IMAGE SEGMENTATION WITH PARTIAL CONVEXITY PRIOR USING DISCRETE CONFORMALITY STRUCTURES

CHUN YIN SIU, HEI LONG CHAN, LOK MING LUI

Abstract. Image segmentation aims to partition an image into meaningful regions and extract important objects therein. In real applications, the given images may contain multiple overlapping objects with noisy background, inducing great challenges to the segmentation task. In these cases, priori information of the target object is essential for an accurate and meaningful segmentation result. In this paper, we present a new framework to achieve image segmentation with convexity prior, guaranteeing the segmented region to be fully or partially convex according to the user's preference. The basic idea is to incorporate a registration-based segmentation model with a specially designed convexity constraint. The convexity constraint is based on the discrete conformality structures of the image mesh. We propose an iterative scheme to solve the segmentation model, which smoothly deforms a template object to trace the boundary of the target object. A projection is carried out to enforce the convexity constraint. The target object is then captured by a (fully or partially) convex region. Experiments have been carried out on both synthetic and real images. Results demonstrate the effectiveness of our proposed framework.

 $\label{eq:constraint} \textbf{Key words.} \ \text{Image segmentation, convexity, conformality structure, registration-based model, } quasi-conformal$

1. Introduction. Image segmentation is an important topic in many aspects as it simplifies higher level image processing, understanding and analysis by highlighting meaningful regions and distinguishing different components in images. It is an important yet challenging task in many scenarios, especially for images with occlusions or corruptions (see Figure 1.1,1.2). For instance, in medical science, MR images may not be clear enough to capture the whole shape of the relevant anatomical structures due to machine and manual artifacts. This causes inaccurate segmentation which in turn induces difficulties to both further analysis and medical operations. In zoology, the occlusion in images of wildlife animals by trees and grass increases the difficulty to estimate and record measurements such as the height and the length of the animals. As for photography, background noise, over-exposure and occlusions may obscure and distort the target object in a video or even an everyday photo, posing great challenge to subsequent processing such as restoration, inpainting and registration.

To overcome these challenges, priori knowledge about the target object is usually introduced to guide the segmentation algorithm to extract the target object with an appropriate shape. However, while some priori information such as the topology of the target object may sometimes be too rough for an accurate segmentation, some other priori information such as the actual shape prior of the target object may not always be available. Therefore, a balance is needed in between. Here, we observe that many objects in real life are indeed convex or partially convex (that is, part of the contour has non-negative curvature) when they are projected onto a 2D image. Therefore, in this work, convexity of the target object is utilized as the prior knowledge to improve segmentation accuracy. By preserving the convexity of the segmented domain, the boundary of the target object can still be captured correctly even if it is occluded and obscured.

In this paper, we propose a topology-preserving registration-based segmentation model, which incorporates the convexity prior. The image domain is discretized by a triangulation mesh. The convexity constraint can then be formulated in terms of the discrete conformality structure, called the dihedral angle, defined on the mesh. Our algorithm iteratively deforms a template binary image to extract the target object in



FIG. 1.1. Examples of mis-segmentation due to occlusion or corruption in images. The left shows the image of a road sign occluded by grass. The right shows a corrupted medical image. The segmentation results are inaccurate using conventional segmentation model without the incorporation of shape information.

the image, by enforcing that the dihedral angles of the deformed mesh satisfies the convexity constraint. As a result, the target object is captured by a domain, which is convex in the prescribed region and preserves the prescribed topology. The accuracy of the segmentation result can be significantly improved with the convexity constraint for degraded images due to noises or occlusions.

Our proposed model has two merits. On one hand, by parametrizing the space of immersions of a topological triangulation using the notion of dihedral angles, the proposed algorithm is developed directly on a discrete setting, hence allowing one to effectively guarantee convexity without the concern of discrepancy from a smooth model to its discretization. As a result, convexity can be guaranteed accurately not only in theory but also in real implementation. On the other hand, our proposed framework allows users to enforce partial convexity, in the sense that non-negativity of curvature may be prescribed only on parts of the boundary curve. To the best of our knowledge, the proposed framework is the first one that explicitly allows users to freely enforce convexity only on disjoint portions of the segmented domain. The implementation of partial convexity allows more flexible use of the algorithm for handling more general images.

The paper is organized as follows. Closely related previous works are reviewed in Section 2. Relevant mathematical background is explained in Section 3. Our proposed segmentation model are explained in details in Section 4. In Section 5, we describe our proposed numerical algorithm to solve the proposed segmentation model. Experimental results are shown in Section 6, and the conclusion follows in Section 7.

2. Related work. In this section, previous works closely related to our proposed framework are reviewed.

2.1. Image segmentation and image registration. The snake model, or active contour model, for segmentation was first introduced in [24] and has been improved since its inception in terms of capture range [13, 52] and removal of dependency on parametrization [26, 6]. A thorough survey on active contour models may be found in [2]. By representing the contour by a level set of a function (very often a *signed-distance function*, i.e. a function with unit gradient) rather than parametrizing the contour explicitly, Chan-Vese model allows for topological change in the contour [11]. Improvements to the model have been made in [40, 12]. A thorough survey on Chan-Vese model may be found in [9].

Prior knowledge is often useful for guiding segmentation. Topological-prior segmentation [21, 45, 28] ensures the segmented object has the prescribed topology. For stronger priors, the statistics of control points of snakes, and statistics of level-set



FIG. 1.2. Examples of fully and partially convex objects. The first row are examples of fully convex objects. The second row shows examples of partially convex objects.

functions or template functions are used in [16] and [31, 15, 10] respectively.

Less stringent than these geometrical priors is the prior of convexity. It was proposed in [43] to segment convex regions by prescribing a set of sufficiently dense set of orientations, as well as their orthogonal complements, and partition the image into regions with linear boundaries with the prescribed orientations, such that the central region will be convex. A graph theory-based method was proposed in [39] to enforce a graph-based discrete convexity constraint via imposing a linear inequality for each path in each segmented region. Gorelic et al. [20] proposed penalizing the number of triplets of collinear pixels (p, q, r) such that p and r lies inside the region, while q lies between p and r but outside the region. In [1], the prior is enforced by penalizing the L^1 norm of the curvature, which necessitates solving a high order PDE. In [53], the prior is enforced by restricting from the space of all level-set function to that of sub-harmonic signed-distance functions, whose level sets are all convex. This is implemented in an alternating manner.

Sometimes, image registration can be used to aid the segmentation process. Many different methods have been developed for image registration, like feature-based methods [19, 17] and mutual information-based methods [50, 44]. A thorough survey may be found in [55]. In [46, 49], the non-parametric registration problem was solved by morphing one image to the other by a vector field, and the action is ensured to be diffeomorphic in the latter paper.

Segmentation and registration are interrelated. In particular, segmentation may be guided by registration by registering with a template [54, 29, 23].

2.2. Discrete Conformal Geometry. Conformal maps be approximated in discrete settings [18, 22, 32, 37], say by approximating Cauchy-Riemann equation. Alternative to the equation-solving approach is the circle packing approach, which is based on the principle that conformal maps map infinitesimal circles to infinitesimal circles. Circle packing was first proposed in [47] for theoretical study of manifolds, and discretized in [48], and implemented in [14]. Circle Pattern, which allows transversely intersecting circles, is a more relaxed setting for computation. Its theory, in the form of dihedral angles¹, was first proposed in [38] for the study of Euclidean simplicial

¹Circle pattern is phrased in terms of the supplement of dihedral angles.

surfaces with cone-like singularities and was extended in [30, 3]. It was applied in [25] for mesh flattening. A similar framework was used in [42] for texture mapping.

Allowing, and accounting for, conformal distortion in a discrete map gives rise to discrete quasi-conformal geometry. It has been applied in diverse setting ranging from shape analysis [34, 7] and map compression [33] to registration [27] and segmentation with topological prior [8]. Quasi-conformal geometry will be reviewed in Section 3.

3. Mathematical background. In this section, the mathematical tools pertinent to the proposed framework are described.

3.1. Quasi-Conformal Geometry. The foundation of the proposed framework is based on quasi-conformal geometry. Quasi-conformal maps generalize conformal maps. Given a domain $\Omega \subset \mathbb{C}$, a mapping $f : \Omega \to \mathbb{C}$ is said to be quasi-conformal if there exists a Lebesgue-measurable $\mu : \Omega \to \mathbb{C}$ such that

$$\frac{\partial f}{\partial \bar{z}}(z) = \mu(z) \frac{\partial f}{\partial z}(z) \tag{3.1}$$

and

$$||\mu||_{\infty} < 1. \tag{3.2}$$

Equation (3.1) is called the Beltrami equation, and μ is called the Beltrami coefficient of f. Roughly speaking, quasi-conformal maps are orientation-preserving homeomorphisms between Riemann surfaces with a bounded conformality distortion, and the distortion can be effectively controlled using the Beltrami coefficient μ of f. The following theorem, whose proof can be found in [33], is a well-established explanation of the relationship between a mapping f and its Beltrami coefficient μ .

THEOREM 3.1. Given a domain $\Omega \subset \mathbb{C}$, let $f : \Omega \to \mathbb{C}$ be a mapping, by defining

$$\mu(z) = \lim_{\hat{z} \to z} \left(\frac{\partial f}{\partial \bar{z}}(\hat{z}) \middle/ \frac{\partial f}{\partial z}(\hat{z}) \right), \tag{3.3}$$

then $||\mu||_{\infty} < 1$ if and only if f is an orientation-preserving homeomorphism.

Therefore, in particular, any diffeomorphic deformation on Ω must be a quasiconformal mapping. This provides an alternative interpretation of diffeomorphic deformations, that requiring a deformation to be diffeomorphic is equivalent to requiring its Beltrami coefficient to have sup norm strictly less than 1.

In the infinitesimal scale, a quasi-conformal mapping f on Ω has its local parametric expression as

$$f(z) \approx f(0) + f_z(0)z + f_{\bar{z}}(0)\bar{z} = f(0) + f_z(0)(z + \mu(0)\bar{z}).$$
(3.4)

From the expression 3.4, it can be seen that the non-conformal part of f comes from the term $D(z) = z + \mu(0)\bar{z}$ which is essentially contributed by the Beltrami coefficient μ of f only. Indeed, the Beltrami coefficient μ has a one-to-one correspondence with the quasi-conformal mapping f. Given a quasi-conformal mapping f, its Beltrami coefficient can be uniquely determined by (3.1). The converse is guaranteed by the following famous theorem.

THEOREM 3.2 (Measurable Riemann Mapping Theorem). Suppose $\mu : \mathbb{C} \to \mathbb{C}$ is Lebesgue measurable satisfying $||\mu||_{\infty} < 1$, then there exists a quasi-conformal

mapping $f: \mathbb{C} \to \mathbb{C}$ in the Sobolev space $W^{1,2}$ that satisfies the Beltrami equation in the distribution sense. Furthermore, assuming that the mapping is stationary at 0, 1and ∞ , then the associated quasi-conformal mapping f is uniquely determined.

The existence and uniqueness of the corresponding quasi-conformal mapping ffrom a given admissible Beltrami coefficient μ is not just guaranteed in theory. In practice, given a Beltrami coefficient $\mu: \Omega \to \mathbb{C}$ with sup-norm strictly less than 1, the corresponding quasi-conformal mapping f can be explicitly determined by the *Linear Beltrami Solver* (LBS), whose details may be found in [33]. With LBS, deformation can be directly controlled by perturbing the Beltrami coefficient and hence deformations can now be prescribed to be diffeomorphic.

Now, let us consider the composition of quasi-conformal maps. Given two quasiconformal mappings $f, g : \Omega \to \Omega$, by using μ_f and μ_g to denote their Beltrami coefficients respectively, the Beltrami coefficient $\mu_{g \circ f}$ of the composite mapping $g \circ f$ is given by

$$\mu_{g \circ f} = \frac{\mu_f + (\mu_g \circ f)\tau}{1 + \bar{\mu}_f (\mu_g \circ f)\tau}, \quad \tau = \frac{\bar{f}_z}{f_z}.$$
(3.5)

This provides a more convenient way to compute the composition of diffeomorphisms. That is, using the above notations, the composite mapping $q \circ f$ on Ω can be obtained by applying the Linear Beltrami Solver on the its Beltrami coefficient $\mu_{g\circ f}$ which can be directly determined via (3.5).

3.2. Discrete Conformality Structure Based on Dihedral Angles. The notion of quasi-conformal maps provides a way to preserve different properties of a mapping, to be diffeomorphic for instance, by perturbing the corresponding Beltrami coefficient. However, it is noted that an image domain is usually represented by a triangular mesh and hence can be regarded as a discrete object. Much geometric information can be effectively described by the discrete conformality structures. In particular, we enforces convexity using the discrete conformality structure based on dihedral angles.

Let M = (V, E, F) be a general triangular mesh with vertex set V, edge set E and face set F. If M is merely a topological simplicial complex, V only has indices rather than coordinates as elements; if M is immersed in \mathbb{R}^3 , then V contains the coordinates. To study angles in a mesh, the notion of plausible angle assignment is in place.

DEFINITION 3.3 (Plausible angle assignment). A function $\varphi: V \times F \to \mathbb{R}$ is called a plausible angle assignment (resp ε -plausible angle assignment, where $\varepsilon > 0$) *if all of the followings hold*

$$\varphi_{v,\Delta} = 0$$
 if v is not a vertex of the face Δ , (3.6)

$$\varphi_{v,\Delta} = 0 \qquad \text{if } v \text{ is not } a \text{ vertex of the face } \Delta, \qquad (3.6)$$

$$\varphi_{v,\Delta} > 0 \ (\text{resp} \ge \varepsilon) \quad \text{if } v \text{ is a vertex of the face } \Delta, \qquad (3.7)$$

$$\sum_{v} \varphi_{v,\Delta} = \pi \qquad \qquad \text{for every face } \Delta \in F, \tag{3.8}$$

where $\varphi_{v,\Delta} = \varphi(v,\Delta)$ for all $v \in V, \Delta \in F$.

The angles of an immersion of M in \mathbb{R}^n defines a plausible angle assignment as follows. Let $\varphi_{v,\Delta}$ to be the interior angle of the face $\Delta \in F$ at the vertex $v \in V$, and $\varphi_{v,\Delta} = 0$ if $v \in V$ is not a vertex of $\Delta \in F$. The $\varphi_{v,\Delta}$ is a plausible angle assignment. However, not all plausible angle assignments are angles of some immersion.



FIG. 3.1. Illustration of dihedral angles. For each mesh, the dihedral angle of the colored edge is given by the sum of colored angles.

Using the plausible angle assignment $\varphi_{v,\Delta}$, the dihedral angle of each edge can be defined as follows.

DEFINITION 3.4 (Dihedral Angle). Let $e \in E$ and φ be a plausible angle assignment. The dihedral angle $\theta_{e;\varphi}$, or simply θ_e , of e is defined by

$$\theta_e = \sum_{v \in V_e} \varphi_{v,(\Delta_{e,v})},\tag{3.9}$$

where V_e is the set of vertices opposite to e, and $\Delta_{e,v}$ is the unique face containing both e and v.

A simple interpretation of dihedral angle is presented in the top row of Figure (3.1), in which for each mesh, the dihedral angle of the colored edge equals to the sum of those colored angles.

Every immersion has dihedral angles. Conversely, under a mild condition, mesh can be recovered from the dihedral angles. To make the converse precise, the following definitions are in place.

DEFINITION 3.5 (Plausible Dihedral Assignment). A function $\theta : E \to \mathbb{R}_+$ is a plausible dihedral assignment if it is the dihedral angles of a plausible angle assignment, or in symbols $\theta(e) = \theta_{e;\varphi}$ for some plausible angle assignment φ .

DEFINITION 3.6 (Delaunay Plausible Dihedral Assignment). A plausible dihedral assignment θ is said to be Delaunay (resp ε -Delaunay) iff $\theta(e) < \pi$ (resp $\theta(e) \le \pi - \varepsilon$). A plausible angle assignment is Delaunay (resp ε -Delaunay) iff its dihedral angles are Delaunay (resp ε -Delaunay).

By the fact that opposite angles in a cyclic quadrilateral are complementary, if M is immersed in the plane, then it is Delaunay iff its dihedral angles of the angles of the immersion is Delaunay.

A plausible dihedral assignment $\theta: E \to \mathbb{R}_+$ is always the dihedral angles of the angles of some immersion under the mild condition of Delaunayness. In fact, we have the following theorem proven in [38].

THEOREM 3.7. If θ is a Delaunay plausible dihedral assignment of a topological triangulation M, then there exists an immersion of M whose dihedral angles are precisely given by θ .

4. Proposed Segmentation Model. In this section, our proposed mathematical model for image segmentation with convexity priors will be discussed in details.



FIG. 3.2. Example of topology prior images with different prescribed topology.



FIG. 3.3. Illustration of the segmentation model using Beltrami coefficient. A prior image is deformed by a diffeomorphism to approximate the boundary of the target object in an image.

4.1. The foundation: Segmentation model using Beltrami coefficient. We first describe the segmentation model in the continuous setting. Suppose $I: \Omega \to \mathbb{R}$ is an image on a rectangular domain $\Omega \subset \mathbb{C}$, and $T \subset \Omega$ is the underlying subdomain in which the target object sits in I. To segment T from Ω with a prescribed topology, Chan et al. [8] proposed a registration-based segmentation model using the Beltrami coefficient. The basic idea is to deform a simple template image J, called the *topological prior image*, to extract the region T of the target object in the image I. The prior image comprises of simple objects, such as a disk, according to the prescribed topology (see Figure 3.2). The segmentation model aims to look for a diffeomorphism $f_{\mu}: \Omega \to \Omega$ with Beltrami coefficient μ , such that the deformed prior image $J \circ f_{\mu}^{-1}$ closely resembles to I (see Figure 3.3). Thus, the desired region T can be estimated by $T \approx f^{\mu}(D)$, where D is the object region in J. More precisely, the overall segmentation model can be formulated as the following optimization problem:

$$\min_{\mu} E(\mu, c_1, c_2) = \int_{\Omega} |\mu|^2 + \eta \int_{\Omega} (I \circ f_{\mu} - J_{c_1, c_2})^2 + \lambda \int_{\Omega} |\nabla \mu|^2 + \sigma \int_{\Omega} (|u|^2 + |\nabla u|^2) \quad (4.1)$$

where $u = f_{\mu} - Id$ and $\eta, \lambda, \sigma > 0$ are weighting parameters. The topological prior image J_{c_1,c_2} is given by

$$J_{c_1,c_2}(x) = \begin{cases} c_1, & \text{if } x \in D\\ c_2, & \text{if } x \in \Omega \backslash D. \end{cases}$$

$$(4.2)$$

The first term of E aims to minimize μ to minimize local geometric distortions under f_{μ} and preserve the bijectivity of f_{μ} . The second term minimizes the discrepancy between $I \circ f_{\mu}$ and J_{c_1,c_2} . The last two terms aim to enhance the smoothness of f_{μ} . Since f_{μ} is bijectivity, the segmented object preserves the topology inherited from the topological prior image.

To simplify the optimization model, the strategy of splitting variables is utilized so that an alternating minimization scheme can be adopted. The modified optimization



FIG. 4.1. Illustration of interior angle Φ and exterior angle K(v) at vertex v on the left. The exterior angle can be used to determine the convexity of a polygon. The right shows a polygon which is partially convex with respect to Γ .

model then reads as follows:

$$\min_{\mu,\nu,c_1,c_2} E(\mu,\nu,c_1,c_2) = \int_{\Omega} |\nu|^2 + \eta (I \circ f_{\mu} - J)^2 + \lambda |\nabla\nu|^2 + \sigma (|u|^2 + |\nabla u|^2) + \delta |\nu - \mu|^2.$$
(4.3)

In practice, a digital image is discretized. In particular, an image can be discretized by a triangular mesh. Suppose Ω is triangulated as $\Omega \approx M = (V, E, F)$, where V is the set of vertices, E is the set of edges and F is the set of faces. The Beltrami coefficients μ and ν can then be discretized by complex-valued functions defined on V, which are respectively denoted by $\mu_V : V \to \mathbb{C}$ and $\nu_V : V \to \mathbb{C}$. Similarly, f_{μ} and u can be discretized on V and respectively denoted as $f_{\mu,V} : V \to \mathbb{C}$ and $u_V : V \to \mathbb{C}$. The deformed image $I \circ f_{\mu_V}$ can be defined on V by interpolation. $\nabla \nu_V$ and ∇u_V are computed by finite element method and are defined on V as well. As such, the segmentation model in the discrete setting is written as:

$$\min_{\mu_V,\nu_V,c_1,c_2} E(\mu_V,\nu_V,c_1,c_2) = \sum_{v \in V} |\nu_V|^2 + \eta \sum_{v \in V} (I \circ f_V^{\mu} - J)^2 + \lambda \sum_{v \in V} |\nabla \nu_V|^2
+ \sigma \sum_{v \in V} (|u_V|^2 + |\nabla u_V|^2) + \delta \sum_{v \in V} |\nu_V - \mu_V|^2$$
(4.4)

4.2. Formulation of the convexity prior. In this work, we propose to impose the convexity prior in our segmentation model. In this subsection, we describe how the convexity prior can be formulated, which can be easily incorporated into the segmentation model (4.4).

In the continuous setting, the convexity or partial convexity of a curve can be defined in the following sense.

DEFINITION 4.1 (Partial Convexity). Let $D \subseteq \mathbb{R}^2$ be a bounded domain with piece-wise smooth boundary. Let Γ be a subset of ∂D . D is said to be partially convex with respect to Γ if the curvature is non-negative on the smooth portions of Γ , and the exterior angles (i.e. the jump in tangent direction) on Γ are non-negative.

In the discrete setting in which $\Omega \approx M = (V, E, F)$, the object domain D is also triangulated as $D \approx (V', E', F')$ for some $V' \subset V$, $E' \subset E$ and $F' \subset F$. The notion of discrete convexity can be formulated by requiring D to be a convex polygon. We have the following definition of discrete (partial) convexity.

DEFINITION 4.2 (Discrete Partial Convexity). Let D be a polygonal domain with triangulation D = (V', E', F') and let ∂D be the boundary of D. The exterior angle

K(v) at a vertex $v \in \partial \tilde{D}$ is defined as $K(v) = \pi - \Phi$, where $\Phi = \sum_{\Delta \in F'_v} \varphi_{v,\Delta}$. Let Γ be a subset of ∂D . D is said to be discrete partially convex with respect to Γ if the exterior angle K(v) is non-negative for every $v \in \Gamma$.

Figure 4.1 shows the exterior angle at a vertex of a polygon, as well as a partially convex polygon. With the above definition, our segmentation model with convexity prior can now be formulated as the following optimization problem:

$$\min_{\mu_V, \nu_V, c_1, c_2} E(\mu_V, \nu_V, c_1, c_2) = \sum_{v \in V} |\nu_V|^2 + \eta \sum_{v \in V} (I \circ f_V^{\mu} - J)^2 + \lambda \sum_{v \in V} |\nabla \nu_V|^2 + \sigma \sum_{v \in V} (|u_V|^2 + |\nabla u_V|^2) + \delta \sum_{v \in V} |\nu_V - \mu_V|^2$$
(4.5)

subject to
$$K(v) = \pi - \sum_{\Delta \in F'_v} \varphi_{v,\Delta} \ge 0$$
 for all $v \in \Gamma$, where $\Gamma \subset \partial D$. (4.6)

The convexity constraint in the above optimization problem is formulated by K(v), which is defined in terms of the vertex angles φ . This constraint is difficult to handle. After adjusting φ to satisfy the convexity constraint, the corresponding deformation map f_V^{μ} has to be found to minimize the energy functional. However, there is no obvious way to obtain f_V^{μ} from φ . The existence of an associated f_V^{μ} from an adjusted φ is also questionable. As such, it is desirable to formulate the convexity constraint in terms of a suitable geometric quantity, which is closely related to an associated deformation map.

In this work, we propose to formulate the discrete convexity based on the dihedral angle. In particular, we will prove that discrete convexity in Definition 4.2 may be equivalently defined by the following inequality.

$$\sum_{e \in E_v} \theta_{f(e)} \ge (|F_v| - 1)\pi \quad \text{for every } v \in \Gamma.$$
(4.7)

In order to observe the above formulation, the relationship between the dihedral angles and the angle sum at a vertex has to be examined. More precisely, in the notations in subsection 3.2, we have the following proposition:

PROPOSITION 4.3. Let φ be the angles of the faces at each vertex of M, and θ be the dihedral angles of φ . Let v be a vertex of M, and E_v and F_v be the sets of edges and faces containing v. Then, for every vertex v of the immersion,

$$\sum_{\Delta} \varphi_{v,\Delta} = |F_v| \pi - \sum_{e \in E_v} \theta_e.$$

Proof. Let ϕ_{Δ}^1 and ϕ_{Δ}^2 be the two angles of the triangular face Δ other than $\varphi_{v,\Delta}$, then clearly we have $\varphi_{v,\Delta} = \pi - \phi_{\Delta}^1 - \phi_{\Delta}^2$. Hence,

$$\sum_{\Delta} \varphi_{v,\Delta} = \sum_{\Delta} (\pi - \phi_{\Delta}^{1} - \phi_{\Delta}^{2})$$
$$= |F_{v}|\pi - \sum_{\Delta} (\phi_{\Delta}^{1} + \phi_{\Delta}^{2})$$
$$= |F_{v}|\pi - \sum_{e \in E_{v}} \theta_{e}$$

10

In this work, the above formulation of discrete convexity in term of the dihedral angle will be adopted, with which the convexity constraint can be handled easily.

4.3. Overall segmentation model with (partial) convexity priors. The overall image segmentation model with (partial) convexity priors can now be reformulated as:

$$\min_{\mu_V,\nu_V} E(\mu_V,\nu_V) = \sum_{v \in V} |\nu_V|^2 + \eta \sum_{v \in V} (I \circ f_V^{\mu} - J)^2 + \lambda \sum_{v \in V} |\nabla \nu_V|^2 + \sigma \sum_{v \in V} (|u_V|^2 + |\nabla u_V|^2) + \delta \sum_{v \in V} |\nu_V - \mu_V|^2$$
(4.8)

subject to:
$$\sum_{e \in E_v} \theta_{f(e)} \ge (|F_v| - 1)\pi \text{ for all } v \in \Gamma, \text{ where } \Gamma \subset \partial D.$$
(4.9)

The constraint in the above optimization problem can be handled. To enforce the convexity constraint, a projection on the dihedral angle $\theta_{f(e)}$ can be carried out. There exists a corresponding deformation map f_V^{μ} associated to the modified dihedral angle $\theta_{f(e)}$, from which the map itself can be reconstructed. Solving the above optimization problem becomes feasible, which will be discussed in details in the next section.

5. Proposed algorithm. In this section, we discuss our proposed algorithm to solve the optimization model as described in subsection 4.3 in details. Our strategy is to apply an alternating minimization scheme, followed by a projection to satisfy the convexity constraint.

5.1. Alternating minimization. In order to minimize E, we adopt an alternating minimization scheme. In each iteration, by fixing ν_V , we minimize over μ_V . Similarly, fixing μ_V , we minimize over ν_V . More specifically, suppose $\mu_{V,n}$ and $\nu_{V,n}$ are obtained at the *n*-th iteration, we proceed to obtain $\mu_{V,n+1}$ and $\nu_{V,n+1}$ by solving the following sub-problems:

$$\mu_{V,n+1} = \operatorname{argmin}_{\mu:V \to \mathbb{C}} \{ \eta \sum_{v \in V} (I \circ f_V^{\mu} - J)^2 + \sigma \sum_{v \in V} (|u_V|^2 + |\nabla u_V|^2) + \delta \sum_{v \in V} |\nu_{V,n} - \mu|^2 \},$$
(5.1)

$$\nu_{V,n+1} = \operatorname{argmin}_{\nu:V \to \mathbb{C}} \{ \sum_{v \in V} |\nu|^2 + \lambda \sum_{v \in V} |\nabla \nu|^2 + \delta \sum_{v \in V} |\nu - \mu_{V,n+1}|^2 \}.$$
(5.2)

We first solve the μ_V -subproblem and ν_V -subproblem, followed by a convexity projection to enforce the convexity constraint. We will now describe the algorithms to solve each subproblem in details.

5.1.1. μ_V -subproblem. It can be proven (see [8]) that the global minimizer of the sub-problem (5.1) can be obtained from the solution to the linear system

$$\left(\eta ||\nabla I||_2^2 + \sigma - \sigma \Delta\right) u_V = \eta (J - I)(\nabla I), \tag{5.3}$$

subjected to $u_V = 0$ on $\partial\Omega$, where ∇, Δ are the discrete gradient and the discrete Laplacian operator on Ω . Then, $\mu_{V,n+1}$ is computed via the finite difference differentiation

$$\mu_{V,n+1}(z) = \frac{\partial f_V^{\mu}}{\partial \overline{z}}(z) / \frac{\partial f_V^{\mu}}{\partial z}(z), \quad \forall z \in \Omega,$$
(5.4)

where $f_V^{\mu} = u_V + Id$. Note that if the denominator of (5.4) is 0 at some $z_0 \in \Omega$, then we set $\mu_{V,n+1}(z_0) = 0$. This completes the sub-problem (5.1).

5.1.2. ν_V -subproblem. To solve the subproblem (5.3), we compute the Euler-Lagrange equation of the energy functional. It ends up with a linear system. More specifically, $\nu_{V,n+1}$ is updated by solving the following linear system

$$(2+2\delta-\lambda\Delta)\nu_{V,n+1} = 2\delta\mu_{V,n+1},\tag{5.5}$$

subjected to $\nu_{V,n+1} = 0$ on $\partial \Omega$. This solves the sub-problem (5.2).

5.1.3. c_1, c_2 -subproblem. Finally, fixing μ_V and ν_V , we optimize over c_1 and c_2 . The optimizers can be written explicitly, which are

$$\begin{cases} c_1 = \frac{\int_D I \circ f_{V,n+1}^{\nu} dz}{\int_D dz}, \\ c_2 = \frac{\int_{\Omega \setminus D} I \circ f_{V,n+1}^{\nu} dz}{\int_{\Omega \setminus D} dz}. \end{cases}$$
(5.6)

5.2. Convexification. After the alternating minimizing steps, a projection is carried out to satisfy the convexity constraint. In this subsection, we will describe the convexification step in details.

5.2.1. Mesh construction from dihedral angles. After the μ_V -subproblem and ν_V -subproblem are solved, the dihedral angles have to be adjusted to satisfy the convexity constraint. After the dihedral angles are modified, it is necessary to have an algorithm to construct a mesh from the dihedral angles. The mesh gives the deformation map f_V^{μ} associated to the dihedral angles. As such, the adjusted dihedral angles can naturally be incorporated to the segmentation model.

In this subsection, the algorithm to construct a mesh from dihedral angles will be explained in details. To this end, the following theorem is important for the development of our algorithm.

THEOREM 5.1. Suppose M is a mesh embedded in \mathbb{R}^n with dihedral angle θ . Then its angles on each face are the solution φ^* of the variational problem

$$\varphi^* = \operatorname{argmax}_{\substack{\varphi \text{ plausible angle assignment} \\ dihedral angles \text{ given by } \theta}} - \sum_{v,\Delta} \int_{0}^{\varphi_{v,\Delta}} \log 2 \sin t dt, \quad (5.7)$$

Further, the objective functional above, denoted by \mathcal{E} , has the following properties.

- 1. Suppose $v \in V$ is a vertex of a face $\Delta \in F$, $\frac{\partial}{\partial \varphi_{v,\Delta}} \mathcal{E} = -\log(2\sin\varphi_{v,\Delta})$ and $\frac{\partial^2}{\partial \varphi_{v,\Delta}^2} \mathcal{E} = -\cot\varphi_{v,\Delta}$. All other first and second order partial derivatives vanish.
- 2. E is concave on the space of plausible angle assignments.

Proof. This theorem has been proven by Rivin in [38]. This theorem can also be illustrated by a more intuitive proof in terms of simple sine law for easier understanding, as described in the Appendix. \Box

Note that the objective functional is independent on θ . Rather, θ appears in the constraint, which reads that the argument φ must satisfy (3.6), (3.7), (3.8) and (3.9), all of which are linear and bound constraints. Theorem 5.1 may be seen as a



FIG. 5.1. Illustration of Equation (5.8)

quantitative version of Theorem 3.7. While the latter asserts the existence of a mesh, the former nails down the shape of the mesh through the specification of its angles.

More specifically, Theorem 5.1 offers a feasible solution to construct a mesh associated to a given dihedral angle. Given a plausible dihedral angle, one can find the angle assignment of the associated immersion by solving the optimization problem (5.7). The vertex positions of the mesh are then obtained from the plausible angle assignment.

On the other hand, to turn the constraint to a bound constraint amenable to numerical computation, the feasible set of the optimization problem (5.7) is further restricted to the space of ε -plausible angle assignment for some small $\varepsilon > 0$. The modified problem is then a concave problem with bound and linear constraints, and hence can be easily solved by any standard convex optimization solver. In this work, the interior-point method [4, 5, 51] is used. To further improve accuracy, a few Newton steps may be added afterwards. For all these algorithms, only the function value of the objective function, as well as its gradient, Hessian are needed. The latter two are available in closed form and the objective values may be efficiently approximated by Taylor expansion.

In addition, if the immersion is planar, the vertex positions can be found by solving a linear equation arising from sine law. This approach allows flexible introduction of regularization to control the mesh position. Before introducing the promised equation, the criteria of planarity in terms of the discrete conformality structures are presented below.

PROPOSITION 5.2. Let M be a connected embedded mesh such that every triangle has an interior vertex. In the notations of Proposition (4.3), the followings are equivalent.

- M is planar.
- For every interior vertex v, Σ_Δ φ_{v,Δ} = 2π.
 For every interior vertex v, Σ_{e∈E_v} θ_e = (|F_v| 2)π.

Proof. The mesh M is planar if and only if the angle sum at every interior angle is 2π , and this is equivalent to the above equation by Proposition 4.3.

The linear equation for solving the vertex positions can then be derived. Suppose the faces are positively oriented on the plane. In other words, for each face $\Delta \in F$, going from the first vertex $v_{\Delta;1}$, through the second $v_{\Delta;2}$ and third $v_{\Delta;3}$, back to the first forms a counter-clockwise loop. Then for each face $\Delta \in F$ and and for each $I \in \{1, 2, 3\}$, the equation

$$\frac{v_{\Delta;I-1} - v_{\Delta;I}}{\sin\varphi_{v_{\Delta;I+1},\Delta}} = e^{\sqrt{-1}\varphi_{f,v_{\Delta;I}}} \frac{v_{\Delta;I+1} - v_{\Delta;I}}{\sin\varphi_{v_{\Delta;I-1},\Delta}}$$
(5.8)

is imposed where v is the vertex positions represented by complex numbers. This ensures the length ratios obey the sine law and the angles are as prescribed. Equivalently, referring to Figure (5.1) with A playing the role of $v_{\Delta,I}$, the equation (5.8) reads

$$\frac{C-A}{\sin\beta} = e^{\sqrt{-1}\alpha} \frac{B-A}{\sin\gamma}$$

By solving equation (5.8), the immersion of the mesh can be obtained as supported by the following proposition.

PROPOSITION 5.3. If the triangulation is connected and has an immersion in the plane, (5.8) has a unique solution up to a linear conformal map.

Proof. Theorem 3.7 guarantees that the equation has nontrivial solutions (i.e. the vertices do not all have the same position). Clearly, translating, rotating and scaling a solution gives another solution.

It remains to show uniqueness. Firstly, for a solution \mathbf{v} , if for some face $\Delta_0 \in F$ and some $I_0 \in \{1, 2, 3\}$, $v_{\Delta_0, I_0} = v_{\Delta_0, I_0+1}$, then $v_{\Delta, I} = v_{\Delta_0, I_0}$ for every face $\Delta \in F$ (in the same connected component) and every $I \in \{1, 2, 3\}$, because (5.8) for Δ_0 and I_0 implies $v_{\Delta_0;1} = v_{\Delta_0;2} = v_{\Delta_0;3}$, and the case for other faces (in the connected component) follows inductively.

For uniqueness, Consider two nontrivial solutions \mathbf{v} and $\tilde{\mathbf{v}}$. The above argument shows that for no face $\Delta_0 \in F$ and index $I_0 \in \{1, 2, 3\}$, $v_{\Delta_0, I_0} = v_{\Delta_0, I_0+1}$ or $\tilde{v}_{\Delta_0, I_0} = \tilde{v}_{\Delta_0, I_0+1}$ holds. Then fixing a face $\Delta_0 \in F$ and an index $I_0 \in \{1, 2, 3\}$ translating, scaling and rotating if necessary, it may be assumed that $v_{\Delta_0, I_0} = \tilde{v}_{\Delta_0, I_0}$ and $v_{\Delta_0, I_0+1} = \tilde{v}_{\Delta_0, I_0+1}$. Then the above argument shows $\tilde{\mathbf{v}} - \mathbf{v}$ is in fact a trivial solution. The result then follows. \Box

Furthermore, vertex positions may be regularized by turning the linear system (5.8) into a variational model and adding a term that penalizes vertices' deviation from desired positions. Let v^{target} be the desired position of vertex v. Then a compromise between the discrete conformality structure and the vertex positions can be achieved by solving the following least-square problem (possibly with linear constraints),

$$\min_{\mathbf{v}} |A\mathbf{v}|^2 + \sum_{v_j \in V} \rho_v |\mathbf{v}_j - v_j^{\text{target}}|^2,$$
(5.9)

where A is the matrix defined in (5.8), and $0 \le \rho_v \le \infty$ is the user-chosen penalty parameter for each vertex².

The algorithm to construct a planar immersion from a plausible dihedral assignment is summarized in Algorithm 1.

5.2.2. Convexity projection. With Algorithm 1, we can now describe our proposed convexity projection to enforce the convexity constraint (4.7).

Recall that the image domain Ω is triangulated as

$$\Omega = (V, E, F), \tag{5.10}$$

with subsets $V' \subset V$, $E' \subset E$ and $F' \subset F$ such that the template object domain can be triangulated by the sub-mesh structure

$$D = (V', E', F'). \tag{5.11}$$

²The convention of $0 \cdot \infty = 0$ is used in case $\rho_v = \infty$.

Algorithm 1 regularized planar dihedral angle-based conformality structure solver Inputs:

1. a triangulation M = (V, E, F) of Ω

- 2. a planar Delaunay plausible dihedral assignment θ
- 3. a target vertex position v^{target} of each vertex v
- 4. a penalty parameter ρ_v for each vertex
- 5. an auxiliary parameter $0 < \varepsilon < \pi$

Output: new vertex positions V^* in complex numbers

1: Solve the following maximization problem for φ^* :

$$\varphi^* = \operatorname{argmax}_{\substack{\varphi \ \epsilon \text{-plausible angle assignment} \\ \text{dihedral angles given by } \theta}} - \sum_{v,\Delta} \int_0^{\varphi_{v,\Delta}} \log 2 \sin t dt$$

2: Solve the following equation for the coefficient matrix A of $v_{\Delta;I}$:

$$\frac{v_{\Delta;I-1} - v_{\Delta;I}}{\sin \varphi^*_{v_{\Delta;I+1},\Delta}} = e^{\sqrt{-1}\varphi^*_{f,v_{\Delta;I}}} \frac{v_{\Delta;I+1} - v_{\Delta;I}}{\sin \varphi^*_{v_{\Delta;I-1},\Delta}}$$

where $I \in \{1, 2, 3\}$, and $\Delta = \{v_{\Delta;1}, v_{\Delta;2}, v_{\Delta;3}\} \in F$ 3: Solve the following functional:

$$\min_{\mathbf{v}} |A\mathbf{v}|^2 + \sum_{v_j \in V} \rho_v |\mathbf{v}_j - v_j^{\text{target}}|^2$$

for V^*

Let Γ be the set of segments on which partial convexity prior is imposed.

Given $f: \Omega \to \Omega$ (we put $f = f^{\mu_n}$ in the main algorithm) with non-convex f(D), we proceed to construct a sequence of diffeomorphisms $\{g_i\}_{i=1}^N$

 $g_i: \Omega \to \Omega, \quad g_i(\partial \Omega) = Id, \quad i = 0, 1, 2, \dots,$ (5.12)

such that g_i gradually convexifies f(D) in the sense that

$$g_N \circ f(D) \tag{5.13}$$

has non-negative exterior angles at prescribed portions eventually.

Denote $f_i = g_i \circ f$. Let $\theta_{\tilde{f}_i(D)}$ be the dihedral angle assignment of the mesh $\tilde{f}_i(D)$. Our goal is to update $\theta_{\tilde{f}_i(D)}$, such that it better satisfies the convexity constraint (4.7). We consider an energy functional defined on the dihedral angle assignment $\theta : E \to \mathbb{R}$ analogous to the one in [25]:

$$E_{proj}(\theta) = E_{fidelity}(\varphi(\theta), \varphi(\theta_{\tilde{f}_i(D)})) + \omega E_{concavity}(\theta), \qquad (5.14)$$

where $E_{concavity}$ measures the derivation of θ from the convexity constraint; $\varphi(\theta)$ and $\varphi(\theta_{\tilde{f}_i(D)})$ are the vertex angle assignments of θ and $\theta_{\tilde{f}_i(D)}$ respectively and $E_{fidelity}$ measures the discrepancy between them. $\omega > 0$ is a balancing parameter. By minimizing E_{proj} iteratively, we search for θ that better satisfies the convexity constraint while its associated mesh is not too deviated from the previous mesh $\tilde{f}_i(D)$.

In this work, we define $E_{concavity}$ as follows:

$$E_{concavity}(\theta) = \sum_{v \in \Gamma} \left(\sum_{e \in E_v} \left(\theta_{\tilde{f}_i(e)} - (|F_v| - 1)\pi \right)^- \right), \tag{5.15}$$

where the super-scripted minus sign denotes the negative part. $E_{fidelity}$ is defined as follows:

$$E_{fidelity}(\varphi(\theta),\varphi(\theta_{\tilde{f}_i(D)})) = \sum_{\Delta \in F, v \in V} \frac{1}{3} \frac{\operatorname{area}(f_i(\Delta))}{\operatorname{area}(\Omega)} |\varphi(\theta)_{\Delta,v} - \varphi(\theta_{\tilde{f}_i(D)})_{\Delta,v}|^2 \quad (5.16)$$

This term penalizes the area-weighted sum of the difference between $\varphi(\theta)$ and $\varphi(\theta_{\tilde{f}_i(D)})$ for all $\Delta \in F, v \in V$, hence diminishing the overall deviation of the shape of the triangulation.

Note that $E_{concavity}$ can also be formulated in term of $\varphi(\theta)$. More precisely,

$$E_{concavity}(\varphi(\theta)) = \sum_{v \in \Gamma} (\pi - \sum_{\Delta \in F} \varphi(\theta)_{\Delta,v})^{-}.$$
 (5.17)

As such, to minimize E_{proj} , we first solve:

$$\hat{\varphi}^{i+1} = \underset{\substack{\varphi \text{ planar } \varepsilon\text{-angle assignment,} \\ \varphi \varepsilon\text{-Delaunay over } D}}{\arg\min} E_{fidelity}(\varphi,\varphi_i) + \omega \sum_{v \in \Gamma} (\pi - \sum_{\Delta \in F} \varphi_{\Delta,v})^-, \quad (5.18)$$

where $\varphi_i = \varphi(\theta_{\tilde{f}_i(D)})$ is the angle assignment of the previous mesh $\tilde{f}_i(D)$ and $0 < \varepsilon < \pi$ is a user-chosen parameter. It is noted that $\hat{\varphi}^{i+1}$ thus obtained may not be the angles of some embedded mesh. Nevertheless, we can compute the associated dihedral angle assignment $\theta(\hat{\varphi}^{i+1})$ according to Definition 3.9. The associated mesh $\tilde{f}_{i+1}(D) = g_{i+1}(f(D))$ can be constructed using Algorithm 1, which better satisfies the convexity constraint. To ensure stability of the position of f(D), penalty parameter ρ_v is chosen as:

$$\rho_v = \begin{cases} \rho L_v/2 & \text{if } v \text{ is on the contour} \\ 0 & \text{otherwise} \end{cases},$$
(5.19)

where $\rho > 0$ is a user-chosen parameter, and L_v is the length of the edges on $f(\partial D)$ containing f(v).

It is remarked that the the angles in $\tilde{f}_{i+1}(D)$ are not the same as $\varphi^{\hat{i}+1}$. However, by Proposition 4.3, they share the same exterior angle as they have the same dihedral angle. Hence, the improvement in convexity from computing $\varphi^{\hat{i}+1}$ is retained in $\tilde{f}_{i+1}(D)$.

Afterwards, the remaining $g_{i+1} \circ f(\Omega \setminus D)$ is constructed by the conformal extension of $g_{i+1} \circ f(D)$ with the fixed boundary condition $g_{i+1}(\partial \Omega) = Id$. Explicitly, by writing $f = u + \sqrt{-1}v$ and $\mu = \zeta + \sqrt{-1}\tau$, the Linear Beltrami Solver (LBS) is applied to solve for f by

$$\nabla \cdot \left(C \begin{pmatrix} u_x \\ u_y \end{pmatrix} \right) = 0, \quad \nabla \cdot \left(C \begin{pmatrix} v_x \\ v_y \end{pmatrix} \right) = 0, \tag{5.20}$$

where

$$C = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}, \quad \alpha_1 = \frac{(\zeta - 1)^2 + \tau^2}{1 - \zeta^2 - \tau^2}, \\ \alpha_2 = -\frac{2\tau}{1 - \zeta^2 - \tau^2}, \\ \alpha_3 = \frac{(\zeta + 1)^2 + \tau^2}{1 - \zeta^2 - \tau^2}, \\ (5.21)$$

with μ replaced by μ_f , the boundary data $(g_{i+1} \circ f)|\partial D$ and $Id|\partial\Omega$ to first obtain $\hat{h}_{i+1}(R^C)$, and then again with $\mu = \operatorname{proj}_{B(0,1-\varepsilon_{\mu})}(\mu_{\hat{h}_{i+1}})$ and the same boundary data to obtain $(g_{i+1} \circ f)(R^C)$.

In this way, the deformations $f(D) \to g_1 \circ f(D) \to g_2 \circ f(D) \to \ldots$ are smooth and $g_i \circ f(D)$ tends to be convex as $i \to \infty$. In practice, a maximum iteration parameter $N' \in \mathbb{N}$ can be prescribed to control the maximum number of iterations involved. While a convex region is usually obtained at the last iteration, and non-convexity is acceptable in intermediate stages, if one would like to guarantee convexity, one may, at the end, by setting $\omega = \infty$, impose tentative target position of two vertices in Algorithm 1, and then find the best-fit rotation and translation with regards to the vertex positions of $f(\partial D)$.

The overall algorithm for convexity projection is summarized in Algorithm 2.

5.3. Overall algorithm. In summary, our proposed segmentation model with convexity priors is solved iteratively. In each iteration, the alternating minimization steps are carried out, which is described in subsection 5.1. Afterwards, a convexification step is performed, which is described in subsection 5.2. The overall image segmentation model with convexity prior can now be described as in Algorithm 3.

6. Experimental results. We have tested our proposed segmentation algorithm on both synthetic and real images. In this section, we show the experimental results to demonstrate the performance of our proposed model.

In our experiments, all computations are done in MATLAB and its optimization toolbox 2016a [36, 35]. The triangulations of Ω and D are computed by [41].

6.1. Full convexity prior. We first test our algorithm with full convexity prior. In other words, we assume f(D) is globally convex.

6.1.1. Synthetic binary images. Figure 6.1 presents a series of experiments on synthetic binary images, each of which consists of a single object with clean boundary. The target objects for the experiments are, respectively, a square, an ellipse, and a hexagon, which are all convex. However, in each experiment, parts of the target object are manually removed in order to simulate occlusions. In particular, the convexity of the target object is lost. The task here is to segment the target object while retrieving a convex object in the segmentation result.

The segmentation results are visualized in Figure 6.1. The 1st column shows the target image I while the 2nd column shows the contour of the template object ∂D in J superimposed on I. The 3rd column presents the segmentation result of the foundation model in [8], or equivalently, the model (4.3). It can be seen that the occluded regions are excluded by the model. The segmentation results are indeed accurate though, if we assume no prior shape knowledge of the target objects. However, according to our setup that the target object has part of its boundary being occluded, the segmentation results are not really satisfying. In the 4th column, the segmentation results using the model in [20], which also aims at segmenting a convex region out of the given image, are presented. It can be observed that although the segmented regions are convex, they are not capturing the target objects accurately, especially in the experiment of ellipse and that of hexagon. Finally, in the last column, the segmentation results

16

Algorithm 2 Convexification Step

Inputs:

- 1. a triangulation M = (V, E, F) on Ω with a template region $D \subseteq \Omega$
- 2. a set of edges $\Gamma \subset \partial D$ to be convexified
- 3. a boundary fixing piecewise-linear homeomorphism $f: \Omega \to \Omega$
- 4. model parameters $0 < \omega < \infty, \rho > 0$
- 5. auxiliary parameters $0 < \varepsilon < \pi, 0 < \varepsilon_{\mu} < 1, N' \in \mathbb{Z}_+$

Output: A piecewise linear homeomorphism $g: \Omega \to \Omega$ such that g(f(D)) is sufficiently convex

- 1: Compute the Beltrami coefficient $\mu_f = \rho + \sqrt{-1}\tau$ of $f = u + \sqrt{-1}v$.
- 2: Set $g_0 = f$.
- 3: for i = 0, 1, 2, ..., N' 1 do
- Solve the variational problem 4:

$$\hat{\varphi}^{i+1} = \operatorname*{arg\,min}_{\substack{\varphi \text{ planar } \varepsilon \text{-angle assignment,} \\ \varphi \text{ } \varepsilon \text{-Delaunay over } D}} E_{fidelity}(\varphi,\varphi_i) + \omega \sum_{v \in \Gamma} (\pi - \sum_{\Delta \in F} \varphi_{\Delta,v})^-, \quad (5.22)$$

for $\hat{\varphi}^{i+1}$ with

$$E_{fidelity}(\varphi(\theta),\varphi(\theta_{\tilde{f}_i(D)})) = \sum_{\Delta \in F, v \in V} \frac{1}{3} \frac{\operatorname{area}(\tilde{f}_i(\Delta))}{\operatorname{area}(\Omega)} |\varphi(\theta)_{\Delta,v} - \varphi(\theta_{\tilde{f}_i(D)})_{\Delta,v}|^2.$$
(5.23)

Compute the dihedral angles θ^i of $\hat{\varphi}^{i+1}$ by 5:

$$\theta^i = \sum_{v \in V_e} \hat{\varphi}^{i+1}.$$
(5.24)

Compute $g_{i+1}(f(D))$ by applying Algorithm 1 with 6:

$$\rho_v = \begin{cases} \rho L_v/2 & \text{if } v \text{ is on the contour} \\ 0 & \text{otherwise} \end{cases},$$
(5.25)

- Obtain $\hat{h}_{i+1}(\mathbb{R}^C)$ by solving Equation (5.20). 7:
- Compute the Beltrami coefficient $\mu_{\hat{h}_{i+1}}$ of \hat{h}_{i+1} and project it to the shrunk 8: unit disc $B(0, 1 - \varepsilon_{\mu})$ to obtain $\hat{\mu}_{i+1}$. Compute $g^{i+1}(f(D^C))$ by the LBS again with $\hat{\mu}_{i+1}$ in place of μ , and boundary
- 9: data given by $g_{i+1}(f(\partial D))$ and $f(\partial \Omega)$.
- Combine $g_{i+1}(f(D))$ and $g_{i+1}(f(D^C))$ to form $g_{i+1}(f(\Omega))$. 10:
- 11: end for

12: **return** $g = g_{N'}$

\mathbf{A}	lgorithm	3	Image	segmentation	with	convexity	priors
--------------	----------	---	-------	--------------	------	-----------	--------

Inputs:

- (a) Given image $I: \Omega \to \mathbb{R}$ in which Ω is triangulated by M = (V, E, F)
- (b) Topological prior image $J: \Omega \to \mathbb{R}$ with known template object $D \subset \Omega$
- (c) Selected segments $\Gamma \subset D$ to be convexified
- (d) model parameters $0 < \omega < \infty, \rho, \sigma, \delta, \lambda > 0$
- (e) auxiliary parameters $0 < \varepsilon < \pi, 0 < \varepsilon_{\mu} < 1, N', K \in \mathbb{Z}_+$

Output: A topological preserving diffeomorphism f_V segmenting the target object by $f_V(D)$ in which $f_V(D)$ is sufficiently convex

- 1: Initialize: $\mu_{V,0} = \nu_{V,0} = 0$
- 2: for $n = 0, 1, \dots, N 1$ do
- 3: μ_V -subproblem: Compute μ_V by solving the linear system

$$\left(\eta ||\nabla I||_2^2 + \sigma - \sigma \Delta\right) u_V = \eta (J - I) (\nabla I),$$

subjected to $u_V = 0$ on $\partial \Omega$. Then compute

$$\mu_{V,n+1}(z) = \frac{\partial f_V^{\mu}}{\partial \overline{z}}(z) / \frac{\partial f_V^{\mu}}{\partial z}(z), \quad \forall z \in \Omega,$$

where $f_V^{\mu} = u_V + Id$.

4: ν_V -subproblem: Compute $\nu_{V,n+1}^1$ by solving

$$(2+2\delta-\lambda\Delta)\nu_{V,n+1}^1 = 2\delta\mu_{V,n+1}$$

subjected to $\nu_{V,n+1}^1 = 0$ on $\partial \Omega$.

- 5: Compute the corresponding mapping $f_{V,n+1}^{\nu'}$ of the Beltrami coefficient $\nu_{V,n+1}^1$ by solving Equation (5.20).
- 6: **if** mod(n, K) = 0 **then**
- 7: Convexification step: Compute g^{n+1} by Algorithm (2) with input $\Omega = (V, E, F), D \subset \Omega, \Gamma, \omega$ and ρ
- 8: Compute the Beltrami coefficient $\nu_{V,n+1}^2$ of g^{n+1} by (3.1).
- 9: Compute $\nu_{V,n+1}$ by the composition of the Beltrami coefficients $\nu_{V,n+1}^1$ and $\nu_{V,n+1}^2$:

$$\nu_{V,n+1} = \frac{\nu_{V,n+1}^1 + (\nu_{V,n+1}^2 \circ f_{V,n+1}^{\nu'})\tau}{1 + \overline{\nu_{V,n+1}^1}(\nu_{V,n+1}^2 \circ f_{V,n+1}^{\nu'})\tau}, \quad \tau = \overline{\left(\frac{\partial f_{V,n+1}^{\nu'}}{\partial z}\right)} / \frac{\partial f_{V,n+1}^{\nu'}}{\partial z}$$

10: **else**

- 11: Set $\nu_{V,n+1} = \nu_{V,n+1}^1$
- 12: **end if**
- 13: c_1, c_2 -subproblem: Update

$$J(z) = \begin{cases} c_1, & \text{if } z \in D \\ c_2, & \text{if } z \in \Omega \setminus D \end{cases} \quad \text{where } \begin{cases} c_1 = \frac{\int_D I \circ f_{V,n+1}^{\nu} dz}{\int_D dz}, \\ c_2 = \frac{\int_{\Omega \setminus D} I \circ f_{V,n+1}^{\nu} dz}{\int_{\Omega \setminus D} dz} \end{cases}$$

14: end for 15: return $f_V = f_{V,N}^{\nu}$.



FIG. 6.1. Segmentation results with full convexity prior on synthetic binary images. The first column shows the input binary images. The second column shows the initial contours. The third column shows the segmentation result of the foundation model without convexity prior. The fourth column shows the segmentation results using the model in [20], which also enforces the convexity prior. The last column shows the segmentation results using our proposed method.

using the proposed model are demonstrated. It is evident that the model achieves high accuracy capturing the target objects while eluding the occlusions. Therefore, these results demonstrate the effectiveness of our proposed model to maintain a balance between convexity of the segmented domain and the accuracy of the segmentation process.

6.1.2. Real images. We also test our proposed segmentation model with full convexity prior on real images. Figure 6.2 shows a series of experiments on real images, each of which shows a target object with occlusion, and hence, having an unclear boundary. The task here is to segment the target object with a natural boundary eluding the occlusions as much as possible. As before, in Figure 6.2, the first and the second column shows the target images and the initial contours of the template object respectively. It is noted that the colored images are compressed into gray-scale images before applying those segmentation models. The third column shows the segmentation results using our foundation model without the convexity prior. The fourth column shows the segmentation results using the model in [20], which also enforces the full convexity prior. The last column shows the segmentation results using our proposed method with full convexity prior enforced.

For the first experiment, the target object is a piece of paper which is covered by many pens. As a result, both the geometry and the topology of the piece of paper is lost due to the occlusion. From the segmentation results, it can be seen that without assuming a convex target object, the foundation model mixes up some pens with the paper, including them into the foreground which is inaccurate. The segmentation result using the model in [20] is better, for at least the model tries to segment a convex region and ignores the pens, yet the accuracy is still unsatisfying. However,



FIG. 6.2. Segmentation results with full convexity prior on real images. The first column shows the input real images. The second column shows the initial contours. The third column shows the segmentation result of the foundation model without convexity prior. The fourth column shows the segmentation results using the model in [20], which also enforces the convexity prior. The last column shows the segmentation results using our proposed method.

using the proposed model, clearly the target paper is extracted accurately without including the occlusions.

In the second experiment, the road sign is partially blocked by some grass. This is one common scenario in real life in which the occlusions destroys the convexity of the target object. Referring to the results, not surprisingly, the foundation model separates all the grass from the road sign and this in turn excludes some hindering part of the road sign. Then, it is observed that the model in [20] does not segment a fully convex region, and the boundary of the road sign is not captured very accurately. Using the proposed model yields the best result among the three attempts, that most part of the road sign is captured with a fully convex domain and the effect of occlusion seems to be diminished.

The third experiment demonstrates one another common type of occlusions destroying convexity of the target object. The target object is still a road sign, yet the occlusion does not come from coverage of other objects but instead is caused by stains on it. From the results, while simply prescribing the segmented domain to be genus-0 using the foundation model does not give a satisfying result, the model in [20] cannot accurately locate the boundary contour of the road sign. It is the contribution of the proposed model to capture the target road sign correctly, with those occlusions being neglected and the true boundary contour is detected more accurately.

Our proposed segmentation model can also handle high-genus domain with full convexity prior. Figure 6.3 demonstrates one medical application of segmentation with convexity prior of high-genus objects. In this experiment, an MR image of the cardiac system including the cardiovascular is given. The goal of this experiment is to simulate the segmentation of the cardiovascular. According to medical research, the



FIG. 6.3. Segmentation results of high-genus domain with full convexity prior on real MR images. The first column shows the input images. The second column shows the initial contours. The third column shows the segmentation result of the foundation model without convexity prior. The last column shows the segmentation results using our proposed method.



FIG. 6.4. Segmentation results with partial convexity prior on synthetic images. The first column shows the input binary images. The second column shows the initial contours and partial convexity priors are enforced on the red segments. The third column shows the segmentation result of the foundation model without convexity prior. The fourth column shows the segmentation results using our model with full convexity prior. The last column shows the segmentation results using our proposed method with partial convexity prior.

cardiovascular should be a hollow object (allowing blood flow) with both the outer and the inner boundary (i.e. the vascular walls) enclosing a convex region. However, the vascular walls in the image are shown to have local concavity (see the first column). In this case, a segmentation tool being able to enforce convexity prior is particularly helpful, aiding the approximation of the true position of the vascular walls in prior to any medical operations. Using the proposed model with a torus as the prior object, it can be observed in the last column that the segmented region matches with the convexity constraint and captures the vascular walls accurately. The segmentation result using the foundation model is also presented in the third column for reference. And it is obvious that the foundation model obeys the intensity difference accurately but does not give a truly helpful result.

6.2. Partial convexity prior. In this subsection, we study our algorithm with partial convexity prior. In other words, we only assume a subset $\Gamma \subset f(D)$ is convex.

6.2.1. Synthetic binary images. We test the proposed algorithm with partial convexity priors on synthetic binary images. The segmentation results are demonstrated in Figure 6.4. The first column shows the input binary images to be segmented. It is noted that in the second column, the red contours indicate the portions of the

boundary of the template object in which convexity is constrained, while the green contours correspond to the boundary of the template object in which not further constraint is enforced. In the first experiment, the target object is a cartoon-moon shaped object. While the left hand side of its boundary should have negative curvature, on the right hand side where non-negativity should be assumed, manual occlusions are added so that the convexity there is destroyed. In the second experiment, the target object is a bin-shaped object, and is assumed to have its outer boundary of entirely non-negative curvature, but not its inner boundary. Similarly, manual obstructions are added around its outer boundary to simulate occlusions. Therefore, convexity should only be imposed on the outer boundary. This renders a segmentation experiment with partial convexity prior on a high-genus object. In both experiments, the tasks are to recover the obscured regions there while preserving the right geometry of the target objects.

The third, fourth and the last columns of Figure 6.4 demonstrate the segmentation results using the foundation model without convexity priors, the proposed model with full convexity being constrained (i.e. convexity is enforced on the whole boundary contours), and the proposed model with partial convexity being constrained in the manner as mentioned before. It can be observed that segmenting the target objects either without using any convexity prior or enforcing fully convex constraint does not give correct results. Only the partial convexity prior can give more relevant solutions for these images having partially occluded objects with much more complicated geometry.

6.2.2. Real images. Indeed, segmentation with partial convexity prior has a wide range of real applications. For instance, in medical science, photography, zoology and even archaeology, it is very common to have a photo capturing a partially occluded object that has a complicated geometry. Figure 6.5 presents some experiments on real images. As before, the first and the second column show the input image and the contour of the template object's boundary superimposed on it, with the red contours corresponding to the portions where convexity is enforced when a partial prior is applied. And, again, the third, fourth and the last columns of Figure 6.4 correspond to the segmentation results using the foundation model without convexity prior, the proposed model with full convexity prior, and the proposed model with partial convexity prior.

In the first experiment, the target object is a piece of leaf which is generally convex. However, part of it is missing due to physical damage. Consider the case if scientists want to recover its original shape, then segmentation with convexity prior is a great tool for advising its actual shape. From this point of view, using simply the foundation model without enforcing the convexity prior cannot retain the missing part as its disk topology is still preserved under the occlusion. The result presented in the fourth column given by the proposed model with full convexity prior seems to be satisfactory. It naturally recovers the missing part of the leaf. However, upon zooming in, one may observe that the there are many local textures that are important to bio-taxonomy and should be preserved. These textures cause fluctuations around the boundary and thus enforcing global convexity on the boundary is indeed not a suitable solution. To accurately keep those minor but important details while naturally recovering the missing part of the leaf, the proposed model should be employed and it can be seen that the result is satisfying. By enforcing partial convexity close to the missing portions of the leaf, the details of the leaf are mostly preserved and the missing part is naturally recovered, which is beneficial for further biological,



FIG. 6.5. Segmentation results with partial convexity prior on real images. The first column shows the input real images. The second column shows the initial contours and partial convexity priors are enforced on the red segments. The third column shows the segmentation result of the foundation model without convexity prior. The fourth column shows the segmentation results using our model with full convexity prior. The last column shows the segmentation results using our proposed method with partial convexity prior.

bio-taxonomical or even archaeological study.

The target object of the second experiment is a wild bear with its bottom part being covered by grass. This is just like the scene that most photos of many other wild animals would capture. In zoology, the height and the size of animals is an interesting study. On top of it, the appearance and the shape of the animals also drive much information for further zoological study. It is therefore important to scientifically estimate the size of the animals captured in the photos. One possible solution is to segmenting the animals from the image while estimating the occluded part by convexity prior. From the segmentation results, clearly both segmentation with convexity prior and solely topology preserving segmentation are not enough for accomplishing the task. They either mess up the occluded part or segment only a very rough and inaccurate convex region. It is only the proposed model with partial convexity prior that can capture all those clear shape details on the top part of the bear, while naturally approximating the bottom part of which is obstructed by grass.

The third experiment is kind of similar to the second one, that the target object is being occluded by grass, but the target object here is a fruit instead of an animal. Actually, in photography it is common to have a photo of occluded objects. Those occlusions could sometimes be considered as the major contributing factor to the beauty and the uniqueness of the photo and hence are important to be kept. However, as for photo editing, it is equally important to be able to extract the object accurately regardless to the obstructions, so that further editing can be applied to a local yet meaningful region. For this purpose, while topology preserving segmentation enforces too loose constraint and hence its result is greatly affected by the occlusions, fully convex segmentation enforces too much constraint on the convexity of the target object and thus some of its details are lost. The proposed algorithm with partial convexity being enforced on the red contour as shown in Figure 6.5 yields the most natural and accurate segmentation of the fruit, capturing both the shape details and also the occluded parts naturally.

In the last subsection, a medical example in which full convexity prior may be utilized to capture the high-genus cardiovascular in case of occlusions is presented. However, it is noted that since not all human tissues are entirely convex. This experiment presents one of those examples in which the full convexity prior is not a suitable assumption. The target tissue has similar geometry as the cartoon-moon shaped object in the previous synthetic experiment. It can be seen that part of the object is missing. This may be caused by disease, e.g. that part may be infected or inflamed. The tissue's structure may hence be damaged and so is the penetrating index. Under this assumption, not only topology preserving segmentation cannot elude the occlusions, but the proposed segmentation model with full convexity prior does not yield a meaningful result either. Applying partial convexity prior is a suitable approach in this situation. By enforcing convexity only around those regions shown to be missing, the resultant contour captures that part very accurately.

7. Conclusion. This paper presents a new model for image segmentation with convexity prior. The convexity constraint can be partially prescribed based on the user's preference. The proposed model implements the convexity constraint in the notion of dihedral angles and incorporates with a topology-preserving mapping-based segmentation model. Beyond inheriting such desirable properties as robustness against topological noise from the topology-preserving segmentation model, the proposed model provides an efficient and effective way to preserve and prescribe convexity of the object domain by manipulation of dihedral angles. The discrete nature of the model render it immune from the discrepancy between a smooth model and the discrete computation. A variety of experiments are presented to demonstrate the effectiveness and efficiency of the proposed framework. It is a natural extension in future work to consider more general shape prior than convexity from the point of view of dihedral angles.

Appendix. Proof of Theorem 5.1. In this Appendix, we give a more intuitive proof for Theorem 5.1 in terms of sine law for easier understanding.

Observe that it suffices to determine the angles of each triangle. Then the triangles can be pieced together one by one to give the immersion. Suppose the length of one edge e_0 is fixed. Since the angles of the triangles, via sine law, determine the other edges in the triangles that contains e_0 , and hence hence all edgelengths in the same connected component, by considering paths of edge-sharing faces. Therefore, the edgelengths will agree upon piecing up, if the edgelengths determined by all paths are the same, or equivalently, for every edge and every loop of edge-sharing faces based at that edge, the length determined by the path is the same as the original length.

More specifically, consider the loop of faces $\Delta_1, ..., \Delta_n$ such that Δ_i and Δ_{i+1} share an edge e_i , where addition is in mod n. This is illustrated in Figure (7.1). Suppose the length of $e_0 = e_n$ is originally assigned to be L. Then letting φ_i^+ and $\varphi_i^$ be the angles in Δ_i opposite to e_i and e_{i-1} respectively, the length of e_n determined Image segmentation with convexity prior



FIG. 7.1. Illustration for the Proof of Theorem 5.1 in Appendix.

by the path (Δ_i) is $L \prod \frac{\sin \varphi_i^+}{\sin \varphi_i^-}$. Therefore, the angles have to be chosen such that

$$\prod \frac{\sin \varphi_i^+}{\sin \varphi_i^-} = 1,$$

or equivalently,

$$\sum \log \sin \varphi_i^+ = \sum \log \sin \varphi_i^- \tag{7.1}$$

for every loop of faces.

Now, the Lagrangian optimality condition of (5.7) implies (7.1) via

$$\sum (\log \sin \alpha_i^+ + \log 2) = \sum \Lambda_{\Delta_i} + \lambda_{e_i}$$
$$\sum (\log \sin \alpha_i^- + \log 2) = \sum \Lambda_{\Delta_i} + \lambda_{e_{i-1}},$$

where Λ_{Δ} is the Lagrange multiplier corresponding to $\sum_{v} \varphi_{v,\Delta} = \pi$ and λ_e is the one corresponding to the constraint of the dihedral angle of e.

Note that the gradients of the constraints are linearly dependent, but since the constraints are linear, this is not an issue.

The assumption that θ is a plausible dihedral angle ensures that the feasible set is nonempty, and Delaunayness ensures the optimum is attained in the interior. We refer the reader to [38] for a detailed proof of the latter claim.

It remains to establish the properties of the objective functional E. The partial derivatives may be computed by direct computation. For concavity, letting Δ be the open simplex spanned by $\pi e_1, \pi e_2, \pi e_3$, where $\{e_1, e_2, e_3\}$ is the standard basis of \mathbb{R}^3 , and $V : \Delta \to \mathbb{R}$ be defined by $V(\varphi_1, \varphi_2, \varphi_3) = -\sum_{I \in \{1,2,3\}} \int_0^{\varphi_I} \log 2 \sin t dt$ for $(\varphi_1, \varphi_2, \varphi_3) \in \Delta$, it suffices to show Hess_V is negative-definite on the tangent space $T_p \Delta$ at every $p = (\alpha, \beta, \pi - \alpha - \beta) \in \Delta$. This is a straight-forward computation.

$$\operatorname{Hess}_{V}(\alpha,\beta,\pi-\alpha-\beta) = -\begin{bmatrix}\cot\alpha & & \\ & \cot\beta & \\ & & \frac{1-\cot\alpha\cot\beta}{\cot\alpha+\cot\beta}\end{bmatrix}$$

Let u = (1, 0, -1) and v = (0, 1, -1). Then $\{u, v\}$ spans $T_p\Delta$. Let $w = \lambda u + \mu v$. Then

$$w^{T}(\operatorname{Hess}_{V}(p))w = -\frac{1}{\cot\alpha + \cot\beta} \begin{bmatrix} \lambda & \mu \end{bmatrix} \begin{bmatrix} \cot^{2}\alpha + 1 & 1 - \cot\alpha \cot\beta \\ 1 - \cot\alpha \cot\beta & \cot^{2}\beta + 1 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix},$$

where monotonicity of cot implies $\cot \alpha + \cot \beta > \cot \alpha + \cot(\pi - \alpha) = 0$. The trace and determinant of the 2 × 2 matrix are, respectively, $2 + \cot^2 \alpha + \cot^2 \beta > 0$ and $(\cot \alpha + \cot \beta)^2 > 0$. The result then follows.

REFERENCES

- E. BAE, X.-C. TAI, AND W. ZHU, Augmented Lagrangian method for an Euler's elastica based segmentation model that promotes convex contours, Inverse Probl. Imaging, 11 (2017), pp. 1–23.
- [2] D. BASWARAJ, A. GOVARDHAN, AND P. PREMCHAND, Active contours and image segmentation: The current state of the art, Global J. Comput. Sci. Tech. Graphics Vis., 12 (2012), pp. 1–9.
- [3] A. L. BOBENKO AND B. SPRINGBORN, Variational principles for circle patterns and Koebe's theorem, Trans. Amer. Math. Soc., 356 (2003), pp. 659–689.
- [4] R. BYRD, M. HRIBAR, AND J. NOCEDAL, An interior point algorithm for large-scale nonlinear programming, SIAM J Optim, 9 (1999), pp. 877–900.
- [5] R. H. BYRD, J. C. GILBERT, AND J. NOCEDAL, A trust region method based on interior point techniques for nonlinear programming, Math. Programming, 89 (2000), pp. 149–185.
- [6] V. CASELLES, R. KIMMEL, AND G. SAPIRO, Geodesic active contours, Int. J. Comput. Vision, 22 (1997), pp. 61–79.
- [7] H. L. CHAN AND L. M. LUI, Detection of n-dimensional shape deformities using n-dimensional quasi-conformal maps, Geom. Imaging Comput., 1 (2014), pp. 395–415.
- [8] H.-L. CHAN, S. YAN, L.-M. LUI, AND X.-C. TAI, Topology-preserving image segmentation by Beltrami representation of shapes, J. Math. Imaging Vision, 60 (2018), pp. 401–421.
- [9] T. CHAN, M. MOELICH, AND B. SANDBERG, Some recent developments in variational image segmentation, in Image Processing Based on Partial Differential Equation, X.-C. Tai, K.-A. Lie, T. F. Chan, and S. Osher, eds., Springer, Berlin, Heidelberg, 2007, pp. 175–210.
- [10] T. CHAN AND W. ZHU, Level set based shape prior segmentation, in 2005 IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR'05), vol. 2, 2005, pp. 1164–1170.
- [11] T. F. CHAN AND L. A. VESE, Active contours without edges, IEEE Trans. Image Process., 10 (2001), pp. 266–277.
- [12] G. CHUNG AND L. A. VESE, Image segmentation using a multilayer level-set approach, Comput. Visual. Sci., 12 (2009), pp. 267–285.
- [13] L. D. COHEN, On active contour models and balloons, Comput. Vis., Graphics Image Process. Image Underst., 53 (1991), pp. 211–218.
- [14] C. R. COLLINS AND K. STEPHENSON, A circle packing algorithm, Comput. Geom., 25 (2003), pp. 233 – 256.
- [15] D. CREMERS, N. SOCHEN, AND C. SCHNÖRR, Towards recognition-based variational segmentation using shape priors and dynamic labeling, in Scale Space Methods in Computer Vision, L. D. Griffin and M. Lillholm, eds., Berlin, Heidelberg, 2003, Springer Berlin Heidelberg, pp. 388–400.
- [16] D. CREMERS, F. TISCHHÄUSER, J. WEICKERT, AND C. SCHNÖRR, Diffusion snakes: Introducing statistical shape knowledge into the mumford-shah functional, Int. J. Comput. Vision, 50 (2002), pp. 295–313.
- [17] X. DAI AND S. KHORRAM, Development of a feature-based approach to automated image registration for multitemporal and multisensor remotely sensed imagery, in IGARSS'97. 1997 IEEE International Geoscience and Remote Sensing Symposium Proceedings. Remote Sensing - A Scientific Vision for Sustainable Development, vol. 1, 1997, pp. 243–245.
- [18] M. DESBRUN, M. MEYER, AND P. ALLIEZ, Intrinsic parameterizations of surface meshes, Comput. Graphics Forum, 21 (2002), pp. 209–218.
- [19] J. FLUSSER AND T. SUK, A moment-based approach to registration of images with affine geometric distortion, IEEE Trans. Geosci. Remote Sensing, 32 (1994), pp. 382–387.
- [20] L. GORELICK, O. VEKSLER, Y. BOYKOV, AND C. NIEUWENHUIS, Convexity shape prior for binary segmentation, IEEE Trans. Pattern Anal. Mach. Intell., 39 (2017), pp. 258–271.
- [21] X. HAN, C. X. XU, AND J. L. PRINCE, A topology preserving level set method for geometric deformable models, IEEE Trans. Pattern Anal. Mach. Intell., 25 (2003), pp. 755–768.
- [22] A. N. HIRANI, Discrete exterior calculus, PhD thesis, California Institute of Technology, Pasadena, CA, 2003.
- [23] M. IBRAHIM, K. CHEN, AND L. RADA, An improved model for joint segmentation and registration based on linear curvature smoother, J. Algorithms Comput. Techno., 10 (2016), pp. 314–324.

- [24] M. KASS, A. WITKIN, AND D. TERZOPOULOS, Snakes: Active contour models, Int. J. Comput. Vision, 1 (1988), pp. 321–331.
- [25] L. KHAREVYCH, B. SPRINGBORN, AND P. SCHODER, Discrete conformal mappings via circle patterns, ACM Trans. Graphics, 25 (2006), pp. 412–438.
- [26] S. KICHENASSAMY, A. KUMAR, P. OLVER, A. TANNENBAUM, AND A. YEZZI, Conformal curvature flows: From phase transitions to active vision, Arch. Ration. Mech. Anal., 134 (1996), pp. 275–301.
- [27] K. C. LAM AND L. M. LUI, Landmark- and intensity-based registration with large deformations via quasi-conformal maps, SIAM J. Imaging Sci., 7 (2014), pp. 2364–2392.
- [28] C. LE GUYADER AND L. A. VESE, Self-repelling snakes for topology-preserving segmentation models, IEEE Trans. Image Process., 17 (2008), pp. 767–779.
- [29] C. LE GUYADER AND L. A. VESE, A combined segmentation and registration framework with a nonlinear elasticity smoother, Comput. Vis. Image Underst., 115 (2011), pp. 1689 – 1709.
- [30] G. LEIBON, Characterizing the Delaunay decompositions of compact hyperbolic surfaces, Geom. Topol., 6 (2002), pp. 361–391.
- [31] M. E. LEVENTON, W. E. L. GRIMSON, AND O. FAUGERAS, Statistical shape influence in geodesic active contours, in Proceedings IEEE Conference on Computer Vision and Pattern Recognition. CVPR 2000 (Cat. No.PR00662), vol. 1, 2000, pp. 316–323.
- [32] B. LÉVY, S. PETITJEAN, N. RAY, AND J. MAILLOT, Least squares conformal maps for automatic texture atlas generation, ACM Trans. Graph., 21 (2002), pp. 362–371.
- [33] L. M. LUI, K. C. CHAN, T. W. WONG, AND X. GU, Texture map and video compression using Beltrami representation, SIAM J. Imaging Sci., 6 (2013), pp. 1880–1902.
- [34] L. M. LUI, T. W. WONG, P. THOMPSON, T. CHAN, X. GU, AND S.-T. YAU, Shape-based diffeomorphic registration on hippocampal surfaces using Beltrami holomorphic flow, in Medical Image Computing and Computer-Assisted Intervention – MICCAI 2010, T. Jiang, N. Navab, J. P. W. Pluim, and M. A. Viergever, eds., Berlin, Heidelberg, 2010, Springer Berlin Heidelberg, pp. 323–330.
- [35] MATHWORKS, Constrained nonlinear optimization algorithms. For more details, please visit https://www.mathworks.com/help/optim/ug/constrained-nonlinear-optimization-algorithms.html.
- [36] Matlab optimization toolbox release 2016a, 2016. The MathWorks, Natick, MA, USA.
- [37] C. MERCAT, Discrete Riemann surfaces and the Ising model, Comm. Math. Phys., 218 (2001), pp. 177–216.
- [38] I. RIVIN, Euclidean structures on simplicial surfaces and hyperbolic volume, Ann. Math. (2), 139 (1994), pp. 553–580.
- [39] L. A. ROYER, D. L. RICHMOND, C. ROTHER, B. ANDRES, AND D. KAINMUELLER, Convexity shape constraints for image segmentation, in Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition (CVPR), 2016.
- [40] B. SANDBERG, T. CHAN, AND L. VESE, A level-set and Gabor-based active contour algorithm for segmenting textured images. CAM Report 02-39, University of Carlifornia, Los Angeles, Los Angeles, CA, 2002.
- [41] J. R. SHEWCHUK, Triangle: Engineering a 2D quality mesh generator and Delaunay triangulator, in Applied Computational Geometry: Towards Geometric Engineering, M. C. Lin and D. Manocha, eds., Springer-Verlag, Berlin, 1996, pp. 203–222.
- [42] B. SPRINGBORN, P. SCHRÖDER, AND U. PINKALL, Conformal equivalence of triangle meshes, ACM Trans. Graph., 27 (2008), pp. 77:1–77:11.
- [43] E. STREKALOVSKIY AND D. CREMERS, Generalized ordering constraints for multilabel optimization, in Proceedings of the IEEE International Conference on Computer Vision (ICCV), 2011.
- [44] C. STUDHOLME, D. L. HILL, AND D. J. HAWKES, An overlap invariant entropy measure of 3D medical image alignment, Pattern Recogn., 32 (1999), pp. 71 – 86.
- [45] G. SUNDARAMOORTHI AND A. YEZZI, Global regularizing flows with topology preservation for active contours and polygons, IEEE Trans. Image Process., 16 (2007), pp. 803–812.
- [46] J.-P. THIRION, Image matching as a diffusion process: an analogy with Maxwell's demons, Med. Image Anal., 2 (1998), pp. 243–260.
- [47] W. THURSTON, Geometry and topology of three-manifolds. Lecture Note, Princeton University, Princeton, NJ, 1980; available at http://library.msri.org/books/gt3m/, 1980.
- [48] W. THURSTON, The finite Riemann mapping theorem. Invited talk, An International Symposium at Purdue University on the occasion of the proof of the Bieberbach conjecture, 1985.
- [49] T. VERCAUTEREN, X. PENNEC, A. PERCHANT, AND N. AYACHE, Diffeomorphic demons: Efficient non-parametric image registration, NeuroImage, 45 (2009), pp. S61–S72.

- [50] P. VIOLA AND W. M. WELLS III, Alignment by maximization of mutual information, Int. J. Comput. Vis., 24 (1997), pp. 137–154.
- [51] R. WALTZ, J. MORALES, J. NOCEDAL, AND D. ORBAN, An interior algorithm for nonlinear optimization that combines line search and trust region steps, Math. Programming, 107 (2006), pp. 391–408.
- [52] C. XU AND J. L. PRINCE, Snakes, shapes, and gradient vector flow, IEEE Trans. Image Process., 7 (1998), pp. 359–369.
- [53] S. YAN, X. TAI, J. LIU, AND H. HUANG, Convexity shape prior for level set based image segmentation method. preprint, arXiv:1805.08676v1 [cs.CV], 2018.
- [54] A. YEZZI, L. ZÖLLEIY, AND T. KAPUR, A variational framework for joint segmentation and registration, in Proceedings IEEE Workshop on Mathematical Methods in Biomedical Image Analysis (MMBIA 2001), 2001, pp. 44–51.
- [55] B. ZITOVA AND J. FLUSSER, Image registration methods: A survey, Image Vis. Comput., 21 (2003), pp. 977–1000.