SHIFT INVARIANT SUBSPACES AND APPLICATIONS TO SIGNAL FRAGMENTATION

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Abstract. We examine the problem of approximating AM radio signals by elements of the principal shift invariant subspace generated by a waveform ϕ. We derive conditions on the Fourier transform of ϕ which characterize which principal shift invariant subspaces have a given approximation order k. Here we provide a new proof of the Strang-Fix conditions derived in [6]. We also consider a variation which is particularly suited to the approximation of AM signals. In addition, we derive lower bounds on the diameter of the support of ϕ when it achieves a given approximation order and characterize those ϕ which achieve this lower bound. Finally, we discuss how to calculate the coefficients of a signal in such an expansion.

1. Introduction

We consider the problem of decomposing a long-wavelength radio signal into compactly supported pieces which we can efficiently transmit over an array of small antennas.

To approximate a low-wavelength signal $f \in L^2(\mathbb{R})$, we fix a compactly supported waveform ϕ and approximate f by

$$f \approx \sum_{-\infty}^{\infty} a_i \phi\left(\frac{x-i}{N}\right) \tag{1.1}$$

here N is a parameter which determines the scale of the approximation. The larger N is, the more pieces we have per unit of time. We wish to investigate how the approximation improves as $N \to \infty$. We will also be considering the improvement of the approximation as f becomes more concentrated about its carrier frequency.

We are particularly interested in the tradeoff between a good approximation and a large overlap between the pieces in our decomposition. This is because for practical purposes the overlap between different pieces controls how many antennas we need to implement the scheme. The overlap of the pieces is in turn controlled by the size of the support of ϕ.

The mathematical formulation of this problem which we consider here is the relationship between the approximation properties of the principal shift invariant subspace generated by ϕ and the size of the support of ϕ.

Our contribution consists of the following. We completely characterize those compactly supported ϕ which achieve a given approximation order. This is related to the work done in [3], but we prove a stronger result for compactly supported functions. In addition, we propose a space of functions for approximating AM radio signals and prove a theorem characterizing compactly supported functions which

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approximate them well. Finally, we prove lower bounds on the support of a function which achieves a given approximation order in both cases, and give a method for explicitly constructing functions which achieve this support bound.

2. Principal Invariant Subspaces

In this section we recall the notion of a principal shift-invariant subspace of $L^2(\mathbb{R})$ and consider in particular the shift invariant subspaces which are generated by a compactly supported functions ϕ.

Definition 2.1. A closed subspace $S \subset L^2(\mathbb{R})$ is shift-invariant if $f \in S$ implies $f(x-n) \in S$ for all $n \in \mathbb{Z}$.

Definition 2.2. Let $\phi \in L^2(\mathbb{R})$. The principal shift-invariant subspace generated by ϕ, denoted $S(\phi)$ is the subspace

$$S(\phi) = \overline{\text{span}\,\{\phi(x-n)\}_{n\in\mathbb{Z}}} \tag{2.1}$$

Shift-invariant subspaces play an important role in approximation theory and the theory of wavelets. An important problem is to determine how well a general function f can be approximated by an element of the scaled subspace

$$S^h = \{f(x/h) \text{ with } f(x) \in S\} \tag{2.2}$$

where S is a shift-invariant subspace. This problem has been considered in [3], among others.

We associate to a shift-invariant subspace S and a function $f \in L^2(\mathbb{R})$ the approximation error

$$E(f,S) = \inf_{x\in S} \|f - x\|_2 \tag{2.3}$$

We wish to obtain bounds on this error for certain classes of functions f. In particular, the class of functions we will initially be interested in is the Sobolev space

$$W_2^k(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \text{ s.t. } \|f\|_{W_2^k} = \|(1+|\cdot|)^k \hat{f}\|_2 < \infty\} \tag{2.4}$$

Later we will define spaces which which better model AM radio signals, but initially we are concerned with the Sobolev spaces. Note that the Soboloev spaces are often defined in terms of the (weak) derivatives of f, but this is not important for the present development.

A shift invariant subspace is said to have approximation order k if the approximation error satisfies

$$E(f,S^h) \leq Ch^k \|f\|_{W_2^k} \tag{2.5}$$

and a function $\phi \in L^2(\mathbb{R})$ is said to have approximation order k if the principal shift invariant subspace $S(\phi)$ has approximation order k. Intuitively, this means that the shifts of ϕ can be used to approximate smooth (i.e. low bandwidth) functions very accurately.

The problem of determining which shift invariant subspaces have approximation order k is addressed in [3]. There the following two theorems are proved (here deliberately only stated for $\mathbb{R}$)

Theorem 2.3 (1.14 in [3]). *Assume that $\hat{\phi}$ is bounded on some neighborhood of the origin. If $S(\phi)$ provides approximation order k, then $\hat{\phi}$ has a zero of order k at every $0 \neq x \in 2\pi\mathbb{Z}$.*

Theorem 2.4 (1.15 in [3])**.** *Assume that $1/\hat{\phi}$ is bounded on some neighborhood of the origin and that, for some $\rho > k + d/2$, all derivatives of $\hat{\phi}$ of order $\leq \rho$ are in $L^2(A)$ where $A = B_\epsilon + (2\pi\mathbb{Z}\backslash 0)$ for some open ball B_ϵ centered at the origin. If $\hat{\phi}^l(x) = 0$ for all $l = 0, ..., k-1$ and $0 \neq x \in 2\pi\mathbb{Z}$, then $S(\phi)$ has approximation order k.*

Also, a necessary and sufficient condition for $S(\phi)$ to have approximation order k is given, however it is a more complicated condition in terms of the Fourier transform of ϕ. In particular, they show that ϕ provides approximation order k iff

$$\left(1 - \frac{|\hat{\phi}(\xi)|^2}{R_\phi(\xi)}\right) = O(\xi^{2k}) \tag{2.6}$$

as $\xi \to 0$, where

$$R_\phi(\xi) = \sum_{k=-\infty}^{\infty} |\hat{\phi}(\xi + k)|^2 \tag{2.7}$$

The theorem which they prove is in fact somewhat more general as it allows ϕ to depend on the scale.

Theorem 2.5 (4.3 in [3])**.** *Let $(\phi_h)_{h\in\mathbb{R}} \subset L^2(\mathbb{R})$. Then the sequence (ϕ_h) provides approximation order k in the sense that $E(f, S(\phi_h)^h) = O(h^k \|f\|_{W_2^k})$ if and only if for small enough ξ we have that*

$$\left(1 - \frac{\hat{\phi}_h(\xi)}{R_{\phi_h}(\xi)}\right) \leq C(h + |\xi|)^{2k} \tag{2.8}$$

for some constant C independent of h and ξ.

Our main result of this section uses this characterization to give a complete description of when ϕ has approximation order k in the case where ϕ has compact support. This condition has already been derived in [6], but we provide a different proof which may be of interest.

Theorem 2.6. *Assume that $\phi \in L^2(\mathbb{R})$ has compact support and $\hat{\phi}(0) \neq 0$. Then ϕ has approximation order k iff $\hat{\phi}^{(l)}(x) = 0$ for $0 \neq x \in 2\pi\mathbb{Z}$ and $l = 0, ..., k-1$.*

Before proving this theorem, we recall the following result of William Fogg Osgood from complex analysis [5].

Theorem 2.7 (Osgood)**.** *Let*

$$\sum_{k=1}^{\infty} f_k(z)$$

be a series of complex analytic functions on an open domain Ω. Assume that the series converges pointwise to a limit function f and the partial sums are uniformly bounded on Ω. Then f is analytic and the convergence is locally uniform.

Proof of Theorem (2.6). Note first that because ϕ has compact support, it is in $L^1(\mathbb{R})$. This, combined with compact support implies that $\hat{\phi}$ is analytic (and thus smooth), so all of the derivates in the statement of the theorem can be evaluated.

Also, the necessity of the condition that $\hat{\phi}^{(l)}(x) = 0$ for $0 \neq x \in 2\pi\mathbb{Z}$ and $l = 0, ..., k-1$ follows from Theorem (2.3). So we only need to show that this condition is also sufficient.

By Theorem (2.5), we must show that

$$\left(1 - \frac{|\hat{\phi}(\xi)|^2}{R_\phi(\xi)}\right) = O(\xi^{2k}) \tag{2.9}$$

as $\xi \to 0$. Now we rewrite the right-hand side of this equation to get

$$\frac{Q_\phi(\xi)}{R_\phi(\xi)} = O(\xi^{2k}) \text{ with } Q_\phi(\xi) = \sum_{0 \neq k \in \mathbb{Z}} |\hat{\phi}(\xi + k)|^2 \tag{2.10}$$

Since $\hat{\phi}$ is analytic (ϕ has compact support) and $\hat{\phi}(0) \neq 0$, we have that $|\hat{\phi}(\xi)|^2$ is bounded below on some open neighborhood of 0. But $R_\phi(\xi) \geq |\hat{\phi}(\xi)|^2$ and so is also bounded below on some neighborhood of 0. Thus it suffices to show that

$$Q_\phi(\xi) = O(\xi^{2k}) \tag{2.11}$$

as $\xi \to 0$.

To this end, consider the periodic function $\Phi_{ab}(x)$ with $a, b \in \mathbb{R}$ on $\mathbb{R}/\mathbb{Z}$ defined by

$$\Phi_{ab}(x) = \sum_{l=-\infty}^{\infty} \phi(x-l) e^{2\pi i(a+ib)(x-l)} = \sum_{l=-\infty}^{\infty} \phi(x-l) e^{2\pi i a(x-l)} e^{-2\pi b(x-l)} \tag{2.12}$$

Since ϕ has compact support and is in $L^2(\mathbb{R})$, it is in $L^1(\mathbb{R})$. Additionally, the compact support of ϕ implies that the above sum is finite, which implies that $\Phi_{ab} \in L^2(\mathbb{R}/\mathbb{Z})$. We now apply the Poisson summation formula (valid because $\phi \in L^1(\mathbb{R})$) to see that the Fourier series of Φ_{ab} is

$$\Phi_{ab}(x) = \sum_{k=-\infty}^{\infty} \hat{\phi}(k + (a+ib)) e^{2\pi i k x} \tag{2.13}$$

here I view the Fourier transform of ϕ as a complex analytic function (essentially the Laplace transform), which is justified by the Paley-Wiener Theorem (see [7]) because ϕ has compact support.

But $\Phi_{ab} \in L^2(\mathbb{R}/\mathbb{Z})$ as well, so we can apply the Plancherel theorem to obtain

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(k + (a+ib))|^2 = \|\Phi_{ab}\|^2_{L^2(\mathbb{R}/\mathbb{Z})} \tag{2.14}$$

Now we bound $\|\Phi_{ab}\|_2$. Let $T \in \mathbb{N}$ be large enough so that ϕ is supported on $[-T, T]$. Then since

$$\Phi_{ab}(x) = \sum_{k=-\infty}^{\infty} \phi(x-k) e^{2\pi i a(x-k)} e^{-2\pi b(x-k)} \tag{2.15}$$

we see that

$$\|\Phi_{ab}(x)\|_2 \leq \sum_{k=-T}^{T-1} e^{2\pi bT} \|\phi\|_{L^2([k,k+1])} \leq (2T+1) e^{2\pi bT} \|\phi\|_{L^2(\mathbb{R})} \tag{2.16}$$

So that

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(k + (a+ib))|^2 \leq (2T+1)^2 e^{4\pi bT} \|\phi\|^2_{L^2(\mathbb{R})} \tag{2.17}$$

Consider now the series of entire analytic functions

$$S_\phi(z) = \sum_{0\neq k=-\infty}^{\infty} \hat{\phi}(z-k)\overline{\hat{\phi}(\overline{z}-k)} \tag{2.18}$$

Each of the terms is an entire function because ϕ is in L^1 and has compact support (we are essentially considering the Laplace transform of ϕ and using the Paley-Wiener Theorem [7]). Note additionally that

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(z-k)\overline{\hat{\phi}(\overline{z}-k)}| \leq \left(\sum_{k=-\infty}^{\infty} |\hat{\phi}(z-k)|^2\right)^{1/2} \left(\sum_{k=-\infty}^{\infty} |\hat{\phi}(\overline{z}-k)|^2\right)^{1/2} \tag{2.19}$$

by Hölder's inequality. Using (2.17) we get

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(z-k)\overline{\hat{\phi}(\overline{z}-k)}| \leq (2T+1)^2 e^{4\pi \mathrm{Im}(z)T} \|\phi\|^2_{L^2(\mathbb{R})} \tag{2.20}$$

This implies that the series $S_\phi(z)$ converges pointwise absolutely and is uniformly bounded in a neighborhood of 0 in the complex plane. We now apply Theorem (2.7) to see that the series $S_\phi(z)$ converges locally uniformly to an analytic function $S_\phi(z)$ (abusing notation). Moreover, on the real line we clearly have that

$$S_\phi(\xi + i0) = Q_\phi(\xi) \tag{2.21}$$

Additionally, the local uniform convergence of the series S_ϕ implies convergence of all of its derivatives at 0. But, by assumption $\hat{\phi}^{(l)} = 0$ for $0 \neq x \in 2\pi\mathbb{Z}$ and $l = 0, ..., k-1$, which implies that $(\hat{\phi}(z-k)\overline{\hat{\phi}(\overline{z}-k)})^{(l)}(0) = 0$ for $l = 0, ..., 2k-1$. Thus $S_\phi^{(l)}(0) = 0$ for $l = 0, ..., 2k-1$, which implies that

$$Q_\phi(\xi) = O(\xi^{2k}) \tag{2.22}$$

as desired. □

3. Approximation of AM Radio Signals

We now investigate the approximation of AM radio signals by shift-invariant subspaces. An AM radio signal has the form

$$f(t) = s(t)\cos(2\pi f_0(t-t_0)) \tag{3.1}$$

where f_0 is the carrier frequency, $s(t)$ is the transmitted signal and t_0 is the delay. We additionally constrain the signal s to be band-limited with bandwidth $\epsilon << k$.

Now, the Fourier transform converts products into convolutions, and the transform of $\cos(2\pi f_0(t-t_0))$ is supported at $\pm f_0$. This, combined with the assumption that s has bandwidth ϵ, implies that

$$\mathrm{supp}(\hat{f}) \subset [f_0-\epsilon, f_0+\epsilon] \cup [-f_0-\epsilon, -f_0+\epsilon] \tag{3.2}$$

This observation motivates the following definition of the space of AM signals.

Definition 3.1. Let $\epsilon << k$ be given and define the AM space of frequency k and bandwidth ϵ to be

$$W^{AM}_{\epsilon,f_0} = \{f \in L^2 \text{ s.t. } \mathrm{supp}(\hat{f}) \subset [f_0-\epsilon, f_0+\epsilon] \cup [-f_0-\epsilon, -f_0+\epsilon]\} \tag{3.3}$$

Note that W^{AM}_{ϵ,f_0} actually contains functions which are not of the form given in (3.1). Nonetheless, in terms of approximation properties not much is lost by considering the larger space W^{AM}_{ϵ,f_0}.

For simplicity, we will assume throughout that $f_0 < 1/2$. When f_0 is larger, we must use principal shift invariant subspaces with a finer scale. The analysis is essentially the same in this case.

Our goal is to find a function ϕ such that $E(f, S(\phi))$ is small. Generally the bandwidth ϵ will be very small with respect to f_0 in practice. This motivates the following definition.

Definition 3.2. Let $\phi \in L^2(\mathbb{R})$. We say that ϕ has AM-approximation order k if for sufficiently small ϵ we have

$$\sup_{f\in W^{AM}_{\epsilon,f_0}} E(f, S(\phi)) \leq C\epsilon^k \|f\|_2 \tag{3.4}$$

for some constant C independent of ϵ.

Notice that in this definition we require the error to decrease as we decrease the bandwidth ϵ.

Note also that this definition depends upon f_0. For us, f_0 will be fixed throughout and so we are not explicit about the f_0 dependence of our notion of AM-approximation order. We will continue to assume that $f_0 < 1/2$.

Our main theorem of this section is the following, which characterizes ϕ with AM-approximation order k when ϕ is compactly supported.

Theorem 3.3. *Assume that $\phi \in L^2(\mathbb{R})$ has compact support and $\hat{\phi}(\pm f_0) \neq 0$. Then ϕ has AM-approximation order k if and only if and $l = 0, ..., k-1$.*

$$\hat{\phi}^{(l)}(\pm f_0 + n) = 0 \tag{3.5}$$

for $l = 0, ..., k-1$ and $0 \neq n \in \mathbb{Z}$.

Proof. Let $g \in L^2(\mathbb{R})$, with $\hat{g}$ supported in $[-1/2, 1/2)$. By Theorem 2.10 in [3], we know that the approximation error of g by $S(\phi)$ is given by

$$E(g, S(\phi)) = \left\| \hat{g}\left(1 - \frac{|\hat{\phi}|^2}{R_\phi}\right)^{1/2} \right\|_2 \tag{3.6}$$

Now let $f \in W^{AM}_{\epsilon,f_0}$. Since $f_0 < 1/2$, for small enough epsilon this formula applies to f. So, letting $A_{\epsilon,f_0} := [f_0 - \epsilon, f_0 + \epsilon] \cup [-f_0 - \epsilon, -f_0 + \epsilon]$, we have that

$$E(f, S(\phi)) = \left\| \hat{f}\left(1 - \frac{|\hat{\phi}|^2}{R_\phi}\right)^{1/2} \right\|_{L^2(A_{\epsilon,f_0})} \tag{3.7}$$

Applying Hölder's inequality, and taking the supremum over $f \in W^{AM}_{\epsilon,f_0}$ with $\|f\|_2 = 1$ yields

$$\sup_{f\in W^{AM}_{\epsilon,f_0},\ \|f\|_2=1} E(f, S(\phi)) = \left\| \left(1 - \frac{|\hat{\phi}|^2}{R_\phi}\right)^{1/2} \right\|_{L^\infty(A_{\epsilon,f_0})} \tag{3.8}$$

Here the equality is due to the fact that Hölder's inequality is sharp.

Thus we obtain that ϕ has AM-approximation order k iff we have

$$\left\| \frac{Q_\phi}{R_\phi} \right\|_{L^\infty(A_{\epsilon,f_0})} = O(\epsilon^{2k}) \tag{3.9}$$

where as before

$$Q_\phi(\xi) = \sum_{0 \neq k \in \mathbb{Z}} |\hat{\phi}(\xi + k)|^2 \tag{3.10}$$

Now the proof proceeds in much the same way as the proof of Theorem (2.6). We use the fact that ϕ is compactly supported and thus in $L^1(\mathbb{R})$ to conclude that $\hat{\phi}$ is analytic (in fact entire when viewed as a function on the complex plane). This, combined with the assumption that $\hat{\phi}(\pm f_0) \neq 0$ implies that $R_\phi(\xi)$ is bounded away from 0 on A_{ϵ,f_0} for small enough ϵ. Hence, (3.9) is equivalent to

$$\|Q_\phi\|_{L^\infty(A_{\epsilon,f_0})} = O(\epsilon^{2k}) \tag{3.11}$$

That the condition on the derivates of $\hat{\phi}$ is necessary now easily follows since

$$|\hat{\phi}(\xi + k)|^2 \leq Q_\phi(\xi) \tag{3.12}$$

for all $0 \neq k \in \mathbb{Z}$.

The same logic as in the proof of Theorem (2.6) implies that

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(k + (a + ib))|^2 \leq (2T + 1)^2 e^{4\pi bT} \|\phi\|^2_{L^2(\mathbb{R})} \tag{3.13}$$

where $\text{supp}(\phi) \subset [-T, T]$.

As before, this combined with Theorem (2.7) implies that the series

$$S_\phi(z) = \sum_{0 \neq k = -\infty}^{\infty} \hat{\phi}(z - k)\overline{\hat{\phi}(\overline{z} - k)} \tag{3.14}$$

converges locally uniformly to an analytic function $S_\phi(z)$. But the the derivatives converge as well and thus $S_\phi^{(l)}(\pm f_0) = 0$ for $l = 0, ..., 2k - 1$. So since

$$S_\phi(\xi + i0) = Q_\phi(\xi) \tag{3.15}$$

we have that

$$\|Q_\phi\|_{L^\infty(A_{\epsilon,f_0})} = O(\epsilon^{2k}) \tag{3.16}$$

as desired. □

4. Support Bounds

In the previous two sections we derived conditions on the Fourier transform which determine when a given compactly supported function ϕ has approximation order k and AM-approximation order k. We will now prove bounds on the support of ϕ in each of these cases.

The type of theorem which we will prove show that if ϕ has approximation order (or AM-approximation order) k, then its support cannot be too small. In particular, the shifts of ϕ must overlap significantly.

The theorems we will prove are the following.

Theorem 4.1. *Let ϕ in $L^2(\mathbb{R})$ have compact support. If ϕ has approximation order k, then*

$$diam(supp(\phi)) \geq k \tag{4.1}$$

Theorem 4.2. *Let ϕ in $L^2(\mathbb{R})$ have compact support. If ϕ has AM-approximation order k, then*

$$diam(supp(\phi)) \geq 2k \tag{4.2}$$

Before proving these theorems, we need a lemma from the theory of entire functions (see, for instance [1] for an introduction).

Lemma 4.3. *Let $t_1, ..., t_l$ be a finite collection of distinct points in $[0, 1)$ and let $k_1, ..., k_l > 0$ be integers. Assume that $f(z)$ is an entire function of exponential type (i.e. $|f(z)| = O(e^{c|z|})$ for some $c \in \mathbb{R}$) which satisfies*

$$|f(iy)| \leq Ce^{\pi\lambda|y|} \tag{4.3}$$

for some constant C and $\lambda < k_1 + ... + k_l$. If additionally we have that for all but finitely many integers n

$$f^{(i)}(t_j + n) = 0 \tag{4.4}$$

for all $j = 1, ..., l$, $i = 0, ..., k_j - 1$, then $f = 0$.

Before proving this, we recall the following lemma ([4], Lemma 1)

Lemma 4.4. *If $f(z)$ is an entire function of exponential type vanishing on the set of integers, then*

$$f(z) = \phi(z)sin(\pi z) \tag{4.5}$$

where ϕ is also of exponential type.

Proof of Lemma (4.3). Assume that f satisfies the assumptions of the theorem. We wish to show that $f = 0$.

The first step will be to multiply f by a polynomial p so that the condition

$$f^{(i)}(t_j + n) = 0 \tag{4.6}$$

for $j = 1, ..., l$, $i = 0, ..., k_j - 1$ holds for all integers n. Since the set of integers where this condition is not already satisfied is finite, p can just be taken as the product of $(x - t_j - n)^{k_j}$ for those n where the condition is not satisfied.

Thus we obtain a function $g(z) = p(z)f(z)$ which is entire, of exponential type and satisfies

$$|g(iy)| \leq C'e^{\pi\lambda'|y|} \tag{4.7}$$

for some $\lambda < \lambda' < k_1 + ... + k_l$. Additionally,

$$g^{(i)}(t_j + n) = 0 \tag{4.8}$$

for all $j = 1, ..., l$, $i = 0, ..., k_j - 1$ and $n \in \mathbb{Z}$.

We proceed to prove that $g = 0$. By repeated application of Lemma (4.4), combined with appropriate changes of variable $z \to z - t_j$, we obtain

$$g(z) = \phi(z) \sin(\pi(z - t_1))^{k_1} \cdots \sin(\pi(z - t_l))^{k_l} \tag{4.9}$$

where ϕ is of exponential type. Also, we now conclude that the growth condition (4.7) on g, together with the rate of growth of sin along the imaginary axis, implies that

$$|\phi(iy)| \leq C' e^{\pi(\lambda' - k_1 - \ldots - k_l)|y|} \tag{4.10}$$

But this means that the indicator function of ϕ, defined by

$$h_\phi(\theta) = \limsup_{r \to \infty} r^{-1} \log(|\phi(re^{i\theta})|) \tag{4.11}$$

satisfies

$$h_\phi(\pi/2), h_\phi(-\pi/2) \leq \pi(\lambda' - k_1 - \ldots - k_l) < 0 \tag{4.12}$$

as $\lambda' < k_1 - \ldots - k_l$. However, if ϕ is a non-zero entire function of exponential type, then it must hold that (see Theorem 5.4.4 of [1])

$$h_\phi(\pi/2) + h_\phi(-\pi/2) \geq 0 \tag{4.13}$$

This contradiction implies that $\phi = 0$, and thus $g = 0$ and $f = 0$ as desired. □

We can now deduce our support bounds for a given approximation order using the Paley-Wiener theorem [7].

Proof of Theorem (4.1). Assume that $\phi \in L^2(\mathbb{R})$ has compact support and achieves approximation order k. Then $\phi \in L^1(\mathbb{R})$ and $\hat{\phi}$ is bounded everywhere by $\|\phi\|_1$ and so we know by Theorem (2.3) that

$$\hat{\phi}^{(k)}(n) = 0 \tag{4.14}$$

for all $0 \neq n \in \mathbb{Z}$. Now, translating ϕ multiplies $\hat{\phi}$ by a phase, which preserves the condition on the derivatives. Thus, we can assume that

$$\text{supp}(\phi) \subset [-\text{diam}(\text{supp}(\phi))/2, \text{diam}(\text{supp}(\phi))/2] \tag{4.15}$$

So we now apply the Paley-Wiener theorem [7] to conclude that $\hat{\phi}$ is an entire function which satisfies

$$|\hat{\phi}(z)| \leq C e^{\pi(\text{diam}(\text{supp}(\phi)))|z|} \tag{4.16}$$

If $\text{diam}(\text{supp}(\phi)) < k$, then Lemma (4.3) with $l = 1$, $j_1 = 0$, and $k_1 = k$ implies that $\hat{\phi} = 0$ which would imply that $\phi = 0$.

As this clearly contradicts the assumption that ϕ has approximation order k, it follows that

$$\text{diam}(\text{supp}(\phi)) \geq k \tag{4.17}$$

as desired. □

Proof of Theorem (4.2). Assume $\phi \in L^2(\mathbb{R})$ has compact support and achieves AM-approximation order k. Then Theorem (3.3) implies that (if $\hat{\phi}(\pm f_0) \neq 0$ then the derivates have to vanish elsewhere to even higher order)

$$\hat{\phi}^{(k)}(\pm f_0 + n) = 0 \tag{4.18}$$

for all $0 \neq n \in \mathbb{Z}$. Now, translating ϕ multiplies $\hat{\phi}$ by a phase, which preserves the condition on the derivatives. Thus, we can assume that

$$\text{supp}(\phi) \subset [-\text{diam}(\text{supp}(\phi))/2, \text{diam}(\text{supp}(\phi))/2] \tag{4.19}$$

So we now apply the Paley-Wiener theorem [7] to conclude that $\hat{\phi}$ is an entire function which satisfies

$$|\hat{\phi}(z)| \leq Ce^{\pi(\mathrm{diam}(\mathrm{supp}(\phi)))|z|} \tag{4.20}$$

If $\mathrm{diam}(\mathrm{supp}(\phi)) < 2k$, then Lemma (4.3) with $l = 2$, $j_1 = -f_0$, $j_2 = f_0$ and $k_1 = k_2 = k$ implies that $\hat{\phi} = 0$ which would imply that $\phi = 0$.

As this clearly contradicts the assumption that ϕ has AM-approximation order k, it follows that

$$\mathrm{diam}(\mathrm{supp}(\phi)) \geq k \tag{4.21}$$

as desired. □

5. Optimal Waveforms

In this section we consider the problem of whether or not the bound derived in Theorem (4.1) is sharp and if so, determining what the waveforms of minimal support are (the situation for a given AM-approximation order is similar). We give a simple method for explicitly constructing them.

Theorem 5.1. *Let $a \in \mathbb{R}$, $k > 0$. Then the set of functions*

$$S_a^k = \{\phi \in L^2(\mathbb{R}) : \mathit{supp}(\phi) \subset [a, a+k], \hat{\phi}^l(x) = 0 \text{ for } 0 \neq x \in \mathbb{Z},\ l = 0, ..., k-1\} \tag{5.1}$$

forms a k-dimensional vector space.

In proving this we explicitly construct the desired vector space.

Proof. Note first that since replacing ϕ by $\phi(x-a)$ scales the the Fourier transform by $e^{2\pi i a}$, it suffices to show the statement for $a = 0$.

So let ϕ have support contained in $[0, k]$. We introduce the pieces $\phi_p \in L^2([0,1))$, $p = 0, ..., k-1$ of ϕ as follows

$$\phi_p(x) = \phi(x+p) \tag{5.2}$$

for $x \in [0,1)$. Also, we introduce periodic functions $g_m \in L^2(\mathbb{R}/\mathbb{Z})$ for $m = 0, ..., k-1$, defined by

$$g_m(x) = \sum_{j\in\mathbb{Z}} (x+j)^m \phi(x+j) \tag{5.3}$$

Since ϕ has compact support, it is in $L^1(\mathbb{R})$ (also $x^k\phi \in L^1(\mathbb{R})$) and thus the Poisson summation formula implies that the Fourier coefficients of g_m are

$$a_{mj} = (-2\pi i)^m \hat{\phi}^{(m)}(j) \tag{5.4}$$

This implies that $\phi \in S_0^k$ iff $a_{mj} = 0$ for all $j \neq 0$, i.e. iff g_m is a constant for each $m = 0, ..., k-1$.

Using the definition of g_m and the ϕ_p, we see that $\phi \in S_0^k$ iff

$$\sum_{p=0}^{k-1} (x+r+p)^m \phi_p(x) = c_m \tag{5.5}$$

But this Vandermonde system is invertible and thus each choice of the c_m gives a unique solution for ϕ_p, which in turn uniquely determine ϕ. Thus we have that S_0^k is a k-dimensional vector space as desired. □

For instance, in the case $k = 2$, we can explicitly calculate the minimal support waveforms as follows. Let the support of ϕ be $[-1, 1]$ so that the pieces of ϕ are

$$\phi_1 : [0, 1) \to \mathbb{R},\ \phi_1(x) = \phi(x - 1)$$

and

$$\phi_2 : [0, 1) \to \mathbb{R},\ \phi_2(x) = \phi(x)$$

Then we have the equations

$$\phi_1 + \phi_2 = c_1,\ (x - 1)\phi_1 + x\phi_2 = c_2 \tag{5.6}$$

Solving this linear system, we see that the minimal support waveforms which are supported in $[-1, 1]$ are spanned by

$$f_1(x) = \begin{cases} 0 & x \notin [-1, 1] \\ (x + 1) & -1 \leq x < 0 \\ (1 - x) & 0 \leq x \leq 1 \end{cases}$$

which is a hat-function and

$$f_2(x) = \begin{cases} 0 & x \notin [-1, 1] \\ -1 & -1 \leq x < 0 \\ 1 & 0 \leq x \leq 1 \end{cases}$$

A similar computation can be done for larger values of k. In fact, the result one obtains in the following.

Theorem 5.2. *The space of waveforms which achieve approximation order k and are supported in $[-k/2, k/2]$ is spanned by $\phi_1, ..., \phi_k$ where*

$$\hat{\phi}_i(\xi) = \text{sinc}^i(\xi)\sin^{k-i}(\xi) \tag{5.7}$$

Proof. Note first that the inverse Fourier transform of sinc is the characteristic function of $[-1/2, 1/2]$ and the Fourier transform of sin is a sum of two multiples of a Dirac delta distribution at $\pm(1/2)$.

This means that the functions ϕ_i can be obtained by convolving the indicator function of $[-1/2, 1/2]$ i-times and the Fourier transform of sin $k-i$ times. Thus the support of ϕ_i will be contained in the interval $[-k/2, k/2]$ and ϕ_i will be a function since $i > 0$ and thus there is at least one indicator function in the convolution.

Moreover, note that the functions ϕ_i are clearly linearly independent in Fourier space. This, along with the previous theorem giving the dimension of the space, completes the proof. □

6. Calculating the Expansion Coefficients

Let $\phi \in L^2(\mathbb{R})$ have compact support. Here we address the question of whether or not there exists a series, absolutely convergent in $L^2(\mathbb{R})$, of the form

$$\sum_{k=-\infty}^{\infty} a_k \phi(x - k) \to P_{S(\phi)} f \tag{6.1}$$

for every function $f \in L^2(\mathbb{R})$ and if so, how one can calculate the coefficients a_k given f.

Central to this development is the theory of frames (see, for instance [2] and the references therein), and we begin with the requisite definition.

Definition 6.1. A sequence of vectors $(f_i)_{i\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is called a frame if there exist constants $0 < A \le B < \infty$ such that for all $f \in \mathcal{H}$

$$A\|f\|_H^2 \le \sum_{k\in\mathbb{Z}} |\langle f, f_i\rangle|^2 \le B\|f\|_H^2 \tag{6.2}$$

Frames have been extensively studied in the literature in relation to wavelets, for instance, and we only recall those results which are relevant to our current development. For more information, see [2] and the references therein.

Definition 6.2. Let $(f_k)_{k\in\mathbb{Z}} \subset \mathcal{H}$ be a frame. We define the pre-frame operator $T : \ell_2(\mathbb{Z}) \to \mathcal{H}$ as

$$T(\{c_k\}_{k\in\mathbb{Z}}) = \sum_{k\in\mathbb{Z}} c_k f_k \tag{6.3}$$

The dual of T, $T^* : \mathcal{H} \to \ell_2(\mathbb{Z})$ is given by

$$T^*(f) = \{\langle f, f_k\rangle\}_{k\in\mathbb{Z}} \tag{6.4}$$

Finally, the composition $S = TT^* : \mathcal{H} \to \mathcal{H}$ is called the frame operator.

Theorem 6.3 (Theorem 5.1.7 in [2])**.** *Let $(f_k)_{k\in\mathbb{Z}} \subset \mathcal{H}$ be a frame. Then the frame operator S is bounded with bounded inverse and for all $f \in \mathcal{H}$ we have*

$$f = \sum_{k\in\mathbb{Z}} \langle f, S^{-1} f_k\rangle f_k \tag{6.5}$$

where the above series converges absolutely. The sequence $(S^{-1}f_k)_{k\in\mathbb{Z}}$ is called the canonical dual frame of (f_k).

Note that this theorem gives us an explicit way of calculating coefficients a_k such that

$$f = \sum_{k\in\mathbb{Z}} a_k f_k \tag{6.6}$$

when (f_k) is a frame. Note however, that the dual frame is not unique. Namely there may exist sequences (g_k) other than $(S^{-1}f_k)$ which satisfy

$$f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle f_k \tag{6.7}$$

This solves our problem if the shifts of ϕ form a frame for their span. In this case, we will have

$$P_{S(\phi)} f = \sum_{k\in\mathbb{Z}} \langle f, \psi(x-k)\rangle \phi(x-k) \tag{6.8}$$

where $(\psi(x-k))$ is the canonical dual frame of $(\phi(x-k))$ (we could take any other dual frame as well).

Luckily, one can characterize both dual frame function ψ as well as whether or not the shifts of ϕ form a frame in terms of the Fourier transform of ϕ.

Theorem 6.4. *Let $\phi \in L^2(\mathbb{R})$. Then the shifts $(\phi(x-k))_{k\in\mathbb{Z}}$ form a frame for their span iff the function $G_\phi(\xi)$ defined by*

$$G_\phi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi - k)|^2 \tag{6.9}$$

satisfies $0 < A \leq G(\xi) \leq B < \infty$. *Moreover, in this case the canonical dual frame* $(\psi(x-k))_{k\in\mathbb{Z}}$ *is characterized by*

$$\hat{\psi}(\xi) = \hat{\phi}(\xi)/G_\phi(\xi) \tag{6.10}$$

Proof. See Section 7 of [2] for a proof. □

It remains to consider the case where $(\phi(x-k))$ doesn't form a frame. However, if ϕ has compact support, then in this case there always must exist a function $f \in \overline{\text{span}\,\{\phi(x-n)\}_{n\in\mathbb{Z}}}$ which cannot be written as an absolutely convergent series

$$f = \sum_{k=-\infty}^{\infty} a_k \phi(x-k) \tag{6.11}$$

This is implied by the following theorem.

Theorem 6.5 (Theorem 5.4.1 in [2])**.** *A sequence* $(f_k)_{k\in\mathbb{Z}}$ *in* $\mathcal{H}$ *is a frame if and only if the operator* $T : \ell_2(\mathbb{Z}) \to \mathcal{H}$ *defined by*

$$T((a_k)_{k\in\mathbb{Z}}) = \sum_{k\in\mathbb{Z}} a_k f_k \tag{6.12}$$

is well-defined and onto $\mathcal{H}$.

Now since ϕ has compact support, the map

$$T((a_k)_{k\in\mathbb{Z}}) = \sum_{k\in\mathbb{Z}} a_k \phi(x-k) \tag{6.13}$$

is well defined and bounded from $\ell_2(\mathbb{Z}) \to \overline{\text{span}\,\{\phi(x-n)\}_{n\in\mathbb{Z}}}$.

So if $(\phi(x-k))$ doesn't form a frame, then this map must not be onto, so there exists an $f \in \overline{\text{span}\,\{\phi(x-n)\}_{n\in\mathbb{Z}}}$ which cannot be written as

$$f = \sum_{k=-\infty}^{\infty} a_k \phi(x-k) \tag{6.14}$$

with $(a_k) \in \ell_2(\mathbb{Z})$.

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