Sparse-data Based 3D Surface Reconstruction for Cartoon and Map

Bin Wu, Talal Rahman and Xue-Cheng Tai

Abstract A model combining the first-order and the second-order variational regularizations for the purpose of 3D surface reconstruction based on 2D sparse data is proposed. The model includes a hybrid fidelity constraint which allows the initial conditions to be switched flexibly between vectors and elevations. A numerical algorithm based on the augmented Lagrangian method is also proposed. The numerical expriments are presented, showing its excellent performance both in designing cartoon characters, as well as in recovering oriented mountain surfaces.

1 Introduction

Image processing has a strong influence and impact on our world, finding applications in almost all areas from nanophysics to astrophysics, from biology to social sciences, from robotics to smart phone applications, etc. 3D surface reconstruction from sparse data is both a challenging and an interesting image processing task.

One area of application of the surface reconstruction has been the sketch based 3D design, which has attracted much attention, cf. [1, 2, 3, 4, 5], because it is intuitive and effective, particularly in applications like cartoon and game design. To a

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sketch based method, the only known informations are information given on sparse lines, for instance in the form of contours [2], without specifying the heights, or in the form of complex sketches with elevation [3], or structured annotations [6]. However, the methods proposed in those papers are limited in their capabilities in reconstructing structures with crease. The crease can be added artificially [7]. However, a simple and automatic method is still necessay when the task becomes large, complex and computationally intensive. Recently, to this end, there is a method been proposed [8] which interpolates the normal vectors under curl-free constraint and then reconstructs the 3D surface based on the obtained vector field. The method [8] is based on the previous work on surface reconstruction from surface gradients [9, 10, 11, 12, 13, 14, 15, 16] and inspired by TV-Stokes method [17, 18, 19, 20] where actually the curl-free constraint comes from. The main difference of this method [8] compared to the other two-step methods [14, 21] is that, instead of the Lapace operator, an TV regularizer is employed which is better in edge preserving. In addition, a more physical constraint, the curl-free constraint is introduced by the method. The numerical results show an excellent performance in preserving edges and crease structures.

Another area of application has been the 3D surface reconstruction based only on height values (contours) or both height values and vectors. The height values are needed because the reconstructed surfaces for such applications are expected to be as precise as possible to the ground truth, e.g. the digital elevation maps and data compression. One way is to use explicit parameterization of given contours with subsequent pointwise matching and interpolation [22, 23, 24]. For such models, the parametrization may be difficult and expensive to compute, and the loss of continuity of slope across contours is a challenge. Another way is to treat the expected surface as a function over the considered domain. A renowned model is the absolutely minimizing Lipschitz extension (AMLE) interpolation model, see [25, 26], based on the PDE theory. The AMLE has a drawback in interpolating slopes. To overcome, one can rely on high-order differential operators or regularizations [27, 28, 29, 30, 31]. The method addressed in [31] introduces a third order anisotropic regularization together with a way to find an auxiliary vector field. The method results in clear surfaces with anisotropic features.

It is however desirable to recover the 3D surface with enough precision at the same time to be able to adjust the shape of the reconstructed surface by tuning vectors. For instance in case of data compression, it may be helpful to store vectors (relative positions) along with sparse elevations instead of single the sparse elevations for correct representations. The aim of this paper is to propose a versatile model incorporating both height and vector information in one place. We thus propose a one-step model with a combination of first-order and second-order variational regularizations under a hybrid fidelity constraint consisting of both elevation and normal vectors. The main contributions of our research can be summarized as follows:

- The model allows for adjusting normal vectors intuitively and a more precise representation of the elevations. It preserves both structures and details.

- A fast and efficient numerical algorithm based on the augmented Lagrangian method [32, 33, 34] is proposed which can be used for 3D surface reconstruction of cartoon and digital map based on very sparse 2D input data.

The paper is organized as follows. In section 2, we propose our model with a second-order regularization and a hybrid constraint. In section 3, we present numerical method based on augmented Lagrangian. Numerical experiments on cartoon design and mountain surface reconstruction are presented in section 4. Finally, in section 5, we give our conclusion.

2 Proposed Model

We first explain the model presented in [8] before we propose ours. We define the surface as the graph of I given by the points $(x,y,I(x,y)) \subset \mathbb{R}^3$ in the space, where I is a function of the coordinates x and y over a two dimensional domain $\Omega \subset \mathbb{R}^2$. The normal vector to the surface or the graph is then given by $(-\partial_x I, -\partial_y I, 1)$. Projecting it to the $xy-$plane, we get the 2D normal vectors as $(-\partial_x I, -\partial_y I)$. Because I is a scalar-valued function, the curl of the gradient of I must be zero. Based on this, a curl-free model has been proposed in [8]. They first interpolated the normal vector $\mathbf{n} := (\partial_x I, \partial_y I)$ by solving the following constrained minimization problem

$$\min_{\mathbf{n}} \left\{ \int_\Omega (1-g)|\nabla \mathbf{n}|_F + g|\nabla \mathbf{n}|_F^2 + \eta \int_\Gamma |\mathbf{n} - \mathbf{n}^*| \right\}, \tag{1}$$

subject to the curl free condition

$$\nabla \times \mathbf{n} = 0,$$

where $\mathbf{n}^*$ is the known normal vector along some given sparse lines or strokes Γ, g is the parameter for a convex combination of the TV and the H^1 norm, and η is the parameter to balance between the regularization terms and the fidelity term. We note that $|\cdot|_F$ is used to denote the standard Frobenius norm [36]. The height map I is then reconstructed by solving the following minimization problem

$$\min_I \left\{ \int_\Omega (1-h)|\nabla I - \mathbf{n}| + h|\nabla I - \mathbf{n}|^2 + \xi \int_\Sigma |I - I_0| \right\}, \tag{2}$$

where $\mathbf{n}$ is the normal vector field obtained from the first minimization step, I_0 is the known elevation along some given sparse lines or strokes Σ, h is the parameter for a convex combination of TV and H^1 norms, ξ is the parameter to balance between the regularization terms and the fidelity term.

It is obvious that reconstructing a 3D surface would require both constraints, the one on the normal vector $\mathbf{n}$ and the one on the height I, corresponding to the fidelity terms of (1) and (2). However, since the model above is not coupled, it is hard to satisfy both constraints simultaneously, and therefore the resulting surface may deviate from the surface actually being sought.

We therefore propose the following one-step model including both the height and the normal vector constraint, that is the hybrid constraint. We note here that because our model is second order it naturally satisfies the curl free condition.

$$\min_I \left\{ \int_\Omega g|\nabla(\nabla I)|_F + h|\nabla I| + \int_\Gamma \eta|\nabla I - \mathbf{n}^0| + \int_\Sigma \theta|I - I^0| \right\}, \tag{3}$$

where h and g are weight parameters for the first and the second variational regularizations,respectively.

3 Augmented Lagrangian Method

For the numerical solution of the problem (4), we propose an augmented Lagrangian method, cf. [32]. Augmented Lagrangian is preferred because it is fast and efficient; for its use in image processing, we refer to e.g. [33, 34].

In order to be able to define our entire minimization problem over the whole domain, we replace the two fidelity parameters with the following parameters,

$$\hat{\eta} = \begin{cases} \eta, & \text{on } \Gamma \\ 0, & \text{in } \Omega \backslash \Gamma, \end{cases} \qquad \text{and} \quad \hat{\theta} = \begin{cases} \theta, & \text{on } \Sigma \\ 0, & \text{in } \Omega \backslash \Sigma. \end{cases}$$

We get our model (3) reformulated as follows,

$$\min_I \left\{ \int_\Omega g|\nabla(\nabla I)|_F + h|\nabla I| + \hat{\eta}|\nabla I - \mathbf{n}^0| + \hat{\theta}|I - I^0| \right\}. \tag{4}$$

For the four L_1-norm terms in the above functional, introducing one new variable to each, we get four new variables $\mathbf{Q} := \nabla \mathbf{E}$, $\mathbf{P} := \nabla I$, $\mathbf{C} := \mathbf{P}$, and $S := I$, corresponding to $|\nabla(\nabla I)|_F$, $|\nabla I|$, $|\nabla I - \mathbf{n}^0|$, and $|I - I^0|$, respectively. In addition, for the term $|\nabla(\nabla I)|_F$, we introduce another variable $\mathbf{E} := \mathbf{P}$ in order to avoid dealing with high order terms. The unconstrained minimization problem (4) is then reformulated as a constrained optimization problem

$$\min_{\mathbf{Q},\mathbf{P},\mathbf{C},S} \left\{ \int_\Omega g|\mathbf{Q}|_F + h|\mathbf{P}| + \hat{\eta}|\mathbf{C} - \mathbf{n}^0| + \hat{\theta}|S - I^0| \right\}$$

such that

$$\mathbf{P} - \nabla I = 0; \;\; \mathbf{E} - \mathbf{P} = 0; \;\; \mathbf{Q} - \nabla \mathbf{E} = 0; \;\; S - I = 0; \;\; \text{and} \;\; \mathbf{C} - \mathbf{P} = 0,$$

where $\mathbf{C}, \mathbf{E}, \mathbf{P} \in \mathbb{R}^2$ are 2-dimensional vectors, and $\mathbf{Q} \in \mathbb{R}^{2\times 2}$ is a 2-by-2 matrix. Using Lagrange multipliers and adding penalty terms for each condition, we get the following augmented Lagrangian functional

$$
\begin{aligned}
\mathscr{L}(\mathbf{Q},\mathbf{P},\mathbf{C},S,I,\mathbf{E};\Lambda_Q,\Lambda_P,\Lambda_C,\lambda_S,\Lambda_E) = \int_\Omega & g|\mathbf{Q}|_F + h|\mathbf{P}| + \hat{\eta}|\mathbf{C}-\mathbf{n}^0| + \hat{\theta}|S-I^0| \\
&+\Lambda_Q\cdot(\mathbf{Q}-\nabla\mathbf{E}) + \frac{c_Q}{2}|\mathbf{Q}-\nabla\mathbf{E}|_F^2 \\
&+\Lambda_P\cdot(\mathbf{P}-\nabla I) + \frac{c_P}{2}|\mathbf{P}-\nabla I|^2 \\
&+\Lambda_C\cdot(\mathbf{C}-\mathbf{P}) + \frac{c_C}{2}|\mathbf{C}-\mathbf{P}|^2 \\
&+\lambda_S\cdot(S-I) + \frac{c_S}{2}|S-I|^2 \\
&+\Lambda_E\cdot(\mathbf{E}-\mathbf{P}) + \frac{c_E}{2}|\mathbf{E}-\mathbf{P}|^2,
\end{aligned}
$$

where Λ_Q, Λ_P, Λ_C, λ_S and Λ_E are Lagrange multipliers, c_Q, c_P, c_C, c_S and c_E are positive penalty parameters. That is, the augmented Lagrangian method is to seek a saddle point of the following problem:

$$
\min_{\mathbf{Q},\mathbf{P},\mathbf{C},S,I,\mathbf{E}} \max_{\Lambda_Q,\Lambda_P,\Lambda_C,\lambda_S,\Lambda_E} \mathscr{L}(\mathbf{Q},\mathbf{P},\mathbf{C},S,I,\mathbf{E};\Lambda_Q,\Lambda_P,\Lambda_C,\lambda_S,\Lambda_E). \tag{5}
$$

For the solution we solve its associated system of optimality conditions with an iterative procedure, see Algorithm 1 and Algorithm 2. For the sake of convenience, we use $\Lambda := (\lambda_S,\Lambda_P,\Lambda_C,\Lambda_Q,\Lambda_E)$ to denote the Lagrange multipliers.

Algorithm 1: The augmented Lagrangian for (5)

Set $k=0$;
Initialize $\mathbf{Q}^0$, $\mathbf{P}^0$, $\mathbf{C}^0$, S^0, I^0, $\mathbf{E}^0$ and Λ^0;
while *not converged* **do**
  Set $k=k+1$;
  Given Λ^{k-1}, solve the minimization problem:

$$
(\mathbf{Q}^k,\mathbf{P}^k,\mathbf{C}^k,S^k,I^k,\mathbf{E}^k) = \arg\min_{\mathbf{Q},\mathbf{P},\mathbf{C},S,I,\mathbf{E}} \mathscr{L}(\mathbf{Q},\mathbf{P},\mathbf{C},S,I,\mathbf{E};\Lambda^{k-1}); \tag{6}
$$

  Update the Lagrange multipliers:

$$
\begin{aligned}
&\lambda_S^k = \lambda_S^{k-1} + c_S(S^k - I^k); && \Lambda_P^k = \Lambda_P^{k-1} + c_P(\mathbf{P}^k - \nabla I^k);\\
&\Lambda_C^k = \Lambda_C^{k-1} + c_C(\mathbf{C}^k - \mathbf{P}^k); && \Lambda_Q^k = \Lambda_Q^{k-1} + c_Q(\mathbf{Q}^k - \nabla(\mathbf{E}^k));\\
&\Lambda_E^k = \Lambda_E^{k-1} + c_E(\mathbf{E}^k - \mathbf{P}^k);
\end{aligned}
$$

end

Because the variables $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{C}$, S, I and $\mathbf{E}$ in $\mathscr{L}(\mathbf{Q},\mathbf{P},\mathbf{C},S,I,\mathbf{E};\Lambda^{k-1})$ are coupled together in the minimization problem (6), it is difficult to solve them simultaneously.

We split the minimization problem into six sub minimization problems, and solve them alternatingly til convergence, see Algorithm 2.

Algorithm 2: Alternating minimization for (6).

Set $l = 0$;

Initialize $\mathbf{Q}^{k,0} = \mathbf{Q}^{k-1}$; $\mathbf{P}^{k,0} = \mathbf{P}^{k-1}$; $\mathbf{C}^{k,0} = \mathbf{C}^{k-1}$; $S^{k,0} = S^{k-1}$; $I^{k,0} = I^{k-1}$; $\mathbf{E}^{k,0} = \mathbf{E}^{k-1}$;

while *not converged and* $l < L$ **do**

Solve the sub-minimization problems:

$$\begin{aligned}
\mathbf{Q}^{k,l+1} &= \arg\min_{\mathbf{Q}} \mathscr{L}(\mathbf{Q}, \mathbf{P}^{k,l}, \mathbf{C}^{k,l}, S^{k,l}, I^{k,l}, \mathbf{E}^{k,l}; \Lambda^{k-1});\\
\mathbf{P}^{k,l+1} &= \arg\min_{\mathbf{P}} \mathscr{L}(\mathbf{Q}^{k,l+1}, \mathbf{P}, \mathbf{C}^{k,l}, S^{k,l}, I^{k,l}, \mathbf{E}^{k,l}; \Lambda^{k-1});\\
\mathbf{C}^{k,l+1} &= \arg\min_{\mathbf{C}} \mathscr{L}(\mathbf{Q}^{k,l+1}, \mathbf{P}^{k,l+1}, \mathbf{C}, S^{k,l}, I^{k,l}, \mathbf{E}^{k,l}; \Lambda^{k-1});\\
S^{k,l+1} &= \arg\min_{S} \mathscr{L}(\mathbf{Q}^{k,l+1}, \mathbf{P}^{k,l+1}, \mathbf{C}^{k,l+1}, S, I^{k,l}, \mathbf{E}^{k,l}; \Lambda^{k-1});\\
I^{k,l+1} &= \arg\min_{I} \mathscr{L}(\mathbf{Q}^{k,l+1}, \mathbf{P}^{k,l+1}, \mathbf{C}^{k,l+1}, S^{k,l+1}, I, \mathbf{E}^{k,l}; \Lambda^{k-1});\\
\mathbf{E}^{k,l+1} &= \arg\min_{\mathbf{E}} \mathscr{L}(\mathbf{Q}^{k,l+1}, \mathbf{P}^{k,l+1}, \mathbf{C}^{k,l+1}, S^{k,l+1}, I^{k,l+1}, \mathbf{E}; \Lambda^{k-1});
\end{aligned}$$

Set $l = l + 1$;

end

Set $(\mathbf{Q}^k, \mathbf{P}^k, \mathbf{C}^k, S^k, I^k, \mathbf{E}^k) = (\mathbf{Q}^{k,L}, \mathbf{P}^{k,L}, \mathbf{C}^{k,L}, S^{k,L}, I^{k,L}, \mathbf{E}^{k,L})$.

The six sub-minimization problems can be formulated as follows.

- The **Q**-subproblem:

$$\mathbf{Q}^\star = \arg\ \min_{\mathbf{Q}} \int_\Omega g|\mathbf{Q}|_F + \Lambda_Q \cdot \mathbf{Q} + \frac{c_Q}{2}|\mathbf{Q} - \nabla \mathbf{E}|_F^2. \tag{7}$$

- The **P**-subproblem:

$$\mathbf{P}^\star = \arg\ \min_{\mathbf{P}} \int_\Omega h|\mathbf{P}| + \tilde{\Lambda} \cdot \mathbf{P} + \frac{c_P}{2}|\mathbf{P} - \nabla I|^2 + \frac{c_E}{2}|\mathbf{E} - \mathbf{P}|^2 + \frac{c_C}{2}|\mathbf{C} - \mathbf{P}|^2, \tag{8}$$

  with $\tilde{\Lambda} := \Lambda_P - \Lambda_E - \Lambda_C$.
- The **C**-subproblem:

$$\mathbf{C}^\star = \arg\ \min_{\mathbf{C}} \int_\Omega \Lambda_C \cdot \mathbf{C} + \frac{c_C}{2}|\mathbf{C} - \mathbf{P}|^2 + \hat{\eta}|\mathbf{C} - \mathbf{n}^0|. \tag{9}$$

- The S-subproblem:

$$S^\star = \arg\min_S \int_\Omega \lambda_S \cdot S + \frac{c_S}{2}|S-I|^2 + \hat{\theta}|S-I^0|. \tag{10}$$

- The I-subproblem:

$$I^\star = \arg\min_I \int_\Omega -\Lambda_P \cdot \nabla I + \frac{c_P}{2}|\mathbf{P}-\nabla I|^2 - \lambda_S \cdot I + \frac{c_S}{2}|S-I|^2. \tag{11}$$

- The $\mathbf{E}$-subproblem:

$$\mathbf{E}^\star = \arg\min_{\mathbf{E}} \int_\Omega \Lambda_E \cdot \mathbf{E} + \frac{c_E}{2}|\mathbf{E}-\mathbf{P}|^2 - \Lambda_q \cdot \nabla \mathbf{E} + \frac{c_Q}{2}|\mathbf{Q}-\nabla \mathbf{E}|_F^2. \tag{12}$$

For the first four sub-minimization problems, we find closed form solutions. Each problem has one L_1-norm term, and either one or more than one quadratic terms in its objective functional. Such problems can be solved using either a subgradient method, cf. [35], or a geometric method, cf. [34]. We propose a simpler approach which is based on the optimality condition of the objective functional, that is the Euler-Lagrange equation, and the remark that follows, which also makes it easier to handle cases with more than one quadratic terms in the functional. For the last two sub-minimization problems, we solve them by the discrete cosine transform.

Remark 1. If A and B are two matrices such that $A = \lambda B$ for some non-negative number λ, in which case we say that A is compatible with B, then it is easy to see that $A/|A|_F = B/|B|_F$.

Solving the Q-subproblem (7): The corresponding optimality condition, that is the Euler-Lagrange equation, is as follows

$$\frac{g}{c_Q}\frac{\mathbf{Q}^\star}{|\mathbf{Q}^\star|_F} + \mathbf{Q}^\star = \nabla \mathbf{E} - \frac{\Lambda_Q}{c_Q}.$$

Since g and c_Q are both positive numbers, the matrices $\mathbf{Q}^\star$ and $(\nabla \mathbf{E} - \frac{\Lambda_Q}{c_Q})$ are both compatible in the sense of Remark 1, according to which, we can replace $\mathbf{Q}^\star/|\mathbf{Q}^\star|_F$ with $(\nabla \mathbf{E} - \frac{\Lambda_Q}{c_Q})/|\nabla \mathbf{E} - \frac{\Lambda_Q}{c_Q}|_F$. Now moving it to the right hand side, we get

$$\mathbf{Q}^\star = \left(1 - \frac{g}{c_Q|\nabla \mathbf{E} - \frac{\Lambda_Q}{c_Q}|_F}\right)\left(\nabla \mathbf{E} - \frac{\Lambda_Q}{c_Q}\right).$$

Again since $\mathbf{Q}^\star$ and $\nabla \mathbf{E} - \frac{\Lambda_Q}{c_Q}$ are compatible, the coefficient on the right hand side, that is $\left(1 - \frac{g}{c_Q|\nabla \mathbf{E} - \frac{\Lambda_Q}{c_Q}|_F}\right)$, must be non-negative, and hence

$$\mathbf{Q}^\star = \max\left\{0, 1 - \frac{g}{c_Q|\nabla\mathbf{E} - \frac{\Lambda_Q}{c_Q}|_F}\right\}\left(\nabla\mathbf{E} - \frac{\Lambda_Q}{c_Q}\right).$$

Solving the P-subproblem (8): The corresponding Euler-Lagrange equation is the following,

$$\frac{h}{c_P + c_E + c_C}\frac{\mathbf{P}^\star}{|\mathbf{P}^\star|} + \mathbf{P}^\star = \frac{c_P\nabla I + c_E\mathbf{E} + c_C\mathbf{C}}{c_P + c_E + c_C} - \frac{\tilde{\Lambda}}{c_P + c_E + c_C}.$$

Use $\mathbf{X}$ to denote $\frac{c_P\nabla I + c_E\mathbf{E} + c_C\mathbf{C}}{c_P + c_E + c_C} - \frac{\tilde{\Lambda}}{c_P + c_E + c_C}$. In the same way as before, since h, c_P, c_E and c_C are positive numbers, both vectors $\mathbf{P}^\star$ and $\mathbf{X}$ are compatible (cf. Remark 1). Accordingly, we replace $\mathbf{P}^\star/|\mathbf{P}^\star|$ with $\mathbf{X}/|\mathbf{X}|$, and move it to the right hand side, to get

$$\mathbf{P}^\star = \left(1 - \frac{h}{(c_P + c_E + c_C)|\mathbf{X}|}\right)\mathbf{X}.$$

Again since $\mathbf{P}^\star$ and $\mathbf{X}$ are compatible, the coefficient $\left(1 - \frac{h}{(c_P + c_E + c_C)|\mathbf{X}|}\right)$ must be non-negative. Hence

$$\mathbf{P}^\star = \max\left\{0, 1 - \frac{h}{(c_P + c_E + c_C)|\mathbf{X}|}\right\}\mathbf{X}.$$

Solving the C-subproblem (9): The corresponding Euler-Lagrange equation is the following,

$$\mathbf{C}^\star - \mathbf{n}^0 + \frac{\hat{\eta}}{c_C}\frac{\mathbf{C}^\star - \mathbf{n}^0}{|\mathbf{C}^\star - \mathbf{n}^0|} = \mathbf{P} - \mathbf{n}^0 - \frac{\Lambda_C}{c_C}.$$

Since $\hat{\eta}$ and c_C are both positive numbers, it follows that both vectors $\mathbf{C}^\star - \mathbf{n}^0$ and $\mathbf{P} - \mathbf{n}^0 - \frac{\Lambda_C}{c_C}$ are compatible (cf. Remark 1). Accordingly, we replace $(\mathbf{C}^\star - \mathbf{n}^0)/|\mathbf{C}^\star - \mathbf{n}^0|$ with $(\mathbf{P} - \mathbf{n}^0 - \frac{\Lambda_C}{c_C})/|\mathbf{P} - \mathbf{n}^0 - \frac{\Lambda_C}{c_C}|$, and move it to the right hand side, to obtain

$$\mathbf{C}^\star - \mathbf{n}^0 = \left(1 - \frac{\hat{\eta}}{c_C|\mathbf{P} - \mathbf{n}^0 - \frac{\Lambda_C}{c_C}|}\right)(\mathbf{P} - \mathbf{n}^0 - \frac{\Lambda_C}{c_C}).$$

Again since $\mathbf{C}^\star - \mathbf{n}^0$ and $\mathbf{P} - \mathbf{n}^0 - \frac{\Lambda_C}{c_C}$ are compatible, the coefficient $\left(1 - \frac{\hat{\eta}}{c_C|\mathbf{P} - \mathbf{n}^0 - \frac{\Lambda_C}{c_C}|}\right)$ must be non-negative. Hence

$$\mathbf{C}^\star = \mathbf{n}^0 + \max\left\{0, 1 - \frac{\hat{\eta}}{c_C|\mathbf{P} - \mathbf{n}^0 - \frac{\Lambda_C}{c_C}|}\right\}\left(\mathbf{P} - \mathbf{n}^0 - \frac{\Lambda_C}{c_C}\right).$$

Solving the S-subproblem (10): The Euler-Lagrange equation is the following,

$$S^\star - I^0 + \frac{\hat{\theta}}{c_S}\frac{S^\star - I^0}{|S^\star - I^0|} = I - I^0 - \frac{\lambda_S}{c_S}.$$

Again using the fact that $\hat{\theta}$ and c_S are both positive numbers, it follows that $S^\star - I^0$ and $I - I^0 - \frac{\lambda_S}{c_S}$ have the same sign. Replacing $(S^\star - I^0)/|S^\star - I^0|$ with $(I - I^0 - \frac{\lambda_S}{c_S})/|I - I^0 - \frac{\lambda_S}{c_S}|$, and moving it to the right hand side, we obtain

$$S^\star - I^0 = \left(1 - \frac{\hat{\theta}}{c_S|I - I^0 - \frac{\lambda_S}{c_S}|}\right)\left(I - I^0 - \frac{\lambda_S}{c_S}\right).$$

Because $S^\star - I^0$ and $I - I^0 - \frac{\lambda_S}{c_S}$ have the same sign, the coefficient $\left(1 - \frac{\hat{\theta}}{c_S|I - I^0 - \frac{\lambda_S}{c_S}|}\right)$ must be non-negative. Therefore

$$S^\star = I^0 + \max\left\{0, 1 - \frac{\hat{\theta}}{c_S|I - I^0 - \frac{\lambda_S}{c_S}|}\right\}\left(I - I^0 - \frac{\lambda_S}{c_S}\right).$$

Solving the I-subproblem (11): The Euler-Lagrange equation is the following inhomogeneous modified Helmholtz equation,

$$\nabla \cdot \nabla I^\star - \frac{c_S}{c_P} I^\star = \nabla \cdot \mathbf{P} + \frac{1}{c_P}\nabla \cdot \Lambda_P - \frac{c_S}{c_P} S - \frac{1}{c_P}\lambda_S,$$

with the Neumann boundary condition,

$$\nabla I^\star \cdot \nu = (\mathbf{P} + \frac{1}{c_P}\Lambda_P) \cdot \nu,$$

where the ν denotes the outward unit normal vector on the boundary of the domain. The Euler-Lagrange equation, with the boundary condition, is solved via the discrete cosine transform, cf. Remark 2.

Solving the E-subproblem (12): The corresponding Euler-Lagrange equation is the following,

$$\nabla \cdot \nabla \mathbf{E}^\star - \frac{c_E}{c_Q}\mathbf{E}^\star = \frac{1}{c_Q}\Lambda_E - \frac{c_E}{c_Q}\mathbf{P} + \frac{1}{c_Q}\nabla \cdot \Lambda_Q + \nabla \cdot \mathbf{Q},$$

which is a set of two inhomogeneous modified Helmholtz equations, one equation for each component of $\mathbf{E} = (E_1, E_2)$, and corresponding Neumann boundary conditions,

$$E_1 \cdot \nu = (\mathbf{Q}_1 + \frac{1}{c_Q}\Lambda_{Q1}) \cdot \nu,$$

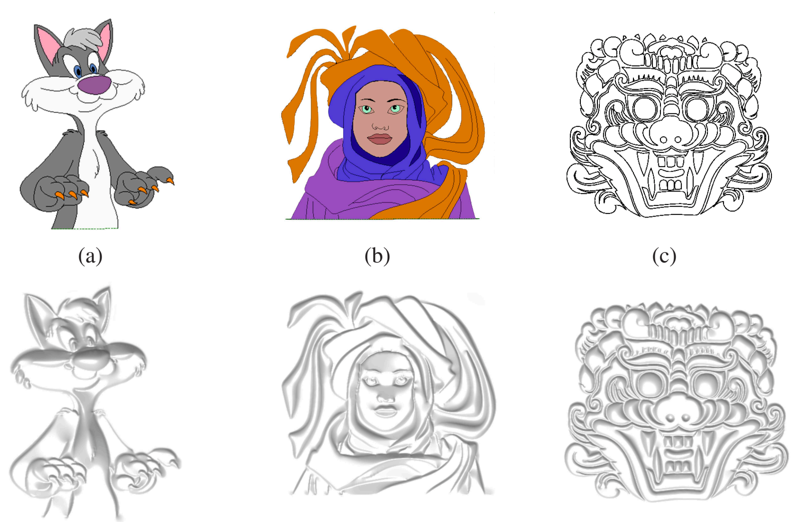
$$E_2 \cdot \nu = (\mathbf{Q}_2 + \frac{1}{c_Q}\Lambda_{Q2}) \cdot \nu,$$

where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are the row vectors of the matrix $\mathbf{Q}$, and Λ_{Q1} and Λ_{Q2} are corresponding Lagrange multipliers, respectively. ν is the outward unit normal on the boundary of the domain. Each equation is solved in the same way as in the I-subproblem, cf. Remark 2.

Remark 2. The last two sub-minimization problems, each reduces to solve a partial differential equation of the form

$$\triangle u(x,y) - \lambda u(x,y) = F(x,y),$$

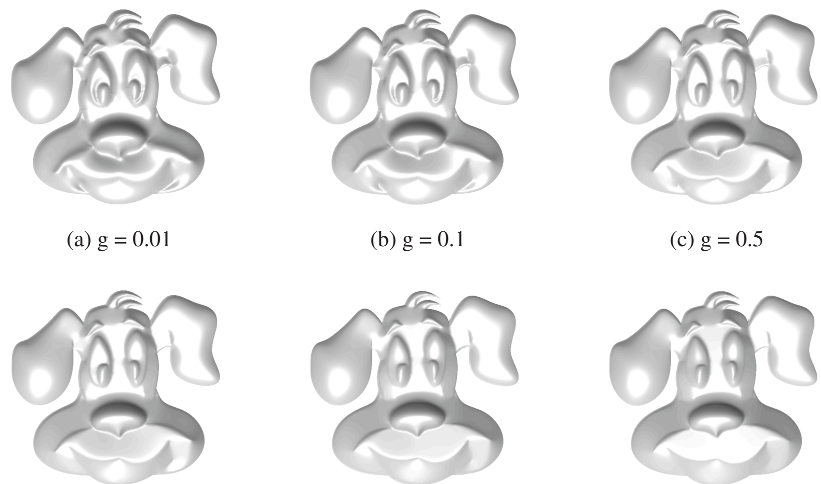
with a Neumann boundary condition and λ a non-negative number, also known as the inhomogeneous modified Helmholtz equation. A fast solver based on discrete cosine transform similar for the Poisson equation, cf. [37], has been developed and will be published in a future paper.



(a) (b) (c)

(d) (e) (f)

Fig. 1: Illustrating the cartoon case, where the input data are vectors along strokes (the top row) drawn by the artist. The vectors are not shown here. The corresponding 3D cartoons generated by our algorithm, are shown in the bottom row. The parameters are $g = 0.5$, $h = 0$, $\theta = 0$, and $\eta = 5.0$.



(a) g = 0.01 (b) g = 0.1 (c) g = 0.5

(d) g = 1.0 (e) g = 1.5 (f) g = 2.0

Fig. 2: Illustrating the effect of second order regularization by varying the parameter g. In these tests, $h = 0$, $\theta = 0$, and $\eta = 5.0$.

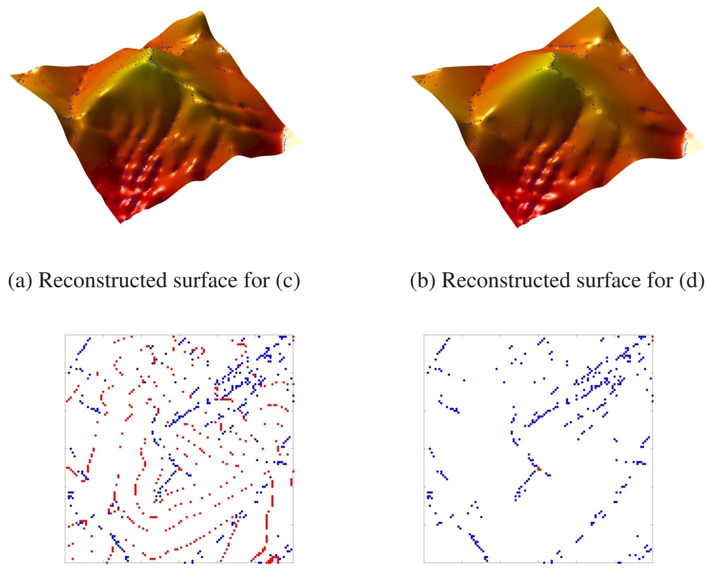
4 Numerical Results

For the numerical results, we consider the two different cases of surface reconstruction, namely, the 3D cartoon generation and the mountain surface reconstruction, where in the first case we are given normal vectors along strokes while in the second case we are given both normal vectors and elevation data along contours and isolated points. The numerical tests are done using the augmented Lagrangian algorithm, Algorithm 1-2. Algorithm 1 stopped when the total energy stabilized. For Algorithm 2, it was enough to use only one iteration.

In the cartoon case, we start with normal vectors along the strokes, which are given by artists. The results are shown in Fig. 1. Since we do not have the elevation data I_0, we set $\theta = 0$. In these experiments, we do not have flat surfaces, and hence we set $h = 0$. For flat surfaces h needs to be nonzero. We have used $g = 0.5$ and $\eta = 5.0$. As we can see from the Fig. 1, the algorithm is effective in preserving both structures and details.

In our next experiment with cartoon, we investigate the effect of the second order regularization by varying g. The results are shown in Fig .2, where the strokes and the normal vectors along strokes are kept the same. As g grows the edges get much sharper.

In our next experiment, we consider the 3D surface reconstruction of mountain for map. The input data are set of contours, with height values, and isolated points with normal vectors. We have experimented with different sparsities. In these experiments, $g = 0.1$, $h = 0$, $\theta = 10^5$ and $\eta = 10^6$. The results are shown in Fig. 3 showing that our model is effective even on extreme sparse data. .



(a) Reconstructed surface for (c)

(b) Reconstructed surface for (d)

(c) Sparsity of input data: 5.18%

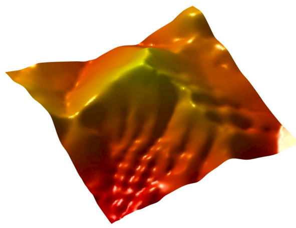
(d) Sparsity of input data: 2.38%

Fig. 3: 3D surface reconstructions with different sparsities of input data. At the red points, the height values are given. At the blue points, the normal vectors are given.

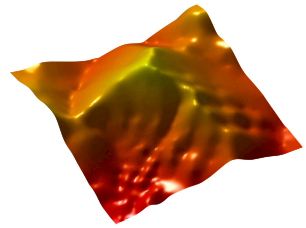
In Fig. 4, we compare the effect of using vector constraint. Fig. 4 (a) shows the result of using the hybrid constraint while Fig.4 (b) shows the result using only the elevation data constraint. As we can see that without the vector constraint, in this test case, there is some loss of structure in the valley. The test cases in Fig. 4 (a) and Fig. 4 (b) have the same input points. The test case in Fig. 4 (b) has only elevation data as input, while in the test case in Fig. 4 (a) the elevation data is replaced with normal vectors for some points. Fig. 4 (d) the same reconstruction is made using the method [31] which is based on 3rd order anisotropic regularization. As we can see that our method manages to preserve the small structure comparatively better even with sparser data, because we have the flexibility to input additional information to our model like the normal vectors.

5 Conclusion

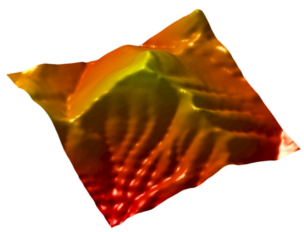
We have proposed a model for 3D surface reconstruction based on 2D sparse hybrid data, that is involving both height values and normal vectors in the same model, allowing for flexible control of their fidelity. A fast and effective algorithm based



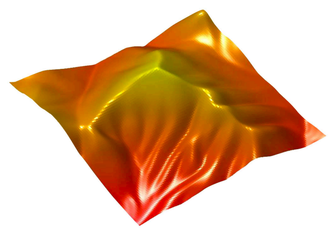
(a) Algorithm 1-2; hybrid data; sparsity 5.24%



(b) Algorithm 1-2; elevation data; sparsity 5.24%



(c) The groundtruth



(d) Algorithm of [31]; countour data; sparsity 7.42%

Fig. 4: Comparing mountain surface reconstructions using different fidelity constraints (hybrid, purely elevation, contour data).

on the augmented Lagrangian has been developed, where we split the minimization problem into six sub minimization problems, each with either a closed form solution or a fast solver. The proposed model is perfect for both 3D cartoon design and digital elevation map. Because it allows for flexible use of both the height data and the vector information, which can be on sparse curves or points, it has the potential to be used in areas where precise reconstruction of surfaces, represented by rather sparse data, are needed, and rather quick, for instance in real time applications like the web-based 3D visualization of maps, 3D GPS navigation, and 3D online gaming.

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