# Efficient Frame Projection of Amplitude-Modulated Signals

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**Abstract**

This is continuation work of "Accuracy and Efficiency of Signal Fragmentation for a Single Frequency Signal". We first re-derive the formula for efficiency and conditions for it to hold. Based on this formula, we optimize the efficiency with respect to wavelet, under some Fourier constraint that corresponds to accuracy. There are two approaches to this optimization problems, which lead to similar results. The resulting wavelet is efficient and can be carried by sets of small antennas. We also hope these mathematical concepts can be used in other areas.

## 1 Introduction

It is a common practice to send an AM(amplitude-modulated) signal through antennas whose length is proportional to its carrier frequency. For a long wavelength signal, however, such antenna might be expensive or even impossible to make. Hence we turn to short antennas, which are not preferred in the literature due to their inefficiency. It turns out that, assuming perfect superposition, we can make them sufficiently accurate and efficient.

To deliver a signal $f(t)$ in the far field, we need to produce current $F(t)$. We would like to come up with an approximation

$$F(t) \approx \sum_{n=1}^{2N} a_n \Phi(t - t_n)$$

where $\Phi$ is called the wavelet. Accuracy of $\phi$ can be described by its order of zeros on some lattice in Fourier space. These are well-established results in the 90s [1] [2]. However, there is less discussion of its efficiency, possibly due to the complicated physics of wave superposition. We attempt to come up with a reasonable formulation and approximation, so that we can measure the efficiency and hopefully maximize it under some accuracy constraints.

We will derive the formula for efficiency in section 2, study the constraints in section 3, and investigate the optimization in section 4. At last, in section 5 we will compare the performance of some typical wavelet with ours.

## 2 Efficiency Formulation

To make it more illustrative, we will from now consider $F(t) = \sin(t)$ (and hence $f(t) = F'(t) = \cos(t)$) over $[0, 2\pi]$, though the analysis is similar for AM-signal.

## 2.1 Current decomposition

We want to approximate $F(t)$ by $2N$ wavelets, distributed over $M$ antennas. In general, $2N \geq M$ since each antenna can be used multiple times, and for simplicity we will assume the $2N$ is an integer multiple of $M$, so each antenna will carry exactly $\frac{2N}{M}$ wavelets. The factor of 2 allows each antenna to have current goes up (counted once) and down (counted once), so the current will be zero after usage. Let $F_m$ represent the current in the $m$-th antenna, so we can write

$$F(t) \approx \sum_{n=1}^{2N} a_n \Phi(t - t_n) = \sum_{m=1}^{M} F_m(t) \tag{2.1}$$

where

$$F_m(t) := \sum_{l=0}^{2N/M-1} a_{lM+m} \Phi(t - t_{lM+m}) \tag{2.2}$$

which means that antennas are used in turns. For example, first, antenna 1 produces $a_1\Phi_1$, then antenna 2 produces $a_2\Phi_2$..., antenna $M$ produces $a_M\Phi_M$, then we go back and use antenna 1 to produce $a_{M+1}\Phi_{M+1}$. Figure 1 shows the cases where $M = 2$ and $M = N$.

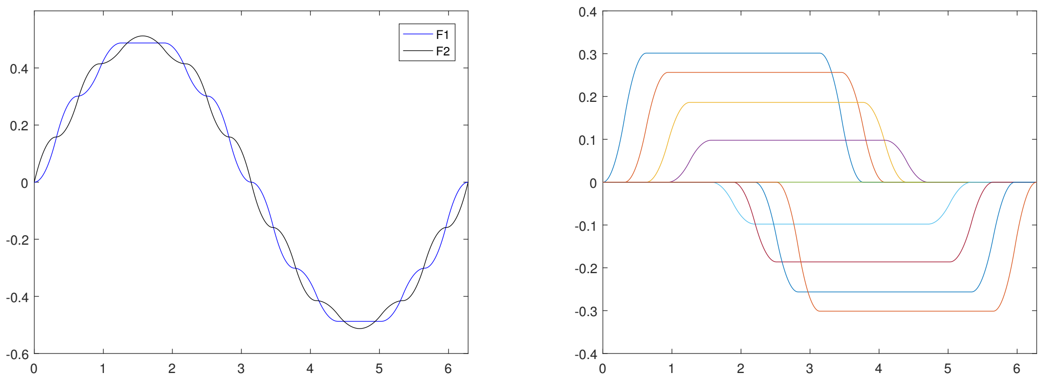


Figure 1: Approximation of a sine wave, left: $M = 2$, right: $M = N$

Large $N$ corresponds to short antennas and high accuracy, and hence we will assume $N \gg 1$. $M$, on the other hand, can be large or small as long as

$$k_\phi \leq M \leq N \tag{2.3}$$

where $k_\phi = \frac{\Omega_\phi}{\Delta}$, $\Delta = \frac{2\pi}{N}$, $\Omega_\phi = \text{supp}(\phi)$. The upper bounds comes from the fact that each antenna must deliver at least two wavelets to ensure that current goes to zero in the end; the lower bound is due to the fact that we are not allowed to reuse an antenna before it finishes its previous task. We will see how to choose $M$ appropriately after the derivation of formula for efficiency. For now, we will assume that $N \gg M \gg k_\phi$.

## 2.2 Derivation of Formula for Efficiency

The energy of antennas consist of the energy lost due to radiation(inductance), which is due to the change in current, and to Ohmic resistance, which is due to current [3]. Given $M$ antennas, the efficiency $E_{\text{eff},M}$ is defined as following:

$$E_{\text{rad},M} := R_{\text{rad}} \sum_m ||f_m||_2^2 \tag{2.4}$$

$$E_{\text{ohm},M} := R_{\text{ohm}} \sum_m ||F_m||_2^2 \tag{2.5}$$

$$E_{\text{far},M} := \left[ \mathcal{F}(\sum_m f_m)(1) \right]^2 \tag{2.6}$$

$$E_{\text{eff},M} := \frac{E_{\text{far},M}}{E_{\text{rad},M} + E_{\text{ohm},M}} \tag{2.7}$$

where $f_m = F'_m$ and $\mathcal{F}(f_m) = \hat{f}_m = \int_0^{2\pi} f_m(t) e^{-i\xi t} dt$. $E_{\text{far},M}$ is defined as such because we are considering the case $f(t) = \cos(t)$. We wish to derive a general expression of $E_{\text{eff},M}$ in terms of $\phi$ or $\Phi$, assuming that $|\Omega_\phi| = O(\Delta)$ and $\max \phi = O(\Delta^{-1})$ (so $\phi$ acts like a delta-function). Decompose $F(t)$ into

$$F(t) \approx \sum_{m=1}^{M} F_m(t) = \sum_{m=1}^{M} \sum_{l=0}^{2N/M-1} \cos(t_{lM+m}) \Phi(t - t_{lM+m}). \tag{2.8}$$

and calculate each terms, we get

$$E_{\text{rad},M} \approx \frac{\pi}{\Delta} R_{\text{rad}} ||\phi||_2^2 \tag{2.9}$$

$$E_{\text{ohm},M} \approx \frac{\pi}{M} R_{\text{ohm}} \tag{2.10}$$

$$E_{\text{far},M} \approx \frac{1}{\Delta^2} ||\phi||_1^2. \tag{2.11}$$

Therefore,

$$E_{\text{eff},M} \approx \frac{||\phi||_1^2}{\pi \Delta R_{\text{rad}} ||\phi||_2^2 + \frac{\pi}{M} R_{\text{ohm}}} \tag{2.12}$$

Since $\phi$ has support of size $O(\Delta)$ and height $O(\Delta^{-1})$, $||\phi||_1 = O(1)$ and $\Delta ||\phi||_2^2 = O(1)$. Hence the numerator and denominator in equation 2.12 are comparable. Note that the contribution of ohmic resistance is $O(M^{-1})$, which is relatively small when $M$ is sufficiently large. However, one must be aware of our assumption that $N \gg M$. If we examine the extreme cases where $M = k_\phi$ and $M = N$, we see

$$E_{\text{ohm},k_\phi} \approx \frac{\pi}{2} R_{\text{ohm}} \tag{2.13}$$

$$E_{\text{ohm},N} \approx \frac{\pi^3}{2M} R_{\text{ohm}} \tag{2.14}$$

This is slightly different from equation 2.10, but the scaling,

$$E_{\text{ohm},M} = O(M^{-1}),$$

seems to hold even outside the regime of $N \gg M$. Hence we from now on we will set $M = N$, i.e. use as many antennas as possible. We will drop the Ohmic resistance term in equation 2.12 since it is independent of $\phi$ and vanishes as $M$ goes large, arriving with a simpler term

$$E_{\text{eff},M} \approx \frac{||\phi||_1^2}{\pi \Delta R_{\text{rad}} ||\phi||_2^2}.$$

Note that the $\Delta$ term in the denominator comes from $\Omega_\phi$. After substitution and dropping some constants we get

$$E_{\text{eff},M} \sim \frac{||\phi||_1^2}{|\Omega_\phi| \cdot ||\phi||_2^2} := E(\phi). \tag{2.15}$$

This is our formula for efficiency.

# 3 Accuracy Constraints

The error of wavelet approximation can be expressed through its Fourier transform, which can be transformed into constraints in regular space as we maximize the efficiency. For simplicity we will introduce the following notation:

$$D_{k,N} = \{\phi \in L_c^2 : \mathcal{F}(\phi)(k + nN) = 0\, \forall n \in \mathbb{Z}\backslash\{0\},\, \mathcal{F}(\phi)(k) = 1\} \tag{3.1}$$

Here is key proposition from frame theory, which will be stated but not proved. For more details see appendix A.

**Proposition 3.1.** *Let $f(t)$ be a band-limited function with bandwidth $\epsilon$ and period $2\pi$. Let $\phi(t)$ be a compactly supported function with $\Omega_\phi \geq \frac{2\pi}{N}$. If*

$$\phi \in D_{1,N} \cap D_{-1,N} \tag{3.2}$$

*then the error of wavelet approximation, using $\phi$, will be $O(\epsilon N^{-1})$, i.e.*

$$\min_{a_1,\ldots,a_n} ||f(t) - \sum_{n=1}^{2N} a_n \phi(t - t_n)||_2 = O(\epsilon N^{-1}) \tag{3.3}$$

According proposition 3.1, equation 3.2 yields a condition where the error can be controlled. However, conditions in Fourier space are in general hard to work with, and hence we replace it with an equivalent condition according to the following proposition.

**Proposition 3.2.** *Suppose $N \geq 3$ and $\phi \in L^2(\mathbb{R})$. Then $\phi \in D_{1,N} \cap D_{-1,N}$ if and only if*

$$Q^\pm_{\phi,p}(t) := \frac{1}{N} \sum_{k=-\infty}^{\infty} Q^\pm_\phi \left(t - \frac{2\pi k}{N}\right) = 1 \tag{3.4}$$

*where $Q^\pm_\phi := e^{\mp it}\phi(t)$.*

*Proof.* Denote $\delta(t - t_0)$ to be the Dirac-delta distribution centered at $t_0$ and recall the Poisson summation formula

$$\mathcal{F}\left(\frac{1}{N} \sum_{k=-\infty}^{\infty} \delta(t - \frac{2\pi k}{N})\right)(\xi) = \sum_{k=-\infty}^{\infty} \delta(\xi - kN) \tag{3.5}$$

Suppose $\phi \in D_{1,N} \cap D_{-1,N}$, then

$$\begin{aligned}
\mathcal{F}(Q^\pm_{\phi,p})(\xi) &= \mathcal{F}\left(e^{\mp it}\phi(t) * \frac{1}{N} \sum_{k=-\infty}^{\infty} \delta(t - \frac{2\pi k}{N})\right)(\xi) \\
&= \mathcal{F}(\phi)(\xi \mp 1) \cdot \left(\sum_{k=-\infty}^{\infty} \delta(\xi - kN)\right) \\
&= \mathcal{F}(\phi)(\pm 1)\delta(\xi) = \delta(\xi) \\
\implies Q^\pm_{\phi,p} &= 1
\end{aligned}$$

so $Q^{\pm}_{\phi,p}$ is a constant function. The proof of converse should easily follow: If $Q^{\pm}_{\phi,p} = 1$, then their Fourier transform must be a Dirac-delta distribution, which implies that $\mathcal{F}(\phi)$ must vanish on other points one the lattices. Hence $\phi \in D_{1,N} \cap D_{-1,N}$. □

An immediate corollary yields the size of minimal support.

**Corollary 3.1.** *For $N \in \mathbb{N}$, $N \geq 3$ and $\Delta = \frac{2\pi}{N}$*

$$\phi_\Delta(t) = \begin{cases} \frac{1}{\Delta}\sin(\Delta - |t|) & \text{if } |t| \leq \Delta \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

*is the unique solution to the optimization problem*

$$\min |\Omega_\phi| \quad \text{s.t. } \phi \in D_{-1,N} \cap D_{1,N} \tag{3.7}$$

*Proof.* Suppose $\phi \in D_{1,N} \cap D_{-1,N}$ with support $\Omega_\phi \subset [-\Delta, \Delta]$. Decompose $\phi$ into $\phi_1(t) = \phi(t - \Delta)\chi_{[0,\Delta]}$ and $\phi_2(t) = \phi(t)\chi_{[0,\Delta]}$ (so $\phi = \phi_1(t + \Delta) + \phi_2(t)$). By proposition 3.2 we have

$$N = e^{-it}(e^{i\Delta}\phi_1 + \phi_2)$$
$$N = e^{it}(e^{-i\Delta}\phi_1 + \phi_2)$$

The solution to the system is

$$\phi_1 = \frac{N}{\sin(\Delta)}\sin(t) \tag{3.8}$$

$$\phi_2 = \frac{N}{\sin(\Delta)}\sin(\Delta - t) \tag{3.9}$$

□

Since $\Omega_\phi$ is proportional to the size of antenna, minimizing it allows us to use small antennas.

# 4 Efficiency Optimization

Recall $E(\phi) = \frac{||\phi||_1^2}{|\Omega_\phi| \cdot ||\phi||_2^2}$. Our objective is to solve

$$\max_\phi E(\phi) \quad \text{s.t. } \phi \in D_{-1,N} \cap D_{1,N} \tag{4.1}$$

a combination of equation 2.15 (for efficiency) and 3.1 (for accuracy). To ensure small support, we will assume $\Omega_\phi = O(\Delta)$, since $2\Delta$ is the lower bound, due to 3.7. We will first perform some analysis on equation 4.1, and solve it with two different approaches. The first one will be an numerical method using convolution, while the second one is an analytic method using periodic extension. It is remarkable that they yield similar results despite the significant difference between those approaches.

Notation-wise, we define $\theta_\Delta(\phi)(t) := \phi(t - \Delta)$ to be the shift operator and $D\phi(t) := \phi'(t)$ to be the differential operator.

## 4.1 Numerical Method

### 4.1.1 Motivation

Given $u \in L^1 \cap L^2$, $E(\phi)$ is bounded by 1 due to Holder inequality:

$$||\phi||_1 \leq ||\phi||_2 ||\chi_{\Omega_\phi}||_2 = ||\phi||_2 \sqrt{|\Omega_\phi|} \implies E(\phi) = \frac{1}{|\Omega_\phi|} \frac{||\phi||_1^2}{||\phi||_2^2} \leq 1.$$

The maximal efficiency is achieved if and only if $\phi$ is a indicator function; however, indicator functions do not satisfy equation 3.2. Hence our best hope is to have some $\phi \in D_{-1,N} \cap D_{1,N}$ that is as close to an indicator function as possible. Since convolution in space preserve the zeros in Fourier space, $\phi_\Delta \in D_{-1,N} \cap D_{1,N}$ implies that $\phi_\Delta * \rho \in D_{-1,N} \cap D_{1,N}$. Hence it is natural to consider the sub-problem

$$\max_{\phi} E(\phi) \quad \text{s.t. } \phi = \phi_\Delta * \rho \tag{4.2}$$

with appropriate restriction on $\rho$. To have $E(\phi_\Delta * \rho)$ well-defined, we need $\phi_\Delta * \rho \in L^1 \cap L^2$. Recall Young's inequality

$$||\phi_\Delta||_r \lesssim ||\phi_\Delta||_p ||\rho||_q \tag{4.3}$$

for $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, 1 \leq p, q, r \leq \infty$. Consider the cases where $(p, q) = (1, 1)$ and $(p, q) = (2, 1)$, and use the fact that $\phi_\Delta \in L^1 \cap L^2$, it is sufficient to have $\rho \in L^1$, which includes some distribution such as Dirac-delta distribution. To ensure small support, we restrict $|\Omega_\rho| = |\Omega_{\phi_\Delta}|$, which implies $\Omega_\phi \leq 2\Delta + 2\Delta = 4\Delta$.

### 4.1.2 Convolution approach

Motivated by 4.2 and the fact that indicator functions are maximizer of $E(u)$, we wish to find a $\rho$ such that $\phi_\Delta * \rho$ is closed to $b = \chi_{[-2\Delta, 2\Delta]}$ (so that its support is of size $4\Delta$), which turns into a least square problem

$$\min_{\rho \in L^1} ||A\rho - b||_2^2 \quad \text{subject to } |\Omega_\rho| = |\Omega_{\phi_\Delta}| \tag{4.4}$$

where we write $\phi_\Delta * \rho = A\rho$ as a matrix multiplication and $b = \chi_{[-2\Delta, 2\Delta]}$. We choose the $L^2$-norm simply because it is simple to solve; one may study the problem using other norms. In later section, we will denote $\rho_\Delta$ to be the minimizer of equation 4.4. To perform a numerical optimization, we set $K$ to be the number of grids used for discretization. According to our numerical simulation, for fixed $N$, $E$ stabilize around 0.9375 as $K$ increases, which is significantly higher than $E(\phi_\Delta)$.

### 4.1.3 Numerical solution

We first plot the efficiency $E$ versus the time discretization $K$. Then plot $\rho_\Delta$ and $\phi_\Delta * \rho_\Delta$. To maintain the assumption $K \gg N \gg 1$, we set $K \geq 10N$ and $N \geq 10$.

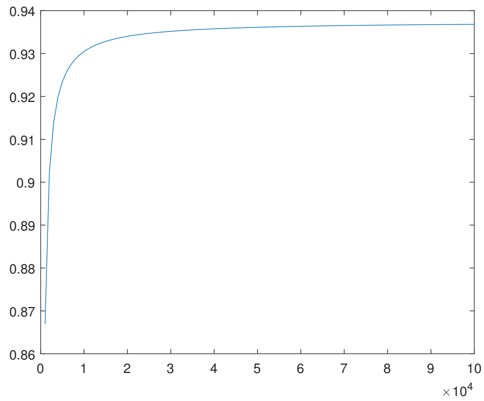


Figure 2: For $N = 100$, efficiency stabilizes around 0.937 as $K$ increases.

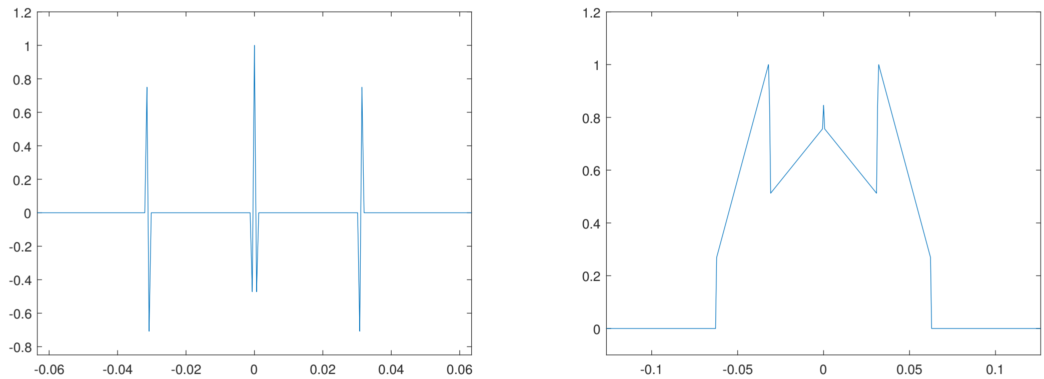


Figure 3: Left: $\rho_\Delta$. Right: $\phi_\Delta * \rho_\Delta$. In $\rho_\Delta$ adjacent spikes have roughly the same magnitude but opposite sign, which suggests that $\rho_\Delta$ might contain derivative operator, in distribution sense. Note that there are discontinuities in $\phi_\Delta * \rho_\Delta$; they are not edges with steep slope.

We investigate this seemingly simple structure of $\rho_\Delta$, with the goal of extracting an analytic solution that has a simple form. With some calculation (see appendix C) we get

$$\rho_\Delta \approx d_0 + d_1(\theta_\Delta + \theta_{-\Delta}) - d_2(\theta_\Delta - \theta_{-\Delta})D \tag{4.5}$$

for some positive constants $d_0, d_1, d_2$. The coefficients converges according to figure 4.

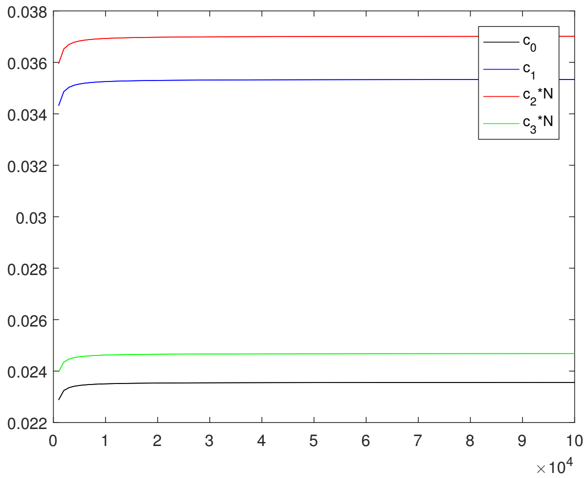


Figure 4: Numerical simulation shows that the coefficients converge as $K \to \infty$.

## 4.2 Analytic Method

We will attempt to solve problem 4.1 analytically. To make a distinction with the numerical solution, we use $\varphi$ instead of $\phi$.

### 4.2.1 Formulation

By applying proposition 3.2 and assuming $||\varphi||_1 = 1$, Our objective becomes

$$\min_{\varphi} ||\varphi||_2^2 \quad \text{subject to } Q^{\pm}_{p,\varphi} = 1 \tag{4.6}$$

Suppose $\varphi$ is supported on $[-2\Delta, 2\Delta]$, where $\Delta = \frac{1}{N}$. Define $\varphi_n(t) := \phi(t + (n-3)\Delta)\chi_{[0,\Delta]}$ for $1 \leq n \leq 4$ (equivalently, $\varphi(t) = \varphi_1(t + 2\Delta) + \varphi_2(t + \Delta) + \varphi_3(t) + \varphi_4(t - \Delta)$). Then the constraints become

$$N = Q^{+}_{u,p} = e^{-it}[e^{2i\Delta}\varphi_1 + e^{i\Delta}\varphi_2 + \varphi_3 + e^{-i\Delta}\varphi_4] \tag{4.7}$$
$$N = Q^{-}_{u,p} = e^{it}[e^{-2i\Delta}\varphi_1 + e^{-i\Delta}\varphi_2 + \varphi_3 + e^{i\Delta}\varphi_4] \tag{4.8}$$

After some algebraic manipulation and setting derivatives equal to zeros, we get the minimizer

$$\begin{bmatrix} \varphi_{1\Delta} \\ \varphi_{2\Delta} \\ \varphi_{3\Delta} \\ \varphi_{4\Delta} \end{bmatrix} = c \begin{bmatrix} 8\cos^3(\Delta) & 4\cos^2(\Delta) - 2 \\ 4\cos^2(\Delta) + 2 & 4\cos(\Delta) \\ 4\cos(\Delta) & 4\cos^2(\Delta) + 2 \\ 4\cos^2(\Delta) - 2 & 8\cos^3(\Delta) \end{bmatrix} \begin{bmatrix} \sin(t) \\ \sin(\Delta - t) \end{bmatrix} \tag{4.9}$$

where $c = \frac{N}{4\sin(\Delta)(1+4\cos^4(\Delta))}$. The solution is shown in figure 4.2.1, which looks fairly similar to figure 3. Let $\varphi_\Delta(t) = \varphi_{1\Delta}(t + 2\Delta) + \varphi_{2\Delta}(t + \Delta) + \varphi_{3\Delta}(t) + \varphi_{4\Delta}(t - \Delta)$. With some computation (see appendix D) we show that

$$\varphi_\Delta = \phi_\Delta * \eta_\Delta \tag{4.10}$$

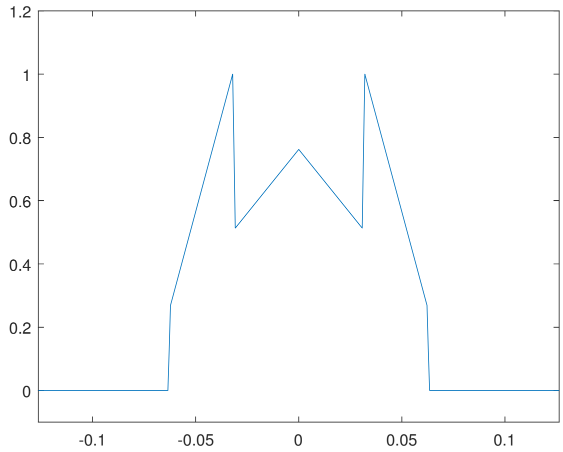


Figure 5: Analytic Minimizer

with

$$\eta_\Delta = c_0 + c_1(\theta_\Delta + \theta_{-\Delta}) - c_2(\theta_\Delta - \theta_{-\Delta})D \tag{4.11}$$

for some positive constants $c_0, c_1, c_2$.

## 4.3 Methods Comparison

Observing the similarity between equation 4.11 and 4.5:

$$\rho_\Delta \approx d_0 + d_1(\theta_\Delta + \theta_{-\Delta}) - d_2(\theta_\Delta - \theta_{-\Delta})D$$
$$\eta_\Delta = c_0 + c_1(\theta_\Delta + \theta_{-\Delta}) - c_2(\theta_\Delta - \theta_{-\Delta})D$$

we suspected that there is some connection between our numerical and analytic scheme. To verify if they are exactly the same, it suffices to check whether $\eta_\Delta$ is a minimizer of

$$\min_{\rho \in L_c^2([-\Delta,\Delta])} ||A\rho - b||_2^2 \quad \text{subject to } |\Omega_\rho| = |\Omega_{\phi_\Delta}|$$

where $A = \phi_\Delta *$ is the convolution operator and $b = \chi_{[-2\Delta,2\Delta]}$. Differentiating $||A\rho - b||_2^2$, we get

$$\frac{d}{d\rho}||A\rho - b||_2^2 = \chi_{[-\Delta,\Delta]} A^*(A\rho - b).$$

Since $A^* = \phi_\Delta(-\cdot)* = \phi_\Delta *$,

$$\begin{aligned}\chi_{[-\Delta,\Delta]} A^*(A\rho - b) &= \chi_{[-\Delta,\Delta]}\phi_\Delta * (\phi_\Delta * \eta_\Delta - b) \\ &= \chi_{[-\Delta,\Delta]}\phi_\Delta * (\varphi_\Delta - b)\end{aligned}$$

Hence it suffices to compute

$$\chi_{[-\Delta,\Delta]}\phi_\Delta * (\varphi_\Delta - b).$$

After some computation (see appendix E) the above expression turns out to be nonzero, meaning that although figure 3 and figure 4.2.1 look quite similar, they are not the same.

The analytic solution $\varphi_\Delta$ is the best solution among the functions in $L^2([-2\Delta, 2\Delta])$, whose expression is explicit and easy to analyzed. However, it is highly sensitive to the support size; if we are considering larger space of candidates, $L^2([-3\Delta, 3\Delta])$ for instance, the analytic method becomes much complicated and the solution will not be unique. On the other hand, although the numerical solution $\phi_\Delta * \rho_\Delta$ has some point-wise defection, it is simple enough for any reasonable support size, and its performance increases as the support size increases. See figure 4.3. Hence the numerical method might be a more practical approach.

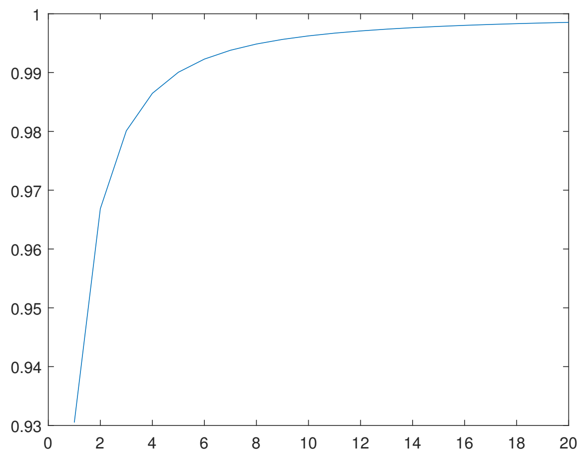


Figure 6: $E(\phi_\Delta * \rho_\Delta)$ versus maximal support size.

# 5 Wavelet Comparison

We compute and compare the efficiency of three different frames.

## 5.1 $\phi_{\text{Gauss}}$

$\phi_{\text{Gauss}}$ is a truncated (for compact support) and shifted (for continuity) Gaussian, which has been implemented quite often. It is defined as

$$\phi_{\text{Gauss}}(t) =: \begin{cases} e^{-\frac{t^2}{2\sigma^2}} - e^{-\frac{\Delta^2}{2\sigma^2}} & \text{if } |t| \le \Delta \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

where to ensure continuity on the boundaries. For simplicity, we will write $\alpha = \frac{\Delta}{\sigma}$ and define error function in the conventional way

$$\text{erf}(t) := \frac{1}{\sqrt{\pi}} \int_{-\tau}^{x} e^{-\tau^2} d\tau.$$

Note that due to change of variable, $\int_{-\Delta}^{\Delta} e^{-\frac{x^2}{\sigma^2}} dx = \int_{-\alpha}^{\alpha} e^{-x^2} \sigma dx = \sqrt{\pi}\sigma \mathrm{erf}(\alpha)$. After some computation we get

$$E(\phi_{\mathrm{Gauss}}(\alpha)) = \frac{1}{2\alpha} \frac{(\sqrt{2\pi}\mathrm{erf}(\frac{\alpha}{\sqrt{2}}) - 2e^{-\frac{\alpha^2}{2}}\alpha)^2}{\sqrt{\pi}\mathrm{erf}(\alpha) - 2e^{-\frac{\alpha^2}{2}}\sqrt{2\pi}\mathrm{erf}(2^{-\frac{1}{2}}\alpha) + 2e^{-\alpha^2}\alpha} \tag{5.2}$$

which only depends on $\alpha$. The maximal efficiency is around 0.83, which is achieved when $\alpha$ is small. Although $\phi_{\mathrm{Gauss}}$ does not satisfy equation 3.1 and hence cannot perfectly reconstruct a sinusoidal signal, we will nevertheless set 0.83 as a bar for our efficiency.

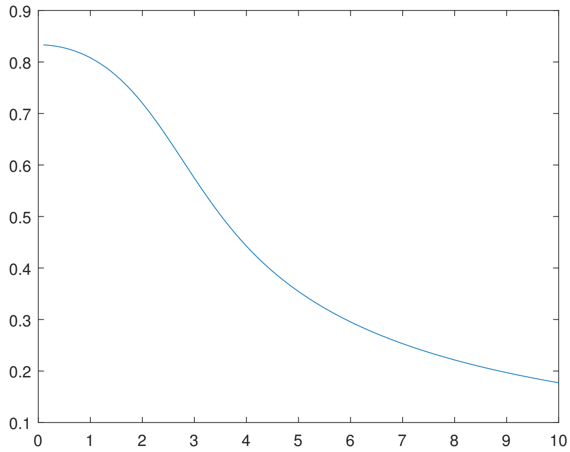


Figure 7: $E(\phi_{\mathrm{Gauss}})$ versus $\alpha$

### 5.2 $\phi_\Delta$

By simple calculus, we get

$$||\phi_\Delta||_1 = \int_{-\Delta}^{\Delta} \sin(\Delta - |t|)dt = 2\int_0^{\Delta} \sin(\Delta - x)dt = 2(1 - \cos(\Delta))$$

$$||\phi_\Delta||_2^2 = \int_{-\Delta}^{\Delta} \sin^2(\Delta - |t|)dt = \int_0^{\Delta} 1 - \cos(2\Delta - 2t)dt = \Delta - \frac{1}{2}\sin(2\Delta)$$

$$E(\phi_\Delta) = \frac{1}{2\Delta}\frac{4(1-\cos(x))^2}{\Delta - \frac{1}{2}\sin(2\Delta)} \approx \frac{4\frac{1}{4}\Delta^4}{2\Delta(\frac{1}{2}\frac{8}{6}\Delta^3)} = \frac{3}{4} \tag{5.3}$$

Compared to $\phi_{\mathrm{Gauss}}$, $\phi_\Delta$ has higher accuracy for band-limited functions but lower efficiency.

### 5.3 $\varphi_\Delta$

With some computation (see appendix F), we get

$$E(\phi_\Delta * \eta) \approx \frac{1}{4\Delta}\frac{15\pi^2\Delta^4}{4\pi^2\Delta^3} = \frac{15}{16} = 0.9375 \tag{5.4}$$

which has efficiency that is much higher than both $\phi_\Delta$ and $\phi_{\mathrm{Gauss}}$, and in fact not too far away from 1, which is the optimal efficiency if we ignore accuracy.

## 6 Conclusion

In this paper we derive a formulation of efficiency, with the condition that we will use many short antennas. We use it as our objective function. Frame theory provides error analysis on the accuracy of our wavelet, which is implemented as constraints in our optimization problem. Two approaches are proposed, and yield similar results. Our wavelet out perform some typical ones in both accuracy and efficiency.

## 7 Acknowledgments

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## Appendix A Accuracy of Frame Projection

**Definition A.1.** Let $\epsilon > 0$, the space of $\epsilon$-band-limited functions $H_\epsilon$ is given by

$$H_\epsilon = \{f \in L^2 \ s.t. \ \mathcal{F}(f)(\xi) = 0 \text{ if } |\xi| \geq \epsilon\} \tag{A.1}$$

**Definition A.2.** Let $\phi \in L^2(\mathbb{R})$ with $\|\phi\|_2 = 1$. The partial wavelet space of $\phi$, $H_\phi$ is defined by

$$S(\phi) = \overline{\text{span}\{\phi(x-n)\}_{n\in\mathbb{Z}}} \tag{A.2}$$

**Definition A.3.** Let $\phi \in L^2(\mathbb{R})$ and $\epsilon > 0$. The accuracy of $\phi$ at bandwidth $\epsilon$ is

$$E_\phi(\epsilon) = \sup_{f\in H_\epsilon,\ \|f\|_2=1} \|P_{S(\phi)}f - f\|_2^2 \tag{A.3}$$

where $P$ stands for projection.

**Theorem A.1.** *Assume that $\phi \in L^2(\mathbb{R})$ has compact support and*

$$R_\phi(\xi) := \sum_{n\in\mathbb{Z}} |\mathcal{F}(\phi)(\xi - n)|^2 \tag{A.4}$$

*is bounded and bounded away from 0. Let $k > 0$ be an integer. Then*

$$E_\phi(\epsilon) = O(\epsilon^{2k}) \tag{A.5}$$

*iff $\mathcal{F}(\phi)^{(l)}(n) = 0$ for all $n \in \mathbb{Z}\backslash\{0\}$ and $l = 0, ..., k-1$, i.e. if and only if the Fourier transform of $\phi$ vanishes to order $k$ at all non-zero integers.*

# Appendix B Efficiency and Number of Antennas

## B.1 $M = 2$

Consider the case where $M = 2$ and $u = u_{\text{sinc2}}$.

$$F(x) = F_1(x) + F_2(x) = \sum_{n=1}^{N} \cos(x_{2n-1}) U(x - x_{2n-1}) + \sum_{n=1}^{N} \cos(x_{2n}) U(x - x_{2n}).$$

Then

$$\begin{aligned}
\frac{E_{\text{ohm}}(F_1)}{R_{\text{ohm}}} &= \int \left( \sum_{n=1}^{N} \cos(x_{2n-1}) U(x - x_{2n-1}) \right)^2 dx \\
&= \sum_{n=1}^{N} \int_{x_{2n-2}}^{x_{2n}} \left[ \cos(x_{2n-1}) U(x - x_{2n-1}) + \left( \sum_{k=0}^{n-1} \cos(x_{2k-1}) ||u||_1 \right) \right]^2 dx \\
&= \sum_{n=1}^{N} \int_{x_{2n-2}}^{x_{2n}} \left[ \cos(x_{2n-1}) U(x - x_{2n-1}) \right]^2 dx \\
&\quad + 2 \sum_{n=1}^{N} \int_{x_{2n-2}}^{x_{2n}} \left( \sum_{k=0}^{n-1} \cos(x_{2k-1}) ||u||_1 \right) \cos(x_{2n-1}) U(x - x_{2n-1}) dx \\
&\quad + \sum_{n=1}^{N} \int_{x_{2n-2}}^{x_{2n}} \left( \sum_{k=0}^{n-1} \cos(x_{2k-1}) ||u||_1 \right)^2 dx \\
&= ||U||_{2,\Delta}^2 \sum_{n=1}^{N} \cos^2(x_{2n-1}) + 2||u||_1 ||U||_{1,\Delta} \sum_{n=1}^{N} \left( \sum_{k=0}^{n-1} \cos(x_{2k-1}) \right) \cos(x_{2n-1}) \\
&\quad + ||u||_1^2 (2\Delta) \sum_{n=1}^{N} \left( \sum_{k=0}^{n-1} \cos(x_{2k-1}) \right)^2 \\
&\approx ||U||_{2,\Delta}^2 \sum_{n=1}^{N} \cos^2(x_{2n-1}) + 2||u||_1 ||U||_{1,\Delta} \underbrace{\sum_{n=1}^{N} -\frac{1}{2\Delta} \sin(x_{2n-3}) \cos(x_{2n-1})}_{=0 \text{ due to periodicity}} \\
&\quad + 2\Delta ||u||_1^2 \sum_{n=1}^{N} \left(\frac{1}{2\Delta} \sin(x_{2n-3})\right)^2 \\
&\approx \underbrace{\frac{\pi}{2\Delta} ||U||_{2,\Delta}^2}_{O(\Delta^2)} + \underbrace{\frac{\pi}{4\Delta^2} ||u||_1^2}_{O(1)} \approx \frac{\pi}{4\Delta^2} ||u||_1^2 \quad \text{(when } \Delta \text{ is small)}
\end{aligned}$$

Hence

$$E_{\text{ohm},\{F_m\}_{m=1}^{M}} \approx 2E_{\text{ohm}}(F_1) = \frac{\pi}{2\Delta^2} R_{\text{ohm}} ||u||_1^2 \approx \frac{\pi}{2} R_{\text{ohm}}$$

## B.2 $M = N$

Consider the case where $M = N$.

$$F(x) = \sum_{m=1}^{N} F_m(x) = \sum_{m=1}^{N} \cos(x_m)[U(x - x_m) - U(x - x_{N+m})]$$

that is, each $F_n$ goes up once, stays constant for (almost) half a period, then goes down. Then

$$\begin{aligned}
\frac{E_{\text{ohm},\{F_m\}_{m=1}^M}}{R_{\text{ohm}}} &= \sum_{m=1}^{M} \int_0^{2\Delta} F_m^2(x)dx + \int_{2\Delta}^{\pi} F_m^2(x)dx + \int_{\pi}^{\pi+2\Delta} F_m^2(x)dx \\
&= \sum_{m=1}^{M} \cos^2(x_m) \left(2||U||_{2,\Delta}^2 + (\pi - 2\Delta)||u||_1^2\right) \\
&\approx \frac{\pi}{2\Delta}(\underbrace{2||U||_{2,\Delta}^2}_{O(\Delta^3)} + \underbrace{(\pi - 2\Delta)||u||_1^2}_{O(\Delta^2)}) \approx \frac{\pi^2}{2\Delta}||u||_1^2 \quad \text{(when } \Delta \text{ is small)}
\end{aligned}$$

Hence

$$E_{\text{ohm},\{F_m\}_{m=1}^M} = \frac{\pi^2}{2\Delta} R_{\text{ohm}} ||u||_1^2 \approx \frac{\pi^3}{2M} R_{\text{ohm}}$$

# Appendix C Analysis on $\rho$

We will investigate this seemingly simple structure of $\rho$, with the goal of extracting an analytic solution that has a simple form. Let $\Delta = \frac{\pi}{K}$, $\Delta = \frac{\pi}{N}$, and $h$ to be the Dirac delta function, we can approximate $\rho$ by

$$\begin{aligned}
\rho(x) \approx &(a + b)h(x + \Delta - \Delta) - ah(x + \Delta - 2\Delta) - ch(x + 2\Delta) + (c + d)h(x + \Delta) \\
&+ (a + b)h(x - \Delta + \Delta) - ah(x - \Delta + 2\Delta) - ch(x - 2\Delta) + (c + d)h(x - \Delta)
\end{aligned}$$

Using Taylor expansion, we get

$$\begin{aligned}
h(x + \Delta - \Delta) - h(x + \Delta - 2\Delta) &\approx \Delta h'(x + \Delta) \\
h(x - \Delta + \Delta) - h(x - \Delta + 2\Delta) &\approx -\Delta h'(x - \Delta) \\
-h(x + 2\Delta) + h(x + \Delta) - h(x - 2\Delta) + h(x - \Delta) &\approx -3\Delta^2 h''(x)
\end{aligned}$$

where $h'$ and $h''$ are defined in the distribution sense. Therefore

$$\rho(x) \approx a\Delta[h'(x + \Delta) - h'(x - \Delta)] - 3c\Delta^2 h''(x) + b[h(x + \Delta) + h(x - \Delta)] + 2dh(x) \tag{C.1}$$

Numerically, $a$ and $c$ scale like $\frac{\Delta}{\Delta}$, which suggests that the effect of $h''$ is much less significant than $h'$. Although $\phi_\Delta(x)$ is not differentiable at $x \in \{-\Delta, 0, \Delta\}$, for now we will define its first and second derivative as:

$$\phi'_\Delta(x) = \begin{cases} 1 & \text{if } x \in (-\Delta, 0) \\ -1 & \text{if } x \in (0, \Delta) \\ 0 & \text{otherwise} \end{cases}$$

$$\phi''_\Delta(x) = h(x + \Delta) - 2h(x) + h(x - \Delta).$$

Furthermore, define the differential operator and shift operator as follows:

$$D\phi(x) := \phi'(x), \quad \theta_\Delta \phi(x) := \phi(x-\Delta).$$

With such notation, we derive an approximation to $\phi_\Delta * \rho(x)$ by

$$\begin{aligned}
\phi_\Delta * \rho(x) &\approx \int \phi_\Delta(x-y) \left\{-a\Delta[\theta_\Delta - \theta_{-\Delta}]D - 3c\Delta^2 D^2 + b[\theta_\Delta + \theta_{-\Delta}] + 2d\right\} h(y)dy \\
&= \left\{-a\Delta[\theta_\Delta - \theta_{-\Delta}]D - 3c\Delta^2 D^2 + b[\theta_\Delta + \theta_{-\Delta}] + 2d\right\} \phi_\Delta(x) \\
&= \{-\tilde{a}[\theta_\Delta - \theta_{-\Delta}]D + b[\theta_\Delta + \theta_{-\Delta}] + 2d\} \phi_\Delta(x) - 3\tilde{c}\Delta[h(x+\Delta) + h(x-\Delta)] \\
&:= [d_0 + d_1(\theta_\Delta + \theta_{-\Delta}) - d_2(\theta_\Delta - \theta_{-\Delta})D]\, \phi_\Delta(x) - 3d_3(1_{\{x=-\Delta\}} - 2_{\{x=0\}} + 1_{\{x=\Delta\}})
\end{aligned}$$

Ignoring the last few terms, which take a finite value on a signal point, we get equation 4.5.

# Appendix D Analysis on $\eta$

To write $\varphi_\Delta = \phi_\Delta * \eta$, we will calculate the Fourier transform of $\phi_\Delta$ and $\varphi_\Delta$. For simplicity, we will drop the normalizing constant. We will first calculate the Fourier transform of elementary pieces $v(\xi)$.

$$\begin{aligned}
v(\xi) &:= \int_0^\Delta e^{-ix\xi} \sin(\Delta - x)dx \\
&= \frac{1}{2i} \int_0^\Delta e^{-ix\xi + i\Delta - ix} - e^{-ix\xi - i\Delta + ix} dx \\
&= \frac{1}{2(\xi^2-1)} \left[(\xi-1)(e^{-i\Delta\xi} - e^{i\Delta}) - (\xi+1)(e^{-i\Delta\xi} - e^{-i\Delta})\right] \\
&= \frac{1}{\xi^2-1}[-i\xi\sin(\Delta) + \cos(\Delta) - e^{-i\Delta\xi}] \\
\mathcal{F}(\phi_\Delta)(\xi) &= v(\xi) + v(-\xi) \\
&= \frac{2}{\xi^2-1}[\cos(\Delta) - \cos(\Delta\xi)] \\
&= \frac{4}{\xi^2-1} \sin\left(\frac{\Delta}{2}(\xi+1)\right) \sin\left(\frac{\Delta}{2}(\xi-1)\right)
\end{aligned}$$

Define $w(\xi) := \int_0^\Delta e^{-ix\xi}\sin(x)dx = e^{-i\Delta\xi}v(-\xi)$, we get

$$
\begin{aligned}
\mathcal{F}(\varphi_\Delta)^+(\xi) &= 4\cos(\Delta)w(\xi) + (4\cos^2(\Delta)+2)v(\xi) + (4\cos^2(\Delta)-2)e^{-i\Delta\xi}w(\xi) + 8\cos^3(\Delta)e^{-i\Delta\xi}v(\xi) \\
&= 4\cos(\Delta)e^{-i\Delta\xi}v(-\xi) + (4\cos^2(\Delta)+2)v(\xi) \\
&\quad + (4\cos^2(\Delta)-2)e^{-2i\Delta\xi}v(-\xi) + 8\cos^3(\Delta)e^{-i\Delta\xi}v(\xi) \\
\mathcal{F}(\varphi_\Delta)(\xi) &= \mathcal{F}(\varphi_\Delta)^+(\xi) + \mathcal{F}(\varphi_\Delta)^+(-\xi) \\
&= \begin{bmatrix} v(\xi)+v(-\xi) & v(\xi)-v(-\xi) \end{bmatrix} \cdot \\
&\begin{bmatrix} 4\cos(\Delta)\cos(\Delta\xi) + 4\cos^2(\Delta) + 2 + (4\cos^2(\Delta)-2)\cos(2\Delta\xi) + 8\cos^3(\Delta)\cos(\Delta\xi) \\ 4i\cos(\Delta)\sin(\Delta\xi) + i(4\cos^2(\Delta)-2)\sin(2\Delta\xi) - 8i\cos^3(\Delta)\sin(\Delta\xi) \end{bmatrix} \\
&= \begin{bmatrix} v(\xi)+v(-\xi) & v(\xi)-v(-\xi) \end{bmatrix} \cdot \\
&\begin{bmatrix} 4 + (4\cos(\Delta) + 8\cos^3(\Delta))\cos(\Delta\xi) + (8\cos^2(\Delta)-4)\cos^2(\Delta\xi) \\ i(4\cos(\Delta) - 8\cos^3(\Delta) + 8\cos^2(\Delta)\cos(\Delta\xi) - 4\cos(\Delta\xi))\sin(\Delta\xi) \end{bmatrix} \\
&= \frac{8}{\xi^2-1}\begin{bmatrix} \cos(\Delta)-\cos(\Delta\xi) & \xi\sin(\Delta)-\sin(\Delta\xi) \end{bmatrix} \cdot \\
&\begin{bmatrix} 1 + \cos(\Delta)(1+2\cos^2(\Delta))\cos(\Delta\xi) + \cos(2\Delta)\cos^2(\Delta\xi) \\ -\cos(2\Delta)(\cos(\Delta)-\cos(\Delta\xi))\sin(\Delta\xi) \end{bmatrix} \\
&= a[1 + \cos(2\Delta) + \cos(\Delta)(1+2\cos^2(\Delta))\cos(\Delta\xi) - \xi\sin(\Delta)\cos(2\Delta)\sin(\Delta\xi)]
\end{aligned}
$$

where $a = \frac{8(\cos(\Delta)-\cos(\Delta\xi))}{\xi^2-1}$. Therefore, adding back the normalization constant, we get

$$
\mathcal{F}(\phi_\Delta)(\xi) = \frac{1}{\Delta^2}\frac{2}{\xi^2-1}[\cos(\Delta)-\cos(\Delta\xi)] \tag{D.1}
$$

$$
\mathcal{F}(\varphi_\Delta)(\xi) = 4c\Delta^2\mathcal{F}(\phi_\Delta)(\xi)[1 + \cos(2\Delta) + \cos(\Delta)(1+2\cos^2(\Delta))\cos(\Delta\xi) - \xi\sin(\Delta)\cos(2\Delta)\sin(\Delta\xi)] \tag{D.2}
$$

If we wish to write $\varphi_\Delta$ in terms of $\phi_\Delta * \eta$, then

$$
\eta = 2c\Delta^2[2 + 2\cos(2\Delta) + \cos(\Delta)(1+2\cos^2(\Delta))(\theta_\Delta + \theta_{-\Delta}) - \sin(\Delta)\cos(2\Delta)(\theta_\Delta - \theta_{-\Delta})D] \tag{D.3}
$$

where $\theta_\Delta(u)(x) := u(x-\Delta)$ is the shift operator and $D\phi(x) := u'(x)$ is the differential operator. $\eta$ is interpreted in the sense of distribution, whose validity is justified by Young's inequality, and such model is again consistent with our numerical approach.

# Appendix E Minimizer Validation

For simplicity, define

$$
\begin{aligned}
c_1 &= 8c\cos^3(\Delta) \\
c_2 &= 4c\cos^2(\Delta) - 2 \\
c_3 &= 4c\cos^2(\Delta) + 2 \\
c_4 &= 4c\cos(\Delta)
\end{aligned}
$$

$$
\phi_\Delta * \varphi_\Delta(x) = \int \phi_\Delta(y)\varphi_\Delta(x-y)dy = \int_{-\Delta}^{-\Delta+x} + \int_{-\Delta+x}^{0} + \int_0^x + \int_x^\Delta = I_1 + I_2 + I_3 + I_4
$$

$$
\begin{aligned}
2I_1 &= 2\int_{-\Delta}^{-\Delta+x} \sin(\Delta+y)[c_1\sin(2\Delta-x+y)+c_2\sin(-\Delta+x-y)]dy \\
&= \int_{-\Delta}^{-\Delta+x} c_1[\cos(\Delta-x)-\cos(3\Delta-x+2y)]+c_2[\cos(2\Delta-x+2y)-\cos(x)]dy \\
&= c_1[x\cos(\Delta-x)-\frac{1}{2}\sin(\Delta+x)+\frac{1}{2}\sin(\Delta-x)] \\
&\quad + c_2[\frac{1}{2}\sin(x)-\frac{1}{2}\sin(-x)-x\cos(x)] \\
&= c_1[x\cos(\Delta-x)-\sin(x)\cos(\Delta)]+c_2[\sin(x)-x\cos(x)]
\end{aligned}
$$

$$
\begin{aligned}
2I_2 &= 2\int_{-\Delta+x}^{0} \sin(\Delta+y)[c_3\sin(\Delta-x+y)+c_4\sin(x-y)]dy \\
&= \int_{-\Delta+x}^{0} c_3[\cos(x)-\cos(2\Delta-x+2y)]+c_4[\cos(\Delta-x+2y)-\cos(\Delta+x)]dy \\
&= c_3[(\Delta-x)\cos(x)-\frac{1}{2}\sin(2\Delta-x)+\frac{1}{2}\sin(x)] \\
&\quad + c_4[\frac{1}{2}\sin(\Delta-x)-\frac{1}{2}\sin(-\Delta+x)-(\Delta-x)\cos(\Delta+x)] \\
&= c_3[(\Delta-x)\cos(x)+\sin(x-\Delta)\cos(\Delta)]+c_4[\sin(\Delta-x)-(\Delta-x)\cos(\Delta+x)]
\end{aligned}
$$

$$
\begin{aligned}
2I_3 &= \int_{0}^{x} \sin(\Delta-y)[c_3\sin(\Delta-x+y)+c_4\sin(x-y)]dy \\
&= \int_{0}^{x} c_3[\cos(x-2y)-\cos(2\Delta-x)]+c_4[\cos(\Delta-x)-\cos(\Delta+x-2y)]dy \\
&= c_3[-\frac{1}{2}\sin(-x)+\frac{1}{2}\sin(x)-x\cos(2\Delta-x)] \\
&\quad + c_4[x\cos(\Delta-x)+\frac{1}{2}\sin(\Delta-x)-\frac{1}{2}\sin(\Delta+x)] \\
&= c_3[\sin(x)-x\cos(2\Delta-x)]+c_4[x\cos(\Delta-x)-\sin(x)\cos(\Delta)]
\end{aligned}
$$

$$
\begin{aligned}
2I_4 &= 2\int_{x}^{\Delta} \sin(\Delta-y)[c_4\sin(-x+y)+c_3\sin(\Delta+x-y)]dy \\
&= \int_{x}^{\Delta} c_4[\cos(\Delta+x-2y)-\cos(\Delta-x)]+c_3[\cos(x)-\cos(2\Delta+x-2y)]dy \\
&= c_4[-\frac{1}{2}\sin(-\Delta+x)+\frac{1}{2}\sin(\Delta-x)-(\Delta-x)\cos(\Delta-x)] \\
&\quad + c_3[(\Delta-x)\cos(x)+\frac{1}{2}\sin(x)-\frac{1}{2}\sin(2\Delta-x)] \\
&= c_4[\sin(\Delta-x)-(\Delta-x)\cos(\Delta-x)]+c_3[(\Delta-x)\cos(x)+\sin(x-\Delta)\cos(\Delta)]
\end{aligned}
$$

Therefore for $x \in [0, \Delta]$,

$$
\begin{aligned}
\frac{2}{c}\phi_\Delta * \varphi_\Delta(x) &= I_1 + I_2 + I_3 + I_4 \\
&= c_1[x(\cos(\Delta)\cos(x) + \sin(\Delta)\sin(x)) - \cos(\Delta)\sin(x)] + c_2[\sin(x) - x\cos(x)] \\
&\quad + c_3[2(\Delta - x)\cos(x) + 2\cos^2(\Delta)\sin(x) - 2\sin(\Delta)\cos(\Delta)\cos(x)] \\
&\quad + c_3[\sin(x) - x(\cos(2\Delta)\cos(x) + \sin(2\Delta)\sin(x))] \\
&\quad + c_4[2(x - \Delta)\cos(\Delta)\cos(x) + 2\sin(\Delta)\cos(x) - 3\cos(\Delta)\sin(x)] \\
&\quad + c_4[x(\cos(\Delta)\cos(x) + \sin(\Delta)\sin(x))] \\
&= [c_1 x\cos(\Delta) - c_2 x + c_3(2(\Delta - x) - 2\sin(\Delta)\cos(\Delta) - x\cos(2\Delta))]\cos(x) \\
&\quad + [c_4(2(x - \Delta)\cos(\Delta) + 2\sin(\Delta) + x\cos(\Delta))]\cos(x) \\
&\quad + [c_1(x\sin(\Delta) - \cos(\Delta)) + c_2 + c_3(1 + 2\cos^2(\Delta) - x\sin(2\Delta))]\sin(x) \\
&\quad + [c_4(-3\cos(\Delta) + x\sin(\Delta))]\sin(x) \\
&= [8x\cos^4(\Delta) - 4x\cos^2(\Delta) + 2x + (\Delta - x)(8\cos^2(\Delta) + 4)]\cos(x) \\
&\quad + [-8\sin(\Delta)\cos^3(\Delta) - 4\sin(\Delta)\cos(\Delta) - 8x\cos^4(\Delta) + 2x]\cos(x) \\
&\quad + [8(x - \Delta)\cos^2(\Delta) + 8\sin(\Delta)\cos(\Delta) + 4x\cos^2(\Delta)]\cos(x) \\
&\quad + [8x\sin(\Delta)\cos^3(\Delta) - 8\cos^4(\Delta) + 4\cos^2(\Delta) - 2 + 4\cos^2(\Delta) + 2]\sin(x) \\
&\quad + [8\cos^4(\Delta) + 4\cos^2(\Delta) - 8x\sin(\Delta)\cos^3(\Delta) - 4x\sin(\Delta)\cos(\Delta)]\sin(x) \\
&\quad + [-12\cos^2(\Delta) + 4x\sin(\Delta)\cos(\Delta)]\sin(x) \\
&= 4[\Delta + \sin(\Delta)\cos(\Delta)(1 - 2\cos^2(\Delta))]\cos(x) + 0 \cdot \sin(x) \\
&= [4\Delta - \sin(4\Delta)]\cos(x)
\end{aligned}
$$

so by symmetry we have

$$
\phi_\Delta * \varphi_\Delta(x) = \frac{N(4\Delta - \sin(4\Delta))}{8\sin(\Delta)(1 + \cos^4(\Delta))}\cos(x) \tag{E.1}
$$

# Appendix F $E(\varphi_\Delta)$

Recall $\varphi_\Delta = \phi_\Delta * \eta$ and

$$
\begin{aligned}
\eta &= 2c\Delta^2[2 + 2\cos(2\Delta) + \cos(\Delta)(1 + 2\cos^2(\Delta))(\theta_\Delta + \theta_{-\Delta}) - \sin(\Delta)\cos(2\Delta)(\theta_\Delta - \theta_{-\Delta})D] \\
&= c_0 + c_1(\theta_\Delta + \theta_{-\Delta}) + c_2(\theta_\Delta - \theta_{-\Delta})D \\
c &= \frac{N}{4\sin(\Delta)(1 + 4\cos^4(\Delta))}
\end{aligned}
$$

Use first order approximation, we have $c_0 \approx \frac{4\pi}{5}$, $c_1 \approx \frac{3\pi}{5}$, $c_2 \approx \frac{\pi}{5}\Delta$. Hence

$$\begin{aligned}
||\phi_\Delta * \eta||_1 &= 2\int_{-2\Delta}^{-\Delta} \phi_\Delta * \eta + 2\int_{-\Delta}^{0} \phi_\Delta * \eta \\
&= 2\int_{-2\Delta}^{-\Delta} c_1 \sin(2\Delta + x) + c_2 dx + 2\int_{-\Delta}^{0} c_0 \sin(\Delta + x) + c_1 \sin(-x) - c_2 dx \\
&= 2(c_1(1-\cos(\Delta)) + c_2\Delta) + 2((c_0 + c_1)(1-\cos(\Delta)) - c_2\Delta) \\
&= (2c_0 + 4c_1)(1-\cos(\Delta)) \\
&\approx \frac{\pi}{5}(8+12)\frac{\Delta^2}{2} = 2\pi\Delta^2
\end{aligned}$$

$$\begin{aligned}
||\phi_\Delta * \eta||_2^2 &= 2\int_{-2\Delta}^{-\Delta} (\phi_\Delta * \eta)^2 + 2\int_{-\Delta}^{0} (\phi_\Delta * \eta)^2 \\
&= 2\int_{-2\Delta}^{-\Delta} c_1^2 \sin^2(2\Delta + x) + c_2^2 + 2c_1c_2 \sin(2\Delta + x) dx \\
&+ 2\int_{-\Delta}^{0} c_0^2 \sin^2(\Delta + x) + c_1^2 \sin^2(x) + c_2^2 dx \\
&+ 4\int_{-\Delta}^{0} -c_0c_1 \sin(\Delta + x)\sin(x) - c_0c_2 \sin(\Delta + x) + c_1c_2 \sin(x) dx \\
&= \frac{1}{2}c_1^2(2\Delta - \sin(2\Delta)) + 2c_2^2\Delta + 4c_1c_2(1-\cos(\Delta)) \\
&+ \frac{1}{2}c_0^2(2\Delta - \sin(2\Delta)) + \frac{1}{2}c_1^2(2\Delta - \sin(2\Delta)) + 2c_2^2\Delta \\
&- c_0c_1(2\cos(\Delta)\Delta - \sin(\Delta) + \sin(-\Delta)) - 4c_0c_2(1-\cos(\Delta)) - 4c_1c_2(1-\cos(\Delta)) \\
&= (\frac{1}{2}c_0^2 + c_1^2)(2\Delta - \sin(2\Delta)) + 4c_2^2\Delta \\
&- 2c_0c_1(\cos(\Delta)\Delta - \sin(\Delta)) - 4c_0c_2(1-\cos(\Delta)) \\
&\approx \frac{\pi^2\Delta^3}{25}[(8+9)\frac{4}{3} + 4 + 24\frac{1}{3} - 16\frac{1}{2}] = \frac{16\pi^2}{15}\Delta^3
\end{aligned}$$

Therefore

$$E(\phi_\Delta * \eta) \approx \frac{1}{4\Delta}\frac{60\pi^2\Delta^4}{16\pi^2\Delta^3} = \frac{15}{16} = 0.9375 \tag{F.1}$$