and the idea of truncated penalization, we propose a truncated l_{1-2} metric, denoted as $l_{t,1-2}$, for sparse (vector) recovery and (matrix) rank minimization. For the vector case, $l_{t,1-2}$ is defined as follows: given $\mathbf{x} \in \mathbb{R}^n$ and $t \leq n$,

$$\|\mathbf{x}\|_{t,1-2} := \sum_{i \notin \Gamma_{\mathbf{x},t}} |x_i| - \sqrt{\sum_{i \notin \Gamma_{\mathbf{x},t}} x_i^2}, \tag{1.3}$$

where $\Gamma_{\mathbf{x},t} \subseteq \{1, \ldots, n\}$ with cardinality t is a set containing the indices of the entries of $\mathbf{x}$ with the t largest magnitudes[1], i.e., for any $i \notin \Gamma_{\mathbf{x},t}$ and $j \in \Gamma_{\mathbf{x},t}$, $|x_i| \leq |x_j|$. In

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[1] One can see that the value of $\|\mathbf{x}\|_{t,1-2}$ is uniquely determined by $\mathbf{x}$ and is independent of the choice of $\Gamma_{\mathbf{x},t}$, if not unique. Throughout the paper, if there are multiple choices of $\Gamma_{\mathbf{x},t}$, we will

the matrix case, $l_{t,1-2}$ is expressed in terms of singular values, i.e., given $\mathbf{X} \in \mathbb{R}^{m \times n}$ and $t \leq m \leq n$,

$$\|\mathbf{X}\|_{t,*-F} := \|\boldsymbol{\sigma}(\mathbf{X})\|_{t,1-2} = \sum_{i=t+1}^{m} \sigma_i(\mathbf{X}) - \sqrt{\sum_{i=t+1}^{m} \sigma_i^2(\mathbf{X})}, \qquad (1.4)$$

where $\boldsymbol{\sigma}(\mathbf{X}) := (\sigma_1(\mathbf{X}), \ldots, \sigma_m(\mathbf{X}))$ is a vector composed of $\mathbf{X}$'s singular values with $\sigma_1(\mathbf{X}) \geq \cdots \geq \sigma_m(\mathbf{X}) \geq 0$. It is clear that when $t = 0$, $l_{t,1-2}$ reduces to l_{1-2}, i.e., $\|\mathbf{x}\|_{0,1-2} = \|\mathbf{x}\|_1 - \|\mathbf{x}\|_2$ and $\|\mathbf{X}\|_{0,*-F} = \|\mathbf{X}\|_* - \|\mathbf{X}\|_F$. The proposed $l_{t,1-2}$ metric, as a natural combination of l_{1-2} and $l_{t,1}$, inherits their merits. On one hand, $l_{t,1-2}$ enjoys the unbiasedness of $l_{t,1}$ (Lemma 3.2 (d)), thus providing a better approximation to l_0 than l_{1-2}. On the other hand, $l_{t,1-2}$ maintains some nice properties of l_{1-2}, such as the sparse properties of local and global minimizers (Theorem 3.8) and the superior performance over $l_{t,1}$ on sparse recovery from coherent sensing matrices.

We consider both constrained and unconstrained formulations of $l_{t,1-2}$ minimization. The constrained minimization problems for sparse recovery and rank minimization are expressed as

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{x}\|_{t,1-2} \quad \text{subject to} \quad \|\mathbf{Ax} - \mathbf{y}\|_2 \leq \tau, \qquad (1.5)$$

$$\min_{\mathbf{X} \in \mathbb{R}^{m \times n}} \|\mathbf{X}\|_{t,*-F} \quad \text{subject to} \quad \|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_2 \leq \tau, \qquad (1.6)$$

where $\tau \geq 0$ is the magnitude of (possible) perturbations on the measurements. The unconstrained formulations are given by

$$\min_{\mathbf{x} \in \mathbb{R}^n} \lambda \|\mathbf{x}\|_{t,1-2} + \frac{1}{2} \|\mathbf{Ax} - \mathbf{y}\|_2^2, \qquad (1.7)$$

$$\min_{\mathbf{X} \in \mathbb{R}^{m \times n}} \lambda \|\mathbf{X}\|_{t,*-F} + \frac{1}{2} \|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_2^2, \qquad (1.8)$$

where $\lambda > 0$ is a regularization parameter balancing the two terms. We shall focus on theoretical analyses of the constrained models and numerical implementations of the unconstrained ones.

The rest of the paper is organized as follows. Section 2 presents notations and a toy example to examine various metrics for exact sparse recovery. In Section 3, we present theoretical results of $l_{t,1-2}$ minimization, including the unbiasedness property, exact and stable recovery conditions, and sparse properties of local and global minimizers. Section 4 applies the difference of convex functions algorithm (DCA) [44, 45] to solve $l_{t,1-2}$ minimization with guaranteed convergence, and proposes an adaptive selection of t without requiring any knowledge of the true sparsity/rank. Experimental results in Section 5 show advantages of our method over the state-of-the-art methods in sparse recovery and matrix completion. Finally, conclusions are given in Section 6.

2. Notation and toy example. We introduce notations of the paper and give a toy example to motivate the use of the proposed $l_{t,1-2}$ metric.

2.1. Notation. We use boldface capital letters for matrices, e.g., $\mathbf{A}$, capital letters for sets, e.g., A, boldface lowercase letters for vectors, e.g., $\mathbf{a}$, and lowercase letters for scalars and matrix/vector entries, e.g., a, $a_{i,j}$, a_i. Given an index set $T \subseteq \{1, \ldots, n\}$,

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always assume that $\Gamma_{\mathbf{x},t}$ is fixed to one of them.

denote $|T|$ be the cardinality of T and T^c be the complement of T. Given a vector $\mathbf{a} \in \mathbb{R}^n$, let $\mathbf{a}_T \in \mathbb{R}^n$ be the vector whose ith entry is equal to a_i if $i \in T$ and 0 if $i \notin T$ for $1 \le i \le n$. Given a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ (we assume that $m \le n$), let $\mathbf{A}_T \in \mathbb{R}^{m \times |T|}$ be the submatrix of $\mathbf{A}$ with column indices T. Let $\mathrm{diag}(\mathbf{a}) \in \mathbb{R}^{n \times n}$ be the diagonal matrix with the entries of $\mathbf{a}$ on its main diagonal, and $\mathrm{diag}(\mathbf{A}) \in \mathbb{R}^m$ be the main diagonal of $\mathbf{A}$. The support of $\mathbf{a}$ is denoted by $\mathrm{supp}(\mathbf{a}) := \{i \mid a_i \ne 0\}$ and any vector with no more than s non-zero entries is called an s-sparse vector. The rank of $\mathbf{A}$ is denoted by $\mathrm{rank}(\mathbf{A})$ and any matrix of rank no greater than r is called an r-rank matrix. Inner products of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$ are denoted by $\langle \mathbf{a}, \mathbf{b} \rangle := \mathbf{a}^T \mathbf{b}$ and $\langle \mathbf{A}, \mathbf{B} \rangle := \mathrm{tr}(\mathbf{A}^T \mathbf{B})$, where $\mathrm{tr}(\cdot)$ is the matrix trace. The Frobenius norm of $\mathbf{A}$ is defined as $\|\mathbf{A}\|_F := \sqrt{\langle \mathbf{A}, \mathbf{A} \rangle}$. The kernel and image of $\mathbf{A}$ are denoted by $\ker(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{0}\}$ and $\mathrm{im}(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^m : \mathbf{x} = \mathbf{A}\mathbf{y}, \mathbf{y} \in \mathbb{R}^n\}$. The full singular value decomposition (SVD) of $\mathbf{A}$ is denoted by $\mathbf{A} = \mathbf{U}[\mathrm{diag}(\boldsymbol{\sigma}(\mathbf{X}))\ \mathbf{0}]\mathbf{V}^T$, where $\mathbf{U} \in \mathbb{R}^{m \times m}$ and $\mathbf{V} \in \mathbb{R}^{n \times n}$ are unitary matrices. If $\mathrm{rank}(\mathbf{A}) = r$, an economy SVD is denoted by $\mathbf{A} = \mathbf{U}_1 \boldsymbol{\Sigma} \mathbf{V}_1^T$, where $\boldsymbol{\Sigma} = \mathrm{diag}([\sigma_1(\mathbf{A}) \ \ldots \ \sigma_r(\mathbf{A})])$ and $\mathbf{U}_1 \in \mathbb{R}^{m \times r}$ and $\mathbf{V}_1 \in \mathbb{R}^{n \times r}$ contain the first r columns of $\mathbf{U}$ and $\mathbf{V}$, respectively. The 2-norm and the nuclear norm of $\mathbf{A}$ are defined as $\|\mathbf{A}\|_2 := \sigma_1(\mathbf{A})$ and $\|\mathbf{A}\|_* := \sum_i \sigma_i(\mathbf{A})$, respectively. The singular value thresholding operator (SVT) [5], i.e., the proximity operator of the nuclear norm, is defined as $\mathrm{SVT}(\mathbf{A}, a) := \mathbf{U}_1 \boldsymbol{\Sigma}_1 \mathbf{V}_1^T$, where a is a given threshold and $\boldsymbol{\Sigma}_1 = \mathrm{diag}([\max(0, \sigma_1(\mathbf{A}) - a) \ \ldots \ \max(0, \sigma_r(\mathbf{A}) - a)])$.

2.2. A toy example. We give a toy example to illustrate the behaviors of sparsity promoting metrics l_1, l_{1-2}, $l_{1/2}$, $l_{t,1}$, and $l_{t,1-2}$. Here $l_{t,1}$ is defined as $\|\mathbf{x}\|_{t,1} := \sum_{i \notin \Gamma_{\mathbf{x},t}} |x_i|$ with $\Gamma_{\mathbf{x},t}$ the same as in (1.3). Consider the following linear system:

$$\begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{2.1}$$

where $a \ne -2$ is a parameter. The general solution of (2.1) is an one-dimensional affine space characterized by k, i.e., $(-ak + 1, 2k + 1, k, k)$, and the unique sparsest solution is $(1, 1, 0, 0)$ obtained by $k = 0$. We plot the objective values of each aforementioned metric for three particular values of a in Figure 2.1. One can see that only $l_{2,1}$ and $l_{1,1-2}$ success in finding $k = 0$ as the unique global minimizer in all cases, whereas the other metrics fail in at least one case, having $k = 0$ as either a local minimizer or one of non-unique global minimizers.

Further calculations give the ranges of a for different metrics to have a local/global minimizer and the unique global minimizer at $k = 0$. As summarized in Table 2.1, one can see that $l_{1/2}$ has a looser condition than l_1 and l_{1-2} to locate the sparsest solution. Two truncated metrics $l_{t,1}$ and $l_{t,1-2}$ exhibit similar behaviors of local/global minimizers. We observe that there exist critical values of t for both of them to yield $k = 0$ as the unique global minimizer, e.g., $t = 2$ for $l_{t,1}$ and $t = 1$ for $l_{t,1-2}$. For t smaller than the critical values, $l_{0,1-2}$ yields a wider range of a than $l_{1,1}$ to find the sparsest solution, while $l_{1,1}$ never succeeds in this regard due to the infinite number of global minimizers. As shown in Section 3.3, $l_{t,1-2}$ with t smaller than the critical value always has finite global minimizers. For t larger than the critical values, both $l_{t,1}$ and $l_{t,1-2}$ have more than one global minimizers and hence fail to select the sparest solution.
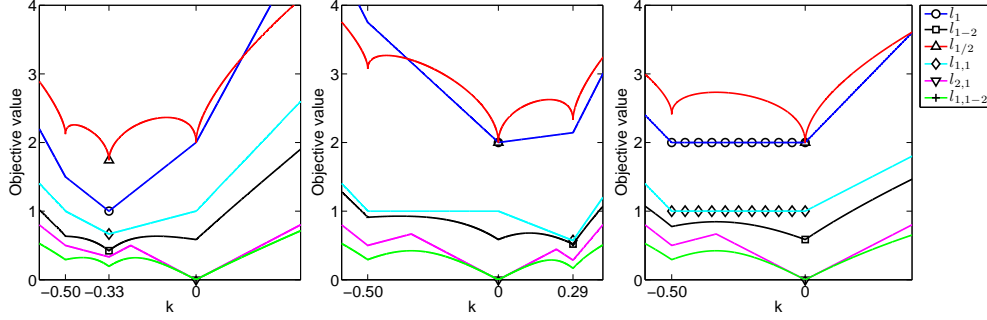
FIG. 2.1. *Objective values of different metrics when $a = -3$ (left), $a = 3.5$ (middle), and $a = 0$ (right) in the toy example (2.1). The global minimizers of different metrics are denoted by different markers. We observe that only $l_{2,1}$ and $l_{1,1-2}$ can find $k = 0$ as the unique global minimizer in all cases, whereas the other metrics fail in at least one case, having $k = 0$ as either a local minimizer or one of non-unique global minimizers. Circles and diamonds in the last plot indicate that there are infinite global minimizers of l_1 and $l_{1,1}$, meaning that they are unable to find the sparsest solution.*

TABLE 2.1
Ranges of a for different metrics when $k = 0$ is a local minimizer, a global minimizer, and the unique global minimizer. Here we assume that $a \neq -2$ and $*$ denotes approximated values.

	Local minimizer	Global minimizer	Unique Global minimizer
l_1 or $l_{0,1}$	$[0, 4]$	$[0, 4]$	$(0, 4)$
l_{1-2} or $l_{0,1-2}$	$(-4.83^*, 8.83^*)$	$[-1.21^*, 3.08^*]$	$(-1.21^*, 3.08^*)$
$l_{1/2}$	$(-\infty, \infty)$	$[-1.31^*, 5.70^*]$	$(-1.31^*, 5.70^*)$
$l_{1,1}$	$[-2, 2]$	$[-1, 2]$	$\emptyset$
$l_{2,1}$	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, \infty)$
$l_{t,1}, t \geq 3$	$(-\infty, \infty)$	$(-\infty, \infty)$	$\emptyset$
$l_{1,1-2}$	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, \infty)$
$l_{t,1-2}, t \geq 2$	$(-\infty, \infty)$	$(-\infty, \infty)$	$\emptyset$

3. Theory of truncated l_{1-2} minimization. Theoretical analysis of $l_{t,1-2}$ minimization is organized into three sections. Section 3.1 provides some preliminary properties of $l_{t,1-2}$, which are helpful to establish exact and stable recovery conditions in Section 3.2. In Section 3.3, we analyze the sparse properties of local and global minimizers for the sparse recovery problems (1.5) and (1.7).

3.1. Preliminary results. We want to show that $l_{t,1-2}$, similarly to its non-truncated origin l_{1-2}, can be expressed as a difference of two convex functions. Given an index set T, we denote $\|\mathbf{x}\|_T := \|\mathbf{x}_T\|_1 + \|\mathbf{x}_{T^c}\|_2$. For $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{X} \in \mathbb{R}^{m \times n}$, we define the following mixed norms:

$$\|\mathbf{x}\|_{t,1+2} := \|\mathbf{x}\|_{\Gamma_{\mathbf{x},t}} = \|\mathbf{x}_{\Gamma_{\mathbf{x},t}}\|_1 + \|\mathbf{x}_{\Gamma_{\mathbf{x},t}^c}\|_2,$$

$$\|\mathbf{X}\|_{t,*+F} := \|\boldsymbol{\sigma}(\mathbf{X})\|_{t,1+2} = \sum_{i=1}^{t} \sigma_i(\mathbf{X}) + \sqrt{\sum_{i=t+1}^{m} \sigma_i^2(\mathbf{X})},$$

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where $\Gamma_{\mathbf{x},t}$ is defined in (1.3). Then $l_{t,1-2}$ can be rewritten as

$$\|\mathbf{x}\|_{t,1-2} = \|\mathbf{x}\|_1 - \|\mathbf{x}\|_{t,1+2} \quad \text{and} \quad \|\mathbf{X}\|_{t,*-F} = \|\mathbf{X}\|_* - \|\mathbf{X}\|_{t,*+F}. \tag{3.1}$$

The following two lemmas give useful properties of the above defined mixed norms and $l_{t,1-2}$. More precisely, Lemma 3.1 (b) establishes the convexity of $\|\cdot\|_{t,1+2}$ and $\|\cdot\|_{t,*+F}$, while Lemma 3.2 (d) states the unbiasedness of $l_{t,1-2}$.

LEMMA 3.1. *The following statements hold:*

(a) *For any $\mathbf{x} \in \mathbb{R}^n$ and $T \subseteq \{1,\ldots,n\}$ with $|T| = t$, one has $\|\mathbf{x}\|_T \leq \|\mathbf{x}\|_{t,1+2}$.*

(b) *$\|\cdot\|_{t,1+2}$ is a norm on $\mathbb{R}^n$, and $\|\cdot\|_{t,*+F}$ is a unitarily invariant norm[2] on $\mathbb{R}^{m\times n}$.*

(c) *Given unitary matrices $\mathbf{U} \in \mathbb{R}^{m\times m}$ and $\mathbf{V} \in \mathbb{R}^{n\times n}$. Partition $\mathbf{U}$ as $[\mathbf{U}_1 \ \mathbf{U}_2]$ and $\mathbf{V}$ as $[\mathbf{V}_1 \ \mathbf{V}_2]$, where $\mathbf{U}_1 \in \mathbb{R}^{m\times t}$ and $\mathbf{V}_1 \in \mathbb{R}^{n\times t}$. Then for any matrix $\mathbf{X} \in \mathbb{R}^{m\times n}$, one has $\|\mathbf{U}_1^T\mathbf{X}\mathbf{V}_1\|_* + \|\mathbf{U}_2^T\mathbf{X}\mathbf{V}_2\|_F \leq \|\mathbf{X}\|_{t,*+F}$.*

Remark 3.1. Note that Lemma 3.1 (a) is a special case of Lemma 3.1 (c) if $\mathbf{X}$ is a diagonal matrix and $\mathbf{U} = \mathbf{V}$ are permutation matrices.

LEMMA 3.2. *Given $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{X} \in \mathbb{R}^{m\times n}$, the following statements hold:*

(a) *([55, Lemma 2.1]) Suppose $\|\mathbf{x}\|_0 = s$, then*

$$(s - \sqrt{s}) \min_{i\in\mathrm{supp}(\mathbf{x})} |x_i| \leq \|\mathbf{x}\|_1 - \|\mathbf{x}\|_2 \leq (\sqrt{s} - 1)\|\mathbf{x}\|_2.$$

(b) *Suppose $\mathrm{rank}(\mathbf{X}) = r$, then*

$$(r - \sqrt{r})\sigma_r(\mathbf{X}) \leq \|\mathbf{X}\|_* - \|\mathbf{X}\|_F \leq (\sqrt{r} - 1)\|\mathbf{X}\|_F.$$

(c) *If $t \leq n - 2$, then $\|\mathbf{x}\|_{t,1-2} \geq (2 - \sqrt{2})\min_{i\in\Gamma_{\mathbf{x},t+2}} |x_i|$. If $t \leq m - 2$, then $\|\mathbf{X}\|_{t,*-F} \geq (2 - \sqrt{2})\sigma_{t+2}(\mathbf{X})$.*

(d) *(unbiasedness) $\|\mathbf{x}\|_{t,1-2} = 0$ if and only if $\mathbf{x}$ is $(t+1)$-sparse, and $\|\mathbf{X}\|_{t,*-F} = 0$ if and only if $\mathbf{X}$ is $(t+1)$-rank.*

Proofs of Lemmas 3.1 and 3.2 are given in Appendix A.

3.2. Exact and stable recovery. The exact and stable recovery theories of $l_{t,1-2}$ minimization are based on the restricted isometry property (RIP) [10] for vectors and its extension to matrices [46]. RIP-based analyses have derived sufficient conditions for sparse recovery and rank minimization using l_1 [10, 9, 7, 46, 41, 8], l_{1-2} [55], and l_q [13, 16, 32]. To make our paper self-contained, the definitions of the vector and matrix RIPs are given as follows.

DEFINITION 3.3. *Given a positive integer s, the restricted isometry constant of a matrix $\mathbf{A}$ is the smallest number δ_s such that*

$$(1 - \delta_s)\|\mathbf{x}\|_2^2 \leq \|\mathbf{A}\mathbf{x}\|_2^2 \leq (1 + \delta_s)\|\mathbf{x}\|_2^2 \tag{3.2}$$

holds for any s-sparse vector $\mathbf{x}$.

DEFINITION 3.4. *Given a positive integer r, the matrix-RIP constant of a linear map $\mathcal{A}$ is the smallest number ρ_r such that*

$$(1 - \rho_r)\|\mathbf{X}\|_F^2 \leq \|\mathcal{A}(\mathbf{X})\|_2^2 \leq (1 + \rho_r)\|\mathbf{X}\|_F^2 \tag{3.3}$$

holds for any r-rank matrix $\mathbf{X}$.

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[2] A matrix norm $\|\cdot\|$ is unitarily invariant if $\|\mathbf{X}\| = \|\mathbf{U}\mathbf{X}\mathbf{V}\|$ for any unitary matrices $\mathbf{U}$ and $\mathbf{V}$ and any $\mathbf{X}$.

We first give the following theorem establishing the existence of optimal solutions of the proposed $l_{t,1-2}$ models.

THEOREM 3.5. *If $\mathbf{A}$ and $\mathcal{A}$ satisfy the conditions $\delta_{t+1} < 1$ and $\rho_{t+1} < 1$, respectively, then each unconstrained model (1.7) and (1.8) has at least an optimal solution. In addition, if the feasible sets in (1.5) and (1.6) are nonempty, then each constrained model (1.5) and (1.6) has at least an optimal solution.*

We then present two RIP-based theorems regarding sufficient conditions for the constrained $l_{t,1-2}$ models (1.5) and (1.6) to guarantee exact and stable sparse recovery and rank minimization.

THEOREM 3.6. *Let $\bar{\mathbf{x}}$ be any s-sparse vector and let $\mathbf{y} := \mathbf{A}\bar{\mathbf{x}} + \mathbf{e}$, where $\mathbf{e} \in \mathbb{R}^m$ is any perturbation with $\|\mathbf{e}\|_2 \leq \tau$. The following statements hold:*

(a) *For $t \geq s - 1$, if $\mathbf{A}$ satisfies the condition*

$$\delta_{s+t+1} < 1, \tag{3.4}$$

   then any solution $\mathbf{x}$ to (1.5) obeys $\|\mathbf{x} - \bar{\mathbf{x}}\|_2 \leq c_1 \tau$ for some positive constant c_1 dependent on δ_{s+t+1}.

(b) *For an arbitrary t, if $s \geq 2$ and $\mathbf{A}$ satisfies the condition*

$$\delta_{2s+t} < \Phi(s), \quad \Phi(s) := \frac{\sqrt{s} - 1}{\sqrt{2s} + \sqrt{s} + \sqrt{2} - 1}, \tag{3.5}$$

   then any solution $\mathbf{x}$ to (1.5) obeys $\|\mathbf{x} - \bar{\mathbf{x}}\|_2 \leq c_2 \tau$ for some positive constant c_2 dependent on δ_{2s+t}.

THEOREM 3.7. *Let $\bar{\mathbf{X}}$ be any r-rank matrix and let $\mathbf{b} := \mathcal{A}(\bar{\mathbf{X}}) + \mathbf{e}$, where $\mathbf{e} \in \mathbb{R}^p$ is any perturbation with $\|\mathbf{e}\|_2 \leq \tau$. The following statements hold:*

(a) *For $t \geq r - 1$, if $\mathcal{A}$ satisfies the condition*

$$\rho_{r+t+1} < 1, \tag{3.6}$$

   then any solution $\mathbf{X}$ to (1.6) obeys $\|\mathbf{X} - \bar{\mathbf{X}}\|_F \leq c_1 \tau$ for some positive constant c_1 dependent on ρ_{r+t+1}.

(b) *For an arbitrary t, if $r \geq 2$ and $\mathbf{A}$ satisfies the condition*

$$\rho_{3r+3t} < \Psi(r,t), \quad \Psi(r,t) := \frac{\sqrt{r} - 1}{2\sqrt{r+t} + \sqrt{r} + \sqrt{2} - 1}, \tag{3.7}$$

   then any solution $\mathbf{X}$ to (1.6) obeys $\|\mathbf{X} - \bar{\mathbf{X}}\|_F \leq c_2 \tau$ for some positive constant c_2 dependent on ρ_{3r+3t}.

Proofs of Theorems 3.5, 3.6, and 3.7 are given in Appendix B. The following remarks regarding Theorems 3.6 and 3.7 are worth noticing.

Remark 3.2. Theorems 3.6 and 3.7 suggest that under certain conditions, $l_{t,1-2}$ minimization can guarantee exact recovery from noiseless measurements ($\tau = 0$); and stable recovery is possible within an error proportional to τ from noisy measurements.

Remark 3.3. We comment on the recovery conditions. By definitions of the RIPs, one has that $\delta_s \leq \delta_{s'}$ for $s \leq s'$ and $\rho_r \leq \rho_{r'}$ for $r \leq r'$. The loosest cases of (3.4) and (3.6) are $\delta_{2s} < 1$ and $\rho_{2r} < 1$ at the critical values $t = s - 1$ and $t = r - 1$, respectively. Note that these conditions for $l_{t,1-2}$ are equivalent to the recovery conditions for l_0 [10, Lemma 1.2] and matrix rank [46, Theorem 3.2], which suggests that $l_{t,1-2}$ has the same recoverability as l_0 in theory. On the other hand, the loosest cases of (3.5) and (3.7) are $\delta_{2s} < \Phi(s)$ and $\rho_{3r} < \Psi(r,0)$, respectively, both achieved at $t = 0$, which

can be considered as extensions of the recovery condition of l_{1-2} in [55]. Compared with other metrics, conditions (3.5) and (3.7) are stronger than $\delta_{2s} < \sqrt{2} - 1$ for l_1 in [7] and $\rho_{3r} < 2\sqrt{5} - 4 \approx 0.4721$ for nuclear norm in [41], since $\Phi(s) < \sqrt{2} - 1$ and $\Psi(r, t) < 1/3$, and also stronger than those for l_q [13, 32].

Remark 3.4. Conditions (3.5) and (3.7) are not tight. We believe that these conditions can be further sharpened, such as the work by Cai and Zhang [6]. Although $l_{t,1-2}$ requires stronger assumptions in theory, we observe empirically that $l_{t,1-2}$ consistently outperforms l_1 and is comparable to some nonconvex metrics, as illustrated by experimental results in Section 5.

3.3. Sparsity of local and global minimizers. We study the properties of local and global minimizers of the sparse recovery problems (1.5) and (1.7). It has been shown that a local minimizer of l_{1-2} [55] and l_q [15] minimizations exhibits certain sparsity, meaning that its support set corresponds to linearly independent columns of $\mathbf{A}$. For $l_{t,1-2}$, this property only holds for global minimizers with nonzero objectives.

THEOREM 3.8. *Let* $\mathbf{x}^*$ *be any vector satisfying* $\|\mathbf{x}^*\|_{t,1-2} \neq 0$. *Denote* $\Lambda^* :=$ $\mathrm{supp}(\mathbf{x}^*)$, $\Lambda_1 := \Gamma_{\mathbf{x}^*, t}$, *and* $\Lambda_2 := \Lambda^* \backslash \Gamma_{\mathbf{x}^*, t}$. *The following statements hold:*

(a) *If* $\mathbf{x}^*$ *is a local minimizer of the constrained model* (1.5), *then the columns of* $\mathbf{A}_{\Lambda_2}$ *are linearly independent and* $\mathrm{im}(\mathbf{A}_{\Lambda_1}) \cap \mathrm{im}(\mathbf{A}_{\Lambda_2}) = \{\mathbf{0}\}$, *where* $\mathrm{im}(\mathbf{A}_\emptyset) = \{\mathbf{0}\}$ *is assumed.*

(b) *If* $\mathbf{x}^*$ *is a global minimizer of the constrained model* (1.5), *then the columns of* $\mathbf{A}_{\Lambda^*}$ *are linearly independent.*

(c) *If* $\mathbf{x}^*$ *is a local (global) minimizer of the unconstrained model* (1.7), *both (a) and (b) are true.*

We have the following conclusions when the inequality constraint in (1.5) becomes exact, i.e., $\tau = 0$.

COROLLARY 3.9. *If* $\tau = 0$ *in* (1.5), *we have*

(a) *the number of local minima of the constrained problem* (1.5) *is finite;*

(b) *if the global minimum of the constrained model* (1.5) *is not zero, then the number of global minimizers is also finite.*

Please refer to Appendix C for proofs of Theorem 3.8 and Corollary 3.9.

Remark 3.5. Unlike l_{1-2} and l_q, there may exist infinite local minimizers of $l_{t,1-2}$. But Corollary 3.9 (b) ensures that the number of global minimizers is finite, as long as the global minimum of the objective function is not zero. Note that l_1 and $l_{t,1}$ may exist infinite global minimizers with nonzero objectives.

4. Numerical algorithm. In this section, we describe a numerical algorithm for solving the unconstrained rank minimization problem (1.8). We begin with Section 4.1 for a difference of convex functions algorithm (DCA) [44, 45] that decomposes the original nonconvex optimization problem into a series of convex subproblems, each of which can be solved efficiently by alternating direction method of multipliers (ADMM) [24, 23], as discussed in Section 4.2. We further develop in Section 4.3 an adaptive selection of t without requiring any knowledge of the true sparsity/rank. At last, we present some implementation details in Section 4.4. Note that all the discussions on rank minimization can be applied to the sparse recovery problem (1.7), considering that (1.7) is a special case of (1.8) by restricting $\mathbf{X}$ to be diagonal; e.g., see (B.11). The details for sparse recovery are omitted here due to space limitation.

4.1. Difference of convex functions algorithm. The DCA [44, 45] is a descent algorithm for solving difference of convex functions (d.c.) optimization

$$\min_{\mathbf{X} \in \mathbb{R}^{m \times n}} F(\mathbf{X}) = G(\mathbf{X}) - H(\mathbf{X}),$$

where G and H are proper lower semicontinuous and strongly convex functions. Starting from an initial point $\mathbf{X}^0$, the DCA iteratively constructs two sequences $\{\mathbf{X}^k\}$ and $\{\mathbf{Y}^k\}$:

$$\begin{cases} \mathbf{Y}^k \in \partial H(\mathbf{X}^k), \\ \mathbf{X}^{k+1} = \arg\min_{\mathbf{X}} G(\mathbf{X}) - \langle \mathbf{Y}^k, \mathbf{X} \rangle, \end{cases} \tag{4.1}$$

where $\mathbf{Y}^k \in \partial H(\mathbf{X}^k)$ means that $\mathbf{Y}^k$ is a subgradient of $H(\mathbf{X})$ at $\mathbf{X}^k$. An important property of the DCA iteration (4.1) is that it leads to monotonically decreasing objective values $\{F(\mathbf{X}^k\}$; see [44, 45] for theoretical analyses of the DCA.

Due to (3.1), it is natural to consider the following d.c. decomposition of the objective in (1.8):

$$G(\mathbf{X}) = \frac{1}{2}\|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_2^2 + \lambda\|\mathbf{X}\|_* + c\|\mathbf{X}\|_F^2,$$
$$H(\mathbf{X}) = \lambda\|\mathbf{X}\|_{t,*+F} + c\|\mathbf{X}\|_F^2, \tag{4.2}$$

where c is a positive constant to ensure strong convexity of G and H. Note that the convexity of H is guaranteed by Lemma 3.1 (b). There is a closed-form solution of $\mathbf{Y}^k$ in (4.1). Denote $\mathbf{X}^k = \mathbf{U}[\text{diag}(\mathbf{s})\ \mathbf{0}]\mathbf{V}^T$ as a full SVD of $\mathbf{X}^k$ and $T := \{1,\ldots,t\}$. Define $\mathbf{d} \in \mathbb{R}^m$ by

$$d_i := \begin{cases} 1 & \text{if} \quad s_i > 0, i \leq t, \\ s_i/\|\mathbf{s}_{T^c}\|_2 & \text{if} \quad s_i > 0, i > t, \\ 0 & \text{if} \quad s_i = 0. \end{cases}$$

It is straightforward that $\mathbf{d} \in \partial\|\mathbf{s}\|_T$. Using the definition of subgradient and Lemma 3.1 (a), we have that for any $\mathbf{s}' \in \mathbb{R}^m$,

$$\|\mathbf{s}'\|_{t,1+2} \geq \|\mathbf{s}'\|_T \geq \|\mathbf{s}\|_T + \langle \mathbf{d}, \mathbf{s}' - \mathbf{s} \rangle = \|\mathbf{s}\|_{t,1+2} + \langle \mathbf{d}, \mathbf{s}' - \mathbf{s} \rangle,$$

which implies that $\mathbf{d} \in \partial\|\mathbf{s}\|_{t,1+2}$. Letting

$$\mathbf{Y}^k := \lambda\mathbf{U}[\text{diag}(\mathbf{d})\quad \mathbf{0}]\mathbf{V}^T + 2c\mathbf{X}^k, \tag{4.3}$$

we have that $\mathbf{Y}^k \in \partial H(\mathbf{X}^k)$ by [52, Theorem 2].

We conclude this section by the convergence analysis of the proposed algorithm.

THEOREM 4.1. *If $\mathcal{A}$ satisfies the condition $\rho_{t+1} < 1$, then $\{\mathbf{X}^k\}$ generated by the DCA (4.1) satisfies*

(a) *$\{F(\mathbf{X}^k)\}$ is monotonically decreasing and convergent.*
(b) *$\{\mathbf{X}^k\}$ is bounded and $\|\mathbf{X}^k - \mathbf{X}^{k+1}\|_F \to 0$ as $k \to \infty$.*
(c) *Any limit point $\mathbf{X}^*$ of $\{\mathbf{X}^k\}$ is a critical point satisfying*

$$\mathbf{0} \in \mathcal{A}^*(\mathcal{A}(\mathbf{X}^*) - \mathbf{b}) + \lambda(\partial\|\mathbf{X}^*\|_* - \partial\|\mathbf{X}^*\|_{t,*+F}),$$

where $\mathcal{A}^*$ denotes the adjoint operator of $\mathcal{A}$.

The results are straightforward by applying the convergence analysis of the general DCA theory [45, Theorem 3.7], and hence the proof is omitted. It is worth noting that the condition $\rho_{t+1} < 1$ in Theorem 4.1 is used to ensure the coercivity of the objective function (see the proof of Theorem 3.5), which implies the boundedness of the sequence $\{\mathbf{X}^k\}$ generated by the DCA.

4.2. Solving the DCA subproblem. We elaborate on how to solve the DCA subproblem in (4.1), i.e.,

$$\min_{\mathbf{X} \in \mathbb{R}^{m \times n}} \frac{1}{2} \|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_2^2 + \lambda \|\mathbf{X}\|_* + c \|\mathbf{X}\|_F^2 - \langle \mathbf{Y}, \mathbf{X} \rangle, \tag{4.4}$$

where the superscript k is omitted without confusion. This is a convex problem, and we apply the ADMM [24, 23] to find the optimal solution. In general, ADMM is based on a variable splitting technique to resolve the original problem into several subproblems, each of which can be solved efficiently. By introducing an auxiliary variable $\mathbf{Z} \in \mathbb{R}^{m \times n}$, we reformulate (4.4) as

$$\min_{\mathbf{X}, \mathbf{Z} \in \mathbb{R}^{m \times n}} \frac{1}{2} \|\mathcal{A}(\mathbf{Z}) - \mathbf{b}\|_2^2 + \lambda \|\mathbf{X}\|_* + c \|\mathbf{Z}\|_F^2 - \langle \mathbf{Y}, \mathbf{Z} \rangle \quad \text{subject to} \quad \mathbf{X} = \mathbf{Z}. \tag{4.5}$$

The augmented Lagrangian function of (4.5) reads

$$L(\mathbf{X}, \mathbf{Z}, \mathbf{W}) = \frac{1}{2} \|\mathcal{A}(\mathbf{Z}) - \mathbf{b}\|_2^2 + \lambda \|\mathbf{X}\|_* + c \|\mathbf{Z}\|_F^2 - \langle \mathbf{Y}, \mathbf{Z} \rangle + \frac{\beta}{2} \|\mathbf{X} - \mathbf{Z} + \mathbf{W}\|_F^2, \tag{4.6}$$

where $\mathbf{W}$ is a Lagrangian multiplier and $\beta > 0$ is a penalty parameter. Denote the (inner) iteration index by l, ADMM iterates as follows:

$$\begin{cases} \mathbf{Z}^{l+1} = \arg \min_{\mathbf{Z}} L(\mathbf{X}^l, \mathbf{Z}, \mathbf{W}^l), \\ \mathbf{X}^{l+1} = \arg \min_{\mathbf{X}} L(\mathbf{X}, \mathbf{Z}^{l+1}, \mathbf{W}^l), \\ \mathbf{W}^{l+1} = \mathbf{W}^l + \mathbf{X}^{l+1} - \mathbf{Z}^{l+1}. \end{cases} \tag{4.7}$$

The minimizations with respect to $\mathbf{Z}$ and $\mathbf{X}$ have closed-form solutions:

$$\mathbf{Z}^{l+1} = (\mathcal{A}^* \mathcal{A} + (2c + \beta)\mathcal{I})^{-1}(\mathcal{A}^* \mathbf{b} + \mathbf{Y} + \beta(\mathbf{X}^l + \mathbf{W}^l)), \tag{4.8}$$

$$\mathbf{X}^{l+1} = \text{SVT}(\mathbf{Z}^{l+1} - \mathbf{W}^l, \frac{\lambda}{\beta}), \tag{4.9}$$

where $\mathcal{I}$ denotes the identity operator. The convergence of ADMM for a convex problem is guaranteed; see [27] for more details.

We discuss the computational complexity of ADMM. We exclude the complexities for computing $(\mathcal{A}^* \mathcal{A} + (2c + \beta)\mathcal{I})^{-1}$ and $\mathcal{A}^* \mathbf{b}$ in (4.8), as they are precomputed before the iterations. The complexities of (4.8) and (4.9) are $\mathcal{O}(m^2 n^2)$ and $\mathcal{O}(m^2 n)$, respectively, for computing a linear map of $\mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$ and a SVD of an $m \times n$ matrix ($m \leq n$), so the total complexity per iteration is $\mathcal{O}(m^2 n^2)$. We then discuss the complexities for two special cases. The first one is the sparse recovery problem (1.7). In this case, (4.8) reduces to a matrix-vector multiplication of an $n \times n$ matrix and an $n \times 1$ vector, which costs $\mathcal{O}(n^2)$; and (4.9) reduces to a soft-thresholding operator on an $n \times 1$ vector, which costs $\mathcal{O}(n)$. Therefore, the total complexity per iteration is $\mathcal{O}(n^2)$. The second case is the matrix completion problem, i.e., $\mathcal{A} = \mathcal{P}_\Omega$ in (1.2). Since (4.8) reduces to an element-wise multiplication between $m \times n$ matrices, which costs $\mathcal{O}(mn)$, and (4.9) remains unchanged, so the total complexity per iteration reduces to $\mathcal{O}(m^2 n)$.

4.3. Adaptive selection of t. The selection of t plays a central role in the effectiveness of the proposed models. Theoretically, taking t as the critical value, i.e., $t = r - 1$ for r-rank matrices, should give the best recovery results. However, the true

rank is often unknown. In addition, it is found experimentally that fixing t as the critical value does not work well for recovering approximately low-rank matrices that contain very small but nonzero singular values, which is typical in real applications. For these reasons, we propose a heuristic scheme for adaptively updating the value of t. The main idea consists of initializing t with a small value and increasing it gradually, similar to the idea of the increasing rank strategy in [53]. In particular, we choose t^k at the kth DCA iteration (4.1) as

$$t^k := \begin{cases} \text{the maximal } t \text{ such that } \sum_{i=1}^{t} \sigma_i(\mathbf{X}^k) \leq \eta^k \|\mathbf{X}^k\|_*, & \text{if } k \leq k_0, \\ t^{k_0}, & \text{if } k > k_0, \end{cases} \quad (4.10)$$

where k_0 indexes the iteration number for the continuation on t and η^k is a thresholding value at the kth iteration given by $\eta^k := \max(0, \theta - \mu(k_0 - k))$, in which $0 \leq \theta < 1$ is the maximal thresholding and $\mu > 0$ is the step length for the increasing of the thresholding. Given parameters (k_0, θ, μ), the proposed scheme can automatically select the value of t without requiring any knowledge of the true rank. In addition, convergence result of the DCA can be applied for this scheme, since t is fixed after finite k_0 iterations. We remark that the proposed scheme, although working well in our experiments, is certainly not optimal. We believe that the selection of t can also benefit from other sparsity/rank estimation schemes such as jump detection [51, 53, 32].

4.4. Implementation details. The pseudo-code of the overall algorithm including the adaptive selection of t is summarized in Algorithm 1. Some implementation details are explained below.

Details for the DCA. The DCA is initialized with $\mathbf{X}^0 = \mathbf{0}$. As a consequence, we have that $t^0 = 0$ from (4.10) and $\mathbf{Y}^0 = \mathbf{0}$ from (4.3). In other words, $\mathbf{X}^1$ is the solution of the l_1 minimization (with a small quadratic term), which produces a good initialization for the subsequent iterations. It has been shown in Theorem 4.1 that $\{F(\mathbf{X}^k)\}$ is monotonically decreasing and convergent and $\|\mathbf{X}^k - \mathbf{X}^{k+1}\|_F \to 0$ as $k \to \infty$, so we terminate the DCA if the following stopping criterion is satisfied:

$$(k = k_{\max}) \vee \left((k > k_0) \wedge \max \left(\frac{\|\mathbf{X}^k - \mathbf{X}^{k-1}\|_F}{\|\mathbf{X}^{k-1}\|_F}, \frac{F(\mathbf{X}^k) - F(\mathbf{X}^{k-1})}{F(\mathbf{X}^{k-1})} \right) \leq \epsilon \right), \quad (4.11)$$

where $k_{\max}$ is the maximum number of iterations and $\epsilon > 0$ is a given tolerance.

Details for ADMM. We adopt a warm-start scheme in ADMM to speed up convergence. Specifically, $\mathbf{X}$ and $\mathbf{W}$ are initialized as their final results in the last outer iteration. The stopping criterion for ADMM is given as follows:

$$\begin{cases} (l = l_{\max}) \vee \left((l \geq 1) \wedge \frac{\|\mathbf{X}^{k,l} - \mathbf{X}^{k,l-1}\|_F}{\|\mathbf{X}^{k,l-1}\|_F} \leq \epsilon \right), & \text{if } k \leq k_0, \\ (l = l_{\max}) \vee \left((l \geq l_{\min}) \wedge \frac{\|\mathbf{X}^{k,l} - \mathbf{X}^{k,l-1}\|_F}{\|\mathbf{X}^{k,l-1}\|_F} \leq \epsilon \right), & \text{if } k > k_0, \end{cases} \quad (4.12)$$

where $l_{\min}$ and $l_{\max}$ are the minimum number and maximum number of iterations, respectively. Here we impose $l_{\min}$ to prevent executing very few iterations, which helps to stabilize the solution process.

5. Experiments. We test the performance of the proposed $l_{t,1-2}$ minimization on sparse recovery and matrix completion, i.e., a special case of (1.2) by taking $\mathcal{A} := \mathcal{P}_\Omega$, and compare it with the state-of-the-art methods. All experiments are conducted under Windows 7 and Matlab R2015b (Version 8.6.0.267246) running on a desktop with an Intel(R) Core (TM) i7-6700 CPU at 3.40GHz and 16GB memory.

Algorithm 1. The DCA for solving (1.8).

Input: $\mathcal{A}$, $\mathbf{b}$, λ, c, β, k_0, θ, μ, $k_{\max}$, $l_{\min}$, $l_{\max}$, and ϵ.

Initialization: Set $\mathbf{X}^0 := \mathbf{0}$, $\mathbf{W}^0 := \mathbf{0}$, and $k := 0$.

Outer loop: while stopping criterion (4.11) is not satisfied **do**

    Compute t^k from (4.10) and $\mathbf{Y}^k$ from (4.3).

      Initialization: Set $\mathbf{X}^{k+1,0} := \mathbf{X}^k$, $\mathbf{W}^{k+1,0} := \mathbf{W}^k$, and $l := 0$.

      Inner loop: while stopping criterion (4.12) is not satisfied **do**

          Compute $\mathbf{Z}^{k+1,l+1}$ by (4.7).

          Compute $\mathbf{X}^{k+1,l+1}$ by (4.7).

          Update $\mathbf{W}^{k+1,l+1}$ by (4.7).

          Set $l := l + 1$.

      End while and **output** $\mathbf{X}^{k+1} := \mathbf{X}^{k+1,l}$ and $\mathbf{W}^{k+1} := \mathbf{W}^{k+1,l}$.

    Set $k := k + 1$.

End while and **output** $\mathbf{X} := \mathbf{X}^k$.

5.1. Sparse recovery. We conduct five sparse recovery tests, as summarized in Table 5.1. We consider two types of sensing matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$. One is independent and identically distributed (i.i.d.) Gaussian random matrix, which is well-conditioned for CS since it has incoherent columns and small RIP constants with high probability. The other is over-sampled discrete cosine transform (DCT) matrix $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$ with

$$\mathbf{a}_j = \frac{1}{\sqrt{m}} \cos\left(\frac{2j\pi\boldsymbol{\xi}}{f}\right), \qquad j = 1, \ldots, n,$$

where $\boldsymbol{\xi} \in \mathbb{R}^m$ is a random vector whose entries are uniformly and independently sampled from $[0,1]$ and f is a positive integer. Over-sampled DCT matrices are considered as ill-conditioned for CS in the sense that their columns become coherent as f increases. It has been shown in [21] that CS under coherent sensing matrices is possible if the nonzeros of the true vector $\bar{\mathbf{x}}$ are sufficiently separated, referred to as minimum separation. In our tests, we generate $\bar{\mathbf{x}}$ as sparse Gaussian signals, i.e., vectors supported on a uniformly random index set with i.i.d. standard Gaussian entries. In addition, we impose the minimum separation condition of $\min_{i,i' \in \mathrm{supp}(\bar{\mathbf{x}})} |i - i'| \geq 2f$ for over-sampled DCT sensing matrices, while no such requirement for random Gaussian sensing matrix. Given $\mathbf{A}$ and $\bar{\mathbf{x}}$, the measurement $\mathbf{y}$ is generated as

$$\mathbf{y} := \mathbf{A}\bar{\mathbf{x}} + \alpha \frac{\|\mathbf{A}\bar{\mathbf{x}}\|_2}{\|\mathbf{e}\|_2} \mathbf{e},$$

where $\mathbf{e}$ is Gaussian white noise and α is the noise level. The relative error (ReErr) is defined as $\|\mathbf{x} - \bar{\mathbf{x}}\|_2 / \|\bar{\mathbf{x}}\|_2$. Each test is repeated over independent trials. For Tests 1 and 2, we record the success rates, where a trial is regarded as successful if ReErr $\leq 10^{-3}$. For Test 3, we record the average execution time to reach the precision ReErr $\leq 10^{-5}$. For Tests 4 and 5, we record the average ReErr values.

    Parameter selection. Our algorithm involves a set of parameters: λ, c, t, k_0, θ, μ, $l_{\min}$, β, $k_{\max}$, $l_{\max}$, and ϵ. The regularization parameter λ balances the data fidelity and regularization terms in (1.7). For noiseless cases, it is set as a very small

TABLE 5.1
Summary of sparse recovery tests. Here $\mathbf{A}$ is the sensing matrix of size $m \times n$, s is the sparsity of the true vector, α is the noise level on measurements, "Trials" is the number of random realizations, and "Objective" is metric for comparison.

#	$\mathbf{A}$	m	n	s	α	Trails	Objective
1	Gaussian	64	256	10:2:32	0	100	Success rate
2	Over-sampled DCT with $f = 20$	100	1500	5:2:35	0	100	Success rate
3	Gaussian	$2^6, \ldots, 2^{12}$	$4m$	$m/8$	0	10	Execution time
4	Gaussian	64	256	15	0.01,0.03, 0.05,0.10	100	Average ReErr
5	Over-sampled DCT with $f = 10$	100	1500	15	0.01,0.03, 0.05,0.10	100	Average ReErr

value depending on the required precision: $\lambda = 10^{-6}$ in Tests 1 and 2 and $\lambda = 10^{-8}$ in Test 3. For noisy cases, we tune λ empirically in Tests 4 and 5 to achieve the lowest ReErr values on average. The positive parameter c introduced in (4.2) ensures the strong convexity of G and H, which theoretically guarantees the convergence of the DCA. We find experimentally that our algorithm still converges at $c = 0$ and a larger c leads to slightly faster convergence but worse performance. So we choose $c = 0$ in all tests. For the selection of t, we adopt the fixed scheme $t = s - 1$ for Test 3 and the adaptive scheme introduced in Section 4.3 for the other tests. For parameters k_0, θ, and μ in the adaptive updating rule (4.10), we adopt a conservative setting $(k_0, \theta, \mu) = (10, 0.95, 0.1)$ corresponding to a slow increase in thresholdings, such that our algorithm is robust to handle vectors with large sparsity. For the minimum iteration number $l_{\min}$ in (4.12), we set $l_{\min} = 0$ in Test 3 for faster speed and $l_{\min} = 1000$ in the other tests for more stability. Other parameters in Algorithm 1 are chosen as $\beta = 100\lambda$, $k_{\max} = 50$, $l_{\max} = 5000$, and $\epsilon = 10^{-5}$, which are shown to provide effective complexity/performance tradeoff.

Figure 5.1 plots the success rates of Algorithm 1 in Test 1, to investigate the effects of different selections of t. We observe from the left plot of Figure 5.1 that the adaptive scheme yields higher success rates than all fixed schemes, especially for vectors with large sparsity. One possible reason is that our algorithm with the adaptive scheme benefits from intermediate recovery results for t going from small to large values, thus largely avoiding local minima, at which a fixed scheme is often being stuck. On the other hand, the right plot of Figure 5.1 shows that the success rates of the adaptive scheme rise as the maximal thresholding θ increases, which is consistent with the fact that $l_{t,1-2}$ with larger t leads to closer approximation to l_0. We can draw the conclusion that the adaptive scheme is more effective and stable than fixed schemes.

Comparison with other methods. We compare the proposed $l_{t,1-2}$ minimization with four state-of-the-art methods for sparse recovery: Lasso (unconstrained l_1 minimization by ADMM) [4], l_{1-2} (unconstrained l_{1-2} minimization by the DCA) [55], IRucLq-v (unconstrained l_q minimization by improved iteratively reweighted least squares algorithm) [32], and ISD (constrained truncated l_1 minimization with iterative support detection) [51]. We use the Matlab codes provided by the authors
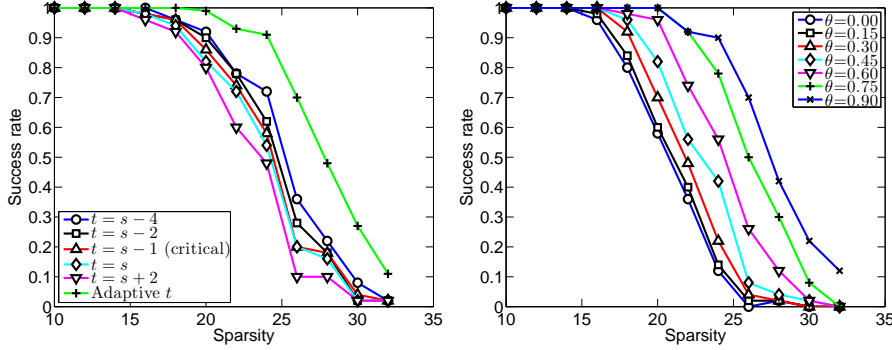
FIG. 5.1. *Comparison between different selections of t for sparse recovery based on Test 1. Left: Fixed schemes and the adaptive scheme with $(k_0, \theta, \mu) = (10, 0.95, 0.1)$. Right: The adaptive scheme with $(k_0, \mu) = (10, 0.1)$ and different θ values.*
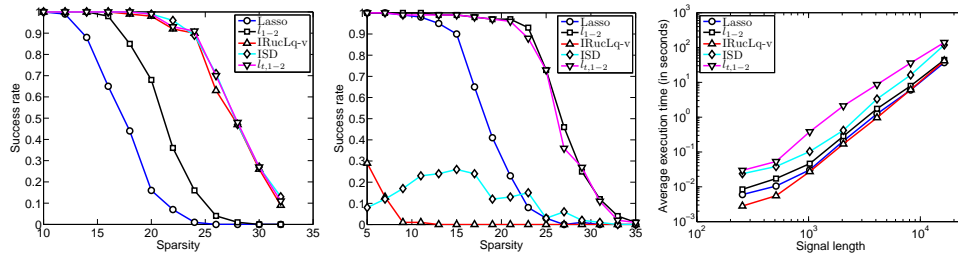

FIG. 5.2. *Comparison results of Tests 1-3. Left: Success rates in Test 1 using incoherent sensing matrices. Middle: Success rates in Test 2 using coherent sensing matrices. Right: Execution time in Test 3 using incoherent sensing matrices.*

for these four methods and give some implementation details as follows. There is a regularization parameter λ in Lasso, l_{1-2}, and IRucLq-v. Same as $l_{t,1-2}$, we set $\lambda = 10^{-6}$ in Tests 1 and 2, $\lambda = 10^{-8}$ in Test 3, and optimize λ for each method in Tests 4 and 5. For IRucLq-v, we set q to 0.5, the tolerance to 10^{-8}, and the maximum iteration number to 5000. We terminate all methods according to their default stopping criteria, except for Test 3, where all methods are terminated once the precision ReErr $\leq 10^{-5}$ is reached (the ReErr of ISD is about 10^{-14} when the support of $\bar{\mathbf{x}}$ is correctly detected). All other settings of the competing methods are left to their default ones.

Figure 5.2 compares the performances of different methods in Tests 1-3, i.e., the noiseless cases. For Test 1 using incoherent sensing matrices, $l_{t,1-2}$ is comparable to IRucLq-v and ISD in recoverability, and these three methods are much better than l_{1-2} and Lasso. For Test 2 using coherent sensing matrices, l_{1-2} and $l_{t,1-2}$ are leading methods in terms of success rates with l_{1-2} slightly better. In contrast, IRucLq-v and ISD perform even worse than the convex Lasso method. Note that $l_{t,1-2}$ is the only method that achieves superior recoverability in both scenarios. Regarding the execution time in Test 3, the last plot of Figure 5.2 shows that IRucLq-v is the fastest for $n \leq 4096$ and is on par with Lasso and l_{1-2} for larger n, while $l_{t,1-2}$ is the slowest in all cases. We remark that $l_{t,1-2}$ is time consuming because it requires a large number of ADMM iterations to solve the DCA subproblems to a high accuracy. We will leave the speed-up of the proposed algorithm in the future.

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TABLE 5.2
Average ReErr values of different methods in Test 4 using incoherent sensing matrices and Test 5 using coherent sensing matrices. The best result for each noise level is highlighted in bold.

#	Noise level	Lasso	l_{1-2}	IRucLq-v	ISD	$l_{t,1-2}$
	0.01	6.22e-2	2.31e-2	**6.51e-3**	1.23e-2	7.81e-3
	0.03	1.07e-1	5.89e-2	**2.33e-2**	3.49e-2	2.38e-2
4	0.05	1.47e-1	9.15e-2	4.43e-2	6.11e-2	**4.01e-2**
	0.10	2.26e-1	1.61e-1	1.17e-1	1.29e-1	**8.71e-2**
	0.01	5.12e-2	1.76e-2	4.19e-2	4.81e-2	**7.88e-3**
	0.03	1.30e-1	5.14e-2	7.16e-2	3.84e-1	**2.45e-2**
5	0.05	1.98e-1	9.41e-2	1.25e-1	6.67e-1	**6.43e-2**
	0.10	3.24e-1	1.91e-1	2.42e-1	3.69e0	**1.73e-1**

Table 5.2 gives the average ReErr values of different methods in Tests 4 and 5, i.e., the noisy cases. For Test 4 using incoherent sensing matrices, IRucLq-v and $l_{t,1-2}$ are considered as the winners, as each of them yields the lowest ReErr values at two noise levels. For Test 5 using coherent sensing matrices, $l_{t,1-2}$ achieves the lowest ReErr values at all noise levels. The phenomenon is consistent with [55, 35] that l_{1-2} performs better than l_q only for coherent sensing matrices.

5.2. Matrix completion. We conduct four tests for matrix completion, summarized in Table 5.3. Two types of testing matrix $\bar{\mathbf{X}}$ are used. One is exactly r-rank matrix generated by $\bar{\mathbf{X}} := \mathbf{X}_L \mathbf{X}_R$, where $\mathbf{X}_L \in \mathbb{R}^{m \times r}$ and $\mathbf{X}_R \in \mathbb{R}^{r \times n}$ are Gaussian random matrices. The other is approximately low-rank matrix considered by Wen et al. [53], which is generated by $\bar{\mathbf{X}} := \mathbf{U}[\mathrm{diag}(\boldsymbol{\sigma}) \ \mathbf{0}]\mathbf{V}^T$, where $\mathbf{U} \in \mathbb{R}^{m \times m}$ and $\mathbf{V} \in \mathbb{R}^{n \times n}$ are random unitary matrices obtained by orthogonalizing Gaussian random matrices, and the singular values σ_i for $i = 1, \ldots, m$ are exponentially decaying, i.e., $\sigma_i = e^{-0.3i}$. We normalize the ground-truth matrix $\bar{\mathbf{X}}$ to have unit 2-norm for the sake of parameter selection. For the linear map $\mathcal{A}$, we sample an index set Ω consisting of p entries uniformly at random and let $\mathcal{A} := \mathcal{P}_\Omega$. We define sampling ratio (SR) as $p/(mn)$ and degree of freedom ratio (FR) as $r(m + n - r)/p$. Introduced by Candès et al. [11], FR is the ratio between the degree of freedom and the number of measurements and a larger FR generally indicates a more difficult completion problem. Note that if FR > 1, there exist infinite r-rank matrices with the given measurements, so a successful recovery is not expected in this case. We obtain the measurement $\mathbf{b}$ as $\mathbf{b} := \mathcal{A}(\bar{\mathbf{X}}) + (\alpha \|\mathcal{A}(\bar{\mathbf{X}})\|_2 / \|\mathbf{e}\|_2)\mathbf{e}$. The relative error (ReErr) for matrix is defined using Frobenius norm, i.e., $\|\mathbf{X} - \bar{\mathbf{X}}\|_F / \|\bar{\mathbf{X}}\|_F$. Similar to sparse recovery, each test is repeated over independent trials to record the success rates under the criterion ReErr $\leq 10^{-3}$ (Test 6), the average execution time to meet the precision ReErr $\leq 10^{-5}$ (Test 7), and the average ReErr values (Tests 8 and 9).

Parameter selection. Parameters involved in our algorithm are λ, c, t, k_0, θ, μ, $l_{\min}$, β, $k_{\max}$, $l_{\max}$, and ϵ. We fix $\lambda = 10^{-6}$ in Tests 6 and 9, $\lambda = 10^{-10}$ in Test 7, and tune λ empirically in Test 8 to achieve the lowest ReErr values on average. Same as sparse recovery, we choose $c = 0$ in all tests. We adopt the fixed scheme $t = r - 1$ for Test 7 and the adaptive scheme for the other tests. We adopt an aggressive setting $(k_0, \theta, \mu) = (5, 0.95, 0.2)$ leading to a rapid increase in thresholdings; this setting provides sufficient robustness to both types of testing matrices. We set $l_{\min} = 0$ in Test 7 and $l_{\min} = 200$ in the other tests. Other parameters in Algorithm 1 are fixed as

TABLE 5.3

Summary of matrix completion tests. Here $\bar{\mathbf{X}}$ is the testing matrix of size $m \times n$, "Exact" denotes exactly low-rank matrix, "Approx." denotes approximately low-rank matrix, r is the rank of $\bar{\mathbf{X}}$, SR stands for sampling ratio, FR stands for degree of freedom ratio, α is the noise level on measurements, "Trials" is the number of random realizations, and "Objective" is the metric for comparison. Note that for Test 9, FR is calculated using $r = 10$.

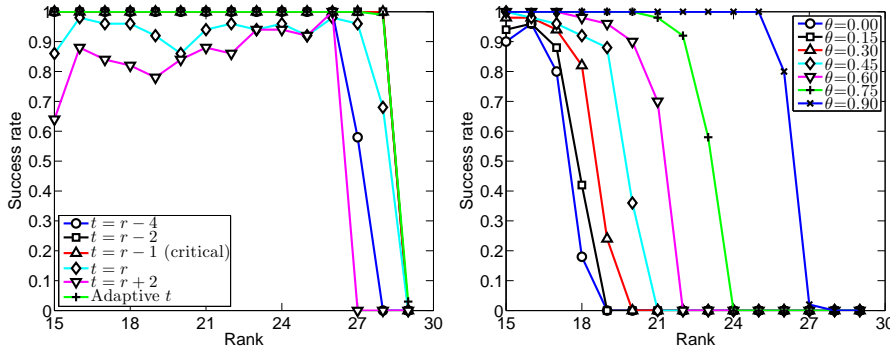| #	$\bar{\mathbf{X}}$	m $(n = m)$	r	SR	FR	α	Trails	Objective
6	Exact	100	15:1:29	0.5	0.56~0.99	0	100	Success rate
7	Exact	$2^6, \ldots, 2^{12}$	$m/16$	0.5	0.24	0	10	Execution time
8	Exact	100	15	0.5	0.56	0.01, 0.03, 0.05, 0.10	100	Average ReErr
9	Approx.	500	500	0.04, 0.08, 0.15, 0.30	0.99, 0.50, 0.26, 0.13	0	50	Average ReErr

FIG. 5.3. *Comparison between different selections of t for matrix completion based on Test 6. Left: Fixed schemes and the adaptive scheme with $(k_0, \theta, \mu) = (5, 0.95, 0.2)$. Right: The adaptive scheme with $(k_0, \mu) = (5, 0.2)$ and different θ values.*

$\beta = 100\lambda$, $k_{\max} = 50$, $l_{\max} = 1000$, and $\epsilon = 10^{-5}$ for tradeoff between complexity and performance.

Figure 5.3 compares the performances of different selections of t in terms of Algorithm 1's success rates in Test 6. The left plot of Figure 5.3 illustrates that the adaptive scheme gives nearly the same recoverability as the fixed schemes $t = r - 2$ and $t = r - 1$, whereas fixed schemes with t larger than $r - 1$ are unstable. We want to point out that the adaptive scheme is more practical than the fixed ones, since it does not require any knowledge of the true rank. On the other hand, the right plot of Figure 5.3 shows that a larger value of the maximal thresholding θ results in better performance, due to the closer approximation to the matrix rank.

Comparison with other methods. We compare $l_{t,1-2}$ with three state-of-the-art methods for matrix completion: FPCA (unconstrained nuclear norm minimization by fix point iterative algorithm with approximate SVD) [39], LMaFit (low-rank factorization by a nonlinear successive over-relaxation algorithm) [53], and IRucLq-M (the matrix version of IRucLq-v) [32]. We use the Matlab codes provided by the authors and detail the parameter setting for each method below. For FPCA, its code offers two parameter settings: "easy" for faster speed and "hard" for higher precision.

We adopt "hard" in all tests. As Test 7 concerns about computational time, we set the maximum iteration number for the inner loop to 10 for efficiency. For LMaFit, we set the tolerance to 10^{-5} and the maximum iteration number to 5000. For IRucLq-M, we fix q to 0.5, the tolerance to 10^{-8}, and the maximum iteration number to 2000; we set the regularization parameter $\lambda = 10^{-8}$ in Tests 6 and 9, $\lambda = 10^{-10}$ in Test 7, and optimize it in Test 8. IRucLq-M also provides an accelerated version in its code, which is used only in Test 7 to record its best efficiency. Both LMaFit and IRucLq-M require an estimation on the rank of $\bar{\mathbf{X}}$, which is set to the ground truth in Test 7 for both methods; in other tests, we adopt adaptively updating rank strategy, which is shown to be robust to the testing matrices [53, 32]. In particular, an *increasing* rank strategy is considered in LMaFit, which starts from a small initial guess r_{ini} and increases the rank gradually until convergence or a maximal rank estimate r_{max} is reached. We set $(r_{\text{ini}}, r_{\text{max}}) = (5, \lfloor 1.5r \rfloor)$ in Tests 6 and 8 and $(r_{\text{ini}}, r_{\text{max}}) = (1, 50)$ in Test 9. For IRucLq-M, a *decreasing* rank strategy is provided by the authors in its implementation, which starts from a large initial guess r_{ini} and decreases the rank once a big jump is detected between two adjacent singular values. Here a user-specified minimum rank estimate r_{min} is imposed. We set $(r_{\text{ini}}, r_{\text{min}}) = (\lfloor 1.5r \rfloor, 5)$ in Tests 6 and 8 and $(r_{\text{ini}}, r_{\text{min}}) = (50, 5)$ in Test 9. We adopt the default stopping criterion of each method, except for Test 7, where all methods iterate until reaching the precision ReErr $\leq 10^{-5}$. Other parameters in the completing methods are left to their default settings.

Figure 5.4 compares the performances of different methods in Tests 6 and 7 using exactly low-rank matrices. It is shown on the left plot of Figure 5.4 that $l_{t,1-2}$ achieves the highest success rates with nearly 100% recoverability for $r \leq 28$ (FR ≤ 0.96). FPCA is the second best, which is largely attributed to an approximated SVD scheme; see [39] for more discussion on this phenomenon. Regarding the execution time in Test 7, we observe from the right plot of Figure 5.4 that LMaFit is the fastest, since its algorithm avoids the expensive SVD computation that is required by all other methods. On the other hand, $l_{t,1-2}$ is the slowest among the three SVD-required methods, which is mainly due to a large number of ADMM iterations.

Table 5.4 reports the average ReErr values of different methods in Test 8 using exactly low-rank matrices and Test 9 using approximately low-rank matrices. We observe that for Test 8, $l_{t,1-2}$ and IRucLq-M achieve comparable ReErr values, which are better than other methods. For Test 9, $l_{t,1-2}$ and LMaFit are the winners for low and high SR cases, respectively. We conclude that $l_{t,1-2}$ is the only method that achieves leading performance in both scenarios.

6. Conclusions. We have presented a truncated l_{1-2} metric for sparse recovery and rank minimization. Compared with its predecessor l_{1-2}, the truncated version $l_{t,1-2}$ significantly neutralizes the bias incurred by leading entries/singular values, thus more effectively recovering sparse vectors and low-rank matrices. Theoretical results of $l_{t,1-2}$ have been established, including exact and stable recovery conditions and sparse properties of local and global minimizers. We have developed a numerical scheme based on the DCA to efficiently solve $l_{t,1-2}$ minimization with guaranteed convergence. Numerical experiments have shown that $l_{t,1-2}$ is highly effective and comparable to state-of-the-art methods in sparse recovery and matrix completion. Future works include speeding-up the numerical algorithm of $l_{t,1-2}$ minimization and applying the $l_{t,1-2}$ methodology to various applications such as image processing [59, 60, 17, 48, 30, 40].
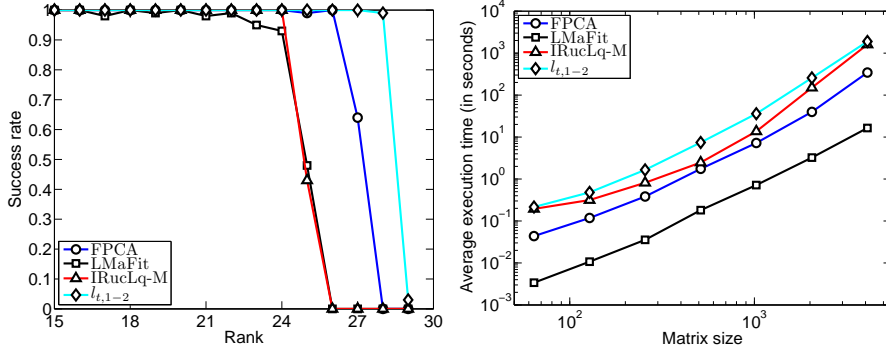
Fig. 5.4. *Comparison results of Tests 6 and 7 using exactly low-rank matrices. Left: Success rates in Test 6. Right: Execution time in Test 7.*

TABLE 5.4

Average ReErr values of different methods in Test 8 using exactly low-rank matrices and Test 9 using approximately low-rank matrices. The best result for each noise level or SR is highlighted in bold.

	Noise level	FPCA	LMaFit	IRucLq-M	$l_{t,1-2}$
	0.01	1.09e-2	1.71e-2	**9.93e-3**	1.00e-2
Test 8	0.03	3.32e-2	4.99e-2	3.06e-2	**3.00e-2**
	0.05	6.82e-2	8.47e-2	5.15e-2	**5.04e-2**
	0.10	1.69e-1	1.73e-1	1.03e-1	**9.99e-2**
	SR	FPCA	LMaFit	IRucLq-M	$l_{t,1-2}$
	0.04	4.40e-1	5.12e-1	5.67e-1	**2.56e-1**
Test 9	0.08	1.05e-1	2.36e-2	2.77e-1	**2.26e-2**
	0.15	5.50e-2	**1.32e-4**	3.77e-2	2.25e-3
	0.30	2.75e-2	**2.10e-5**	3.27e-3	5.52e-5

Appendix A. Proofs of Lemmas 3.1 and 3.2. To prove Lemma 3.1, we need the following lemma to show that a pinching, or diagonal cell operator, reduces all unitarily invariant norms; see, e.g., [28, 3].

LEMMA A.1. *Let $\| \cdot \|$ be a unitarily invariant norm and let*

$$\mathbf{A} := \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \quad and \quad \mathbf{B} := \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix},$$

then one has $\|\mathbf{A}\| \geq \|\mathbf{B}\|$.

Proof of Lemma 3.1. It is straightforward that (a) is true if $T = \Gamma_{\mathbf{x},t}$. If $|T\backslash\Gamma_{\mathbf{x},t}| = t_1 > 0$, we denote $T\backslash\Gamma_{\mathbf{x},t} = \{i_1, \ldots, i_{t_1}\}$ and $\Gamma_{\mathbf{x},t}\backslash T = \{j_1, \ldots, j_{t_1}\}$. Then for any $i \in T\backslash\Gamma_{\mathbf{x},t}$ and $j \in \Gamma_{\mathbf{x},t}\backslash T$, we have that $i \neq j$ and $|x_i| \leq |x_j|$. We define $T_k := (T \cap \Gamma_{\mathbf{x},t}) \cup \{j_1, \ldots, j_k, i_{k+1}, \ldots, i_{t_1}\}$ for any $k \in \{0, \ldots, t_1\}$, then we have $T = T_0$ and $\Gamma_{\mathbf{x},t} = T_{t_1}$. Simple calculation gives that

$$\|\mathbf{x}\|_{T_k} - \|\mathbf{x}\|_{T_{k+1}} = \frac{x_{i_{k+1}}^2 - x_{j_{k+1}}^2}{|x_{i_{k+1}}| + |x_{j_{k+1}}|} + \frac{x_{j_{k+1}}^2 - x_{i_{k+1}}^2}{\sqrt{\|\mathbf{x}_S\|_2^2 + x_{j_{k+1}}^2} + \sqrt{\|\mathbf{x}_S\|_2^2 + x_{i_{k+1}}^2}} \leq 0,$$

holds for any $0 \leq k \leq t_1 - 1$, where $S := T_k^c \cap T_{k+1}^c$. Therefore, we have that $\|\mathbf{x}\|_{T_0} \leq \|\mathbf{x}\|_{T_1} \leq \cdots \leq \|\mathbf{x}\|_{T_{t_1}}$, which implies that $\|\mathbf{x}\|_T \leq \|\mathbf{x}\|_{t,1+2}$.

For (b), to prove that $\| \cdot \|_{t,1+2}$ is a norm, it only needs to show the triangle inequality. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have that $\|\mathbf{x} + \mathbf{y}\|_{t,1+2} \leq \|\mathbf{x}\|_{\Gamma_{\mathbf{x}+\mathbf{y},t}} + \|\mathbf{y}\|_{\Gamma_{\mathbf{x}+\mathbf{y},t}} \leq \|\mathbf{x}\|_{t,1+2} + \|\mathbf{y}\|_{t,1+2}$, where the last inequality follows from (a). Note that $\| \cdot \|_{t,1+2}$ is also a symmetric gauge function [28, Definition 3.5.17], i.e., a norm that is invariant with respect to sign-changes and entry-permutations. Since $\|\mathbf{X}\|_{t,*+F} = \|\boldsymbol{\sigma}(\mathbf{X})\|_{t,1+2}$, we deduce that $\| \cdot \|_{t,*+F}$ is a unitarily invariant norm by [28, Theorem 3.5.18].

For (c), we have that

$$\|\mathbf{U}_1^T \mathbf{X} \mathbf{V}_1\|_* + \|\mathbf{U}_2^T \mathbf{X} \mathbf{V}_2\|_F = \|\boldsymbol{\sigma}(\mathbf{U}_1^T \mathbf{X} \mathbf{V}_1)\|_1 + \|\boldsymbol{\sigma}(\mathbf{U}_2^T \mathbf{X} \mathbf{V}_2)\|_2$$

$$\leq \|[\boldsymbol{\sigma}(\mathbf{U}_1^T \mathbf{X} \mathbf{V}_1) \quad \boldsymbol{\sigma}(\mathbf{U}_2^T \mathbf{X} \mathbf{V}_2)]\|_{t,1+2} = \left\| \begin{bmatrix} \mathbf{U}_1^T \mathbf{X} \mathbf{V}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_2^T \mathbf{X} \mathbf{V}_2 \end{bmatrix} \right\|_{t,*+F}$$

$$\leq \left\| \begin{bmatrix} \mathbf{U}_1^T \mathbf{X} \mathbf{V}_1 & \mathbf{U}_1^T \mathbf{X} \mathbf{V}_2 \\ \mathbf{U}_2^T \mathbf{X} \mathbf{V}_1 & \mathbf{U}_2^T \mathbf{X} \mathbf{V}_2 \end{bmatrix} \right\|_{t,*+F} = \|\mathbf{X}\|_{t,*+F},$$

where the two inequalities follow from (a) and Lemma A.1, respectively. □

Proof of Lemma 3.2. A proof of (a) can be found in [55, Lemma 2.1], and (b) can be obtained by applying (a) to the vector of singular values of $\mathbf{X}$. For (c) and (d), it is sufficient to prove the vector cases. For (c), we assume without loss of generality that $|x_1| \geq |x_2| \geq \cdots \geq |x_n|$ and $x_{t+2} \neq 0$. Denote $T_1 := \{t+1, t+2\}$ and $T_2 := \{t+3, \ldots, n\}$. Then we have that

$$\|\mathbf{x}\|_{t,1-2} \geq \|\mathbf{x}_{T_1}\|_1 + \|\mathbf{x}_{T_2}\|_1 - \|\mathbf{x}_{T_1}\|_2 - \|\mathbf{x}_{T_2}\|_2 \geq \|\mathbf{x}_{T_1}\|_1 - \|\mathbf{x}_{T_1}\|_2 \geq (2 - \sqrt{2})|x_{t+2}|,$$

where the last inequality follows from (a). For (d), the statement naturally holds for $t \geq n-1$. Suppose $t \leq n-2$. If $\|\mathbf{x}\|_{t,1-2} = 0$, (c) implies that $\min_{i \in \Gamma_{\mathbf{x},t+2}} |x_i| = 0$, and thus $\mathbf{x}$ is $(t+1)$-sparse. The other direction is trivial. □

Appendix B. Proofs of Theorems 3.5, 3.6, and 3.7.

Proof of Theorem 3.5. For brevity, we only prove the case of the unconstrained rank minimization model (1.8); the other three models are analogous.

First of all, if $t \geq m-1$, we have that $\|\mathbf{X}\|_{t,*-F} = 0$ and then (1.8) has at least one optimal solution $\mathcal{A}^\dagger(\mathbf{b})$, where $\mathcal{A}^\dagger$ denotes the Moore-Penrose pseudoinverse of $\mathcal{A}$. We then discuss the case of $t \leq m-2$. Let $F(\mathbf{X})$ denote the objective in (1.8). It follows from (3.1) and Lemma 3.1 (b) that $F(\mathbf{X})$ is proper and continuous. Let $\{\mathbf{X}^k\}$ be a sequence such that $\{F(\mathbf{X}^k)\}$ is bounded. We want to show that $\{\|\mathbf{X}^k\|_F\}$ is bounded, thus leading to the coercivity of $F(\mathbf{X})$. For each k, we express $\mathbf{X}^k$ in full-SVD form: $\mathbf{X}^k = \mathbf{U}^k[\text{diag}(\mathbf{s}^k)\ \mathbf{0}](\mathbf{V}^k)^T$. Let $T := \{1, \ldots, t+1\}$, $\mathbf{X}_1^k := \mathbf{U}^k[\text{diag}(\mathbf{s}_T^k)\ \mathbf{0}](\mathbf{V}^k)^T$, and $\mathbf{X}_2^k = \mathbf{X}^k - \mathbf{X}_1^k$, then

$$F(\mathbf{X}^k) = \frac{1}{2}\|\mathcal{A}(\mathbf{X}^k) - \mathbf{b}\|_2^2 + \lambda\|\mathbf{X}^k\|_{t,*-F} \geq \frac{1}{2}(\|\mathcal{A}(\mathbf{X}_1^k)\|_2 - \|\mathcal{A}(\mathbf{X}_2^k) - \mathbf{b}\|_2)^2 + \lambda(2 - \sqrt{2})s_{t+2}^k,$$

where the last inequality is due to Lemma 3.2 (c). Since $\{F(\mathbf{X}^k)\}$ is bounded, we have that $\{s_{t+2}^k\}$, or $\{\|\mathbf{X}_2^k\|_2\}$, is bounded and hence $\{\|\mathbf{X}_2^k\|_F\}$, $\{\|\mathcal{A}(\mathbf{X}_2^k) - \mathbf{b}\|_2\}$, and $\{\|\mathcal{A}(\mathbf{X}_1^k)\|_2\}$ are bounded. Since $\rho_{t+1} < 1$, the definition of matrix-RIP suggests that $\{\|\mathbf{X}_1^k\|_F\}$ is bounded, and thus $\{\|\mathbf{X}^k\|_F\}$ is bounded. We complete the proof by using the Weierstrass' theorem [2], which guarantees that a proper, continuous, and coercive function has at least one minimizer. □

The proofs of Theorems 3.6 and 3.7 generally follow the approaches in [9, 46, 41, 8, 55]. For brevity, we present the proof of Theorem 3.7 in detail, while providing

some key steps in proving Theorems 3.6. To prove Theorem 3.7, we need the following four lemmas.

LEMMA B.1 ([46, Lemma 2.3]). *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices of the same dimensions.* *If* $\mathbf{A}\mathbf{B}^T = \mathbf{0}$ *and* $\mathbf{A}^T\mathbf{B} = \mathbf{0}$*, then* $\|\mathbf{A} + \mathbf{B}\|_* = \|\mathbf{A}\|_* + \|\mathbf{B}\|_*$.

LEMMA B.2 ([8, Lemma 3.3]). *For all* $\mathbf{X}$*,* $\mathbf{X}'$ *obeying* $\langle \mathbf{X}, \mathbf{X}'\rangle = 0$*, and* $\text{rank}(\mathbf{X}) \leq r$*,* $\text{rank}(\mathbf{X}') \leq r'$*, one has* $|\langle \mathcal{A}(\mathbf{X}), \mathcal{A}(\mathbf{X}')\rangle| \leq \rho_{r+r'}\|\mathbf{X}\|_F\|\mathbf{X}'\|_F$.

LEMMA B.3. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ *with* $\text{rank}(\mathbf{A}) \leq r$*. Consider a matrix* $\mathbf{C} := \mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{Q}$*, where* $\mathbf{P} \in \mathbb{R}^{m\times m}$*,* $\mathbf{Q} \in \mathbb{R}^{n\times n}$*, and* $\text{rank}(\mathbf{P}) = \text{rank}(\mathbf{Q}) = t$ *with* $r + t < m$*. Then one can find a full SVD of* $\mathbf{C}$*, denoted as* $\mathbf{C} = \mathbf{U}\mathbf{S}\mathbf{V}^T$*, such that the last* $m - r - t$ *columns of* $\mathbf{U}$ *are orthogonal to columns of* $\mathbf{P}$ *and the last* $n - r - t$ *columns of* $\mathbf{V}$ *are orthogonal to rows of* $\mathbf{Q}$*.*

Proof. We decompose $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$ by the orthogonal projection onto $\text{im}(\mathbf{C})$, where $\text{im}(\mathbf{P}_1) \subseteq \text{im}(\mathbf{C})$ and $\text{im}(\mathbf{P}_2)\perp\text{im}(\mathbf{C})$. Letting $r_1 := \text{rank}(\mathbf{C})$ and $r_2 := \text{rank}(\mathbf{P}_2)$, we write an economy SVD of $\mathbf{C}$ as $\mathbf{C} = \mathbf{U}_1\mathbf{S}_1\mathbf{V}_1^T$, where $\mathbf{U}_1 \in \mathbb{R}^{m\times r_1}$, and let $\mathbf{U}_2 \in \mathbb{R}^{m\times r_2}$ be a matrix whose columns form an orthonormal basis of $\text{im}(\mathbf{P}_2)$. Then we deduce that $\text{im}(\mathbf{U}_1)\perp\text{im}(\mathbf{U}_2)$ from $\text{im}(\mathbf{U}_1) = \text{im}(\mathbf{C})$, $\text{im}(\mathbf{U}_2) = \text{im}(\mathbf{P}_2)$, and $\text{im}(\mathbf{C})\perp\text{im}(\mathbf{P}_2)$. Since

$$r_1 + r_2 = \text{rank}([\mathbf{C} \quad \mathbf{P}_1 + \mathbf{P}_2]) = \text{rank}\left([\mathbf{A} \quad \mathbf{P}]\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{B}\mathbf{Q} & \mathbf{I} \end{bmatrix}\right) \leq r + t,$$

one can find matrices $\mathbf{U}_3 \in \mathbb{R}^{m\times(r+t-r_1-r_2)}$ and $\mathbf{U}_4 \in \mathbb{R}^{m\times(m-r-t)}$ such that $\mathbf{U} = [\mathbf{U}_1 \ \mathbf{U}_2 \ \mathbf{U}_3 \ \mathbf{U}_4]$ is a unitary matrix. Then

$$\text{im}(\mathbf{U}_4) \subseteq (\text{im}(\mathbf{U}_1) + \text{im}(\mathbf{U}_2))^\perp = (\text{im}(\mathbf{C}) + \text{im}(\mathbf{P}_2))^\perp \subseteq (\text{im}(\mathbf{P}_1) + \text{im}(\mathbf{P}_2))^\perp = (\text{im}(\mathbf{P}))^\perp,$$

where $(\cdot)^\perp$ denotes the orthogonal complement, which implies that $\text{im}(\mathbf{U}_4)\perp\text{im}(\mathbf{P})$. Similarly, by considering the matrix $\mathbf{C}^T = \mathbf{A}^T - \mathbf{Q}^T\mathbf{B}^T\mathbf{P}^T$, we can extend the columns of $\mathbf{V}_1$ to a unitary matrix $\mathbf{V}$ such that the last $n - r - t$ columns of $\mathbf{V}$ are orthogonal to rows of $\mathbf{Q}$. Thus $\mathbf{C} = \mathbf{U}\begin{bmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{V}^T$ is the desired full SVD of $\mathbf{C}$. $\quad\square$

The following lemma is a variant of [46, Lemma 3.4] and establishes a key decomposition for proving Theorem 3.7.

LEMMA B.4. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ *with* $\text{rank}(\mathbf{A}) \leq r$*. Consider matrices* $\mathbf{A}_c := \mathbf{A} - \mathbf{U}\mathbf{U}^T\mathbf{A}\mathbf{V}\mathbf{V}^T$ *and* $\mathbf{B}_a := \mathbf{U}\mathbf{U}^T\mathbf{B}\mathbf{V}\mathbf{V}^T$*, where* $\mathbf{U} \in \mathbb{R}^{m\times t}$*,* $\mathbf{V} \in \mathbb{R}^{n\times t}$*, and* $\mathbf{U}^T\mathbf{U} = \mathbf{V}^T\mathbf{V} = \mathbf{I}$*. Then one can find* $\mathbf{B}_b$ *and* $\mathbf{B}_c$ *such that*

   (a) $\mathbf{B} = \mathbf{B}_a + \mathbf{B}_b + \mathbf{B}_c$,
   (b) $\text{rank}(\mathbf{B}_b) \leq 2r + 2t$,
   (c) $\mathbf{A}_c^T\mathbf{B}_c = \mathbf{0}$ *and* $\mathbf{A}_c\mathbf{B}_c^T = \mathbf{0}$,
   (d) $\langle \mathbf{B}_a, \mathbf{B}_b\rangle = 0$*,* $\langle \mathbf{B}_a, \mathbf{B}_c\rangle = 0$*, and* $\langle \mathbf{B}_b, \mathbf{B}_c\rangle = 0$*.*

Moreover, given an integer $k > 0$*, one can find* l *matrices* $\{\mathbf{B}_i\}_{i=1}^l$ *such that*

   (e) $\mathbf{B}_c = \sum_{i=1}^l \mathbf{B}_i$,
   (f) $\text{rank}(\mathbf{B}_i) \leq k$ *for* $1 \leq i \leq l$ *and* $\sigma_1(\mathbf{B}_{i+1}) \leq \sigma_k(\mathbf{B}_i)$ *for* $1 \leq i \leq l - 1$,
   (g) $\mathbf{B}_i^T\mathbf{B}_j = \mathbf{0}$ *and* $\mathbf{B}_i\mathbf{B}_j^T = \mathbf{0}$ *for* $1 \leq i, j \leq l, i \neq j$,
   (h) $\langle \mathbf{B}_a, \mathbf{B}_i\rangle = 0$ *and* $\langle \mathbf{B}_b, \mathbf{B}_i\rangle = 0$ *for* $1 \leq i \leq l$*.*

Proof. The proof follows the idea in [46, Lemma 3.4]. If $r + t \geq m$, it is easy to check that $\mathbf{B}_b := \mathbf{B} - \mathbf{B}_a$ and $\mathbf{B}_c := \mathbf{0}$ satisfy (a)–(d). If $r + t < m$, it follows from Lemma B.3 that there exists a full SVD $\mathbf{A}_c = [\mathbf{U}_1 \ \mathbf{U}_2]\begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}[\mathbf{V}_1 \ \mathbf{V}_2]^T$ satisfying $\mathbf{U}_2^T\mathbf{U}\mathbf{U}^T = \mathbf{0}$, where $\mathbf{U}_2 \in \mathbb{R}^{m\times(m-r-t)}$. Let $\hat{\mathbf{B}} := [\mathbf{U}_1 \ \mathbf{U}_2]^T(\mathbf{B} - \mathbf{B}_a)[\mathbf{V}_1 \ \mathbf{V}_2]$ and partition $\hat{\mathbf{B}}$ as $\hat{\mathbf{B}} = \begin{bmatrix} \hat{\mathbf{B}}_{11} & \hat{\mathbf{B}}_{12} \\ \hat{\mathbf{B}}_{21} & \hat{\mathbf{B}}_{22} \end{bmatrix}$, where $\hat{\mathbf{B}}_{11} \in \mathbb{R}^{(r+t)\times(r+t)}$. It is easy to verify that $\mathbf{B}_b$

and $\mathbf{B}_c$ defined as

$$\mathbf{B}_b := [\mathbf{U}_1 \quad \mathbf{U}_2] \begin{bmatrix} \hat{\mathbf{B}}_{11} & \hat{\mathbf{B}}_{12} \\ \hat{\mathbf{B}}_{21} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^T \\ \mathbf{V}_2^T \end{bmatrix} \quad \text{and} \quad \mathbf{B}_c := [\mathbf{U}_1 \quad \mathbf{U}_2] \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{B}}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^T \\ \mathbf{V}_2^T \end{bmatrix},$$

satisfy (a)–(c) and $\langle \mathbf{B}_b, \mathbf{B}_c \rangle = 0$. Moreover, one has that $\langle \mathbf{B}_a, \mathbf{B}_c \rangle = 0$ from $\mathbf{U}_2^T \mathbf{U} \mathbf{U}^T = \mathbf{0}$, and then $\langle \mathbf{B}_a, \mathbf{B}_b \rangle = \langle \mathbf{B}_a, \mathbf{B} - \mathbf{B}_a \rangle = 0$.

For (e)–(h), we assume that $\mathbf{B}_c \neq \mathbf{0}$; otherwise, one can set $l = 1$ and $\mathbf{B}_1 = \mathbf{0}$. Let $\hat{\mathbf{B}}_{22} = \hat{\mathbf{U}}\mathrm{diag}(\hat{\boldsymbol{\sigma}})\hat{\mathbf{V}}^T$ be an economy SVD of $\hat{\mathbf{B}}_{22}$. For each $i \geq 1$, define the index set $I_i := \{k(i-1)+1, \cdots, ki\} \cap \{1, \ldots, \mathrm{rank}(\hat{\mathbf{B}}_{22})\}$. Assuming that there are l such nonempty sets, we define, for $i = 1, \ldots, l$,

$$\mathbf{B}_i := [\mathbf{U}_1 \quad \mathbf{U}_2] \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{U}}\mathrm{diag}(\hat{\boldsymbol{\sigma}}_{I_i})\hat{\mathbf{V}}^T \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^T \\ \mathbf{V}_2^T \end{bmatrix}.$$

It is easy to check that $\{\mathbf{B}_i\}_{i=1}^l$ satisfy (e)–(h). □

Then we proceed to prove Theorem 3.7.

Proof of Theorem 3.7. For (a), we have $\|\bar{\mathbf{X}}\|_{t,*-F} = 0$, due to $r \leq t + 1$ and Lemma 3.2 (d). Let $\mathbf{X}$ be a solution to (1.6), then $\mathbf{X}$ satisfies $\|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_2 \leq \tau$ and $\|\mathbf{X}\|_{t,*-F} = 0$. So $\mathbf{X}$ must be $(t+1)$-rank. Let $\mathbf{R} := \mathbf{X} - \bar{\mathbf{X}}$, then $\mathbf{R}$ is $(r+t+1)$-rank and $\|\mathcal{A}(\mathbf{R})\|_2 \leq 2\tau$. Since $\rho_{r+t+1} < 1$, the matrix-RIP of $\mathcal{A}$ gives that $\|\mathbf{R}\|_F \leq 1/\sqrt{1 - \rho_{r+t+1}}\|\mathcal{A}(\mathbf{R})\|_2 \leq c_1 \tau$, where $c_1 := 2/\sqrt{1 - \rho_{r+t+1}} > 0$.

Next we prove (b). Let $\mathbf{X}$ be a solution to (1.6), then $\|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_2 \leq \tau$ and $\|\mathbf{X}\|_{t,*-F} \leq \|\bar{\mathbf{X}}\|_{t,*-F}$. Let $\mathbf{R} := \mathbf{X} - \bar{\mathbf{X}}$, one has that $\|\mathcal{A}(\mathbf{R})\|_2 \leq 2\tau$. Unlike (a), there is no upper bound of $\mathrm{rank}(\mathbf{R})$. We decompose $\mathbf{R}$ into a series of matrices, each with an upper bound on rank so that the matrix-RIP can be applied. We write a full SVD of $\mathbf{X}$ as $\mathbf{X} = [\mathbf{U}_1 \ \mathbf{U}_2]\mathbf{S}[\mathbf{V}_1 \ \mathbf{V}_2]^T$, where $\mathbf{U}_1 \in \mathbb{R}^{m \times t}$ and $\mathbf{V}_1 \in \mathbb{R}^{n \times t}$. Denote $\mathbf{R}_a := \mathbf{U}_1\mathbf{U}_1^T\mathbf{R}\mathbf{V}_1\mathbf{V}_1^T$, $\bar{\mathbf{X}}_a := \mathbf{U}_1\mathbf{U}_1^T\bar{\mathbf{X}}\mathbf{V}_1\mathbf{V}_1^T$, and $\bar{\mathbf{X}}_c := \bar{\mathbf{X}} - \bar{\mathbf{X}}_a$. Using Lemma B.4 with $k = r$, one can find $\mathbf{R}_b$, $\mathbf{R}_c$, and $\{\mathbf{R}_i\}_{i=1}^l$ satisfying the following properties:

(P1) $\mathbf{R} = \mathbf{R}_a + \mathbf{R}_b + \mathbf{R}_c$,
(P2) $\mathrm{rank}(\mathbf{R}_b) \leq 2r + 2t$,
(P3) $\bar{\mathbf{X}}_c^T\mathbf{R}_c = \mathbf{0}$ and $\bar{\mathbf{X}}_c\mathbf{R}_c^T = \mathbf{0}$,
(P4) $\langle \mathbf{R}_a, \mathbf{R}_b \rangle = 0$, $\langle \mathbf{R}_a, \mathbf{R}_c \rangle = 0$, and $\langle \mathbf{R}_b, \mathbf{R}_c \rangle = 0$,
(P5) $\mathbf{R}_c = \sum_{i=1}^l \mathbf{R}_i$,
(P6) $\mathrm{rank}(\mathbf{R}_i) \leq r$ for $1 \leq i \leq l$ and $\sigma_1(\mathbf{R}_{i+1}) \leq \sigma_r(\mathbf{R}_i)$ for $1 \leq i \leq l-1$,
(P7) $\mathbf{R}_i^T\mathbf{R}_j = \mathbf{0}$ and $\mathbf{R}_i\mathbf{R}_j^T = \mathbf{0}$ for $1 \leq i, j \leq l, i \neq j$,
(P8) $\langle \mathbf{R}_a, \mathbf{R}_i \rangle = 0$ and $\langle \mathbf{R}_b, \mathbf{R}_i \rangle = 0$ for $1 \leq i \leq l$.
Using $\mathbf{U}_1^T\mathbf{X}\mathbf{V}_2 = \mathbf{0}$ and $\mathbf{U}_2^T\mathbf{X}\mathbf{V}_1 = \mathbf{0}$, direct calculations give that

$$\|\mathbf{U}_2^T\mathbf{X}\mathbf{V}_2\|_* = \left\| \mathbf{U} \begin{bmatrix} \mathbf{0} & \mathbf{U}_1^T\mathbf{X}\mathbf{V}_2 \\ \mathbf{U}_2^T\mathbf{X}\mathbf{V}_1 & \mathbf{U}_2^T\mathbf{X}\mathbf{V}_2 \end{bmatrix} \mathbf{V}^T \right\|_* = \|\bar{\mathbf{X}}_c + \mathbf{R}_b + \mathbf{R}_c\|_*, \qquad (B.1)$$

$$\|\mathbf{U}_2^T\mathbf{R}\mathbf{V}_2\|_F \leq \left\| \mathbf{U} \begin{bmatrix} \mathbf{0} & \mathbf{U}_1^T\mathbf{R}\mathbf{V}_2 \\ \mathbf{U}_2^T\mathbf{R}\mathbf{V}_1 & \mathbf{U}_2^T\mathbf{R}\mathbf{V}_2 \end{bmatrix} \mathbf{V}^T \right\|_F \leq \|\mathbf{R}_b\|_F + \|\mathbf{R}_c\|_F. \qquad (B.2)$$

According to Lemma B.1, (P3) and (P7) imply that

$$\|\bar{\mathbf{X}}_c + \mathbf{R}_c\|_* = \|\bar{\mathbf{X}}_c\|_* + \|\mathbf{R}_c\|_*, \qquad (B.3)$$

$$\|\mathbf{R}_c\|_* = \sum_{i=1}^l \|\mathbf{R}_i\|_*. \qquad (B.4)$$

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Using the above analyses, we deduce that

$$
\begin{aligned}
\|\bar{\mathbf{X}}\|_* - \|\bar{\mathbf{X}}\|_{t,*+F} &= \|\bar{\mathbf{X}}\|_{t,*-F} \geq \|\mathbf{X}\|_{t,*-F} = \|\mathbf{U}_2^T \mathbf{X} \mathbf{V}_2\|_* - \|\mathbf{U}_2^T \mathbf{X} \mathbf{V}_2\|_F \\
&\stackrel{\text{(B.1)}}{=} \|\bar{\mathbf{X}}_c + \mathbf{R}_b + \mathbf{R}_c\|_* - \|\mathbf{U}_2^T(\bar{\mathbf{X}} + \mathbf{R})\mathbf{V}_2\|_F \\
&\stackrel{\text{(B.3,B.2)}}{\geq} \|\bar{\mathbf{X}}_c\|_* - \|\mathbf{R}_b\|_* + \|\mathbf{R}_c\|_* - \|\mathbf{U}_2^T \bar{\mathbf{X}} \mathbf{V}_2\|_F - \|\mathbf{R}_b\|_F - \|\mathbf{R}_c\|_F \\
&= (\|\bar{\mathbf{X}}_c\|_* + \|\bar{\mathbf{X}}_a\|_*) - (\|\bar{\mathbf{X}}_a\|_* + \|\mathbf{U}_2^T \bar{\mathbf{X}} \mathbf{V}_2\|_F) - \|\mathbf{R}_b\|_* + \|\mathbf{R}_c\|_* - \|\mathbf{R}_b\|_F - \|\mathbf{R}_c\|_F \\
&\geq \|\bar{\mathbf{X}}\|_* - \|\bar{\mathbf{X}}\|_{t,*+F} - \|\mathbf{R}_b\|_* + \|\mathbf{R}_c\|_* - \|\mathbf{R}_b\|_F - \|\mathbf{R}_c\|_F,
\end{aligned}
$$

where the last inequality follows from $\|\bar{\mathbf{X}}_a\|_* = \|\mathbf{U}_1^T \bar{\mathbf{X}} \mathbf{V}_1\|_*$ and Lemma 3.1 (c). So we obtain the following inequality:

$$
\|\mathbf{R}_c\|_* - \|\mathbf{R}_c\|_F \leq \|\mathbf{R}_b\|_* + \|\mathbf{R}_b\|_F. \tag{B.5}
$$

Let $\mathbf{R}_0 := \mathbf{R}_a + \mathbf{R}_b + \mathbf{R}_1$, then $\text{rank}(\mathbf{R}_0) \leq 3r + 3t$ and $\langle \mathbf{R}_0, \mathbf{R}_i \rangle = 0$ for $i = 2, \dots, l$. Next we set upper bounds of $\sum_{i=2}^l \|\mathbf{R}_i\|_F$ and $\|\mathbf{R}\|_F$ using $\|\mathbf{R}_0\|_F$. Using (P6) and Lemma 3.2 (b), we have that for $i \geq 2$, $\sigma_1(\mathbf{R}_i) \leq \sigma_r(\mathbf{R}_{i-1}) \leq (\|\mathbf{R}_{i-1}\|_* - \|\mathbf{R}_{i-1}\|_F)/(r - \sqrt{r})$, then

$$
\|\mathbf{R}_i\|_F \leq \frac{\|\mathbf{R}_{i-1}\|_* - \|\mathbf{R}_{i-1}\|_F}{\sqrt{r} - 1}. \tag{B.6}
$$

Putting things together, we obtain the following upper bounds:

$$
\begin{aligned}
\sum_{i=2}^l \|\mathbf{R}_i\|_F &\stackrel{\text{(B.6)}}{\leq} \frac{\sum_{i=1}^{l-1} \|\mathbf{R}_i\|_* - \sum_{i=1}^{l-1} \|\mathbf{R}_i\|_F}{\sqrt{r} - 1} \stackrel{\text{(B.4)}}{\leq} \frac{\|\mathbf{R}_c\|_* - \|\mathbf{R}_c\|_F}{\sqrt{r} - 1} \stackrel{\text{(B.5)}}{\leq} \frac{\|\mathbf{R}_b\|_* + \|\mathbf{R}_b\|_F}{\sqrt{r} - 1} \\
&\stackrel{\text{(P2)}}{\leq} \frac{\sqrt{2r + 2t} + 1}{\sqrt{r} - 1} \|\mathbf{R}_b\|_F \stackrel{\text{(P4,P8)}}{\leq} \frac{\sqrt{2r + 2t} + 1}{\sqrt{r} - 1} \|\mathbf{R}_0\|_F = c_0 \|\mathbf{R}_0\|_F, \quad \text{(B.7)}
\end{aligned}
$$

where $c_0 := (\sqrt{2r + 2t} + 1)/(\sqrt{r} - 1)$, and

$$
\|\mathbf{R}\|_F = \sqrt{\|\mathbf{R}_0\|_F^2 + \sum_{i=2}^l \|\mathbf{R}_i\|_F^2} \leq \sqrt{\|\mathbf{R}_0\|_F^2 + \left(\sum_{i=2}^l \|\mathbf{R}_i\|_F\right)^2} \stackrel{\text{(B.7)}}{\leq} \sqrt{1 + c_0^2} \|\mathbf{R}_0\|_F. \tag{B.8}
$$

Using the matrix-RIP of $\mathcal{A}$, we get

$$
\begin{aligned}
(1 - \rho_{3r+3t}) \|\mathbf{R}_0\|_F^2 &\leq \|\mathcal{A}(\mathbf{R}_0)\|_2^2 = \langle \mathcal{A}(\mathbf{R}_0), \mathcal{A}(\mathbf{R}) \rangle - \sum_{i=2}^l \langle \mathcal{A}(\mathbf{R}_a + \mathbf{R}_b) + \mathcal{A}(\mathbf{R}_1), \mathcal{A}(\mathbf{R}_i) \rangle \\
&\leq \|\mathcal{A}(\mathbf{R}_0)\|_2 \|\mathcal{A}(\mathbf{R})\|_2 + (\rho_{3r+3t} \|\mathbf{R}_a + \mathbf{R}_b\|_F + \rho_{2r} \|\mathbf{R}_1\|_F) \sum_{i=2}^l \|\mathbf{R}_i\|_F \\
&\leq 2\tau \sqrt{1 + \rho_{3r+3t}} \|\mathbf{R}_0\|_F + \sqrt{(\rho_{3r+3t}^2 + \rho_{2r}^2)(\|\mathbf{R}_a + \mathbf{R}_b\|_F^2 + \|\mathbf{R}_1\|_F^2)} \sum_{i=2}^l \|\mathbf{R}_i\|_F \\
&\leq 2\tau \sqrt{1 + \rho_{3r+3t}} \|\mathbf{R}_0\|_F + \sqrt{2} c_0 \rho_{3r+3t} \|\mathbf{R}_0\|_F^2,
\end{aligned}
$$

$$
\tag{B.9}
$$

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where the second inequality uses Lemma B.2, the third inequality uses $\|\mathcal{A}(\mathbf{R})\|_2 \leq 2\tau$, and the last inequality uses $\rho_{2r} \leq \rho_{3r+3t}$ and (B.7). Note that condition (3.7) implies that $1 - (1 + \sqrt{2}c_0)\rho_{3r+3t} > 0$, then (B.9) reads

$$\|\mathbf{R}_0\|_F \leq \frac{2\sqrt{1 + \rho_{3r+3t}}}{1 - (1 + \sqrt{2}c_0)\rho_{3r+3t}}\tau. \tag{B.10}$$

Finally, combining (B.8) and (B.10), we obtain the desired result $\|\mathbf{X} - \bar{\mathbf{X}}\|_F \leq c_2\tau$, where $c_2 := 2\sqrt{(1 + c_0^2)(1 + \rho_{3r+3t})}/(1 - (1 + \sqrt{2}c_0)\rho_{3r+3t}) > 0$. $\square$

We briefly mention the proof of Theorem 3.6. Note that (1.5) can be considered as a special case of (1.6) if $\mathbf{X}$ is diagonal, i.e.,

$$\min_{\mathbf{X} \in \mathbb{R}^{n \times n}} \|\mathbf{X}\|_{t,*-F} \quad \text{subject to} \quad \mathbf{X} \text{ is diagonal and } \|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_2 \leq \tau, \tag{B.11}$$

where $\mathcal{A}(\mathbf{X}) = \mathbf{A}\mathrm{diag}(\mathbf{X})$. If both $\mathbf{X}$ and $\bar{\mathbf{X}}$ are diagonal, Theorem 3.6 (a) can be proved by the same argument in the proof of Theorem 3.7 (a). Similarly, for Theorem 3.6 (b), we let all the components in the decomposition of $\mathbf{R}$ be diagonal matrices whose main diagonals have mutually disjoint support sets. Then (P2) becomes $\mathrm{rank}(\mathbf{R}_b) \leq s$. By setting the maximum rank of $\mathbf{R}_i$ as s, we have that $\mathrm{rank}(\mathbf{R}_0) \leq 2s + t$. Letting $c_0 := (\sqrt{s} + 1)/(\sqrt{s} - 1)$ and $c_2 := 2\sqrt{(1 + c_0^2)(1 + \delta_{2s+t})}/(1 - (1 + \sqrt{2}c_0)\delta_{2s+t}) > 0$, we obtain the desired result $\|\mathbf{x} - \bar{\mathbf{x}}\|_2 = \|\mathbf{X} - \bar{\mathbf{X}}\|_F \leq c_2\tau$.

Appendix C. Proofs of Theorem 3.8 and Corollary 3.9.

Proof of Theorem 3.8. Since $\|\mathbf{x}^*\|_{t,1-2} \neq 0$, Lemma 3.2 (d) suggests that $|\Lambda_2| \geq 2$ and $\Lambda_1 = \Lambda^* \backslash \Lambda_2$. To prove (a), it is sufficient to show that $\mathbf{r}_{\Lambda_2} = \mathbf{0}$, for any vector $\mathbf{r}$ satisfying $\mathbf{r} \in \ker(\mathbf{A})$ and $\mathrm{supp}(\mathbf{r}) \subseteq \Lambda^*$. Since $\mathbf{x}^*$ is a local minimizer, there exists $\epsilon > 0$ such that for any $\mathbf{x}$ satisfying $\|\mathbf{x} - \mathbf{x}^*\|_2 \leq \epsilon$ and $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \leq \tau$, we have $\|\mathbf{x}\|_{t,1-2} \geq \|\mathbf{x}^*\|_{t,1-2}$. We scale $\mathbf{r}$ such that

$$\|\mathbf{r}\|_2 < \min\left\{\min_{i \in \Lambda^*} |x_i^*|, \epsilon\right\}. \tag{C.1}$$

Therefore, for any index $i \in \Lambda^*$, we have that $|r_i| < |x_i^*|$ and

$$|x_i^* + r_i| + |x_i^* - r_i| = 2|x_i^*|. \tag{C.2}$$

Denote $\mathbf{x}_1 := \mathbf{x}^* + \mathbf{r}$ and $\mathbf{x}_2 := \mathbf{x}^* - \mathbf{r}$, then we have that $\|\mathbf{x}_1 + \mathbf{x}_2\|_1 = \|\mathbf{x}_1\|_1 + \|\mathbf{x}_2\|_1$ from (C.2). In addition, both $\mathbf{x}_1$ and $\mathbf{x}_2$ are feasible vectors and satisfy $\|\mathbf{x}_1\|_{t,1-2} \geq \|\mathbf{x}^*\|_{t,1-2}$ and $\|\mathbf{x}_2\|_{t,1-2} \geq \|\mathbf{x}^*\|_{t,1-2}$. Therefore, we obtain that

$$\|\mathbf{x}_1\|_{t,1-2} + \|\mathbf{x}_2\|_{t,1-2} \geq 2\|\mathbf{x}^*\|_{t,1-2} = \|\mathbf{x}_1 + \mathbf{x}_2\|_{t,1-2} = \|\mathbf{x}_1\|_1 + \|\mathbf{x}_2\|_1 - \|\mathbf{x}_1 + \mathbf{x}_2\|_{\Lambda_1}$$
$$\geq \|\mathbf{x}_1\|_1 + \|\mathbf{x}_2\|_1 - \|\mathbf{x}_1\|_{t,1+2} - \|\mathbf{x}_2\|_{t,1+2} = \|\mathbf{x}_1\|_{t,1-2} + \|\mathbf{x}_2\|_{t,1-2},$$

where the last inequality uses Lemma 3.1 (a). Hence, we have the following equalities:

$$\|\mathbf{x}^*\|_{t,1-2} = \|\mathbf{x}_1\|_{t,1-2} \quad \text{and} \quad \|\mathbf{x}^*\|_{t,1-2} = \|\mathbf{x}_2\|_{t,1-2}, \tag{C.3}$$
$$\|\mathbf{x}_1 + \mathbf{x}_2\|_{\Lambda_1} = \|\mathbf{x}_1\|_{\Lambda_1} + \|\mathbf{x}_2\|_{\Lambda_1}, \tag{C.4}$$
$$\|\mathbf{x}_1\|_{\Lambda_1} = \|\mathbf{x}_1\|_{t,1+2} \quad \text{and} \quad \|\mathbf{x}_2\|_{\Lambda_1} = \|\mathbf{x}_2\|_{t,1+2}. \tag{C.5}$$

Note that (C.2) also implies that

$$2\|\mathbf{x}_{\Lambda_1}^*\|_1 = \|\mathbf{x}_{\Lambda_1}^* + \mathbf{r}_{\Lambda_1}\|_1 + \|\mathbf{x}_{\Lambda_1}^* - \mathbf{r}_{\Lambda_1}\|_1. \tag{C.6}$$

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Subtracting (C.4) with (C.6) gives that

$$\|(\mathbf{x}^*_{\Lambda_2} + \mathbf{r}_{\Lambda_2}) + (\mathbf{x}^*_{\Lambda_2} - \mathbf{r}_{\Lambda_2})\|_2 = \|\mathbf{x}^*_{\Lambda_2} + \mathbf{r}_{\Lambda_2}\|_2 + \|\mathbf{x}^*_{\Lambda_2} - \mathbf{r}_{\Lambda_2}\|_2,$$

which implies that $\mathbf{x}^*_{\Lambda_2} + \mathbf{r}_{\Lambda_2}$ and $\mathbf{x}^*_{\Lambda_2} - \mathbf{r}_{\Lambda_2}$ are collinear, and thus $\mathbf{x}^*_{\Lambda_2}$ and $\mathbf{r}_{\Lambda_2}$ are collinear. Since $\mathbf{x}^*_{\Lambda_2} \neq \mathbf{0}$, there exists k such that $k\mathbf{x}^*_{\Lambda_2} = \mathbf{r}_{\Lambda_2}$, and (C.3) becomes

$$\|\mathbf{x}^*\|_{t,1-2} \overset{(C.5)}{=} \|\mathbf{x}_1\|_1 - \|\mathbf{x}_1\|_{\Lambda_1} = \|\mathbf{x}^*_{\Lambda_2} + \mathbf{r}_{\Lambda_2}\|_1 - \|\mathbf{x}^*_{\Lambda_2} + \mathbf{r}_{\Lambda_2}\|_2 = |1 + k|\|\mathbf{x}^*\|_{t,1-2},$$

and similarly $\|\mathbf{x}^*\|_{t,1-2} = |1 - k|\|\mathbf{x}^*\|_{t,1-2}$. Since $\|\mathbf{x}^*\|_{t,1-2} \neq 0$, we have that $k = 0$ and thus $\mathbf{r}_{\Lambda_2} = \mathbf{0}$, which concludes the proof of (a).

For (b), it remains to show that the columns of $\mathbf{A}_{\Lambda_1}$ are linearly independent, since a global minimizer $\mathbf{x}^*$ is also a local minimizer that satisfies (a). We prove by contradiction. Suppose there exists $\mathbf{r} \in \ker(\mathbf{A})\backslash\{\mathbf{0}\}$ with $\mathrm{supp}(\mathbf{r}) \subseteq \Lambda_1$. Since $\mathbf{r} \neq \mathbf{0}$, one can find $j_1 \in \Lambda_1$ such that $r_{j_1} \neq 0$. We choose any $j_2 \in \Lambda_2$ and scale $\mathbf{r}$ so that

$$|x^*_{j_1} + r_{j_1}| < |x^*_{j_2}|. \tag{C.7}$$

Denote $\mathbf{x}' := \mathbf{x}^* + \mathbf{r}$, $\Lambda'_1 := (\Lambda_1\backslash\{j_1\}) \cup \{j_2\}$, and $\Lambda'_2 := (\Lambda_2\backslash\{j_2\}) \cup \{j_1\}$, then $\Lambda'_1 = \Lambda^*\backslash\Lambda'_2$. Using (C.7), Lemma 3.1 (a), and $\mathbf{x}^*_{\Lambda_2\backslash\{j_2\}} \neq \mathbf{0}$, we have that

$$\|\mathbf{x}'\|_{t,1-2} \leq \|\mathbf{x}'\|_1 - \|\mathbf{x}'\|_{\Lambda'_1} = \|\mathbf{x}^*_{\Lambda_2\backslash\{j_2\}}\|_1 - \frac{\|\mathbf{x}^*_{\Lambda_2\backslash\{j_2\}}\|_2^2}{|x'_{j_1}| + \sqrt{\|\mathbf{x}^*_{\Lambda_2\backslash\{j_2\}}\|_2^2 + (x'_{j_1})^2}}$$

$$< \|\mathbf{x}^*_{\Lambda_2\backslash\{j_2\}}\|_1 - \frac{\|\mathbf{x}^*_{\Lambda_2\backslash\{j_2\}}\|_2^2}{|x^*_{j_2}| + \sqrt{\|\mathbf{x}^*_{\Lambda_2\backslash\{j_2\}}\|_2^2 + (x^*_{j_2})^2}} = \|\mathbf{x}^*\|_{t,1-2},$$

which contradicts the assumption that $\mathbf{x}^*$ is a global minimizer.

Finally, assume that $\mathbf{x}^*$ is a local (global) minimizer of the unconstrained model (1.7). Using the same argument in the proof of [55, Theorem 2.4], one can show that $\mathbf{x}^*$ is also a local (global) minimizer of the following constrained problem:

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{x}\|_{t,1-2} \quad \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{x}^*.$$

Thus, (a) and (b) also hold for $\mathbf{x}^*$. □

Proof of Corollary 3.9. For (a), let $X \subset \mathbb{R}^n$ be the set of local minimizers of (1.5) with $\tau = 0$ and $Y := \{\|\mathbf{x}\|_{t,1-2} \mid \mathbf{x} \in X\}$ be the corresponding local minima. For any $\Lambda \subseteq \{1,\ldots,n\}$, define

$$X_\Lambda := \{\mathbf{x} \mid \mathbf{x} \in X, \mathrm{supp}(\mathbf{x}) = \Lambda\} \quad \text{and} \quad Y_\Lambda := \{\|\mathbf{x}\|_{t,1-2} \mid \mathbf{x} \in X_\Lambda\}.$$

Then $|Y| \leq \sum_{\Lambda \subseteq \{1,\ldots,n\}} |Y_\Lambda|$. Since the number of Λ is finite, it remains to show that $|Y_\Lambda| < \infty$ for every Λ. For a fixed Λ, if $|\Lambda| \leq t + 1$, we have that $Y_\Lambda = \{0\}$. If $|\Lambda| \geq t + 2$, for any $T \subset \Lambda$ with $|T| = t$, define

$$X_{\Lambda,T} := \{\mathbf{x} \mid \mathbf{x} \in X_\Lambda, |x_i| \geq |x_j|, \forall i \in T, j \notin T\} \quad \text{and} \quad Y_{\Lambda,T} := \{\|\mathbf{x}\|_{t,1-2} \mid \mathbf{x} \in X_{\Lambda,T}\}.$$

Then $|Y_\Lambda| \leq \sum_{T \subset \Lambda} |Y_{\Lambda,T}|$. Since the number of T is finite, we just need to show that $|Y_{\Lambda,T}| < \infty$ for every T. In fact, we have $|Y_{\Lambda,T}| \leq 1$. If not, there exists $\hat{\mathbf{x}}, \check{\mathbf{x}} \in X_{\Lambda,T}$

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satisfying $\|\hat{\mathbf{x}}\|_{t,1-2} \neq \|\check{\mathbf{x}}_2\|_{t,1-2}$, which implies that $\hat{\mathbf{x}}_{\Lambda \setminus T} \neq \check{\mathbf{x}}_{\Lambda \setminus T}$. Consider $\mathbf{r} := \hat{\mathbf{x}} - \check{\mathbf{x}}$, one has $\mathbf{r} \in \ker(\mathbf{A})$, $\mathrm{supp}(\mathbf{r}) \subset \Lambda$, and $\mathbf{r}_{\Lambda \setminus T} \neq \mathbf{0}$, which contradicts Theorem 3.8 (a).

For (b), redefine $X \subset \mathbb{R}^n$ be the set of global minimizers of (1.5) with $\tau = 0$ and $X_\Lambda := \{\mathbf{x} \mid \mathbf{x} \in X, \mathrm{supp}(\mathbf{x}) = \Lambda\}$ for any $\Lambda \subset \{1, \ldots, n\}$. Since the global minimum is not zero, it is straightforward that $|X_\Lambda| \leq 1$ for every Λ from Theorem 3.8 (b), which completes the proof. □

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