

Algorithm for Overcoming the Curse of Dimensionality for State-dependent Hamilton-Jacobi equations

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Abstract

In this paper, we develop algorithms to overcome the curse of dimensionality in possibly non-convex state-dependent Hamilton-Jacobi partial differential equations (HJ PDEs) arising from optimal control and differential game problems. The subproblems are independent and they can be implemented in an embarrassingly parallel fashion. This is an ideal setup for perfect scaling in parallel computing. The algorithm is proposed to overcome the curse of dimensionality [1, 2] when solving HJ PDE.

The major contribution of the paper is to change an optimization problem over a space of curves to an optimization problem of a single vector, which goes beyond the work of [23]. We extend the method in [5, 6, 8]. We *conjecture* a (Lax-type) minimization principle to solve *state-dependent* HJ PDE when the Hamiltonian is convex, as well as *conjecture* a (Hopf-type) maximization principle to solve *state-dependent* HJ PDE when the Hamiltonian is *non-convex*. In particular the conjectured (Hopf-type) maximization principle is a generalization of the well-known Hopf formula in [11, 16, 30]. The optimization problems are of the same dimension as the dimension of the HJ PDE. The evaluation of the functional inside the minimization/maximization principles comes along with numerical ODE solvers and numerical quadrature rules. We suggest a coordinate descent method for the minimization procedure in the generalized Lax/Hopf formula, and numerical differentiation is used to compute the derivatives. This method is preferable since the evaluation of the function value itself requires some computational effort, especially when we handle higher dimensional optimization problem. Similar to [6], numerical errors come in because we use a numerical quadrature rule for computing integrals within the minimization/maximization principles, numerical differentiation to minimize the number of calculation procedures in each iteration, and in addition, with the choice of the numerical ODE solver. These errors can be effectively controlled by choosing an appropriate mesh-size in time and the method does not use a mesh in space. The use of multiple initial guesses is suggested to overcome possibly multiple local extrema since the optimization process is no longer convex.

Our method is expected to have wide application in control theory and differential game problems, and elsewhere.

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1 Introduction

Hamilton-Jacobi-Isaacs partial differential equations (HJ PDE) are crucial in analyzing continuous/differential dynamic games, control theory problems, and dynamical systems coming from the physical world, e.g. [13]. An important application is to compute the evolution of geometric objects [25], which was first used for reachability problems in [19, 20], to our knowledge.

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Numerical solutions to HJ PDE have attracted a lot of attention. Most of the methods involve the introduction of a grid and a finite difference discretization of the Hamiltonian. Some of these well-known methods using discretization include ENO/WENO-type methods [24] and Dijkstra-type [10] methods such as fast marching [32] and fast sweeping [31]. However, with their discretization nature, these numerical approaches of HJ PDE suffer from poor scaling with respect to dimension d , hence rendering them impossible to be applied to problems in high dimensions.

Research has therefore been conducted by several groups in search of possible algorithms that can scale reasonably with dimension. Some new algorithms are introduced in e.g. [7, 17, 18]. In [8, 5, 6], the authors proposed a causality-free method for solving possibly non-convex and time dependent HJ PDE based on the generalized Hopf-Lax formula. Using the Hopf-Lax formula, the PDE becomes decoupled and the solution at each point can be effectively calculated by d -dimensional minimization, with d the space dimension.

In this work, we propose to extend the method in [8, 5, 6]. The major contribution of the paper is to change an optimization problem over a space of curves to an optimization problem of a single vector. We *conjecture* the (Lax-type) minimization principle to solve *state-dependent* HJ PDE when the Hamiltonian is convex. We also *conjecture* a (Hopf-type) maximization principle to solve *state-dependent* HJ PDE when the Hamiltonian is *non-convex* but when the initial data is *convex*. In particular the conjectured (Hopf-type) maximization principle is a generalization of the well-known Hopf formula in [11, 16, 30]. The optimization problems are of the same dimension as the dimensions of the HJ PDE. A coordinate descent method is suggested for the minimization procedure in the generalized Lax/Hopf formula, and numerical differentiation is used to compute the derivatives. This method is preferable since the evaluation of the function value itself requires some computational effort, especially when we handle higher dimensional optimization problems. The use of multiple initial guesses is suggested to overcome possibly multiple local extrema since the optimization process is no longer convex. A simple numerical ODE solver is used to compute the bi-characteristics in the Hamiltonian system. A numerical quadrature rule is used for computing integrals with respect to time inside the minimization/maximization principles. Coordinate descent is also used and a numerical differentiation is performed to minimize the number of calculation procedures in each iteration steps. In this paper we illustrate the practicality of our method using the simplest ODE solvers, quadrature rules and finite difference methods, namely the forward Euler, rectangular rule, and forward difference methods. Nonetheless, all these components of the optimization can be improved by using better numerical methods, e.g. pseudo-spectral methods for ODE, Gauss-Lobatto quadrature rules, etc. These choices, together with choosing an appropriate mesh-size in time, minimizes errors effectively. Our method does not use a mesh in space, and solutions can be computed at each point (x, t) in a totally decoupled and embarrassingly parallel manner. Moreover, the solution to the HJ PDE is evaluated at each point (x, t) by coordinate minimization described by the minimization/maximization principles at that point.

As for high dimensional control, we would like to compare our work with [23], which is concerned with discrete approximations of a particular set of control, namely the discrete linear-convex control problems involving a quadratic control. In the paper, both the final parameter v as well as the whole curve representing controls are unknowns. The major contribution of our paper are to remove the optimization over the control (getting the HJ PDE as in [5, 6, 8]), as well as extending our method in [5, 6, 8] to a much more general setting, i.e. differential games/nonconvex problems and the Hamilton-Jacobi limit. Also, our algorithm is faster comparing with [23] even at the discrete level because we remove the optimization over the control. We have also proposed both the Lax and the Hopf version, while [23] focuses only on the analogue of a Lax type formula. We would like to remark that by considering and working on the dual, the algorithm can be made much faster in many cases, especially when the initial data is convex.

Our formal statements of the conjectures that are used for computation will be given in section 3. However, before we give an exact formulation of our conjectures, let us briefly provide our main conjectures. In what follows, we denote p as the co-state variable and $H(x, p, t)$ as the Hamiltonian, as well as $\varphi(x, t)$ as the value function satisfying (2.1)-(2.2). We also denote $(\gamma(t), p(t))$ as the bi-characteristic curve in the phase space that shall satisfy the constraints in the following formulae. Then we conjecture the followings:

1. **Minimization principle** (Lax Formula) when $H(x, p, t)$ is smooth and convex w.r.t. p and possibly under some further mild assumptions:

$$\begin{aligned} \varphi(x, t) = \min_{v \in \mathbb{R}^d} & \left\{ g(\gamma(v, 0)) + \int_0^t \{ \langle p(v, s), \partial_p H(\gamma(v, s), p(v, s), s) \rangle - H(\gamma(v, s), p(v, s), s) \} ds : \right. \\ & \left. \begin{aligned} \dot{\gamma}(v, s) &= \partial_p H(\gamma(v, s), p(v, s), s), \\ \dot{p}(v, s) &= -\partial_x H(\gamma(v, s), p(v, s), s), \\ \gamma(v, t) &= x, p(v, t) = v \end{aligned} \right\} \end{aligned}$$

and its discrete approximation given a small δ ,

$$\begin{aligned} \varphi(x, t) \approx \min_{v \in \mathbb{R}^d} & \left\{ g(x_0(v)) + \delta \sum_{n=1}^{N-1} \{ \langle p_n(v), \partial_p H(x_n(v), p_n(v), t_n) \rangle - H(x_n(v), p_n(v), t_n) \} : \right. \\ & \left. \begin{aligned} x_{n+1}(v) - x_n(v) &= \delta \partial_p H(x_n(v), p_n(v), t_n), \\ p_{n+1}(v) - p_n(v) &= \delta \partial_x H(x_n(v), p_n(v), t_n), \\ x_N &= x, p_N = v \end{aligned} \right\} \end{aligned}$$

2. **Maximization principle** (Hopf Formula) when $H(x, p, t)$ is smooth and $g(p)$ is convex w.r.t. p and possibly under some further mild assumptions:

$$\begin{aligned} \varphi(x, t) = \sup_{v \in \mathbb{R}^d} & \left\{ \langle x, v \rangle - g^*(p(v, 0)) - \int_0^t \left\{ H(\gamma(v, s), p(v, s), s) - \langle \partial_x H(\gamma(v, s), p(v, s), s), \gamma(v, s) \rangle \right\} ds : \right. \\ & \left. \begin{aligned} \dot{\gamma}(v, s) &= \partial_p H(\gamma(v, s), p(v, s), s), \\ \dot{p}(v, s) &= -\partial_x H(\gamma(v, s), p(v, s), s), \\ \gamma(v, t) &= x, p(v, t) = v \end{aligned} \right\} \end{aligned}$$

and its discrete approximation given a small δ

$$\begin{aligned} \varphi(x, t) \approx \max_{v \in \mathbb{R}^d} & \left\{ \langle x_N, v_N \rangle - g^*(p_0(v)) - \delta \sum_{n=1}^{N-1} \{ H(x_n(v), p_n(v), t_n) - \langle x_n(v), \partial_x H(x_n(v), p_n(v), t_n) \rangle \} : \right. \\ & \left. \begin{aligned} x_{n+1} - x_n &= \delta \partial_p H(x_n(v), p_n(v), t_n), \\ p_{n+1} - p_n &= \delta \partial_x H(x_n(v), p_n(v), t_n), \\ x_N &= x, p_N = v \end{aligned} \right\} \end{aligned}$$

and in both cases the argument v attaining the minimum or the maximum in the respective formula is $\partial_x \varphi(x, t)$ when φ is smooth at (x, t) . These are the key conjectures from a practical point of view. When $H(x, p, t)$ is non-smooth (e.g. in the case when $H(x, p, t)$ is homogeneous of degree 1 w.r.t. p), more general forms of the respective conjectures are also available. We must emphasize that these modifications of the formulas are **necessary** when H is non-smooth or otherwise some part of sub-gradient flow will be missed and the formula will be incorrect. We notice that, in particular, our conjectured (Hopf-type) maximization principle is a generalization of the well-known Hopf formula in [11, 16, 30].

The rest of our paper is organized as follows: in subsection 2.1 we introduce the general class of HJ-PDE that we are interested in, and then we briefly explain the connection between HJ-PDE and differential games subsection 2.2. Then in section 3 we discuss the major formulae that we use in our work: the conjectured minimization principle (Lax formulation) and the conjectured maximization principle (Hopf formulation). We then go on to explain briefly our numerical techniques in section 4. Numerical examples are given in section 5.

2 Review of Hamilton-Jacobi Equations and Differential Games

2.1 Hamilton-Jacobi Equations

In this work, we are concerned with an approximation scheme for solving the following HJ PDE:

$$\frac{\partial}{\partial t}\varphi(x, t) + H(x, \nabla_x \varphi(x, t), t) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty), \quad (2.1)$$

where $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous Hamiltonian function bounded from below by an affine function, $\frac{\partial}{\partial t}\varphi$ and $\nabla_x \varphi$ respectively denote the partial derivatives with respect to t and the gradient vector with respect to x of the function $\varphi : \mathbb{R}^d \times (0, \infty) \rightarrow \mathbb{R}$. We are also given the initial data

$$\varphi(x, 0) = g(x) \quad \text{in } \mathbb{R}^d. \quad (2.2)$$

For the sake of simplicity, we only consider functions H and g that are finite everywhere. Results presented in this paper can be generalized to functions with the extended value $+\infty$ under suitable assumptions. We wish to compute the viscosity solution to (2.1)-(2.2) [3, 4] at a given point $x \in \mathbb{R}^d$ and time $t \in (0, \infty)$.

The viscosity solution to (2.1)-(2.2) is explicitly given by the Hopf-Lax formulae when H is x independent, which holds because the integral curves of the Hamiltonian vector field (i.e., the bi-characteristics in the phase space) are straight lines when projected to x -space.

In general, when H is x -dependent and is convex, we also postulate the minimization can only be performed along the bi-characteristics in the phase which are obtained numerically (i.e. minimization over the integral curves generated by the Hamiltonian flow in the phase space, which by definition preserves H). We also postulate a maximization principle when the Hamiltonian is non-convex.

2.2 Differential Games and its Connection with Non-convex Hamilton-Jacobi Equations

In this subsection, we give a brief introduction to the specific optimal control and differential game problems we are considering, and a brief explanation as to how we recast them as problems of solving HJ PDE's. We follow discussions in [5, 6, 7, 13], see also [11], about optimal control and also for differential games, and their links with HJ PDE.

To start with, we first consider two convex compact sets C and D , in which control parameters lie. Then let us denote $\mathcal{A} = \{a : [t, T] \rightarrow C : a \text{ is measurable}\}$, which is referred to as the admissible set of Player I; and $\mathcal{B} = \{b : [t, T] \rightarrow D : b \text{ is measurable}\}$, which is referred to as the admissible set of Player II. We call the measurable functions $a : [t, T] \rightarrow C$ in the set \mathcal{A} and the function $b : [t, T] \rightarrow D$ in the set \mathcal{B} as *controls* performed by Players I and II respectively.

We start with a system of differential equations given as follows. Fix $0 \leq t < T$, $x \in \mathbb{R}^d$. We consider

$$\begin{cases} \frac{dx}{ds}(s) = f(s, x(s), a(s), b(s)) & t \leq s \leq T, \\ x(t) = x, \end{cases}$$

We assume that the function

$$f : [0, T] \times \mathbb{R}^d \times A \times B \rightarrow \mathbb{R}^m$$

is uniformly continuous and

$$\begin{cases} |f(t, x, a, b)| \leq C_1 \\ |f(t, x, a, b) - f(t, y, a, b)| \leq C_1 |x - y|, \end{cases}$$

for some constant C_1 and for all $0 \leq t \leq T$, $x, y \in \mathbb{R}^m$, $a \in A$, $b \in B$. The unique solution to (2.3) is called the response of the controls $a(\cdot), b(\cdot)$. Then we introduce the *payoff* functional for a given pair of

(x, t) :

$$P(a, b) := P_{t,x}(a(\cdot), b(\cdot)) := \int_t^T h(s, x(s), a(s), b(s)) ds + g(x(T)),$$

where $g : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies

$$\begin{cases} |g(x)| \leq C_2 \\ |g(x) - g(y)| \leq C_2|x - y|, \end{cases}$$

and h satisfies

$$\begin{cases} |h(t, x, a, b)| \leq C_3 \\ |h(t, x, a, b) - h(t, y, a, b)| \leq C_3|x - y|, \end{cases}$$

for some constants C_2, C_3 and all $0 \leq t \leq T$, $x, y \in \mathbb{R}^m$, $a \in A$, $b \in B$. In a differential game, the goal of player I is to maximize the functional P by choosing his control a whereas that of player II is to minimize P by choosing his control b .

Now we are ready to define the lower and upper values of the differential game, based on the notation introduced above. We first define the two sets containing the respective controls of players I and II:

$$\begin{aligned} M(t) &:= \{a : [t, T] \rightarrow A : a \text{ is measurable.}\}, \\ N(t) &:= \{b : [t, T] \rightarrow B : b \text{ is measurable.}\}. \end{aligned}$$

Define a strategy for player I as the map

$$\alpha : N(t) \rightarrow M(t)$$

for each $t \leq s \leq T$ and $b, \hat{b} \in B$ such that

$$b(\tau) = \hat{b}(\tau) \text{ for a.e. } t \leq \tau \leq s \quad \Rightarrow \quad \alpha[b](\tau) = \alpha[\hat{b}](\tau) \text{ for a.e. } t \leq \tau \leq s.$$

Therefore a strategy for player I $\alpha[b]$ is the control of player I given that of player II as b . Similarly, let us define a strategy for player II as

$$\beta : M(t) \rightarrow N(t)$$

for each $t \leq s \leq T$ and $a, \hat{a} \in A$ such that

$$a(\tau) = \hat{a}(\tau) \text{ for a.e. } t \leq \tau \leq s \quad \Rightarrow \quad \beta[a](\tau) = \beta[\hat{a}](\tau) \text{ for a.e. } t \leq \tau \leq s.$$

Again a strategy for player II $\beta[a]$ is the control of player II given that of player I as a .

Now let $\Gamma(t)$ denote the set of all strategies for I and $\Delta(t)$ for II beginning at time t . We are well equipped to define the upper and lower values of the differential game. The lower value $V(x, t)$ is defined as

$$\begin{aligned} V(x, t) &:= \inf_{\beta \in \Delta(t)} \sup_{a \in M(t)} P_{t,x}(a, \beta[a]) \\ &:= \inf_{\beta \in \Delta(t)} \sup_{a \in M(t)} \left\{ \int_t^T h(s, x(s), a(s), \beta[a](s)) ds + g(x(T)) \right\}, \end{aligned}$$

where $x(\cdot)$ solves (2.3) for a given pair of (x, t) . Likewise, the upper value $U(x, t)$ is defined as

$$\begin{aligned} U(x, t) &:= \sup_{\alpha \in \Gamma(t)} \inf_{b \in N(t)} P_{t,x}(\alpha[b], b) \\ &:= \sup_{\alpha \in \Gamma(t)} \inf_{b \in N(t)} \left\{ \int_t^T h(s, x(s), \alpha[b](s), b(s)) ds + g(x(T)) \right\}, \end{aligned}$$

where $x(\cdot)$ again solves (2.3) for a given pair of (x, t) .

In fact, derived from dynamic programming optimality conditions in [13], the lower and upper values V and U are the viscosity solutions of a certain possibly nonconvex HJ PDE. For the sake of exposition, we first define the following two Hamiltonians:

$$\begin{aligned}\tilde{H}^+(x, p, t) &= \max_{b \in B} \min_{a \in A} \{ -\langle f(t, x, a, b), p \rangle - h(t, x, a, b) \}, \\ \tilde{H}^-(x, p, t) &= \min_{a \in A} \max_{b \in B} \{ -\langle f(t, x, a, b), p \rangle - h(t, x, a, b) \}.\end{aligned}$$

A very important case of this class of Hamiltonian is when $\tilde{H}^\pm(t, x, p)$ are homogeneous of degree 1, which is highlighted in this work. In fact, in the case where

$$\begin{aligned}f(t, x, a, b) &= a - b \\ h(t, x, a, b) &= -c_1(x, t)\mathcal{I}_{A(x, t)}(p) + c_2(x, t)\mathcal{I}_{B(x, t)}(p),\end{aligned}$$

where \mathcal{I}_Ω is the indicator functions of the sets Ω , and $A(x, t)$ and $B(x, t)$ are balanced, then it holds that \tilde{H}^+ and \tilde{H}^- coincide, as well as the following relationship:

$$\begin{aligned}\tilde{H}^\pm(x, p, t) &= \max_{b \in B(x, t)} \min_{a \in A(x, t)} \{ c_1(x, t)\langle a, p \rangle - c_2(x, t)\langle b, p \rangle \} \\ &= c_1(x, t) \min_{a \in A(x, t)} \{ \langle a, p \rangle \} - c_2(x, t) \max_{b \in B(x, t)} \{ \langle b, p \rangle \} \\ &= -c_1(x, t)\mathcal{I}_{A(x, t)}^*(p) + c_2(x, t)\mathcal{I}_{B(x, t)}^*(p).\end{aligned}$$

In this case, $H^\pm(x, p, t)$ can be written as a difference of two positively homogeneous (of degree 1) Hamiltonians $\Phi_1(x, \cdot, t)$, $\Phi_2(x, \cdot, t)$, namely,

$$\tilde{H}^\pm(x, p, t) = -c_1(x, t)\Phi_1(x, p, t) + c_2(x, t)\Phi_2(x, p, t)$$

where $\Phi_1(x, \cdot, t)$ and $\Phi_2(x, \cdot, t)$ have their respective Wulff sets as $A(x, t)$ and $B(x, t)$ (see [14, 15, 26] for more details of the Wulff set.)

Now, for a general pair of $\tilde{H}^\pm(x, p, t)$, we have the following well-known theorem:

Theorem 2.1. [13] *The function U is the viscosity solution to the HJ PDE :*

$$\begin{cases} \frac{\partial}{\partial t}U - \tilde{H}^+(x, \nabla_x U, t) = 0 & \text{on } \mathbb{R}^d \times (-\infty, T), \\ U(x, T) = g(x) & \text{on } \mathbb{R}^d. \end{cases}$$

Similarly, the function V is the viscosity solution to the HJ PDE :

$$\begin{cases} \frac{\partial}{\partial t}V - \tilde{H}^-(x, \nabla_x V, t) = 0 & \text{on } \mathbb{R}^d \times (-\infty, T), \\ V(x, T) = g(x) & \text{on } \mathbb{R}^d. \end{cases}$$

It is worth mentioning again that, in a general setting where h is possibly nonconvex, the two Hamiltonians $\tilde{H}^+(x, p, t)$ and $\tilde{H}^-(x, p, t)$ may not coincide. But, when they do, there is the following corollary:

Corollary 2.2. [13] *If*

$$\tilde{H}^+(x, p, t) = \tilde{H}^-(x, p, t) \text{ on } [t, T] \times \mathbb{R}^d \times \mathbb{R}^d,$$

then it holds that $U = V$.

Hereafter, when $U = V$, we write $\varphi(x, t) := U(x, T - t) = V(x, T - t)$, and write $H(x, p, t) = H^\pm(x, p, T - t)$, then

$$\begin{cases} \frac{\partial}{\partial t}\varphi + H(x, \nabla_x \varphi, t) = 0 & \text{on } \mathbb{R}^d \times (0, \infty), \\ \varphi(x, 0) = g(x) & \text{on } \mathbb{R}^d. \end{cases}$$

Note that in general, the Hamiltonians H can be nonconvex and/or nonconcave, and this is one very important occasion in which nonconvex HJ PDE arises.

In the next section, we will discuss possible representation formulae of the HJ-PDE equation, which may help us to compute the solution quickly and in parallel.

3 Representation formulae for viscosity solution of HJ PDE

In this section, we introduce our optimization formulae. We first conjecture a (Lax-type) minimization principle for the viscosity solution to (2.1)-(2.2) when H is convex. Then we go on to (2.1)-(2.2) when H is non-convex but when g is convex, and make a conjecture that a (Hopf-type) maximization principle shall hold in this case. In several special cases, when a Hopf formula is known for the solution, our conjectured representation reduces to these known formulae.

Before we provide our formal statements of the conjectures, for the sake of exposition, let us repeat our brief statements of the conjectures as in section 1 as follows:

1. **Minimization principle** (Lax Formula) when $H(x, p, s)$ is smooth and convex w.r.t. p :

$$\begin{aligned} \varphi(x, t) = \min_{v \in \mathbb{R}^d} & \left\{ g(\gamma(v, 0)) + \int_0^t \{ \langle p(v, s), \partial_p H(\gamma(v, s), p(v, s), s) \rangle - H(\gamma(v, s), p(v, s), s) \} ds : \right. \\ & \left. \begin{aligned} \dot{\gamma}(v, s) &= \partial_p H(\gamma(v, s), p(v, s), s), \\ \dot{p}(v, s) &= -\partial_x H(\gamma(v, s), p(v, s), s), \\ \gamma(v, t) &= x, p(v, t) = v \end{aligned} \right\} \end{aligned}$$

and its discrete approximation given a small δ ,

$$\begin{aligned} \varphi(x, t) \approx \min_{v \in \mathbb{R}^d} & \left\{ g(x_0(v)) + \delta \sum_{n=1}^{N-1} \{ \langle p_n(v), \partial_p H(x_n(v), p_n(v), t_n) \rangle - H(x_n(v), p_n(v), t_n) \} : \right. \\ & \left. \begin{aligned} x_{n+1}(v) - x_n(v) &= \delta \partial_p H(x_n(v), p_n(v), t_n), \\ p_{n-1}(v) - p_n(v) &= \delta \partial_x H(x_n(v), p_n(v), t_n), \\ x_N &= x, p_N = v \end{aligned} \right\} \end{aligned}$$

2. **Maximization principle** (Hopf Formula) when $H(x, p, s)$ is smooth and $g(p)$ is convex w.r.t. p :

$$\begin{aligned} \varphi(x, t) = \sup_{v \in \mathbb{R}^d} & \left\{ \langle x, v \rangle - g^*(p(v, 0)) - \int_0^t \left\{ H(\gamma(v, s), p(v, s), s) - \langle \partial_x H(\gamma(v, s), p(v, s), s), \gamma(v, s) \rangle \right\} ds : \right. \\ & \left. \begin{aligned} \dot{\gamma}(v, s) &= \partial_p H(\gamma(v, s), p(v, s), s), \\ \dot{p}(v, s) &= -\partial_x H(\gamma(v, s), p(v, s), s), \\ \gamma(v, t) &= x, p(v, t) = v \end{aligned} \right\} \end{aligned}$$

and its discrete approximation given a small δ

$$\begin{aligned} \varphi(x, t) \approx \max_{v \in \mathbb{R}^d} & \left\{ \langle x_N, v_N \rangle - g^*(p_0(v)) - \delta \sum_{n=1}^{N-1} \{ H(x_n(v), p_n(v), t_n) - \langle x_n(v), \partial_x H(x_n(v), p_n(v), t_n) \rangle \} : \right. \\ & \left. \begin{aligned} x_{n+1} - x_n &= \delta \partial_p H(x_n(v), p_n(v), t_n), \\ p_{n-1} - p_n &= \delta \partial_x H(x_n(v), p_n(v), t_n), \\ x_N &= x, p_N = v \end{aligned} \right\} \end{aligned}$$

and in both cases the argument v attaining the minimum or the maximum in the respective formula is $\partial_x \varphi(x, t)$ when φ is smooth at (x, t) . These are the key conjectures from a practical point of view. When $H(x, p, t)$ is non-smooth, more general forms of the respective conjectures are necessary. The precise statements and formal derivations will be given in the following subsections. One point to remark is that the conjectured Hopf formula is a generalization of the well-known Hopf formula in [11, 16, 30].

3.1 A simplified system

In order to get to these two formulae, for the sake of exposition, let us consider first a **simplified** system of differential equations: Fix $0 \leq t < T$, $x \in \mathbb{R}^d$. We consider

$$\begin{cases} \frac{dx}{ds}(s) = f(s, x(s), u(s)) & t \leq s \leq T, \\ x(t) = x, \end{cases}$$

where $u(\cdot) \in \mathcal{U}(t) = \{u : [t, T] \rightarrow U : u \text{ is measurable}\}$ is again a control and U is the admissible set. We also consider the following simplified payoff function

$$P(u) := P_{t,x}(u(\cdot)) := \int_t^T h(s, x(s), u(s)) ds + g(x(T)),$$

We consider also the value function

$$U(x, t) := \inf_{u \in \mathcal{U}(t)} P_{t,x}(u) = \inf_{u \in \mathcal{U}(t)} \left\{ \int_t^T h(s, x(s), u(s)) ds + g(x(T)) \right\},$$

and the Hamiltonians

$$\tilde{H}(t, x, p) = \max_{u \in U} \{-\langle f(t, x, u), p \rangle - h(t, x, u)\} \text{ as well as } H(x, p, t) = \tilde{H}(x, p, T - t).$$

This is actually the special case of the setting as discussed in Section 2.2 when the set $C = \{0\}$ is a singleton, after we denote $b(\cdot)$ as $u(\cdot)$ instead.

One point to note is that, the argument to get either a Lax and a Hopf formula in the general case with differential games as discussed in sec 2.2 shall be the same with the standard assumption on the set of strategy following the causality.

3.2 Convex/Concave Hamiltonian

We first describe how we get to a Lax formula for:

$$\begin{aligned} \varphi(x, t) := U(x, t) &:= \inf_{u \in \mathcal{U}(T-t)} \left\{ \int_{T-t}^T h(s, x(s), u(s)) ds + g(x(T)) \right\} \\ &:= \inf_{u \in \mathcal{U}(T-t)} \left\{ \int_0^t h(T-s, x(T-s), u(T-s)) ds + g(x(T)) \right\}, \end{aligned}$$

We derive our formula formally as follows. Following [23], we shall first consider the following discretization (approximation) for a given δ such that $\delta N = t$, by denoting $s_n = \delta n$ and $x_N = x$ (and flipping the sign),

$$\begin{aligned} &\varphi(x, t) \\ &\approx \min_{\{u_n\}_{n=0}^N \in U} \left\{ g(x_0) + \delta \sum_{n=1}^{N-1} h(T-s_n, x_n, u_n) : x_{n+1} - x_n = -\delta f(T-s_n, x_n, u_n) \text{ for } n = 1, \dots, N-1, x_N = x \right\}, \\ &= \min_{\{u_n\}_{n=0}^N \in U} \max_{\{x_n\}_{n=0}^{N-1}} \left\{ g(x_0) + \delta \sum_{n=1}^{N-1} h(T-s_n, x_n, u_n) + \sum_{n=1}^{N-1} \langle p_n, x_{n+1} - x_n \rangle + \delta \sum_{n=1}^{N-1} \langle p_n, f(T-s_n, x_n, u_n) \rangle \right\}. \end{aligned}$$

Assuming some appropriate conditions for the Von-Neumann minimax principle [22] to hold, we have

$$\begin{aligned} \varphi(x, t) &\approx \min_{\{x_n\}_{n=0}^{N-1}} \max_{\{p_n\}_{n=0}^N} \min_{\{u_n\}_{n=0}^N} \left\{ g(x_0) + \delta \sum_{n=1}^{N-1} h(T-s_n, x_n, u_n) + \sum_{n=1}^{N-1} \langle p_n, x_{n+1} - x_n \rangle + \delta \sum_{n=1}^{N-1} \langle p_n, f(T-s_n, x_n, u_n) \rangle \right\} \\ &= \min_{\{x_n\}_{n=0}^{N-1}} \max_{\{p_n\}_{n=0}^N} \left\{ g(x_0) - \delta \sum_{n=1}^{N-1} \max_{u \in U} \{-\langle p_n, f(T-s_n, x_n, u) \rangle - h(T-s_n, x_n, u)\} + \sum_{n=1}^{N-1} \langle p_n, x_{n+1} - x_n \rangle \right\}, \\ &= \min_{\{x_n\}_{n=1}^N} \max_{\{p_n\}_{n=0}^N} \left\{ g(x_0) + \delta \sum_{n=1}^{N-1} \langle p_n, \frac{x_{n+1} - x_n}{\delta} \rangle - \delta \sum_{n=1}^{N-1} \tilde{H}(x_n, p_n, T-s_n) \right\}, \\ &= \min_{\{x_n\}_{n=1}^N} \max_{\{p_n\}_{n=0}^N} \left\{ g(x_0) + \delta \sum_{n=1}^{N-1} \langle p_n, \frac{x_{n+1} - x_n}{\delta} \rangle - \delta \sum_{n=1}^{N-1} H(x_n, p_n, s_n) \right\}, \end{aligned}$$

If we further make appropriate assumptions that leads to a possible replacement of the mini-max optimization by the KKT condition, then we have this last equality as

$$\begin{aligned} \varphi(x, t) \approx \min_{v \in \mathbb{R}^d} \left\{ g(x_0) + \delta \sum_{n=1}^{N-1} \langle p_n, \frac{x_{n+1} - x_n}{\delta} \rangle - \delta \sum_{n=1}^{N-1} H(x_n, p_n, s_n) : \right. \\ \left. \begin{aligned} x_{n+1}(v) - x_n(v) &= \delta \partial_p H(x_n(v), p_n(v), s_n), \\ p_{n-1}(v) - p_n(v) &= \delta \partial_x H(x_n(v), p_n(v), s_n), \\ x_N &= x, p_N = v \end{aligned} \right\}, \end{aligned}$$

Notice that if we assume some appropriate assumptions (related to convexity) on $h(s, x, u)$ and $f(s, x, u)$ w.r.t. u , then there exists a saddle point for the saddle point problem and thus one may apply Von-Neumann minimax theorem [22] in the first step. In the second step, if we H is convex w.r.t. p , concave w.r.t. x , again there exists a saddle point for the saddle point problem and one may replace the mini-max optimization by the KKT condition. (This is exactly the case when one can directly write the formula as a minimization of a Lagrangian as the dual of the Hamiltonian H .)

However, we notice that the resulting formula that we conjectured seems to be correct beyond these assumptions, as the numerical results show (especially when we take the minimum over all the paths satisfying the KKT conditions). We hope to get rigorous criteria for these formula to hold in the future.

Passing to the limit, in the special case when $H(x, p, t)$ is smooth also w.r.t p , we arrive at the following Lax formula:

$$\begin{aligned} \varphi(x, t) = \min_{v \in \mathbb{R}^d} \left\{ g(\gamma(v, 0)) + \int_0^t \{ \langle p(v, s), \partial_p H(\gamma(v, s), p(v, s), s) \rangle - H(\gamma(v, s), p(v, s), s) \} ds : \right. \\ \left. \begin{aligned} \dot{\gamma}(v, s) &= \partial_p H(\gamma(v, s), p(v, s), s), \\ \dot{p}(v, s) &= -\partial_x H(\gamma(v, s), p(v, s), s), \\ \gamma(v, t) &= x, p(v, t) = v \end{aligned} \right\} \end{aligned}$$

Allowing a more general case when H is non-smooth w.r.t. p , we postulate the following minimization principle. In what follows, we denote $\partial_x^- f(x)$ as the (regularized) subdifferential of f for a given f .

Conjecture 3.1. *When $H(x, p, t)$ is smooth w.r.t. x and convex w.r.t. p (and perhaps under some other mild conditions on $H(x, p, t)$ and $g(p)$), the viscosity solution to (2.1)-(2.2) can be represented as*

$$\begin{aligned} \varphi(x, t) = \inf_{v \in \mathbb{R}^d} \inf_{\gamma \in C^\infty} \left\{ g(\gamma(v, 0)) + \int_0^t \{ \langle p(v, s), \dot{\gamma}(v, s) \rangle - H(\gamma(v, s), p(v, s), s) \} ds : \right. \\ \left. \begin{aligned} \dot{\gamma}(v, s) &\in \partial_p^- H(\gamma(v, s), p(v, s), s), \\ \dot{p}(v, s) &= -\partial_x H(\gamma(v, s), p(v, s), s), \\ \gamma(v, t) &= x, p(v, t) = v \end{aligned} \right\} \end{aligned} \quad (3.1)$$

for small time t . In here we always use the convention that the infimum of an empty set is minus infinity, $\inf \emptyset = -\infty$. If furthermore that ϕ is differentiable at a neighbourhood of (x, t) , the minimum argument in the above formula shall coincide with $\partial_x \varphi(x, t)$.

We refer to it as the **minimization principle**, or the generalized Lax formula. In below there are several examples for the conjecture, that we only provide a brief account.

Example 1 For a restricted class of Hamiltonians $H(x, p, t)$ (convex w.r.t. p , concave w.r.t. x and satisfying a finite concavity-convexity assumption, c.f. Assumptions (A) and Theorem 2.3 in [28]), the following minimization principle holds for the viscosity solution to (2.1)-(2.2) (see Theorem 4.8 in [29]):

$$\varphi(x, t) = \inf_{\gamma \in C^\infty, \gamma(t)=x, \gamma(0)=y} \{ g(y) + \int_0^t L(\gamma(s), \dot{\gamma}(s), s) ds \}$$

where the Lagrangian L is defined as

$$L(x, q, s) = \sup_p \{ \langle p, q \rangle - H(x, p, t) \}.$$

Example 2 On the other hand, when $H(x, p, t)$ is a convex homogeneous degree-1 functional w.r.t. p of the following special form

$$H(x, p, t) = c(x)\Phi(p),$$

where Φ is homogeneous of degree 1 with its Wulff set $W(x)$ convex and containing 0, together with the fact that realizing the following set (a.k.a. domain of dependence)

$$C(x, t) := \bigcup \left\{ \gamma(0) : \begin{array}{l} \dot{\gamma}(s) \in W(x) \\ \gamma(t) = x, \end{array} \right\}$$

has its boundary exactly coinciding with

$$\partial[C(x, t)] = \bigcup_{v \in \mathbb{R}^d \setminus \{0\}} \left\{ \gamma(v, 0) : \begin{array}{l} \dot{\gamma}(v, s) = \partial_p H(\gamma(v, s), p(v, s)), \\ \dot{p}(v, s) = -\partial_x H(\gamma(v, s), p(v, s)), \\ \gamma(v, t) = x, p(v, t) = v \end{array} \right\}.$$

we have (after a further change of variable from v to $-v$ and from s to $t - s$)

$$\varphi(x, t) = \min_{v \in \mathbb{R}^d \setminus \{0\}} \min_{0 \leq r \leq t} \left\{ g(\gamma(v, r)) : \begin{array}{l} \dot{\gamma}(v, s) = -c(\gamma(v, s))\partial_p \Phi(p(v, s)), \\ \dot{p}(v, s) = \partial_x c(\gamma(v, s))\Phi(p(v, s)), \\ \gamma(v, 0) = x, p(v, 0) = v \end{array} \right\} \quad (3.2)$$

Notice since $v \neq 0$, $p(v, s) \neq 0$, and hence $\partial_p \Phi(p(v, s))$ exists as a derivative, e.g. if $\Phi(p) = |p|_2$, we have $\partial_p \Phi(p(v, s)) = p(v, s)/|p(v, s)|_2$.

Example 3 On the other hand, for a concave $H(x, p, t)$ (concave w.r.t. p), and in the special case when

$$H(x, p, t) = -c(x)\Phi(p),$$

we again have

$$\varphi(x, t) = \max_{v \in \mathbb{R}^d \setminus \{0\}} \max_{0 \leq r \leq t} \left\{ g(\gamma(v, r)) : \begin{array}{l} \dot{\gamma}(v, s) = -c(\gamma(v, s))\partial_p \Phi(p(v, s)), \\ \dot{p}(v, s) = \partial_x c(\gamma(v, s))\Phi(p(v, s)), \\ \gamma(v, 0) = x, p(v, 0) = v \end{array} \right\}. \quad (3.3)$$

When $g(x)$ is convex, we have

$$\varphi(x, t) = \max_{v \in \mathbb{R}^d \setminus \{0\}} \left\{ g(\gamma(v, t)) : \begin{array}{l} \dot{\gamma}(v, s) = -c(\gamma(v, s))\partial_p \Phi(p(v, s)), \\ \dot{p}(v, s) = \partial_x c(\gamma(v, s))\Phi(p(v, s)), \\ \gamma(v, 0) = x, p(v, 0) = v \end{array} \right\}$$

In fact the two formulae (3.2) and (3.3) are the Hugyens principles in disguise, under the domain of dependence and domain of influence respectively. Note also that the last formula will also coincide with the Hopf formula that we will propose in the next subsection.

3.3 Hamiltonian H that are neither convex nor concave, but the initial data g is convex

When the Hamiltonian H is are neither convex nor concave, but the initial data g is convex, the aforementioned conjectured minimization principle does not seem to hold any longer. In light of the fact that in some special cases a Hopf formula holds, we conjecture that a generalized Hopf-type maximization principle shall hold for a wide class of problem.

We now describe how we get to a Hopf formula using summation by parts and mini-max theorem:

$$\begin{aligned}\varphi(x, t) := U(x, t) &:= \inf_{u \in \mathcal{U}(T-t)} \left\{ \int_{T-t}^T h(s, x(s), u(s)) ds + g(x(T)) \right\} \\ &:= \inf_{u \in \mathcal{U}(T-t)} \left\{ \int_0^t h(T-s, x(T-s), u(T-s)) ds + g(x(T)) \right\},\end{aligned}$$

We again derive our formula formally as follows. Again, following [23], we shall first consider the following discretization (approximation) for a given δ such that $\delta N = t$, by denoting $s_n = \delta n$ and $x_N = x$

$$\begin{aligned}\varphi(x, t) &\approx \min_{\{u_n\}_{n=0}^N \in U} \left\{ g(x_0) + \delta \sum_{n=1}^{N-1} h(T-s_n, x_n, u_n) : x_{n+1} - x_n = -\delta f(T-s_n, x_n, u_n) \text{ for } n = 1, \dots, N-1, x_N = x \right\}, \\ &= \min_{\{u_n\}_{n=0}^N \in U} \max_{\{x_n\}_{n=0}^{N-1}} \left\{ g(x_0) + \delta \sum_{n=1}^{N-1} h(T-s_n, x_n, u_n) + \sum_{n=1}^{N-1} \langle p_n, x_{n+1} - x_n \rangle + \delta \sum_{n=1}^{N-1} \langle p_n, f(T-s_n, x_n, u_n) \rangle \right\}.\end{aligned}$$

We again now assume some appropriate conditions for the Von-Neumann minimax principle [22] to hold, then we have

$$\begin{aligned}\varphi(x, t) &\approx \max_{\{p_n\}_{n=0}^N} \min_{\{x_n\}_{n=0}^{N-1}} \min_{\{u_n\}_{n=0}^N} \left\{ g(x_0) + \delta \sum_{n=1}^{N-1} h(T-s_n, x_n, u_n) + \sum_{n=1}^{N-1} \langle p_n, x_{n+1} - x_n \rangle + \delta \sum_{n=1}^{N-1} \langle p_n, f(T-s_n, x_n, u_n) \rangle \right\} \\ &= \max_{\{p_n\}_{n=0}^N} \min_{\{x_n\}_{n=0}^{N-1}} \left\{ g(x_0) - \delta \sum_{n=1}^{N-1} \max_{u \in U} \{ -\langle p_n, f(T-s_n, x_n, u) \rangle - h(T-s_n, x_n, u) \} + \sum_{n=1}^{N-1} \langle p_n, x_{n+1} - x_n \rangle \right\}, \\ &= \max_{\{p_n\}_{n=0}^N} \min_{\{x_n\}_{n=1}^{N-1}} \min_{x_0} \left\{ g(x_0) - \langle p_0, x_0 \rangle - \delta \sum_{n=1}^{N-1} \tilde{H}(x_n, p_n, T-s_n) + \sum_{n=1}^N \langle p_{n-1} - p_n, x_n \rangle + \langle p_N, x \rangle \right\},\end{aligned}$$

where the last equality comes from summation by part. Hence

$$\begin{aligned}\varphi(x, t) &\approx \max_{\{p_n\}_{n=0}^N} \min_{\{x_n\}_{n=1}^{N-1}} \left\{ \langle p_N, x \rangle - g^*(p_0) - \delta \sum_{n=1}^{N-1} \tilde{H}(x_n, p_n, T-s_n) + \delta \sum_{n=1}^N \left\langle \frac{p_{n-1} - p_n}{\delta}, x_n \right\rangle \right\}, \\ &= \max_{\{p_n\}_{n=0}^N} \min_{\{x_n\}_{n=1}^{N-1}} \left\{ \langle p_N, x \rangle - g^*(p_0) - \delta \sum_{n=1}^{N-1} H(x_n, p_n, s_n) + \delta \sum_{n=1}^N \left\langle \frac{p_{n-1} - p_n}{\delta}, x_n \right\rangle \right\}, \\ &= \max_{v \in \mathbb{R}^d} \left\{ \langle p_N, x \rangle - g^*(p_0) - \delta \sum_{n=1}^{N-1} H(x_n, p_n, s_n) + \delta \sum_{n=1}^N \left\langle \frac{p_{n-1} - p_n}{\delta}, x_n \right\rangle : \right. \\ &\quad \left. \begin{aligned} x_{n+1}(v) - x_n(v) &= \delta \partial_p H(x_n(v), p_n(v), t_n), \\ p_{n-1}(v) - p_n(v) &= \delta \partial_x H(x_n(v), p_n(v), t_n), \\ x_N &= x, p_N = v \end{aligned} \right\},\end{aligned}$$

where the last equality may again come from the KKT condition if we have additional assumptions that leads to a possible dropping of the mini-max optimization problem.

Notice that in the first step, we again need to assume some appropriate convexity assumptions on $h(s, x, u)$ and $f(s, x, u)$ w.r.t. x and u , as well as concavity assumption w.r.t. p to guarantee the existence of a saddle point for the saddle point problem and thus Von-Neumann minimax theorem [22] can be applied. In the second step, one may again need concavity of $H(x, p, t)$ w.r.t. x and convexity of $H(x, p, t)$ w.r.t. p to replace the optimization problem by the KKT condition.

Again, nonetheless, we notice that the resulting formula that we conjectured seems to be correct beyond these assumptions, as the numerical results show (again especially when we take the maximum

over all the paths satisfying the KKT conditions). We hope to get some rigorous criteria for these formula to hold in the future.

Passing to the limit, in the special case when $H(x, p, t)$ is smooth also w.r.t p , we arrive at the following conjectured Hopf formula:

$$\begin{aligned} \varphi(x, t) = \sup_{v \in \mathbb{R}^d} \left\{ \langle x, v \rangle - g^*(p(v, 0)) - \int_0^t \left\{ H(\gamma(v, s), p(v, s), s) - \langle \partial_x H(\gamma(v, s), p(v, s), s), \gamma(v, s) \rangle \right\} ds : \right. \\ \left. \begin{aligned} \dot{\gamma}(v, s) &= \partial_p H(\gamma(v, s), p(v, s), s), \\ \dot{p}(v, s) &= -\partial_x H(\gamma(v, s), p(v, s), s), \\ \gamma(v, t) &= x, \quad p(v, t) = v \end{aligned} \right\} \end{aligned}$$

Moreover, in this case, we shall observe that if $\phi(x, t)$ is smooth w.r.t. x at the point x and the infimum is attained by v , then we have

$$\partial_x \varphi(x, t) = v.$$

Allowing a more general case when H is non-smooth w.r.t. p , we postulate the following maximization principle/Hopf formula. In what follows, we denote $\partial_x^- f(x)$ as the (regularized) subdifferential of f for a given function f .

Conjecture 3.2. *When $H(x, p, t)$ is smooth w.r.t. x , and $g(p)$ is convex w.r.t. p (and perhaps under some other mild conditions on $H(x, p, t)$ and $g(p)$), the viscosity solution to (2.1)-(2.2) can be represented as*

$$\begin{aligned} \varphi(x, t) = - \inf_{v \in \mathbb{R}^d} \sup_{\gamma \in C^\infty} \left\{ g^*(p(v, 0)) + \int_0^t \left\{ H(\gamma(v, s), p(v, s), s) + \langle \dot{p}(v, s), \gamma(v, s) \rangle \right\} ds - \langle x, v \rangle : \right. \\ \left. \begin{aligned} \dot{\gamma}(v, s) &\in \partial_p^+ H(\gamma(v, s), p(v, s), s), \\ \dot{p}(v, s) &= -\partial_x H(\gamma(v, s), p(v, s), s), \\ \gamma(v, t) &= x, \quad p(v, t) = v \\ p(v, 0) &\in \partial_y^- g(\gamma(v, 0)) \end{aligned} \right\} \end{aligned} \quad (3.4)$$

for small time t (at least) such that the differential $\partial_v p(0)$ is a non-singular matrix. Such a mild condition might be some convexity assumption of $H(x, p, t)$ w.r.t. the convex hull of the set of minimizers in the variable p . (see [30] for predicting this technical assumption). In here we again always use the convention that the infimum of an empty set is minus infinity $\inf \emptyset = -\infty$. If furthermore that ϕ is differentiable at a neighbourhood of (x, t) , the maximum argument in the above formula shall coincide with $\partial_x \varphi(x, t)$.

We may refer to it as the **maximization principle**, or the generalized Hopf formula. In practice, the formula (3.4) below is more easy to be used in practice. This conjecture is made also under private communication with Prof. Wilfrid Gangbo.

One point to remark is that the conjectured Hopf formula is a generalization of the well-known Hopf formula in [11, 16, 30].

A rigorous approach toward this formula might be following an approach of [30] to show the postulated formula is a minimax viscosity solution following the notations in [30] and references therein.

Below we present several examples where this conjecture holds.

Example 1 First, we notice that in the case when $H(x, p, t) = H(p, t)$, we have $\dot{p}(v, s) = \partial_x H(\gamma(v, s), p(v, s), s) = 0$, and therefore the conjectured formula gives

$$\begin{aligned} \varphi(x, t) &= - \min_{v \in \mathbb{R}^d} \left\{ g^*(p(v, 0)) + \int_0^t \left\{ H(p(v, s), s) - \langle 0, \gamma(v, s) \rangle \right\} ds - \langle x, v \rangle : \right. \\ &\quad \left. \begin{aligned} \dot{\gamma}(v, s) &\in \partial_p^- H(p(v, s), s), \\ \dot{p}(v, s) &= 0, \\ \gamma(v, t) &= x, \\ p(v, t) &= v \end{aligned} \right\} \\ &= - \min_{v \in \mathbb{R}^d} \left\{ g^*(v) + \int_0^t H(v, s) ds - \langle x, v \rangle \right\} \end{aligned}$$

which gets us back to the Hopf formula.

Example 2 Now in the case when $H(x, p, t)$ is a **non-convex** homogeneous degree-1 functional w.r.t. p in the following form, we have

$$H(x, p, t) = c_1(x, t)\Phi_1(p) - c_2(x, t)\Phi_2(p)$$

then from non-convexity, together with the convention $\sup \emptyset = +\infty$, we have:

$$\begin{aligned} \varphi(x, t) = & - \inf_{v \in \mathbb{R}^d \setminus \{0\}} \left\{ g^*(p(v, 0)) + \int_0^t \left\{ \left(c_1(\gamma(v, s), s) - \langle \partial_p \Phi_1(p(v, s)), \gamma(v, s) \rangle \right) \Phi_1(p(v, s)) \right. \right. \\ & \left. \left. - \left(c_2(\gamma(v, s), t - s) - \langle \partial_p \Phi_2(p(v, s)), \gamma(v, s) \rangle \right) \Phi_2(p(v, s)) \right\} ds - \langle x, v \rangle : \right. \\ & \left. \begin{aligned} \dot{\gamma}(v, s) &= c_1(\gamma(v, s), s) \partial_p \Phi_1(p(v, s)) - c_2(\gamma(v, s), s) \partial_p \Phi_2(p(v, s)) \\ \dot{p}(v, s) &= -\partial_x c_1(\gamma(v, s), s) \Phi_1(p(v, s)) - \partial_x c_2(\gamma(v, s), s) \Phi_2(p(v, s)), \\ \gamma(v, t) &= x, p(v, t) = v \end{aligned} \right\}. \end{aligned} \quad (3.5)$$

In the special case of

$$H(x, p, t) = -c(x)|p|_2$$

we have, from (3.5) and tracing back the integration by parts, that

$$\varphi(x, t) = \max_{v \in \mathbb{R}^d \setminus \{0\}} \left\{ g(\gamma(v, t)) : \begin{aligned} \dot{\gamma}(v, s) &= c_2(\gamma(v, s))p(v, s)/|p(v, s)|_2 \\ \dot{p}(v, s) &= -\partial_x c_2(\gamma(v, s))|p(v, s)|_2, \\ \gamma(v, 0) &= x, p(v, 0) = v \end{aligned} \right\}$$

this gets us back to (3.3) in subsection 3.2 after we are aware of the fact that $g(x)$ is convex w.r.t. x .

When $H(x, p, t)$ is, on the other hand, a **convex** homogeneous degree-one functional w.r.t. p , i.e. when $c_2(x, t) = 0$, and hence $H(x, p, t) = c(x, t)\Phi(p)$ for some $c > 0$ and Φ , then the formula reads

$$\begin{aligned} \varphi(x, t) = & - \inf_{v \in \mathbb{R}^d \setminus \{0\}} \left\{ g^*(p(v, 0)) + \int_0^t \left\{ \left(c(\gamma(v, s), s) - \langle \dot{p}(v, s), \gamma(v, s) \rangle \right) \Phi(p(v, s)) \right\} ds - \langle x, v \rangle : \right. \\ & \left. \begin{aligned} \dot{\gamma}(v, s) &\in c(\gamma(v, s), s) \partial_p^- \Phi(p(v, s)) \\ \dot{p}(v, s) &= -\partial_x c(\gamma(v, s), s) \Phi(p(v, s)), \\ \gamma(v, t) &= x, p(v, t) = v \end{aligned} \right\}. \end{aligned}$$

Example 3 In the special case when the HJ PDE comes from an ODE system in a differential game, we have the following finite horizon problem with initial state $x \in \mathbb{R}^d$. We consider the (Lipschitz) solution $x : [t, T] \rightarrow \mathbb{R}^d$ of the following linear dynamic system with initial condition x at time t :

$$\begin{cases} \frac{dx}{ds}(s) = Mx(s) + N_C(s)a(s) + N_D(s)b(s) & \text{in } (t, T) \\ x(t) = x \end{cases}$$

where M is a given $d \times d$ matrix independent of time, and $\{N_C(s)\}_{t < s < T}$, $\{N_D(s)\}_{t < s < T}$ are two families of $d \times d$ matrices with real entries. Using the notation in section 2.2, if we let

$$\begin{aligned} f(t, x, a, b) &= Mx(s) + N_C(s)a + N_D(s)b \\ h(t, x, a, b) &= -\mathcal{I}_{C(t)}(p) + \mathcal{I}_{D(t)}(p), \end{aligned}$$

for some family of convex sets $\{C(s)\}_{t < s < T} \subset C$, $\{D(s)\}_{t < s < T} \subset D$, our Hamiltonians read:

$$\tilde{H}^\pm(x, p, t) = \max_{b \in D(t)} \min_{a \in C(t)} \{-\langle Mx + N_C(t)a + N_D(t)b, p \rangle\} = -\langle Mx, p \rangle + \Phi_{C(t)}(-N_C^*(t)p) - \Phi_{D(t)}(-N_D^*(t)p).$$

where

$$\Phi_W(p) = \max_{x \in W} \langle x, p \rangle.$$

After a change of variable to get from a finite time PDE problem $(-\infty, T)$ to an initial time PDE problem $(0, \infty)$, we have the upper/lower values $\varphi(x, t) := U(x, T - t) = V(x, T - t)$ satisfy:

$$\begin{cases} \frac{\partial}{\partial t} \varphi + H(x, \nabla_x \varphi, t) = 0 & \text{on } \mathbb{R}^d \times (0, \infty), \\ \varphi(x, 0) = g(x) & \text{on } \mathbb{R}^d \end{cases}$$

where the Hamiltonian H is now

$$H(x, p, t) = \tilde{H}^\pm(x, p, T - t) = -\langle Mx, p \rangle + \Phi_{C(T-t)}(-N_C^*(T-t)p) - \Phi_{D(T-t)}(-N_D^*(T-t)p).$$

Now, notice that H is smooth w.r.t. x , from the proposed generalized Hopf formula applied to Hamiltonian H and initial data g , together with the fact that $\partial_x H(x, p, t) = -M^*p$, together with some manipulations and change of variable, we can get back to same formula as in (2.5) in [6]. In fact, The generalized Hopf formula gives

$$\begin{aligned} & \varphi(x, t) \\ = & - \min_{p \in \mathbb{R}^d} \left\{ g^*(e^{-M^*T}p) + \int_0^t \left\{ \Phi_{C(T-s)} \left([-e^{-M(T-s)}N_C(T-s)]^*p \right) - \Phi_{D(T-s)} \left([-e^{-M(T-s)}N_D(T-s)]^*p \right) \right\} ds - \langle e^{-M(T-t)}x, p \rangle \right\} \end{aligned}$$

If we write $z = e^{-M(T-t)}x$ and write $J(z) = g(e^{MT}z)$, then this change of coordinate in the state variable by e^{MT} gives rise to a symplectic change of coordinate in the phase variables by $\text{diag}(e^{MT}, e^{-M^*T})$, hence

$$J^*(p) = g^*(e^{-M^*T}p),$$

and therefore we get to

$$\begin{aligned} & \varphi(x(z), t) \\ = & - \min_{p \in \mathbb{R}^d} \left\{ J^*(p) + \int_0^t \left\{ \Phi_{C(T-s)} \left([-e^{-M(T-s)}N_C(T-s)]^*p \right) - \Phi_{D(T-s)} \left([-e^{-M(T-s)}N_D(T-s)]^*p \right) \right\} ds - \langle z, p \rangle \right\} \end{aligned}$$

which is the same formula as in (2.5) in [6].

4 Numerical methods

4.1 Optimization Methods: Coordinate Descent

In order for computation of optimization (in either the Lax formulation or the Hopf formulation) to be efficient, we have recasted the initial value HJ PDE problem to minimization problem in d dimensions, where the curves γ and p inside the function evaluation (given a vector v) are defined explicitly as the solutions to the bicharacteristic equation. We suggest to solve the ODE's numerically, given a pair of (x, v) , up to time t using any ODE solver. This way the minimization/maximization problem reduces to optimization of a finite-dimensional problem (as a function of v .) Similar to [6], we suggest to apply coordinate descent to the following functionals with argument v for a given pair (x, t) :

$$\mathcal{F}_{x,t}^1(v) := g(\gamma(v, 0)) + \int_0^t \{ \langle p(v, s), \partial_p H(\gamma(v, s), p(v, s), s) \rangle - H(\gamma(v, s), p(v, s), s) \} ds$$

or (noticed we omitted r in this functional, but r is actually the final time of the ODE (γ, p) stated below)

$$\mathcal{F}_{x,t}^2(v) := \min_{0 \leq r \leq t} \{ g(\gamma(v, 0)) \}$$

or

$$\mathcal{G}_{x,t}(v) := g^*(p(v, 0)) + \int_0^t \left\{ H(\gamma(v, s), p(v, s), s) - \langle \partial_x H(\gamma(v, s), p(v, s), s), \gamma(v, s) \rangle \right\} ds - \langle x, v \rangle$$

where in either case, the pair $\gamma(v, s), p(v, s)$ solves the following final value problem for the given pair of x, t (that we omitted):

$$\begin{cases} \dot{\gamma}(v, s) = \partial_p H(\gamma(v, s), p(v, s), s), \\ \dot{p}(v, s) = -\partial_x H(\gamma(v, s), p(v, s), s), \\ \gamma(v, t) = x, \\ p(v, t) = v \end{cases}$$

Now, to minimize $\mathcal{F}_{x,t}^1(\cdot)$, $\mathcal{F}_{x,t}^2(\cdot)$ or $\mathcal{G}_{x,t}(\cdot)$, we utilize a cyclic coordinate descent algorithm. We illustrate our algorithm with the functional $\mathcal{G}_{x,t}(\cdot)$:

Algorithm 1. Take an initial guess of the Lipschitz constant L , and set $\text{count} := 0$. Initialize $j_1 := 1$ and a parameter $\alpha := 1/L$. For $k = 1, \dots, M$, do:

1:

$$\begin{cases} v_i^{k+1} = v_i^k - \alpha \partial_i \mathcal{G}_{x,t}(v^k) & \text{if } i = j_k, \\ v_i^{k+1} = v_i^k & \text{otherwise.} \end{cases}$$

2:

$$j_{k+1} := j_k + 1.$$

If $j_{k+1} = d + 1$, then reset $j_{k+1} = 1$.

3: If $|v_i^{k+1} - v_i^k| > \varepsilon$, then set $\text{count} := 0$. If $k = M$, then reset $k := 0$ and set $\alpha := \alpha/2$, (i.e. let $L := 2L$.)

4: If $|v^{k+1} - v^k| < \varepsilon$, set $\text{count} := \text{count} + 1$.

5: If $\text{count} = d$, stop.

Return $v_{\text{final}} = v^{k+1}$.

We minimize $\mathcal{F}_{x,t}^1(\cdot)$, $\mathcal{F}_{x,t}^2(\cdot)$ in a similar fashion. In this algorithm, we will need to discuss how to evaluate the functional values and also their numerical derivatives. This will be discussed in the next subsection.

4.2 Evaluation of functional and its derivatives: ODE solver, numerical differentiation and integration

In this subsection, we discuss several numerical approximation used in our numerical experiments. First, in our step, we need to devise by ODE solvers. We suggest to use the standard forward Euler solver for a given stepsize Δs . Of course the performance can be improved by using more advanced solvers, such as the pseudo-spectral method, etc. We also need to deal with approximating integrals. Similar to [6], we suggest to evaluate either the derivatives of $\mathcal{F}_{x,t}^1$ or $\mathcal{G}_{x,t}$ by numerical quadrature rules for integral computations. As for $\mathcal{F}_{x,t}^2$, we compute the maximum by directly choosing the maximum around the computed grid from the ODE solver in the case when $H(x, p, t) = H(x, p)$, since the ODE will be the same in this case and the ODE solver will get to the function values of $\gamma(v, s)$ easily.

$$\mathcal{F}_{x,t}^1(v) \approx \min_{r_i = i\Delta s, i=0, \dots, t/\Delta s} \{g(\gamma(v, 0))\} \quad (4.1)$$

with a same choice of Δs as the ODE solver. We also suggest, as in [6], a numerical differentiation rule for derivative computations. We integrate using a standard rectangular quadrature rule (we use $\mathcal{F}_{x,t}^1$ to illustrate):

$$\mathcal{F}_{x,t}^1(v) \approx g(\gamma(v, 0)) + \sum_i \{ \langle p(v, i\Delta s), \partial_p H(\gamma(v, i\Delta s), p(v, i\Delta s), i\Delta s) \rangle - H(\gamma(v, i\Delta s), p(v, i\Delta s), i\Delta s) \} \Delta s \quad (4.2)$$

again with the same choice of Δs as in the ODE solver. We suggest approximating the partial derivative $\partial_i \mathcal{F}_{x,t}(p)$ (which means the differentiation of the function w.r.t. the direction e_i) by a finite difference:

$$\partial_i \mathcal{F}_{x,t}(v) \approx \frac{\mathcal{F}_{x,t}^1(v + \sigma e_i) - \mathcal{F}_{x,t}^1(v)}{\sigma} \quad (4.3)$$

with a given choice of σ . By using numerical differentiation, we have the advantage of not necessarily handling tedious analytic computations of the derivative of Hamiltonian which might be singular at times. Also, we only have two evaluations of the function value per iteration. By performing numerical approximations, either ODE solvers, differentiation or integration, we are bound to introduce numerical errors. These errors introduced by numerical approximation can be effectively controlled by choosing appropriately small sizes of Δs and σ .

5 Numerical Results

In this section, we provide numerical experiments which compute viscosity solutions to HJ PDE with a time-dependent Hamiltonian arising from control system. For a given set of points (t, z) , we use **Algorithm 1** to compute (2.1)-(2.2). We set $M = 500$ and have a different initial guess of the Lipschitz constant L in each example. We evaluate (x, t) in a given set of grid points over some 2 dimensional cross-sections of the form $[-3, 3]^2 \times \{0\}^{d-2}$. We choose our error tolerance in the coordinate descent iteration as $\varepsilon = 0.5 \times 10^{-7}$, which acts as our stopping criterion. The step-size in the numerical quadrature rule in (4.1), (4.2) as well as the forward Euler ODE solver is set to be different in each example, and they are all denoted as Δs . The step-size for numerical differentiation in (4.3) denoted by σ , and the Lipschitz constants L in **Algorithm 1** are also chosen differently in each in each example. In all our examples, we set random initial starting points uniformly distributed in $[-2, 2]^d$. We always consider the initial value to be a function with zero level set as an ellipse enclosed by the equation $\langle x, Ax \rangle = 1$ where $A = \text{diag}(2.5, 1, 0.5, 0.5, 0.5, \dots, 0.5)$, i.e. our initial condition for the HJ PDE is $g(x) = \frac{1}{2}(\langle x, Ax \rangle - 1)$. For a convex Hamiltonian, we make one initial guess, and for a non-convex Hamiltonian, we perform at most 20 independent trials of random initial guesses to get rid of possible local minima or in places when the derivative of the viscosity solution does not exist. We present our run-time in the format of $a \times 10^{-c} \text{ s} \times k$ where k is the number of initial guesses made in the respective example. When $d = 2$, clear comparison is performed with our solution to the solution computed by a first order Lax-Friedrichs monotone scheme [24] with $\Delta t = 0.001$ and $\Delta x = 0.005$. Our algorithm is implemented in C++ on an 1.7 GHz Intel Core i7-4650U CPU. In what follows, we present some examples.

Example 1 We solve for the state-dependent Hamiltonian of the linear form

$$H(x, p, t) = 0.2 c(x) - \nabla c(x) \cdot p,$$

where

$$c(x) = 1 + 3 \exp(-4|x - (1, 1, 0, 0, \dots, 0)|_2^2).$$

The minimization principle (3.1) is used to compute the solution φ . The temporal stepsize is chosen as $\Delta s = 0.02$. In this example, since the Hamiltonian is linear, there exists only a **unique** v such that the constraints of (γ, p) in the bi-characteristics are satisfied, and therefore there is nothing to optimize, thus no choices of σ and L are necessary. Figure 1 (left) gives the solutions when $d = 2$, $T = 0.12$. The runtime

using C++ is $9.929 \times 10^{-7} s \times 1$ per point. Figure 1 (right) is the solution computed by the Lax-Friedrichs scheme, which illustrates that the two solutions coincide. Figure 2 gives the solutions when $d = 1024$, $T = 0.5$. The runtime using C++ is $4.342 \times 10^{-2} s \times 1$ per point. The computation time is reasonably fast for a 1024 dimensional problem (although in this toy example, there is no optimization to solve).

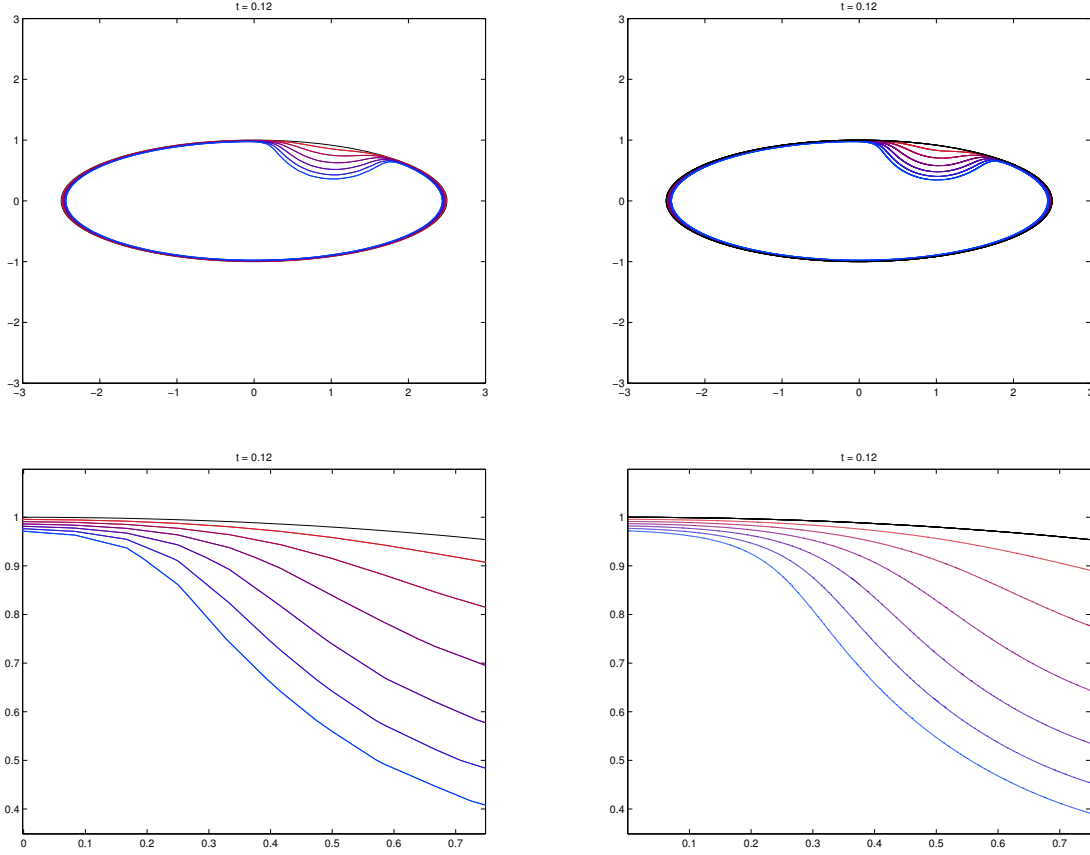
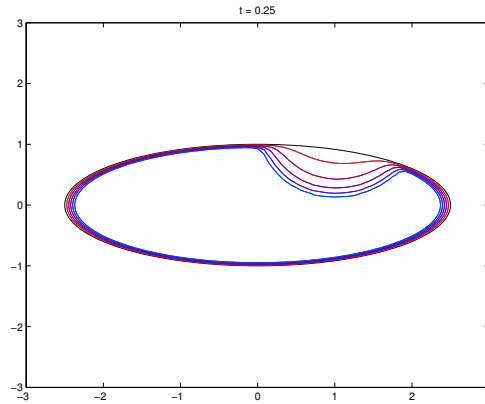


Figure 1: Level sets of the solution in **Example 1** with $d = 2$; left: minimization/maximization principle, right: Lax-Friedrichs.



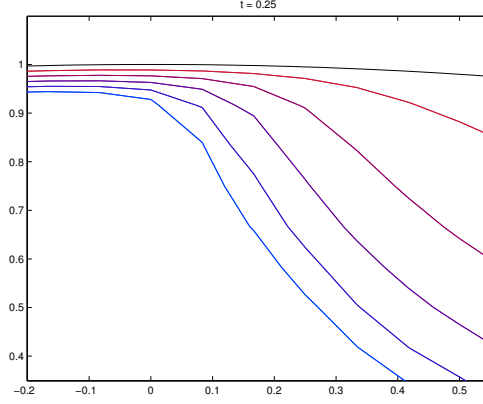


Figure 2: Level sets of the solution by minimization/maximization principle in **Example 1** with $d = 1024$; top: full-size; bottom: close-up.

Example 2 Next we solve for the state-dependent Hamiltonian, the well-known harmonic oscillator:

$$H^\pm(x, p, t) = \pm \frac{1}{2}(|p|_2^2 + |x|_2^2),$$

The maximization principle (3.4) is used to compute the solution φ for both of the \pm cases. The temporal stepsize is chosen as $\Delta s = 0.02$. The other constants are chosen as follows: stepsize $\sigma = 0.001$ and $L = 3$. Figure 3 (left) gives the solutions when $d = 2$ and sign of H is negative, i.e. H^- , $T = 0.5$. The runtime using C++ is $2.366 \times 10^{-5} s \times 5$ per point. Figure 3 (right) is the solution computed by the Lax-Friedrichs scheme for comparison. Figure 4 (left) gives the solutions when $d = 2$ and sign of H is positive, i.e. H^+ , $T = 0.5$. The runtime using C++ is $2.605 \times 10^{-5} s \times 5$ per point. Figure 4 (right) is the solution computed by the Lax-Friedrichs scheme for comparison. Figure 5 gives the solutions when $d = 7$ and sign of H is negative, i.e. H^- , $T = 0.2$. The runtime using C++ is $3.717 \times 10^{-4} s \times 5$ per point. The computation time is minimal for a 7 dimensional problem.

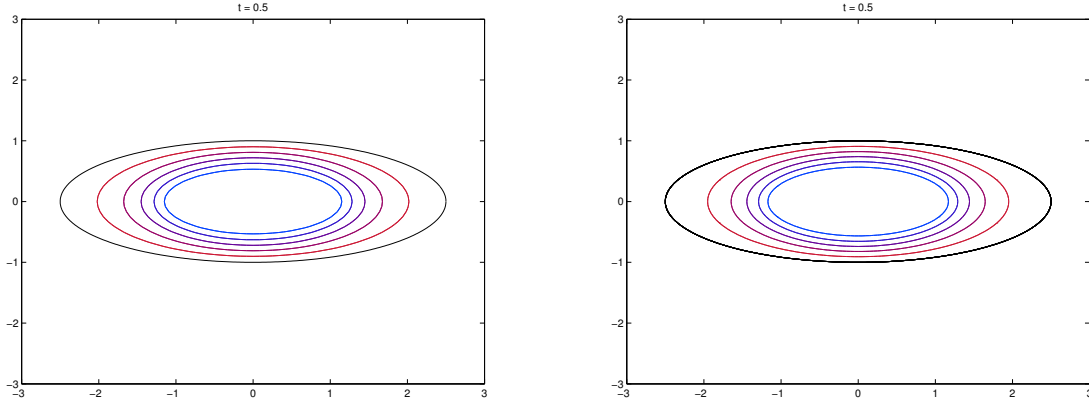


Figure 3: Level sets of the solution by minimization/maximization principle in **Example 2** with H^- and $d = 2$; top: full-size; bottom: close-up.

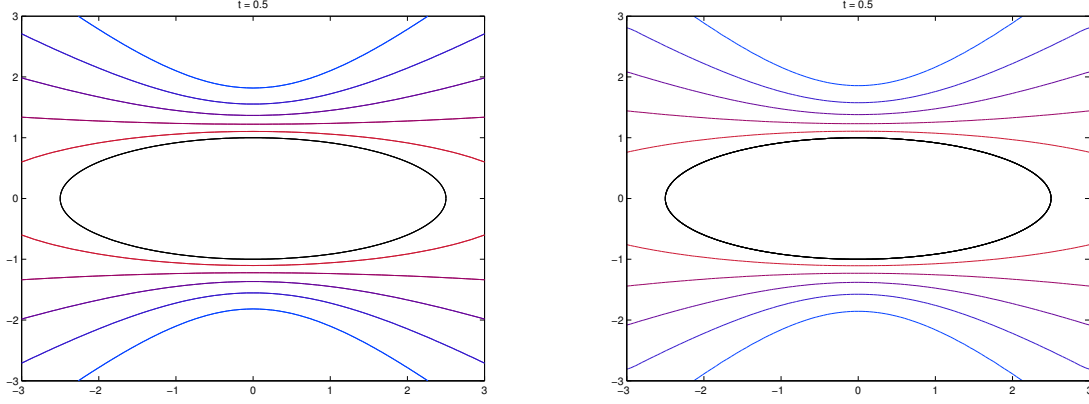


Figure 4: Level sets of the solution by minimization/maximization principle in **Example 2** with H^+ and $d = 2$; top: full-size; bottom: close-up.

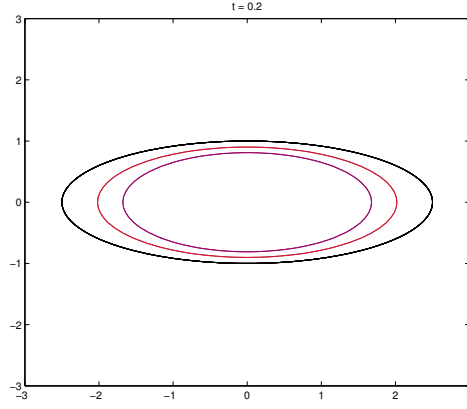


Figure 5: Level sets of the solution by minimization/maximization principle in **Example 2** with H^- and $d = 7$.

Example 3 We solve for the state-dependent Hamiltonian of the form

$$H^\pm(x, p, t) = \pm c(x)|p|_2,$$

where

$$c(x) = 1 + 3 \exp(-4|x - (1, 1, 0, 0, \dots, 0)|_2^2).$$

The minimization principle (3.1) is used to compute the solution φ for both of the \pm cases. The temporal stepsize is chosen as $\Delta s = 0.02$. The other constants are chosen as follows: stepsize $\sigma = 0.001$ and $L = 0.02$. Figure 6 gives the solutions when $d = 2$ and sign of H is positive, i.e. H^+ , $T = 0.3$. The runtime using C++ is $6.865 \times 10^{-4}s \times 5$ per point. For comparison, Figure 6 (right) is the solution computed by Lax-Friedrichs scheme. Figure 7 gives the solutions when $d = 2$ and sign of H is negative, i.e. H^- , $T = 0.5$. The runtime using C++ is $1.417 \times 10^{-3}s \times 5$ per point. Figure 7 (right) is the solution computed by Lax-Friedrichs scheme for comparison. The angles where a discontinuity of derivative is present are computed more accurately using our maximization principle. Figure 8 gives the solutions when $d = 10$ and sign of H is negative, i.e. H^- , $T = 0.4$. The runtime using C++ is $2.470 \times 10^{-2}s \times 5$ per point. The computation time is still excellent for a 10 dimensional problem.

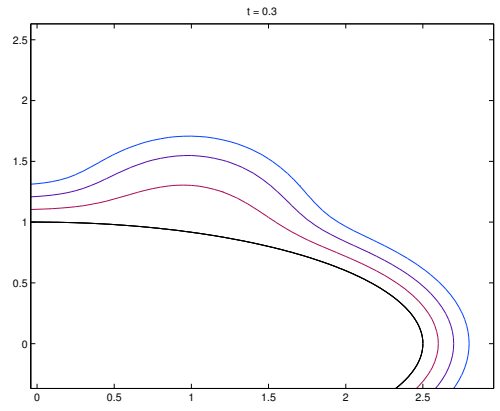
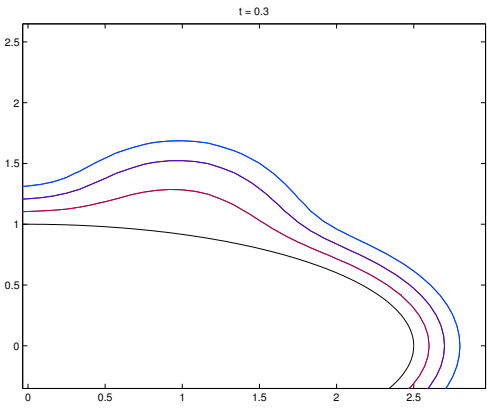
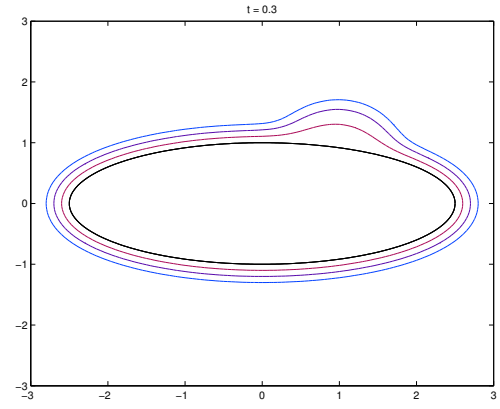
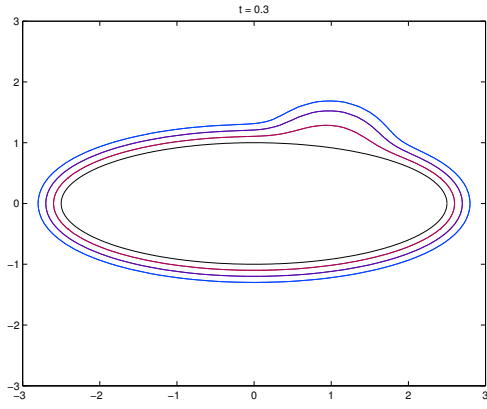
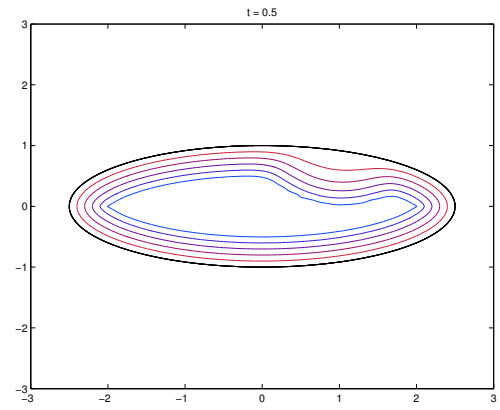
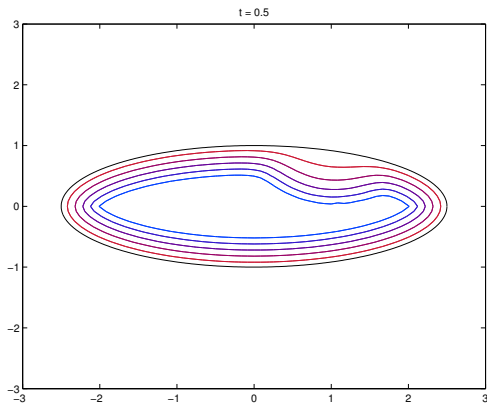


Figure 6: Level sets of the solution in **Example 3** with H^+ and $d = 2$; left: minimization/maximization principle, right: Lax-Friedrichs.



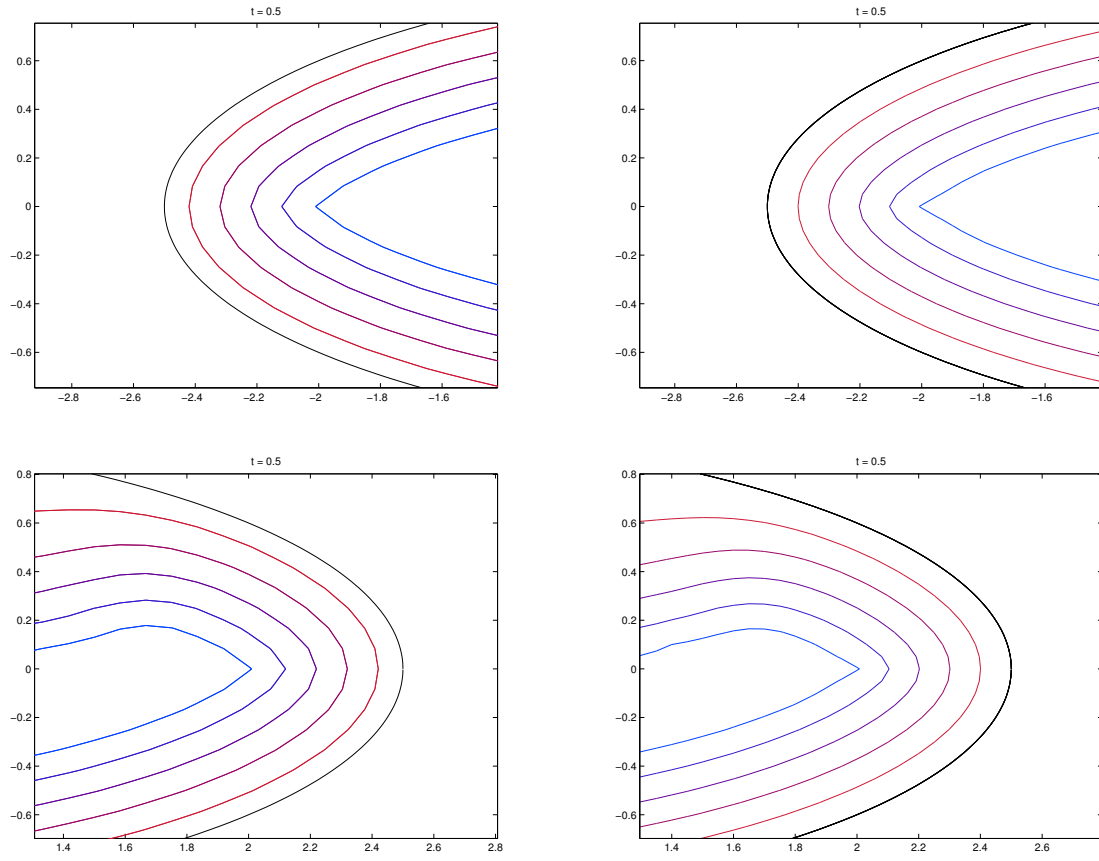
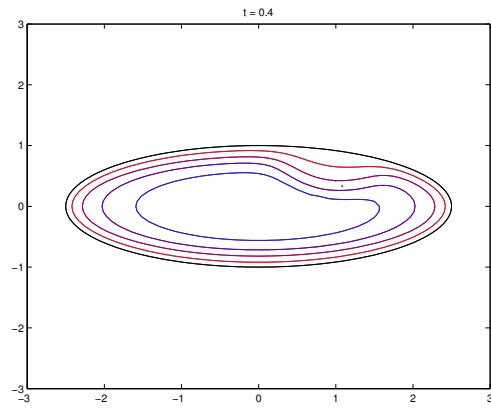


Figure 7: Level sets of the solution in **Example 3** with H^- and $d = 2$; left: minimization/maximization principle, right: Lax-Friedrichs.



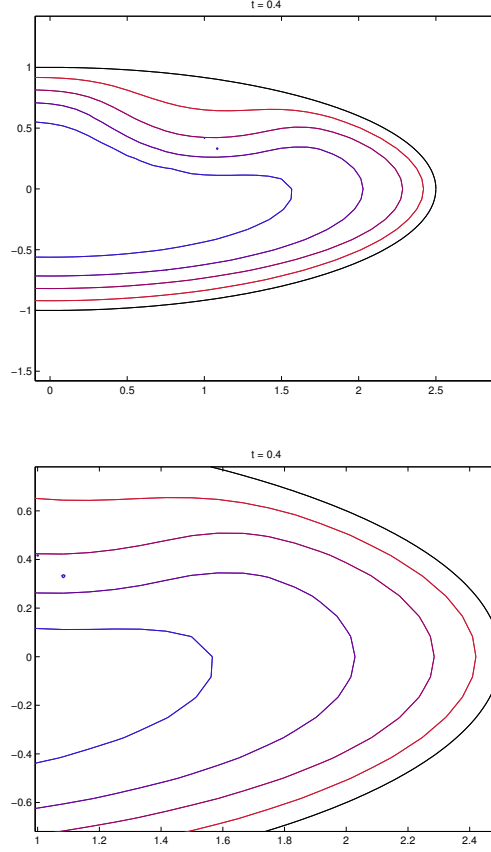


Figure 8: Level sets of the solution by minimization/maximization principle in **Example 3** with H^- and $d = 10$; top: full-size; middle and bottom: close-up.

Example 4 To test our maximization principle for a general state-dependent non-convex Hamilton-Jacobi equation, we use a state-dependent non-convex Hamiltonian of the following form given in [12] but a different problem from [12]:

$$H(x, p, t) = -c(x)p_1 + 2|p_2| - \sqrt{|p_1|^2 + |p_2|^2} - 1,$$

where we write $p = (p_1, p_2)$ and

$$c(x) = 2 \left(1 + 3 \exp(-4|x - (1, 1)|_2^2) \right).$$

The maximization principle (3.4) is used to compute the solution φ . The temporal stepsize is chosen as $\Delta s = 0.005$. The other constants are chosen as follows: stepsize $\sigma = 0.001$ and $L = 4$. Figure 9 gives the solutions with $T = 0.1$. The runtime using C++ is $7.279 \times 10^{-2} s \times 5$ per point. Figure 9 (right) is the solution computed by the Lax-Friedrichs scheme for comparison. This example is to illustrate that the maximization principle coincide with the Lax-Friedrichs solution in a quite general case.

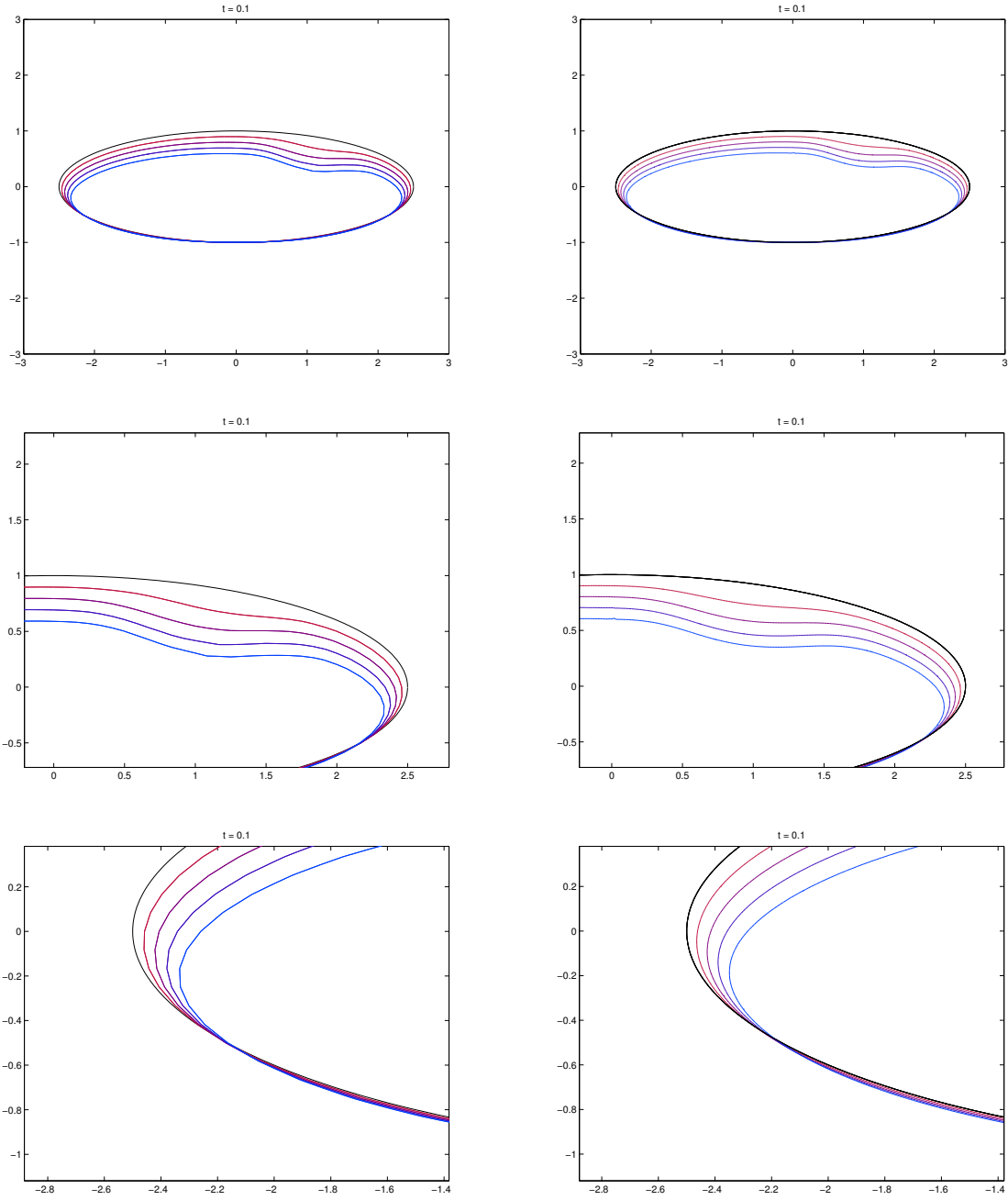


Figure 9: Level sets of the solution in **Example 4** with $d = 2$; left: minimization/maximization principle, right: Lax-Friedrichs.

Example 5 We solve for the state-dependent non-convex Hamiltonian of the form

$$H(x, p, t) = c_1(x)|p_1, \dots, p_k|^2 - c_2(x)|p_{k+1}, \dots, p_d|^2,$$

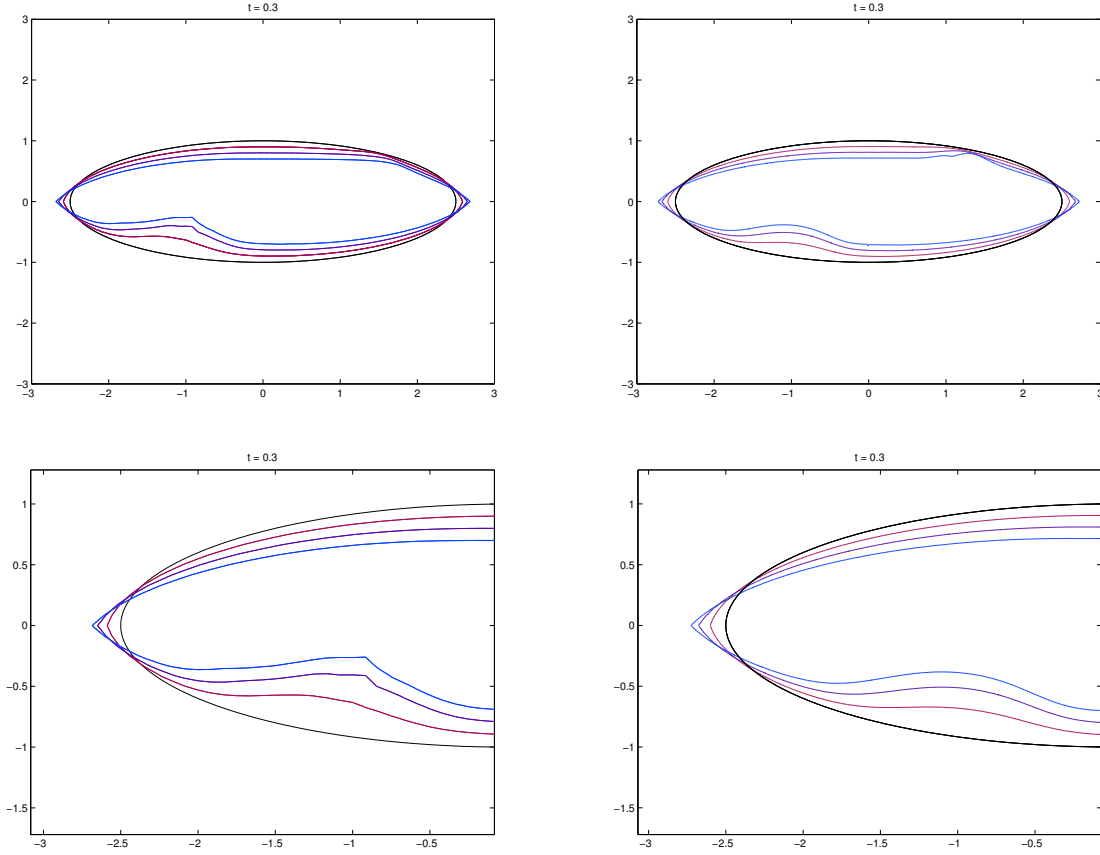
where

$$c_1(x) = c(x), \quad c_2(x) = c(-x), \quad c(x) = 2 \left(1 + 3 \exp(-4|x - (1, 1)|_2^2) \right).$$

In this case The maximization principle (3.4) is used to compute the solution φ . The constants are chosen as follows: the temporal stepsize $\Delta s = 0.02$, stepsize in numerical differentiation $\sigma = 0.001$ and $L = 50$.

Figure 10 gives the solutions when $d = 2$ and $k = 1$, $T = 0.3$. The runtime using C++ is $9.094 \times 10^{-2} s \times 20$ per point. To compare, Figure 10 (right) is the solution computed by Lax-Friedrichs scheme for comparison. Now there is a small defect in the solution computed by the maximization principle at one point of the wave-front close to $(-1, 0.4)$, possibly either because of the high non-convexity of the functional around that point, and more random initial guesses are needed, or that the time variable t is so large that it has reached to a point where the Hopf formula breaks down around $(-1, 0.4)$.

Figure 11 gives the solutions when $d = 7$ and $k = 1$, $T = 0.3$. The runtime using C++ is $5.428 \times 10^{-1} s \times 20$ per point. The computation time is still acceptable for a 7 dimensional problem which is fully non-convex and state-dependent.



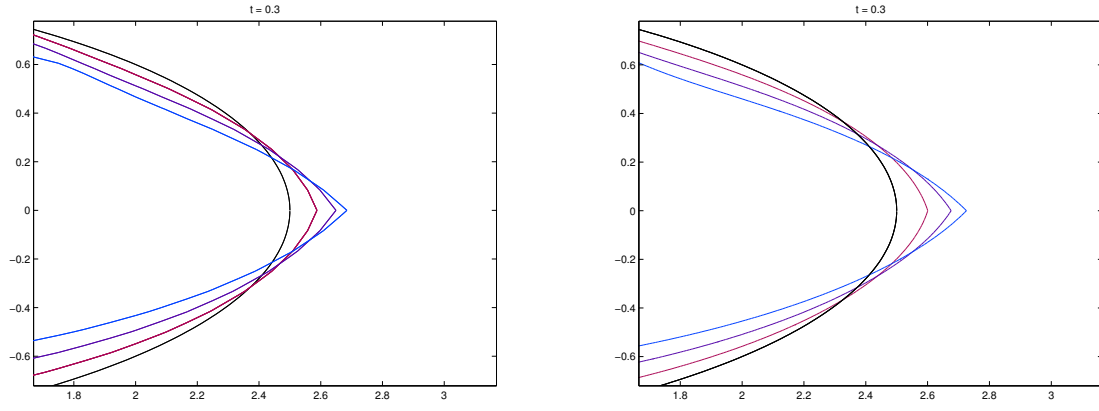
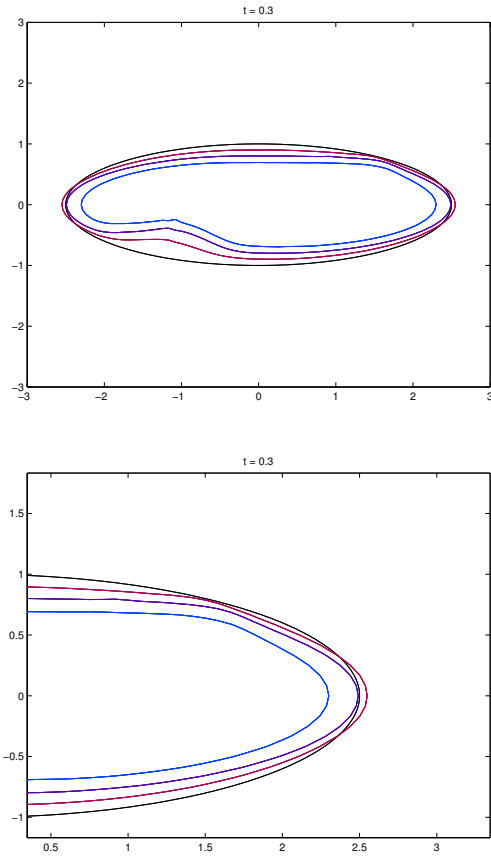


Figure 10: Level sets of the solution; left: minimization/maximization principle in **Example 5** with $k = 1$ and $d = 2$, right: Lax-Friedrichs.



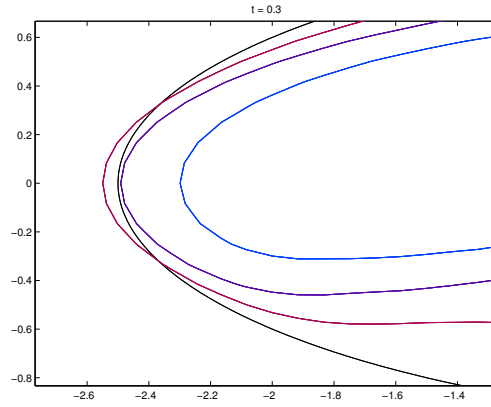


Figure 11: Level sets of the solution by minimization/maximization principle in **Example 5** with $k = 1$ and $d = 7$; top: full-size; middle and bottom: close-up.

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