

Tribracket Modules

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Abstract

Niebrzydowski tribrackets are ternary operations on sets satisfying conditions obtained from the oriented Reidemeister moves such that the set of tribracket colorings of an oriented knot or link diagram is an invariant of oriented knots and links. We introduce *tribracket modules* analogous to quandle/biquandle/rack modules and use these structures to enhance the tribracket counting invariant. We provide examples to illustrate the computation of the invariant and show that the enhancement is proper.

KEYWORDS: Niebrzydowski tribrackets, enhancements, oriented knot and link invariants, tribracket modules

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1 Introduction

In [10] and [12], algebraic structures called *quandles* (or *distributive groupoids*) were introduced as an abstraction of the Wirtinger presentation of the fundamental group of the complement of a knot in \mathbb{R}^3 . Colorings of knot diagrams by elements of finite quandles define an integer-valued invariant known as the *quandle counting invariant*. Invariants of quandle-colored knots, e.g. Boltzmann weights defined from quandle 2-cocycles, can be used to strengthen this invariant, defining new invariants known as *enhancements*. See [7] for more.

In [4], algebraic structures called *quandle modules* were used to enhance the quandle counting invariant, inspiring later generalizations by one of the authors to the cases of rack modules in [9], biquandle modules in [3] and birack shadow modules in [14], among others. In each of these cases, a counting invariant is enhanced with secondary colorings by elements of a commutative ring with identity obeying an Alexander-style relation which depends on the quandle colors at the crossing.

In [16], the notion of using sets with ternary operations to define knot invariants was considered, with colorings of regions in the planar complement of a knot or link diagram by elements of structures known as *ternary quasigroups*. These structures can be seen as an abstraction of the Dehn presentation of the knot group analogous to the way quandles abstract the Wirtinger presentation. In [17], ternary quasigroup invariants were enhanced with a homology theory. A related structure called *biquasiles* was introduced in [13] by two of the authors with applications to surface-links in [11] by one of the authors. Recently ternary quasigroup operations known as *Niebrzydowski tribrackets* have been studied with additional generalizations to the cases of virtual knots in [15] and trivalent spatial graphs in [8].

In this paper we apply the idea behind quandle modules to the case of Niebrzydowski tribrackets, obtaining an infinite family of enhancements of the tribracket counting invariant. The paper is organized as follows. In Section 2 we recall the basics of Niebrzydowski tribrackets and see some examples, and introduce an enhancement for Alexander tribrackets. In Section 3 we define tribracket modules and introduce the tribracket module enhancement of the counting invariant. We compute some examples to show that the enhancement is nontrivial. We conclude in Section 4 with some questions for future work.

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2 Tribrackets

We begin with a defintion.

Definition 1. Let X be a set. A *horizontal tribracket* on X is a map $[\ , \] : X \times X \times X \rightarrow X$ satisfying

- (i) For any subset $\{a, b, c, d\} \subset X$, any three elements uniquely determine the fourth such that $[a, b, c] = d$, and
- (ii)

$$[b, [a, b, c], [a, b, d]] = [c, [a, b, c], [a, c, d]] = [d, [a, b, d], [a, c, d]].$$

Example 1. Let X be any module over a commutative ring wth identity R . Then any pair of units $x, y \in R^\times$ defines a tribracket structure on X by setting

$$[a, b, c] = -xya + xb + yc.$$

We call this an *Alexander tribracket* and denote it by $X = (R, x, y)$. Let us verify axiom (ii):

$$\begin{aligned} [b, [a, b, c], [a, b, d]] &= -xyb + x(-xya + xb + yc) + y(-xya + xb + yd) \\ &= (-x^2y - xy^2)a + x^2b + xyc + y^2d \\ [c, [a, b, c], [a, c, d]] &= -xyc + x(-xya + xb + yc) + y(-xya + xc + yd) \\ &= (-x^2y - xy^2)a + x^2b + xyc + y^2d \\ [d, [a, b, d], [a, c, d]] &= -xyd + x(-xya + xb + yd) + y(-xya + xc + yd) \\ &= (-x^2y - xy^2)a + x^2b + xyc + y^2d. \end{aligned}$$

Example 2. Let G be a group. Then G has the structure of a tribracket by setting

$$[a, b, c] = ba^{-1}c.$$

We call this a *Dehn tribracket*. As with the Alexander case, let us verify axiom (ii):

$$\begin{aligned} [b, [a, b, c], [a, b, d]] &= [a, b, c]b^{-1}[a, b, d] \\ &= ba^{-1}cb^{-1}ba^{-1}d \\ &= ba^{-1}ca^{-1}d \\ [c, [a, b, c], [a, c, d]] &= [a, b, c]c^{-1}[a, c, d] \\ &= ba^{-1}cc^{-1}ca^{-1}d \\ &= ba^{-1}ca^{-1}d \\ [d, [a, b, d], [a, c, d]] &= [a, b, d]d^{-1}[a, c, d] \\ &= ba^{-1}dd^{-1}ca^{-1}d \\ &= ba^{-1}ca^{-1}d \end{aligned}$$

as required.

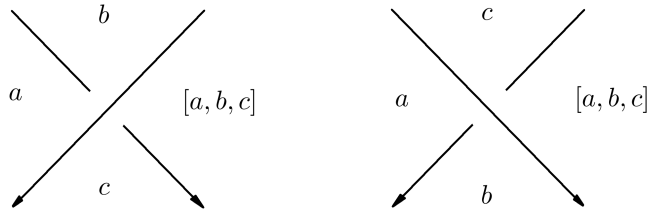
Example 3. We can specify a tribracket structure on a finite set $X = \{1, 2, \dots, n\}$ with an operation 3-tensor, i.e. an ordered list of $n \times n \times n$ matrices with elements in X such that the element in matrix a , row b , column c is $[a, b, c]$. This notation enables us to compute with tribrackets for which we lack algebraic formulas. For example, the set $X = \{1, 2, 3\}$ has tribracket structures including

$$\left[\left[\begin{array}{ccc} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{array} \right], \left[\begin{array}{ccc} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{array} \right], \left[\begin{array}{ccc} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{array} \right] \right].$$

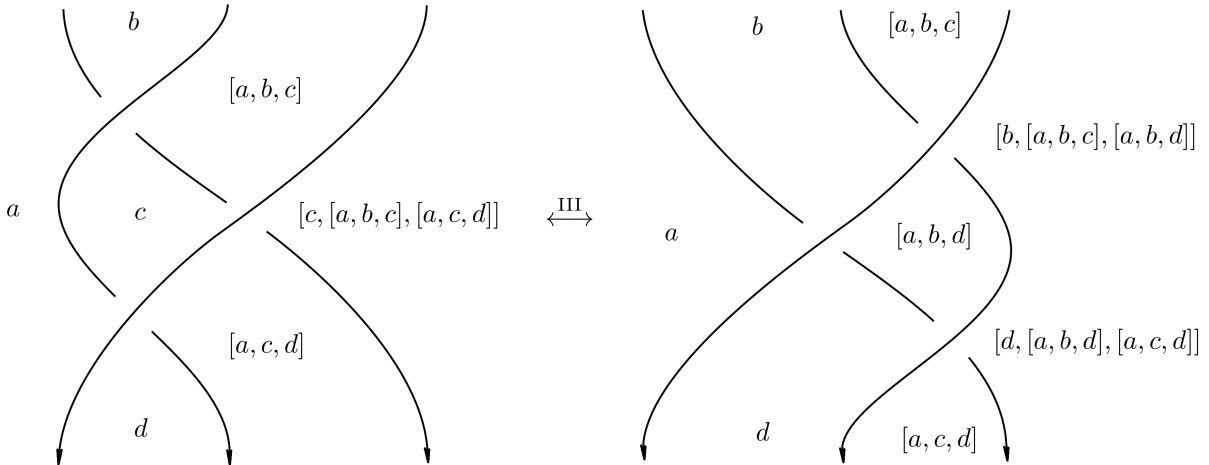
In this case for example, we verify axiom (ii) for the case $a = 1, b = 2, c = 3, d = 1$ by the computation

$$\begin{aligned} [2, [1, 2, 3], [1, 2, 1]] &= [2, 3, 2] = 3, \\ [3, [1, 2, 3], [1, 3, 1]] &= [3, 3, 3] = 3 \quad \text{and} \\ [1, [1, 2, 1], [1, 3, 1]] &= [1, 2, 3] = 3. \end{aligned}$$

The tribracket axioms are motivated by the Reidemeister moves using the following region coloring rule:



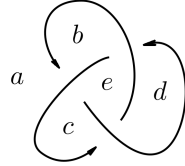
We call the invertibility conditions in axiom (i) *left*, *center* and *right invertibility* for the ability to uniquely recover a, b , and c respectively in $[a, b, c] = d$ given the other three. These are the conditions required to guarantee that for every coloring on one side of an oriented Reidemeister I or II move, there is a *unique* coloring of the diagram on the other side of the move which agrees with the original coloring outside the neighborhood of the move. Axiom (ii) is the condition required by the Reidemeister III move needed to complete a generating set of oriented Reidemesiter moves:



It follows that for any tribracket X , the number of X -colorings of an oriented knot or link L diagram is an integer-valued link invariant, which we call the *tribracket counting invariant*, denoted $\Phi_X^{\mathbb{Z}}(L)$. We will denote the set of X -colorings of L as $\mathcal{C}_X(L)$, and we have $\Phi_X^{\mathbb{Z}}(L) = |\mathcal{C}_X(L)|$.

Example 4. If X is an Alexander tribracket, we can compute $\Phi_X^{\mathbb{Z}}(L)$ using linear algebra. Let X be the Alexander tribracket on \mathbb{Z}_3 with $x = 1$ and $y = 2$, so we have $[a, b, c] = a + b + 2c$. Then the trefoil knot 3_1

below has system of coloring equations



$$\begin{aligned} [a, b, c] &= e \\ [a, c, d] &= e \\ [a, d, b] &= e \end{aligned}$$

and after row-reduction mod 3,

$$\begin{bmatrix} 1 & 1 & 2 & 0 & 2 \\ 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we see that $\Phi_X^{\mathbb{Z}}(3_1) = 3^3 = 27$. This distinguishes the trefoil from the unknot 0_1 , which has $\Phi_X^{\mathbb{Z}}(0_1) = 3^2 = 9$ X -colorings.

Remark 1. Writing the region Alexander tribracket coloring equations from an oriented link diagram as a system of linear equations yields a matrix from which an Alexander matrix and the Alexander polynomial of the knot or link can be derived.

An *enhancement* of a counting invariant is a generally stronger invariant from which we can recover the counting invariant. Any invariant ϕ of X -colored knots and links defines an enhancement by taking the multiset of ϕ -values over the set of colorings of the knot or link,

$$\Phi_X^{\phi, M}(K) = \{\phi(L_f) \mid L_f \in \mathcal{C}_X(L)\}.$$

Remark 2. For Alexander tribrackets $X = (R, x, y)$, we can enhance the tribracket counting invariant by setting $\phi(L_f)$ equal to the rank of the image submodule of the coloring, analogous to the (t, s) -rack enhancements in [5]. The multiset of the ranks of these image submodules over the complete set of colorings is the *Alexander image enhancement* of the tribracket counting invariant,

$$\Phi_X^A(L) = \{\text{rank}(\text{Span}(\text{Im}(f))) \mid f \in \mathcal{C}_X(L)\}.$$

3 Tribracket Modules

We would like to enhance the tribracket counting invariant by finding an invariant of X -colored oriented knot and link diagrams. To this end, we make the following definition, analogous to the cases of quandles, racks, and biracks in papers such as [3, 4, 9, 14].

Definition 2. Let X be a tribracket and R a commutative ring with identity. A *tribracket module* structure on R , also called an X -*module*, is a choice of units $x_{a,b,c}$ and $y_{a,b,c}$ for each triple of elements of X satisfying

the conditions

$$\begin{aligned}
x_{c,[a,b,c],[a,c,d]}x_{a,b,c} &= x_{d,[a,b,d],[a,c,d]}x_{a,b,d} \\
&= x_{b,[a,b,c],[a,b,d]}x_{a,b,c} + y_{b,[a,b,c],[a,b,d]}x_{a,b,d} \\
&\quad - x_{b,[a,b,c],[a,b,d]}y_{b,[a,b,c],[a,b,d]} \\
y_{c,[a,b,c],[a,c,d]}y_{a,c,d} &= y_{b,[a,b,c],[a,b,d]}y_{a,b,d} \\
&= x_{d,[a,b,d],[a,c,d]}y_{a,b,d} + y_{d,[a,b,d],[a,c,d]}y_{a,c,d} \\
&\quad - x_{d,[a,b,d],[a,c,d]}y_{d,[a,b,d],[a,c,d]} \\
x_{b,[a,b,c],[a,b,d]}y_{a,b,c} &= y_{d,[a,b,d],[a,c,d]}x_{a,c,d} \\
&= x_{c,[a,b,c],[a,c,d]}y_{a,b,c} + y_{c,[a,b,c],[a,c,d]}x_{a,c,d} \\
&\quad - x_{c,[a,b,c],[a,c,d]}y_{c,[a,b,c],[a,c,d]} \\
x_{c,[a,b,c],[a,c,d]}x_{a,b,c}y_{a,b,c} + y_{c,[a,b,c],[a,c,d]}x_{a,c,d}y_{a,c,d} &= x_{b,[a,b,c],[a,b,d]}x_{a,b,c}y_{a,b,c} + y_{b,[a,b,c],[a,b,d]}x_{a,b,d}y_{a,b,d} \\
&= x_{d,[a,b,d],[a,c,d]}x_{a,b,d}y_{a,b,d} + y_{d,[a,b,d],[a,c,d]}x_{a,c,d}y_{a,c,d}
\end{aligned}$$

for all $a, b, c, d \in X$.

A tribracket module over a tribracket $X = \{1, 2, \dots, n\}$ is specified with a pair $V = (x, y)$ of 3-tensors such that the entries in matrix a , row b , column c are $x_{a,b,c}$, $y_{a,b,c}$ respectively.

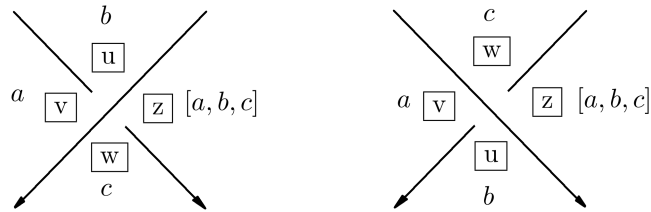
Example 5. A *constant tribracket module* is one in which the $x_{a,b,c}$ and $y_{a,b,c}$ -values do not depend on $a, b, c \in X$. In this case, V is an Alexander tribracket on R and the sticker colorings are independent of the X colorings. For instance the tribracket

$$X = \left[\left[\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right], \left[\begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right] \right]$$

has constant tribracket modules with \mathbb{Z}_3 coefficients including

$$\begin{aligned}
V_1 &= \left[\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right], \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] \right], \quad \left[\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right], \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] \right], \\
V_2 &= \left[\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right], \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] \right], \quad \left[\left[\begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array} \right], \left[\begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array} \right] \right], \\
V_3 &= \left[\left[\begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array} \right], \left[\begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array} \right] \right], \quad \left[\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right], \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] \right] \quad \text{and} \\
V_4 &= \left[\left[\begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array} \right], \left[\begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array} \right] \right], \quad \left[\left[\begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array} \right], \left[\begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array} \right] \right].
\end{aligned}$$

The tribracket module axioms are motivated by the Reidemeister moves with the coloring scheme below. Given an oriented knot or link diagram with a tribracket coloring, we put a secondary labeling on the regions with a *sticker* in each region.



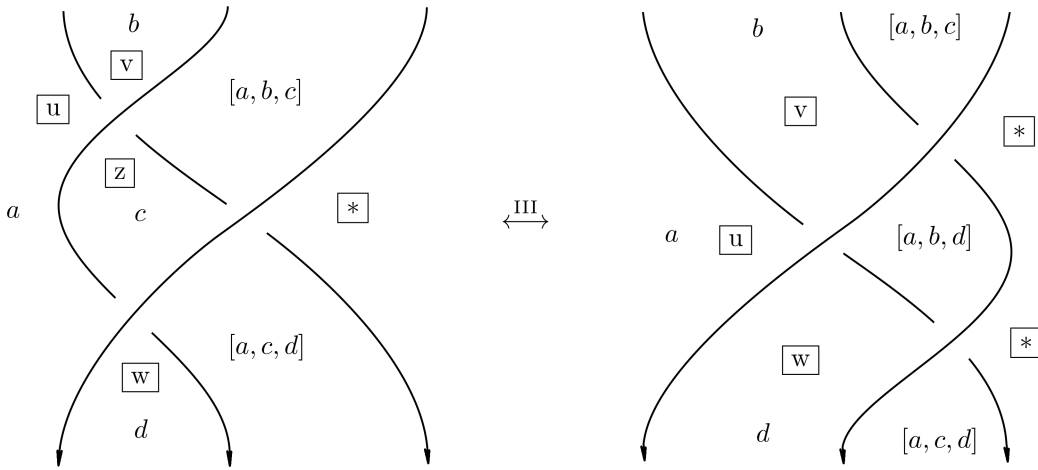
The sticker colorings must then satisfy the rule

$$z = -x_{a,b,c}y_{a,b,c}v + x_{a,b,c}u + y_{a,b,c}w,$$

a customized Alexander tribracket-style coloring with coefficients depending on the X -colors at the crossing.

Proposition 1. *Let X be a tribracket, R a commutative ring with identity and V an X -module over R . Then sticker colorings of the regions in an oriented knot diagram's planar complement are in one-to-one correspondence before and after X -colored Reidemeister moves.*

Proof. Invertibility of $x_{a,b,c}$ and $y_{a,b,c}$ satisfies the claim for Reidemeister I and II moves, so it remains only to verify for the all-positive Reidemeister III move.



The region marked $\boxed{*}$ gets three sticker colorings – one on the left side of the move and two on the right – which must all agree. Each of these is an expression in the independent variables u, v, z, w , so we can compare these coefficients to obtain the necessary equations, i.e., the conditions in Definition 2. \square

Definition 3. Let L be an oriented link diagram, X a tribracket, R a commutative ring with identity and $V = (x, y)$ an X -module structure on R . For each X -coloring $f \in \mathcal{C}_X(L)$ of L , let A_f be the coefficient matrix of the homogeneous system of sticker coloring equations over R determined by f and V . Then the *tribracket module multiset enhancement* of the tribracket counting invariant is the multiset

$$\Phi_X^{V,M}(L) = \{|\text{Ker } A_f| : f \in \mathcal{C}_X(L)\}$$

or if R is infinite,

$$\Phi_X^{V,M}(L) = \{\text{rank}(\text{Ker } A_f) : f \in \mathcal{C}_X(L)\}.$$

We can optionally convert these multisets to “polynomial” form by replacing multiplicities with coefficients and elements with exponents of a formal variable u

$$\Phi_X^V(L) = \sum_{f \in \mathcal{C}_X(L)} u^{|\text{Ker } A_f|}$$

or if R is infinite,

$$\Phi_X^V(L) = \sum_{f \in \mathcal{C}_X(L)} u^{\text{rank}(\text{Ker } A_f)}.$$

This notation has the advantage that evaluation at $u = 1$ yields the original counting invariant and provides easier visual comparison of invariant values.

By construction, we have the following proposition:

Proposition 2. For any X -module V over a tribracket X and commutative ring with identity R , $\Phi_X^{V,M}(L)$ and $\Phi_X^V(L)$ are invariants of oriented knots and links.

Example 6. If V is a constant X -module, then $|\text{Ker } A_f|$ is just the number of colorings of L by the Alexander tribracket A on R with parameters (x, y) and we have

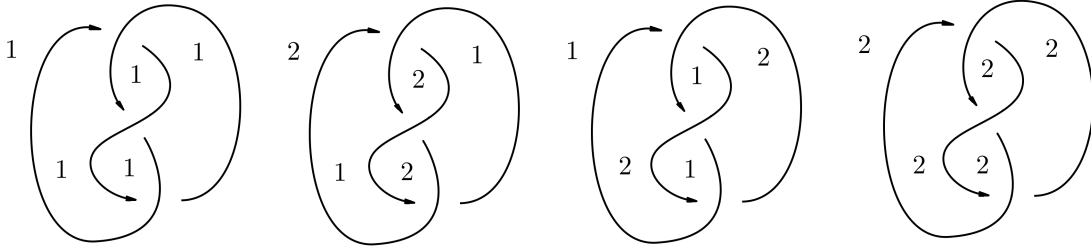
$$\Phi_X^V(L) = |\Phi_X^{\mathbb{Z}}(L)|u^{|\Phi_A^{\mathbb{Z}}(L)|}.$$

Example 7. Let V be an X -module with coefficients in a finite ring R . The unlink of n components has $n+1$ regions with no crossings and hence no restrictions on X -colorings, so there are $|X|^{n+1}$ region colorings. Each of these has similarly no restrictions on sticker colorings, so there are $|R|^{n+1}$ sticker colorings for each region coloring. Hence, the value of $\Phi_X^V(L)$ on the unlink of n components is $\Phi_X^V(L) = |X|^{n+1}u^{|R|^{n+1}}$.

Example 8. Let X be the set $\{1, 2\}$ with tribracket operation given by

$$\left[\left[\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right], \left[\begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right] \right].$$

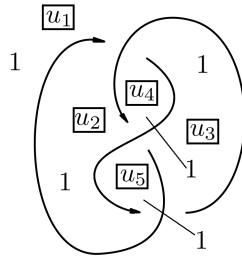
The trefoil knot 3_1 has four X -colorings:



Then we compute via python that X has modules with \mathbb{Z}_3 coefficients including

$$V = \left[\left[\begin{array}{cc} 2 & 2 \\ 2 & 1 \end{array} \right], \left[\begin{array}{cc} 1 & 2 \\ 2 & 2 \end{array} \right] \right], \quad \left[\left[\begin{array}{cc} 1 & 2 \\ 2 & 2 \end{array} \right], \left[\begin{array}{cc} 2 & 2 \\ 2 & 1 \end{array} \right] \right].$$

Consider the X -coloring of the trefoil with all regions colored $1 \in X$:



Let us compute the set of sticker colorings. We obtain linear system of coloring equations over $R = \mathbb{Z}_3$

$$\begin{aligned} -x_{11}y_{11}u_2 + x_{11}u_1 + y_{11}u_4 &= u_3 & -(1)(2)u_2 + 1u_1 + 2u_4 &= u_3 \\ -x_{11}y_{11}u_2 + x_{11}u_1 + y_{11}u_4 &= u_3 & \Rightarrow -(1)(2)u_2 + 1u_4 + 2u_5 &= u_3 \\ -x_{11}y_{11}u_2 + x_{11}u_1 + y_{11}u_4 &= u_3 & -(1)(2)u_2 + 1u_5 + 2u_1 &= u_3 \end{aligned}$$

which via row-reduction over \mathbb{Z}_3

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has kernel of dimension 3. Similarly, the other X -colorings have $|R|^3 = 3^3 = 27$ colorings. In particular the trefoil has $\Phi_X^V(3_1) = 4u^{27}$ which is different from the unknot's value of $4u^9$, and the enhancement detects the difference between the unknot and the trefoil.

Example 9. Let X be the tribracket in example 7. Via `python` computation we selected tribracket modules V_1 and V_2 with coefficients in \mathbb{Z}_3 and V_3 with coefficients in \mathbb{Z}_8 ,

$$\begin{aligned} V_1 &= \left[\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \right], \quad \left[\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \right], \\ V_2 &= \left[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right], \quad \left[\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \right] \quad \text{and} \\ V_3 &= \left[\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}, \begin{bmatrix} 7 & 1 \\ 3 & 1 \end{bmatrix} \right], \quad \left[\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix} \right], \end{aligned}$$

and computed the $\Phi_X^V(L)$ values for the prime links of up to seven crossings at the knot atlas [2]. The results are collected in the tables.

| $\Phi_X^{V_1}(L)$ | L |
|----------------------------|--------------------------------------|
| $4u^9$ | $L6a4$ |
| $2u^9 + 6u^{27}$ | $L2a1, L4a1, L5a1, L6a2, L7a4, L7a6$ |
| $2u^9 + 4u^{27} + 2u^{81}$ | $L7a2, L7a3, L7n1, L7n2$ |
| $8u^{27}$ | $L6a1, L6a3, L7a1, L7a5$ |
| $8u^{27} + 8u^{81}$ | $L6a5$ |
| $2u^9 + 6u^{27} + 8u^{81}$ | $L6n1, L7a7$ |

| $\Phi_X^{V_2}(L)$ | L |
|----------------------------|--|
| $4u^9$ | $L6a4$ |
| $6u^9 + 2u^{27}$ | $L2a1, L62, L7a6$ |
| $2u^9 + 6u^{27}$ | $L4a1, L5a1, L7a2, L7a3, L7a4, L7n1, L7n2$ |
| $4u^9 + 4u^{27}$ | $L6a3, L7a5$ |
| $8u^{27}$ | $L6a1, L7a1$ |
| $6u^9 + 8u^{27} + 2u^{81}$ | $L6a5$ |
| $8u^9 + 6u^{27} + 2u^{81}$ | $L6n1, L7a7$ |

| $\Phi_X^{V_3}(L)$ | L |
|--|--------------------------------|
| $4u^{64}$ | $L6a4$ |
| $2u^{128} + 4u^{256} + 2u^{512}$ | $L2a1, L6a2, L6a3, L7a5, L7a6$ |
| $2u^{256} + 6u^{512}$ | $L4a1, L6a1, L7a2, L7n1$ |
| $8u^{512}$ | $L5a1, L7a1, L7a3, L7a4, L7n2$ |
| $2u^{256} + 6u^{1024} + 6u^{2048} + 2u^{4096}$ | $L6a5, L6n1, L7a7$ |

Example 10. For our final example let X be the 4-element tribracket

$$\left[\begin{bmatrix} 4 & 3 & 2 & 1 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 4 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 3 & 1 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \\ 2 & 4 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix} \right]$$

with the module V with \mathbb{Z}_3 coefficients specified by

$$\left[\begin{array}{c} \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{array} \right], \left[\begin{array}{cccc} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{array} \right], \left[\begin{array}{cccc} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{array} \right], \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{array} \right] \end{array} \right],$$

$$\left[\begin{array}{c} \left[\begin{array}{cccc} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{array} \right], \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array} \right], \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array} \right], \left[\begin{array}{cccc} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{array} \right] \end{array} \right].$$

We computed the invariant on prime knots with up to eight crossings and links with up to seven crossings. The results are collected in the table. In particular the knots in the table all have counting invariant value 16 but are sorted into three classes by the enhancement, while the invariant is quite effective at distinguishing the links in the table.

| $\Phi_X^V(L)$ | L |
|-------------------------------|---|
| $16u^9$ | $4_1, 5_1, 5_2, 6_2, 6_3, 7_1, 7_2, 7_3, 7_5, 7_6, 8_1, 8_2, 8_3, 8_4,$ $8_6, 8_7, 8_8, 8_9, 8_{12}, 8_{13}, 8_{14}, 8_{16}, 8_{17}$ |
| $8u^9 + 8u^{27}$ | $3_1, 6_1, 7_4, 7_7, 8_5, 8_{10}, 8_{11}, 8_{15}, 8_{19}, 8_{20}, 8_{21}$ |
| $8u^9 + 8u^{81}$ | 8_{18} |
| $16u^9$ | $L6a4$ |
| $16u^9 + 16u^{27}$ | $L2a1, L6a2, L7a6$ |
| $16u^9 + 48u^{27}$ | $L4a1, L5a1$ |
| $32u^{27}$ | $L6a3, L7a5$ |
| $16u^9 + 32u^{27} + 16u^{81}$ | $L7a3, L7n1, L7n2$ |
| $16u^9 + 48u^{27}$ | $L7a4$ |
| $32u^9 + 32u^{81}$ | $L6n1, L7a7$ |
| $32u^{27} + 32u^{81}$ | $L6a5$ |
| $64u^{27}$ | $L6a1, L7a1$ |

4 Questions

We conclude with some questions and directions for future work.

First and foremost, more efficient methods than our (relatively) brute-force axiom testing for finding tribracket modules would be highly desirable. While even the relatively small examples we have found are fairly good at distinguishing classical knots and links, we expect that modules over larger finite rings or infinite rings should yield even stronger invariants.

What is the relationship between tribracket modules and tribracket cocycles? In the case of racks, rack modules are closely related to structures known as *dynamical cocycles* [1]; what is the appropriate definition for tribracket dynamical cocycles?

As in the case of [6], we can consider tribracket modules with coefficients in a polynomial algebra as defining a kind of tribracket-colored Alexander polynomial for each coloring. Do these invariants satisfy skein relations? Do their coefficients define Vassiliev invariants?

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