Mathematical Models and Methods in Applied Sciences

A Continuum PDE Model for Agents on a Lattice with Applications to Homelessness

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In this paper we develop a continuum model for the movement of agents on a lattice, taking into account location desirability, local and far-range migration, and localized entry and exit rates. Specifically, our motivation is to qualitatively describe the homeless population in Los Angeles. The model takes the form of a fully nonlinear, nonlocal, non-degenerate parabolic partial differential equation. We derive the model and prove useful properties of smooth solutions, including uniqueness and L^2-stability under certain hypotheses. We also illustrate numerical solutions to the model and find that a simple model can be qualitatively similar in behavior to observed homeless encampments.

Keywords: agent-based model; nonlocal PDE; nonlinear diffusion; solution properties; homelessness.

AMS Subject Classification: 35B10, 35B35, 35B50, 35B99, 92B99, 91C99, 65N99

1. Introduction

Homelessness is a growing problem for society, with large cities like Los Angeles having homeless populations of nearly 40,000 people [24]. To date, correlations have

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Fig. 1. Part of a homeless encampment in a part of Los Angeles called "Skid Row." Homeless individuals have set up tents and makeshift shelters on the sidewalk.

been established between homeless populations in different cities; factors such as the cost of rent, number of affordable housing units, and poverty rates [14,12] are relevant. But outside of a few anecdotal examples in the social science literature [39], little is known about the daily lives of the homeless, how they became homeless, and how long they may remain homeless. There is also the question of how the homeless population changes over time and space, which is essential for planning social support services to counteract this problem. A few studies address the dynamics of the homeless population [27,14], but there are still many unanswered questions. Within Los Angeles, the homeless population is a big concern. Complaints from Los Angeles residents regarding homeless encampments (see figure 1) are among the most common reason residents call the city [1]. The encampments can form quickly and require some careful consideration in terms of clean up [33]. But the encampments may also serve as places where the homeless themselves live preferentially, feeling safer [16].

Human interactions with their environment can be studied both from a social science perspective and through mathematical and statistical methodologies. Pedestrian travel has been studied experimentally and modelled with mathematics[30], even taking into account personal objectives of minimal travel time and avoiding high density areas [17]; and statistics can be employed to describe large crowds at special events, including the fatal pressures that can build up in the center, asphyxiating standing people [25]. Mathematical models for riots have also been developed [3]. On a larger scale, agent-based models have been used to study residential burglary, taking into account history-dependent target attractiveness [35,34], truncated travel distances [31], and agents burglarizing according to independent Poisson clocks [38]. More broadly, urban crime has received a lot of attention in models that incorporate police into the models [32,18,40,10,37,15]; models that predict locations of offenders [29]; rigorous proofs of solution properties of such equations [28,5]; and the study of emergent patterns and behavior [19,4,11]. Species aggregation has also been thoroughly studied [7], with many authors modelling such phenomena by nonlocal differential equations [9,2,36]. On even larger scales, we can see the natural emergence of pack formation in predator-prey systems, for sufficiently strong intraspecies competition [6]; descriptions of human territorial conquest [13]; and ecology models for a species

in an environment with patchy resources [26]. But to our knowledge, little has been done directly to model homelessness, which is a growing societal concern.

In our study, we address the dynamics of the homeless population on small geographic scales, as has been studied with neural networks for Los Angeles[27]. We develop an agent based model for the homeless population living on the streets, which, in the continuum limit, yields a partial differential equation for the population density. As in the neural network study, our data sources for the homeless population are a collection of annual point-in-time estimates broken down by census tract from the Los Angeles Homeless Services Authority [20,21,22,23]. We remark that accurate counting of the homeless population is challenging at best, with some authors suggesting the counts could be significant underestimates [8]. Our work focuses upon building a qualitative model so absolute precision is not necessary.

This paper makes three main contributions to the scientific community: firstly, the derivation of a new continuum partial differential equation to describe homeless populations; secondly, rigorous analysis of basic properties of solutions; and finally, in using the model to yield qualitatively consistent behaviors observed in the true homeless population. The paper structure is as follows: in section 2, we derive the model; in section 3 we prove useful properties of solutions; in section 4, we demonstrate how the model can qualitatively describe encampment formation; and finally, in section 5, we summarize our work and discuss future research.

2. Model Formulation

In describing the homeless population, we consider several processes taking place at each location (see figure 2):

- *Entry into population* - due to local features of the area such as cost of living, etc., individuals may become homeless;
- *Exit from population* - through localized social services, family support, death, or other processes, individuals may cease to be homeless;
- *Diffusive movement* - a homeless individual who does not like their current area due to lack of resources or other factors may venture to a surrounding neighbourhood; and
- *Nonlocal movement* - they may intentionally travel to another part of the city, such as through public transit.

The features of the cityscape vary by location so some areas are more residential, some are more industrial, etc.

2.1. *Derivation*

We use an Eulerian frame of reference discretized by a regular lattice of points S in $\mathbb{R}^d$, with spacing of δx in each coordinate direction and volume δx^d. We also discretize time into time steps of δt with $t_k = k\delta t, k = 0, 1, 2, ...$ For modelling

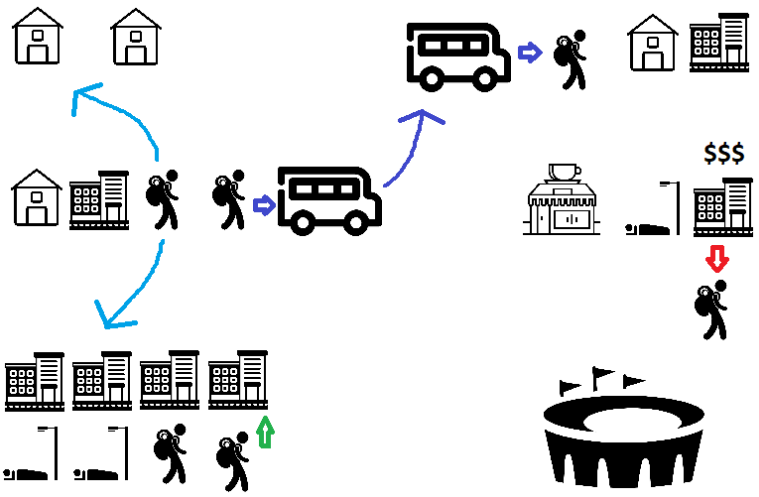

Fig. 2. In this model, homeless individuals may stay where they are, walk to an adjacent neighborhood (turquoise), take a bus to a distant neighborhood (blue-purple), become homeless due to external factors (red), or cease to become homeless (green). The features of the city change by location. [b]

homeless populations, our main focus is $d = 2$, but $d = 1$ could describe populations of people (or even other species) in a long/narrow geometry and $d = 3$ could describe populations that can freely move in 3-dimensions such as birds or fish.

We concern ourselves with the evolution of $\{N_i^k\}$, the number of individuals occupying site $i \in \mathcal{S}$ at time t_k. Intrinsic to each site i at time t_k, we denote an "attractiveness" $A_i^k \in (0, 1)$. This could be influenced by the resources available at the location. In general we denote subscripts for the spatial location index and superscripts for the time index. Over each very small time interval δt, we assume:

- people enter the population at a Poisson rate E_i^k;
- people leave the entire domain at a Poisson rate L_i^k;
- individuals who do not leave the population will remain at their current location with probability A_i^k;
- individuals who do not leave the population and choose to leave their current lattice point will make deliberate travel, such as with public transit in the case of homelessness, at a Poisson rate Γ_i^k; and
- if an individual remains in the population, does not stay at their location, and does not travel deliberately from their site to another, they will travel to each neighbouring site with equal probability.

[b]Image credits are as follows: backpacking by Gan Khoon Lay, homeless by Ed Harrison, Bus by Andre Buand, Cafe by Creative Mania, House by Vectors Point, and Apartment by priyanka - all from the Noun Project.

We incorporate all terms, allowing them to potentially be "in balance" so that transportation and movement may be comparable to arrivals/exists. In reality some of these effects may be negligible in describing the actual homeless population but we opt for generality.

At each site i, *up to* $O(\delta t)$ the following probabilities describe an individual:

$$\Pr(\text{leave } \mathcal{S}) = L_i^k \delta t \tag{2.1}$$
$$\Pr(\text{stay in } \mathcal{S}) = 1 - L_i^k \delta t \tag{2.2}$$
$$\Pr(\text{stay at i} \mid \text{stay in } \mathcal{S}) = A_i^k \tag{2.3}$$
$$\Pr(\text{stay at i and stay in } \mathcal{S}) = A_i^k(1 - L_i^k \delta t) \tag{2.4}$$
$$\Pr(\text{deliberate travel} \mid \text{stay in } \mathcal{S}) = (1 - A_i^k)\Gamma_i^k \delta t \tag{2.5}$$
$$\Pr(\text{deliberate travel and stay in } \mathcal{S}) = (1 - A_i^k)\Gamma_i^k \delta t \tag{2.6}$$
$$\Pr(\text{leave i but not travel far} \mid \text{stay in } \mathcal{S}) = (1 - A_i^k)(1 - \Gamma_i^k \delta t) \tag{2.7}$$
$$\Pr(\text{leave i but not travel far and stay in } \mathcal{S}) = (1 - A_i^k)(1 - (\Gamma_i^k + L_i^k)\delta t) \tag{2.8}$$

so that on average at site i starting at time t_k, over each δt:

- $E_i^k \delta t$ people enter the population;
- $L_i^k N_i^k \delta t$ people leave the population;
- $A_i^k(1 - L_i^k \delta t)N_i^k$ people remain at their current location;
- $(1 - A_i^k)(1 - L_i^k \delta t)N_i^k$ people move to a neighbouring lattice point;
- $(1 - A_i^k)\Gamma_i^k N_i^k \delta t$ people travel deliberately to another point; and
- $(1 - A_i^k)(1 - (\Gamma_i^k + L_i^k)\delta t)N_i^k$ people move to a neighbouring lattice point.

We make one further assumption that there exists a transition matrix $\mathcal{T}^k = (T_{ji}^k)_{(j,i)\in S^2}$ describing the probability an agent deliberately travels from site j to site i. We denote $j \sim i$ to signify that $i \neq j$ are neighbours and $j \to i$ to signify that $i \neq j$ and there is deliberate (potentially far-range) movement from j to i. We also assume that each lattice point has $n = 2d$ neighbours. Then:

$$\begin{aligned} N_i^{k+1} - N_i^k = {} & \overbrace{E_i^k \delta t}^{\text{entry at site}} + \overbrace{\sum_{j\sim i} \frac{1}{n}(1 - A_j^k)(1 - (\Gamma_j^k + L_j^k)\delta t)N_j^k}^{\text{from neighbours}} + \overbrace{\sum_{j\to i}(1 - A_j^k)\Gamma_j^k N_j^k T_{ji}^k \delta t}^{\text{deliberate moves in}} \\ & + \overbrace{A_i^k(1 - L_i^k \delta t)N_i^k}^{\text{remain at location}} - \overbrace{(1 - A_i^k)\Gamma_i^k N_i^k \delta t}^{\text{deliberate moves out}} - N_i^k \\ = {} & \frac{1}{n}\left(\sum_{j\sim i}(1 - A_j^k)N_j^k - n(1 - A_i^k)N_i^k\right) + \delta t\left(E_i^k + \sum_{j\to i}(1 - A_j^k)\Gamma_j^k N_j^k T_{ji}^k\right) \\ & + \delta t\left[\frac{-1}{n}\left(\sum_{j\sim i}(\Gamma_j^k + L_j^k)(1 - A_j^k)N_j^k - n(\Gamma_i^k + L_i^k)(1 - A_i^k)N_i^k\right) - (1 - A_i^k)\Gamma_i^k N_i^k - L_i^k N_i^k\right] \end{aligned}$$

The terms of the form $\sum_{j\sim i} \bullet_j^k - n\bullet_i^k$ are the discrete second-order centered

difference Laplacian at position i and time t_k multiplied by δx^2. Denote the discrete Laplacian by Δ. Dividing the equation by δt we have:

$$\frac{N_i^{k+1}-N_i^k}{\delta t} = \frac{\delta x^2}{n\delta t}\Delta((1-A_i^k)N_i^k) + E_i^k + \sum_{j\to i}(1-A_j^k)\Gamma_j^k N_j^k T_{ji}^k - L_i^k N_i^k$$
$$- (1-A_i^k)\Gamma_i^k N_i^k - \frac{\delta x^2}{n}\Delta(\Gamma_i^k + L_i^k)(1-A_i^k)N_i^k$$

Under a diffusive scaling, such that $\frac{\delta x^2}{n\delta t} = D = O(1)$, in taking the limit we furnish the partial differential equation

$$\rho_t = D\Delta((1-a)\rho) + \eta - \omega\rho + \int \tau(y,\cdot,t)\left((1-a)\gamma\rho\right)|_y \mathrm{d}y - (1-a)\gamma\rho, \qquad (2.9)$$

where ρ is a spatial density of agents; η is an entry rate per unit area; ω is an exit rate; τ governs transitions from y to x at time t; γ is a travel rate term; and a is a continuous attractiveness field. We can express (2.9) more nicely by defining the unattractiveness

$$u = 1 - a \qquad (2.10)$$

so that over a domain Ω:

$$\rho_t(x,t) = D\Delta(\rho(x,t)u(x,t)) + \eta(x,t) - \omega(x,t)\rho(x,t)$$
$$+ \left(\int_\Omega \tau(y,x,t)\gamma(y,t)\rho(y,t)u(y,t)\mathrm{d}y - \gamma(x,t)\rho(x,t)u(x,t)\right) \qquad (2.11)$$

subject to the normalization of $\tau \geq 0$:

$$\int_\Omega \tau(y,x,t)\mathrm{d}x = 1 \qquad (2.12)$$

2.2. *Nondimensionalization*

We begin with the PDE

$$\rho_t = D\Delta(\rho u) + \eta - \omega\rho + \left(\int_\Omega \tau(y,x,t)\gamma(y,t)\rho(y,t)u(x,t)\mathrm{d}y - \gamma\rho u\right)$$
$$(x,t) \in \Omega \times \mathbb{R}_{\geq 0}$$

We write $x = \bar{x}x^*$, $t = \bar{t}t^*$, $\rho(x,t) = \bar{\rho}\rho^*(x^*,t^*)$, $u(x,t) = u^*(x^*,t^*)$, $\tau(y,x,t) = \bar{\tau}\tau^*(y^*,x^*,t^*)$, $\eta(x,t) = \bar{\eta}\eta^*(x^*,t^*)$, $\omega(x,t) = \bar{\omega}\omega^*(x^*,t^*)$, $\gamma(x,t) = \bar{\gamma}\gamma^*(x^*,t^*)$, where the bars are scales and the asterisk variables are dimensionless. We also define $\Omega^* = \frac{1}{\bar{x}^d}\Omega$.

Noting that if $y = \bar{x}y^*$ then $\mathrm{d}y = \bar{x}^d \mathrm{d}y^*$, we have:

$$\rho^*_{t^*} = \frac{D\bar{t}}{\bar{x}^2}\Delta^*(u^*\rho^*) + (\frac{\bar{\eta}\bar{t}}{\bar{\rho}})\eta^* - (\bar{\omega}\bar{t})\omega^*\rho^* + (\bar{t}\bar{\gamma}\bar{\tau}\bar{x}^d)\int_{\Omega^*}\tau^*(y^*,x^*,t^*)\gamma^*(y^*,t^*)\rho(y^*,t^*)u(y^*,t^*)\mathrm{d}y^* - (\bar{\gamma}\bar{t})\gamma^*\rho^*u^*.$$

We can adopt scalings relevant homeless count data: given the homeless counts are done annually, we choose $\bar{t}$ to be 1 year. We also choose $\bar{x}$ as a characteristic length within our region of interest (later we use the geometric mean of the length/width of the spatial domain). There is also a natural scaling for $\bar{\rho}$ as the initial average population density over the region. These choices suggest the derived scales $\bar{\eta} = \bar{\rho}/\bar{t}$, $\bar{\omega} = \bar{\gamma} = 1/\bar{t}$, and $\bar{\tau} = 1/\bar{x}^d$. Then for a constant $\delta = D\bar{t}/\bar{x}^2$ and after removing the asterisks we have:

$$\rho_t = \overbrace{\delta\Delta(\rho u)}^{\text{diffusive movement}} + \overbrace{\eta}^{\text{entry}} - \overbrace{\omega\rho}^{\text{loss}} + \overbrace{I[\gamma\rho u]}^{\text{nonlocal entry}} - \overbrace{\gamma\rho u}^{\text{loss from nonlocal movement}}, \tag{2.13}$$

$$I[q](\cdot) := \int_\Omega \tau(y,\cdot,t)q(y,t)\mathrm{d}y, \tag{2.14}$$

$$\int_\Omega \tau(\cdot,x,\cdot)\mathrm{d}x = 1, \qquad \tau \geq 0. \tag{2.15}$$

In general, the unattractiveness u may depend upon ρ and other localized features. In section 2.3.2 we choose a particular form for u (equation (2.16)). Because of this potential density-dependence, the system of equations is nonlinear. We can interpret (2.13)-(2.15) as consisting of a nonlinear diffusion operator where ρu is diffusing; a local source term η; a localized exit rate ω; a nonlocal operator $I[\gamma\rho u](\cdot)$ describing intentional travel, and a local loss due to intentional travel $\gamma\rho u$. Note that the nonlocal operator is not a convolution.

2.3. *Mathematical Formulation*

2.3.1. *Notation*

To denote solution spaces, we may explicitly label a function's argument and specify the continuity/differentiability assumed with a subscript with that label. We denote $f(x,t) \in \mathcal{C}^2_x \cap \mathcal{C}^1_t(\Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0})$ to be a function f that is twice continuously differentiable in x and once in t. We use similar notation for continuity in higher derivatives. The notation $\mathcal{C}$ represents continuity in all arguments, possibly with superscripts to denote the number of derivatives, e.g. $\mathcal{C}^\infty$.

2.3.2. *Assumptions*

We denote our spatial domain to be Ω and assume it is bounded and simply connected. For simplicity in the analysis, we will choose Ω to be the torus

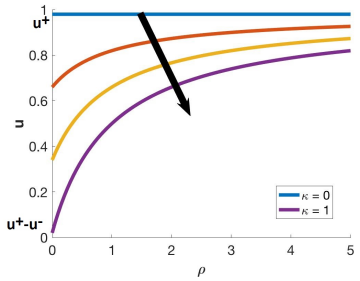

Fig. 3. Plots of unattractiveness vs density with $\tilde{\rho} = 1$. The unattractiveness increases with density for all fixed κ. It is always at least $u^+ - u^-$ and never exceeds u^+. As κ increases, the unattractiveness decreases.

$\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \cong [0,1)^d$ for $d \in \mathbb{N}$. Periodic boundary conditions actually exist in cities e.g. with perimeter roads, however here the choice is more for convenience of the mathematics. Also cities like Los Angeles can have repeating patches of residential (both affluent and disadvantaged) neighborhoods and commercial regions, which can give the impression of a repeating pattern. For numerical studies we sometimes use no-flux boundaries in $\mathbb{R}^2$, identifying a flux of $-\delta\nabla(\rho u)$.

We denote:

- $\rho(x,t) : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the population density (people per unit area);
- $\theta(x,t) : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}^p$, a features vector of size p, which can vary over space;
- $\kappa(\theta) : \mathbb{R}^p \to [0,1]$, the (dimensionless) relative capacity, to be used below;
- $$u(\kappa,\rho) : [0,1] \times \mathbb{R}_{\geq 0} \to (0,1)$$
$$(\kappa,\rho) \mapsto u^+ - \frac{\kappa u^-}{1+\rho/\tilde{\rho}}, \tag{2.16}$$
the (dimensionless) unattractiveness for $0 < u^- < u^+ < 1, 0 < \tilde{\rho}$ (see figure 3 for qualitative depiction);
- $\tau(y,x,t) : \Omega \times \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the travel term (per unit area probability), such that
$$\forall y \in \Omega, \qquad \int_\Omega \tau(y,x,t)\mathrm{d}x = 1;$$
- $\eta(\theta,t) : \mathbb{R}^p \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the entry rate (people per unit area per unit time);
- $\omega(\theta,t) : \mathbb{R}^p \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ the exit rate (per unit time); and
- $\gamma(\theta,t) : \mathbb{R}^p \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the intentional travel rate (per unit time).

The fact that unattractiveness is density-dependent is motivated by a study of homeless populations on census tracts that found homeless populations are more

likely to decrease if they begin at an above average value [27]. The potential for including more general τ to include preferential travel to locations with a higher population density is briefly discussed in section 3.3 as this could be a possible mechanism [16]. For simpler analysis, however, we do not consider such τ.

We operate under the hypotheses that

Hypothesis 1. The functions $\rho, \theta, \kappa, \tau, \eta, \omega, \gamma \in \mathcal{C}^\infty$ for all their respective arguments.

Hypothesis 2. We have $0 \leq \kappa \leq 1$ and there is a $\Xi > 0$ so that $|\nabla\kappa|, |\Delta\kappa|, |\omega|, |\eta|, |\gamma|, |\tau| \leq \Xi$ for all $x \in \Omega, t \geq 0$.

We then consider the evolution of ρ given by equations (2.13)-(2.15), (2.16).

2.4. *Model Exploration*

To build a basic understanding of the model features, we study the solutions numerically and vary the parameters. We do this in one dimension on the torus with length 1. The numerical method employed is explained in Appendix 6.1.

We wish to qualitatively understand the effects of the different model features. As part of this work, we locally perturb some features sometimes making use of the bump function

$$\Upsilon(x) = \begin{cases} 0, |x| \geq 1 \\ \frac{\mathrm{e}^{-1/(1-x^{20})}}{\int_{-1}^{1} \mathrm{e}^{-1/(1-x^{20})} \mathrm{d}x}, |x| < 1. \end{cases} \tag{2.17}$$

The function $\Upsilon \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}_{\geq 0})$ is compactly supported on the interval $[-1, 1]$ with a wide range of x-values where it is nearly constant (owing to the large power of 20). We remark that Υ has a support of length 2, which is larger than the unit torus, but by shifting and rescaling its argument, we ensure our use of Υ does not violate the properties of the torus.

We perform a series of numerical experiments upon the model, as seen in figure 2.4 with explicit parameter listings in table 1. In particular, we study the following:

- **baseline**: we study how the population density evolves from an initial distribution when all the parameter functions are constants, observing a steady approach of the population density to a spatially constant solution at steady-state.
- **enhanced local entry**: from the baseline, we increase the entry rate η in a region and at steady-state find the population is largest where there is more entry.
- **enhanced local exit**: from the baseline, we increase the exit rate ω in a region and at steady-state find the population is reduced where there is a greater exit rate.

- **enhanced local to far migration:** from the baseline, we increase the travel rate γ in a region and at steady-state find the population has diminished due to a higher rate of moving away from the area.
- **biased transfer:** from the baseline, we choose τ to be biased to relocate individuals to one particular region; at steady-state, the population density is higher at this destination.
- **exponential decay transfer:** from the baseline, we choose τ such that from y, the probability density in moving to x decays exponentially with $|x-y|$. Compared to the baseline, the change is quite small, but we observe the population decreases slightly slower in more concentrated regions since the people are not travelling as far.
- **textured relative capacity:** from the baseline, we vary the relative capacity κ in an oscillatory fashion; we find that at steady-state, the population density fluctuates with this varying relative capacity.
- **enhanced diffusion:** from the baseline, we raise the diffusion coefficient and find the behavior can quickly be dominated by diffusion, resulting in flat density profiles.
- **more density sensitivity**: from the baseline, we make the unattractiveness more sensitive to density variations by decreasing $\tilde{\rho}$ in (2.16). We observe the population disperses more rapidly in regions where the population density was initially highest.
- **narrow unattractiveness range**: from the baseline, we modify u^- and u^+ in (2.16) so that the unattractiveness is almost constant, regardless of ρ or κ. We also perturb κ as in the "textured relative capacity" experiment. Here we find the steady state population is much less influenced by the relative capacity.
- **region becomes uninhabitable**: from the baseline, we make a region Q, say, uninhabitable by making the relative capacity zero in Q, biasing τ to move people away from Q, only allowing entries parts of the domain outside of Q, and greatly increasing the travel rate in Q. We find the population relocates to the other, more habitable part of the domain.

Description	**Parameters**	**Observations**
baseline	$\eta = 2, \omega = 1, \gamma = \frac{1}{2}$, $\tau = 1, \kappa = \frac{1}{2}, \delta = 0.02$, $u^+ = 0.99$, $u^- = 0.98$, $\tilde{\rho} = 20$	tends to flat solution
enhanced local entry near $x = \frac{3}{4}$	$\eta \mapsto 2 + 5\Upsilon(4(x - \frac{3}{4}))$	population largest at $x = \frac{3}{4}$
enhanced local exit near $x = \frac{1}{4}$	$\omega \mapsto 1 + 5\Upsilon(4(x - \frac{1}{4}))$	population minimal at $x = \frac{1}{4}$
enhanced local to far migration at $x = \frac{1}{4}$	$\gamma \mapsto \frac{1}{2} + 4\Upsilon(4(x - \frac{3}{4}))$	population moves away from $x = \frac{1}{4}$
biased transfer	$\tau \mapsto 4\Upsilon(4(x - \frac{3}{4}))$	buildup of population at bias destination
exponential decay transfer	$\tau(y, x, t) \mapsto \mathrm{e}^{-\mathrm{dist}(y,x)}/(\int_{\mathbb{T}} \mathrm{e}^{-\mathrm{dist}(0,x')}\mathrm{d}x')$, $\lambda = 0.05$	population slower to decrease near concentration region
textured relative capacity	$\kappa \mapsto \sin^2(4\pi x)$	spatial frequencies observed in density
enhanced diffusion	$\delta \mapsto 1$	population levels out faster
more density sensitivity	$\tilde{\rho} \mapsto \frac{1}{10}$	population spreads out faster
narrow unattractiveness range	$\kappa \mapsto \sin^2(4\pi x)$, $u^+ \mapsto 0.51$, $u^- \mapsto 0.01$	relative capacity has smaller effect
region becomes uninhabitable near $x = \frac{1}{4}$	$\kappa \mapsto \Upsilon(4(x - \frac{3}{4}))/\Upsilon(0)$ $\tau(y, x, t) \mapsto 4\Upsilon(4(x - \frac{3}{4}))$ $\eta \mapsto 8\Upsilon(4(x - \frac{3}{4}))$ $\gamma \mapsto \frac{1}{2} + 20\Upsilon(4(x - \frac{1}{4}))$	population drastically drops near $x = \frac{1}{4}$

Table 1. Numerical experiments that are run on the model in one dimension with Υ defined as in (2.17). Plots of these experiments are given in figure 2.4.

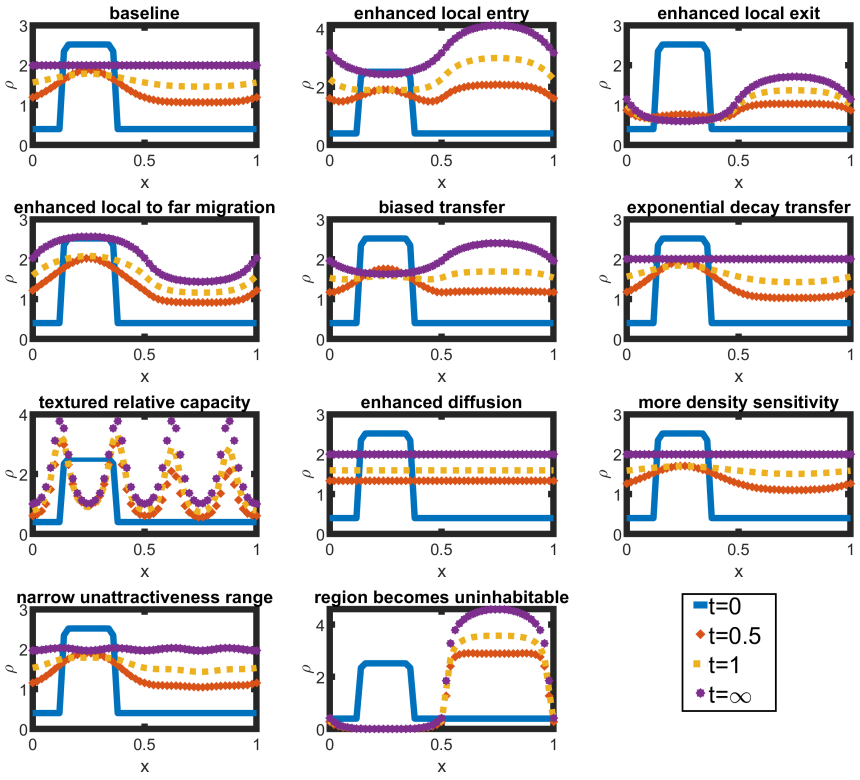

Fig. 4. Initial conditions are that $\rho(x,0) = 0.4 + 4\Upsilon(8(x - \frac{1}{4}))$. The scenarios depicted correspond to those of table 1. Row 1 left: baseline; row 1 center: enhanced entry near $x = 3/4$; row 1 right: enhanced exit rate near $x = 1/4$; row 2 left: enhanced deliberate travel rate near $x = 1/4$; row 2 center: transfer kernel moves people to near $x = 3/4$; row 2 right: transfer kernel decays exponentially with distance; row 3 left: textured relative capacity; row 3 middle: enhanced diffusion; row 3 right: unattractiveness becomes more sensitive to density; row 4 left: same textured relative capacity as in row 3 left but with smaller variance in unattractiveness; row 4 center: region near $x = 1/4$ becomes uninhabitable; row 4 right: time legend.

3. Properties of Smooth Periodic Solutions

We study the solutions to

$$\rho_t = \delta\Delta(\rho u) + \eta - \omega\rho + I[\gamma\rho u] - \gamma\rho u, \tag{3.1}$$

$$I[q](\cdot) := \int_\Omega \tau(y,\cdot,t)q(y,t)\mathrm{d}y, \tag{3.2}$$

$$\int_\Omega \tau(\cdot,x,\cdot)\mathrm{d}x = 1, \qquad \tau \geq 0 \tag{3.3}$$

$$u = u(\kappa,\rho) = u^+ - \frac{\kappa u^-}{1+\rho/\tilde{\rho}} \tag{3.4}$$

for $0 < u^- < u^+ < 1, 0 < \tilde{\rho}, 0 < \delta$.

Here we work in a spatial domain $\Omega = \mathbb{T}^d$, the $d-$dimensional torus. For $T > 0$, we denote

$$\mathcal{U}_T \equiv \{\rho(x,t) | \rho(x,t) \in \mathcal{C}_x^2 \cap \mathcal{C}_t^1(\mathbb{T}^d \times [0,T] \to \mathbb{R}_{\geq 0})\}$$

with $t-$derivatives understood to be right-, respectively, left-derivatives at $t = 0$ and $t = T$. In general, we will be assuming the existence of solutions to (3.1)-(3.4) within the space $\mathcal{U}_T$. If we write $\mathcal{U}_\infty$ we refer to a solution that exists globally for all time. *At all times, we assume that hypotheses 1 and 2 are satisfied.*

As the equations (3.1)-(3.4) are parabolic and nondegenerate (u is never zero), we anticipate global existence of smooth solutions, without the formation of shocks or phenomena such as finite-time blowup. However, such proofs are beyond this paper.

Remark 3.1. The solutions we consider are continuous on the torus $\mathbb{T}^d$, which is compact, so we can use sup and max, respectively, inf and min, interchangeably.

3.1. *Useful Properties of ρu*

The term ρu appears many times in our analysis and we make a list of some useful properties. Note that $\rho u = u^+\rho - \kappa M(\rho)$ where we define

$$M(\rho) = \frac{u^-\rho}{(1+\rho/\tilde{\rho})}. \tag{3.5}$$

Observe

$$M'(\rho) = \frac{u^-}{(1+\rho/\tilde{\rho})^2}, \tag{3.6}$$

$$M''(\rho) = \frac{-2u^-/\tilde{\rho}}{(1+\rho/\tilde{\rho})^3}. \tag{3.7}$$

Thus, for $\rho \geq 0$,

$$0 \leq (u^+ - u^-)\rho \leq \rho u \tag{3.8}$$

$$0 \leq M(\rho) \leq u^-\rho \tag{3.9}$$

$$0 \leq M(\rho) \leq \tilde{\rho}u^- = \text{constant} \tag{3.10}$$

$$0 \leq M'(\rho) \leq u^-, \tag{3.11}$$

$$M''(\rho) \leq 0. \tag{3.12}$$

We also have that

$$\nabla(\rho u) = (u^+ - \kappa M'(\rho))\nabla\rho - M(\rho)\nabla\kappa \tag{3.13}$$

$$\Delta(\rho u) = (u^+ - \kappa M'(\rho))\Delta\rho - \kappa M''(\rho)|\nabla\rho|^2 - 2M'(\rho)\nabla\kappa\cdot\nabla\rho - M(\rho)\Delta\kappa \tag{3.14}$$

In one spatial dimension, we also have:

$$\begin{aligned}(\rho u)_{xxx} = (u^+ - \kappa M'(\rho))\rho_{xxx} - 3M'(\rho)\kappa_x\rho_{xx} - 3M'(\rho)\kappa_{xx}\rho_x - M(\rho)\kappa_{xxx} \\ - 3\kappa M''(\rho)\rho_x\rho_{xx} - 3M''(\rho)\kappa_x\rho_x^2 - \kappa M'''(\rho)\rho_x^3\end{aligned} \tag{3.15}$$

Owing to (3.11), hypothesis 2, and $0 \leq \kappa \leq 1$, we have:

$$\rho \mapsto u^+\rho - \kappa M(\rho) \in \mathrm{Lip}_{u^+}(\mathbb{R}_{\geq 0}) \tag{3.16}$$

$$\rho \mapsto \nabla \kappa M(\rho) \in \mathrm{Lip}_{\Xi u^-}(\mathbb{R}_{\geq 0}) \tag{3.17}$$

$$\rho \mapsto \kappa M'(\rho) \in \mathrm{Lip}_{2\Xi u^-/\tilde{\rho}}(\mathbb{R}_{\geq 0}) \tag{3.18}$$

where the subscripts in the Lipschitz spaces denote the bounding constants.

3.2. *Results*

We have a series of results of solution properties below. All proofs are provided in the paper, but the following lemma and subsequent three propositions are proved in Appendix 6.2.

Lemma 3.1. *Let $0 \leq t_0 < T \leq \infty$ and define $S = \mathbb{T}^d \times (t_0, T]$.*
Let $q(x,t) \in \mathcal{C}_x^2 \cap \mathcal{C}_t(\mathbb{T}^d \times [0,T] \to \mathbb{R}_{\geq 0})$ solve

$$q_t - \alpha \Delta F(q,t) + \lambda q + \Phi(\nabla q, x, t) = \Theta$$

where $\alpha \in \mathcal{C}(S \to \mathbb{R}_{\geq 0})$,
$\Phi \in \mathcal{C}(\mathbb{R}^d \times \mathbb{T}^d \times \mathbb{R}_{\geq 0} \to \mathbb{R})$ with $\Phi(0,\cdot,\cdot) = 0$,
$\lambda(x,t) \in \mathcal{C}(S \to \mathbb{R}_{\geq 0})$, resp., $\lambda \in \mathcal{C}(S \to \mathbb{R}_{\leq 0})$,
$F(q,t) \in \mathcal{C}(\mathbb{R}_{\geq 0} \times [0,T] \to \mathbb{R}_{\geq 0})$ with F monotonically nondecreasing with respect to q, and
$\Theta[q](x,t)$ is an operator depending on q, $x \in \mathbb{T}^d$, and $t \in \mathbb{R}_{\geq 0}$ such that if q achieves a global maximum over $\mathbb{T}^d$ at x then $\Theta[q](x,t) \leq 0$, resp., if q achieves a global minimum over $\mathbb{T}^d$ at x then $\Theta[q](x,t) \geq 0$.
The preceding conditions force $\sup_S q = \max_{\partial^* S} q$, resp., $\inf_S q = \min_{\partial^* S} q$ where $\partial^* S \equiv \mathbb{T}^d \times \{t_0\}$.

Remark 3.2. The zero operator can be used in lemma 3.1 and the result clearly holds.

Proposition 3.1 (Positivity). *Let $T > 0$ and suppose that $\rho \in \mathcal{U}_T$ is a solution to (3.1)-(3.4) with $\Omega = \mathbb{T}^d$ and $\rho(x,0) > 0$ for all $x \in \mathbb{T}^d$. Then $\forall t \in [0,T], \rho > 0$. In particular if $\min_{\mathbb{T}^d} \rho(\cdot,0) = \rho_m$ then $\rho(x,t) \geq \rho_m \mathrm{e}^{-(2\Xi + \delta \Xi u^-)t} > 0$ for all $x \in \mathbb{T}^d, t \in [0,T]$.*

Proposition 3.2 (Explicit Bound on Total Population). *Let $T > 0$ and suppose $\rho \in \mathcal{U}_T$ is a solution to (3.1)-(3.4) with $\Omega = \mathbb{T}^d$ and $\rho(x,0) > 0$ for all $x \in \mathbb{T}^d$. Then $||\rho(\cdot,t)||_{L^1(\mathbb{T}^d)} \leq ||\rho(\cdot,0)||_{L^1(\mathbb{T}^d)} + \Xi t < \infty$ for all $t \in [0,T]$.*

Proposition 3.3 (Explicit Bound on Population Density). *Let $T > 0$ and suppose that $\rho \in \mathcal{U}_T$ is a solution to (3.1)-(3.4) with $\Omega = \mathbb{T}^d$, $\rho(x,0) > 0$ for all $x \in \mathbb{T}^d$, and $||\rho(\cdot,0)||_{L^\infty(\mathbb{T}^d)} = \rho_M < \infty$. Then $\forall t \in [0,T], ||\rho(\cdot,t)||_{L^\infty(\mathbb{T}^d)} \leq \rho_M + \frac{\Xi^3}{2} t^2 + (\Xi^2 ||\rho(\cdot,0)||_{L^1(\mathbb{T}^d)} + \Xi + \Xi \delta \tilde{\rho} u^-)t < \infty$.*

Theorem 3.1 (Uniqueness of Smooth Solutions). *Let $T > 0$ and suppose $\rho_1, \rho_2 \in \mathcal{U}_T$ are two solutions to (3.1)-(3.4) with $\Omega = \mathbb{T}^d$. If ρ_1 and ρ_2 have identical and strictly positive initial conditions at $t = 0$, then $\rho_1 = \rho_2$ on $0 \le t \le T$.*

Proof. From subtracting the respective PDEs, we have that

$$\begin{aligned}(\rho_1-\rho_2)_t &= \delta u^+\Delta(\rho_1-\rho_2) - \delta\Delta\left(\kappa(M(\rho_1)-M(\rho_2))\right) - \omega(\rho_1-\rho_2)\\ &\quad + I[\gamma(u^+(\rho_1-\rho_2)-\kappa(M(\rho_1)-M(\rho_2))]\\ &\quad - \gamma(u^+(\rho_1-\rho_2)-\kappa(M(\rho_1)-M(\rho_2)).\end{aligned}$$

By multiplying the equation by $(\rho_1 - \rho_2)$, integrating over Ω, and integrating by parts once, we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}^d}(\rho_1-\rho_2)^2\mathrm{d}x &= \delta\int_{\mathbb{T}^d}\big(-u^+|\nabla(\rho_1-\rho_2)|^2 + \\ &\overbrace{(M(\rho_1)-M(\rho_2))\nabla\kappa\cdot\nabla(\rho_1-\rho_2)+\kappa(M'(\rho_1)\nabla\rho_1-M'(\rho_2)\nabla\rho_2)\cdot\nabla(\rho_1-\rho_2)}^{\nabla(\rho_1-\rho_2)\cdot\nabla(\kappa(M(\rho_1)-M(\rho_2)))}\big)\mathrm{d}x\\ &\quad -\int_{\mathbb{T}^d}\left(\omega(\rho_1-\rho_2)^2+(\rho_1-\rho_2)\gamma(u^+(\rho_1-\rho_2)-\kappa(M(\rho_1)-M(\rho_2)))\right)\mathrm{d}x\\ &\quad +\int_{\mathbb{T}^d}(\rho_1-\rho_2)\int_\Omega\tau(y,x,t)\gamma(u^+(\rho_1-\rho_2)-\kappa(M(\rho_1)-M(\rho_2))|_y\mathrm{d}y\mathrm{d}x\\ &\le \delta\int_{\mathbb{T}^d}\big(-u^+|\nabla(\rho_1-\rho_2)|^2+(M(\rho_1)-M(\rho_2))\nabla\kappa\cdot\nabla(\rho_1-\rho_2)\\ &\qquad +\kappa(M'(\rho_1)\nabla\rho_1-M'(\rho_2)\nabla\rho_2)\cdot\nabla(\rho_1-\rho_2)\big)\mathrm{d}x\\ &\quad +\int_{\mathbb{T}^d}|\rho_1-\rho_2|\int_{\mathbb{T}^d}\tau(y,x,t)\gamma|(u^+(\rho_1-\rho_2)-\kappa(M(\rho_1)-M(\rho_2))|\mathrm{d}y\mathrm{d}x,\end{aligned}$$

where we used the fact that $(\rho_1 - \rho_2)$ and $\gamma(\rho_1 u(\kappa, \rho_1) - \rho_2 u(\kappa, \rho_2))$ will have the same sign (because ρu is positive and monotonically increasing in ρ). Now, by (3.16), $||\rho_1-\rho_2||_{L^1(\mathbb{T}^d)} \le |\mathbb{T}^d|^{1/2}||\rho_1-\rho_2||_{L^2(\mathbb{T}^d)} = ||\rho_1-\rho_2||_{L^2(\mathbb{T}^d)}$, and $\gamma(y,t)\tau(y,x,t) \le \Xi^2$, we have that

$$\begin{aligned}&\int_{\mathbb{T}^d}|\rho_1-\rho_2|\int_{\mathbb{T}^d}\tau(y,x,t)\gamma\overbrace{|(u^+(\rho_1-\rho_2)-\kappa(M(\rho_1)-M(\rho_2))|}^{\le u^+|\rho_1-\rho_2|}\mathrm{d}y\mathrm{d}x\\ &\le u^+\Xi^2||\rho_1-\rho_2||^2_{L^2(\mathbb{T}^d)}\end{aligned}\tag{3.19}$$

Working with the derivative terms, we have

$$\begin{aligned}
&\int_{\mathbb{T}^d} \big(-u^+|\nabla(\rho_1-\rho_2)|^2 + (M(\rho_1)-M(\rho_2))\nabla\kappa\cdot\nabla(\rho_1-\rho_2)\\
&+ \kappa(M'(\rho_1)\nabla\rho_1 - M'(\rho_2)\nabla\rho_2)\cdot\nabla(\rho_1-\rho_2)\big)\,\mathrm{d}x\\
&= \int_{\mathbb{T}^d} \big(-u^+|\nabla(\rho_1-\rho_2)|^2 + (M(\rho_1)-M(\rho_2))\nabla\kappa\cdot\nabla(\rho_1-\rho_2)\\
&+ \kappa(M'(\rho_1)\nabla(\rho_1-\rho_2) + (M'(\rho_1)-M'(\rho_2))\nabla\rho_2)\cdot\nabla(\rho_1-\rho_2)\big)\,\mathrm{d}x\\
&= \int_{\mathbb{T}^d} \big[-\big(u^+ - \kappa M'(\rho_1)\big)|\nabla(\rho_1-\rho_2)|^2\\
&+ \Big(\kappa(M'(\rho_1)-M'(\rho_2))\nabla\rho_2 + \nabla\kappa(M(\rho_1)-M(\rho_2))\Big)\cdot\nabla(\rho_1-\rho_2)\big]\mathrm{d}x\\
&= \int_{\mathbb{T}^d} \Big[-\big(u^+ - \kappa M'(\rho_1)\big)\times\\
&\Big|\Big(\nabla(\rho_1-\rho_2) - \frac{\kappa(M'(\rho_1)-M'(\rho_2))\nabla\rho_2 + \nabla\kappa(M(\rho_1)-M(\rho_2))}{2(u^+-\kappa M'(\rho_1))}\Big)\Big|^2\\
&+ \frac{|\kappa(M'(\rho_1)-M'(\rho_2))\nabla\rho_2 + \nabla\kappa(M(\rho_1)-M(\rho_2))|^2}{4(u^+-\kappa M'(\rho_1))}\Big]\mathrm{d}x\\
&\le K||\rho_1-\rho_2||^2_{L^2(\mathbb{T}^d)},
\end{aligned} \tag{3.20}$$

for some K that depends on $\max_{t\in[0,T]}||\nabla\rho_2(\cdot,t)||_{L^\infty(\mathbb{T}^d)}$. It is finite as solutions are twice continuously differentiable in $\mathbb{T}^d$ so the gradient cannot blow up. The last inequality stems from the final (and positive) term in the integrand being Lipschitz: since $\nabla\rho_2$ is bounded on Ω (ρ_2 is $\mathcal{C}^2$ in x) and using (3.17)-(3.18), the term being squared in the numerator is Lipschitz. Also, the denominator is bounded below by $4(u^+-u^-)$. Whence, by combining inequalities (3.19) and (3.20):

$$\frac{\mathrm{d}}{\mathrm{d}t}||\rho_1-\rho_2||^2_{L^2(\mathbb{T}^d)} \le 2(u^+\Xi^2+K)||\rho_1-\rho_2||^2_{L^2(\mathbb{T}^d)}$$

and by a standard application of Grönwall's inequality, the result is proven. □

Corollary 3.1 (Continuous Dependence on Initial Conditions Given the Solutions Exist). *Let $T>0$ and $\rho\in\mathcal{U}_T$ be a solution to* (3.1)-(3.4) *with $\Omega=\mathbb{T}^d$ and $\rho(\cdot,0)=\rho_0(\cdot)>0$. Then $\forall\epsilon>0$, $\exists\zeta(T)>0$ s.t. if $\varrho\in\mathcal{U}_T$ is a solution to* (3.1)-(3.4) *also with $\Omega=\mathbb{T}^d$, $\varrho(\cdot,0)=\varrho_0(\cdot)>0$ and $||\varrho_0-\rho_0||_{L^2(\mathbb{T}^d)}<\zeta(T)$ then $\forall 0\le t\le T$, we have $||\varrho(\cdot,t)-\rho(\cdot,t)||_{L^2(\mathbb{T}^d)}<\epsilon$.*

Proof. The proof is trivial by using the arguments that prove 3.1: replace ρ_1 with ϱ and ρ_2 by ρ. □

Theorem 3.2 (Spatially Homogeneous Forcing Induces Convergence to Homogeneous Spatial Density in Measure). *Let $\rho\in\mathcal{U}_\infty$ be a solution to* (3.1)-(3.4) *with $\Omega=\mathbb{T}^d$, $\rho(x,0)>0$ for all $x\in\mathbb{T}^d$, $\theta=\theta(t)$ (making κ constant in space), $\eta=\eta(t)\ge 0$, $\omega=\omega(t)>\omega^-$ with ω^- a constant, $\gamma=\gamma(t)\ge 0$,*

and $\tau = 1$. *Then for all* $\rho(\cdot,0) \in \mathcal{C}^2$, $\rho(\cdot,t) \to \bar{\rho}(t)$ *in measure where* $\bar{\rho}(t) = \mathrm{e}^{-\int_0^t \omega(s)\mathrm{d}s} \int_0^t \eta(r)\mathrm{e}^{\int_0^r \omega(s)\mathrm{d}s}\mathrm{d}r$.

Remark 3.3. We are assuming global existence here in order for the limit to make sense.

Proof. [**Proof of theorem 3.2**] Our argument will come from two parts: first, we will establish a uniform upper bound for the solution over $\mathbb{T}^d$ that converges to $\bar{\rho}$ as $t \uparrow \infty$. We will then show that the solution cannot remain smaller than $\bar{\rho}$ except on sets whose measure vanishes. With κ constant in space, $\rho u = F(\rho,t)$ is monotonically increasing with ρ for each fixed t. Then (3.1)-(3.4) become

$$\rho_t = \delta\Delta F + \eta - \omega\rho + \gamma(\bar{F} - F) \tag{3.21}$$

where $\bar{F} = \int_{\mathbb{T}^d} F\mathrm{d}x$ is the average value of $F(\rho)$ on $\mathbb{T}^d$. We wish to change variables to remove as many terms from (3.21) as possible so as to apply the maximum principle supplied by lemma 3.1. For smooth ξ_1 and ξ_2, write $\rho = \mathrm{e}^{\xi_1(t)}q(x,t) + \xi_2(t)$ so that

$$\dot{\xi_1}q + q_t + \dot{\xi_2}\mathrm{e}^{-\xi_1} = \delta\Delta F + \eta - \omega q - \omega\xi_2\mathrm{e}^{-\xi_1} + \gamma(\bar{F} - F)\mathrm{e}^{-\xi_1}.$$

The dot signifies a time derivative. We choose

$$\dot{\xi_1} = -\omega, \quad \xi_1(0) = 0 \tag{3.22}$$

$$\dot{\xi_2} = -\omega\xi_2 + \eta\mathrm{e}^{\xi_1}, \quad \xi_2(0) = 0 \tag{3.23}$$

so that in simpler terms

$$q_t - \delta\Delta F = \gamma(\bar{F} - F)\mathrm{e}^{-\xi_1}.$$

Note that $\gamma\mathrm{e}^{-\xi_1}(\bar{F} - F)$ is an operator that satisfies the conditions of Θ in lemma 3.1. By lemma 3.1, the global maximum for q in $\mathcal{U}_T$ is achieved at $t = 0$. By choice of initial conditions $(3.22)_2$ and $(3.23)_2$, $\rho(\cdot,0) = q(\cdot,0)$ and thus

$$\rho = \mathrm{e}^{\xi_1}q + \xi_2 \leq \mathrm{e}^{\xi_1}\max_{\mathbb{T}^d} q(\cdot,0) + \xi_2 = \mathrm{e}^{\xi_1}\max_{\mathbb{T}^d}\rho(\cdot,0) + \xi_2.$$

By recalling η, ω are constant in space and solving the ODEs (3.22)-(3.23), we can precisely state that

$$\begin{aligned}\rho(x,t) &\leq \mathrm{e}^{-\int_0^t \omega(s)\mathrm{d}s}\max_{\mathbb{T}^d}\rho(\cdot,0) + \mathrm{e}^{-\int_0^t \omega(s)\mathrm{d}s}\int_0^t \eta(r)\mathrm{e}^{\int_0^r \omega(s)\mathrm{d}s}\mathrm{d}r \\ &= \mathrm{e}^{-\int_0^t \omega(s)\mathrm{d}s}\max_{\mathbb{T}^d}\rho(\cdot,0) + \bar{\rho}(t).\end{aligned} \tag{3.24}$$

This proves directly that the set of points upon which ρ exceeds $\bar{\rho} + \epsilon$ must have vanishing measure as $t \uparrow \infty$ for all $\epsilon > 0$. Having established an upper bound, we now work on the second part of the proof. Integrating (3.21) over $\mathbb{T}^d$ we have (as in proof of proposition 3.2 in the Appendix) that

$$\frac{\mathrm{d}}{\mathrm{d}t}||\rho||_{L^1(\mathbb{T}^d)} = \eta - \omega||\rho||_{L^1(\mathbb{T}^d)}$$

so

$$||\rho||_{L^1(\mathbb{T}^d)}(t) = \mathrm{e}^{-\int_0^t \omega(s)\mathrm{d}s}\left(||\rho||_{L^1(\mathbb{T}^d)}(0) + \int_0^t \eta(r)\mathrm{e}^{\int_0^r \omega(s)\mathrm{d}s}\mathrm{d}r\right). \tag{3.25}$$

Let $\epsilon > 0$ and define $S_\epsilon(t) = \{x \in \mathbb{T}^d | \rho(x,t) \leq \bar{\rho}(t) - \epsilon\}$. We show that in measure, $\rho \to \bar{\rho}(t)$.

Observe that by equations (3.25) and (3.24)

$$\begin{aligned}
||\rho||_{L^1(\mathbb{T}^d)}(t) &= \mathrm{e}^{-\int_0^t \omega(s)\mathrm{d}s}||\rho||_{L^1(\mathbb{T}^d)}(0) + \bar{\rho}(t) \\
&= \int_{\mathbb{T}^d \setminus S_\epsilon} \rho \mathrm{d}x + \int_{S_\epsilon} \rho \mathrm{d}x \\
&\leq |\mathbb{T}^d \setminus S_\epsilon|\left(\mathrm{e}^{-\int_0^t \omega(s)\mathrm{d}s}\max_{\mathbb{T}^d}\rho(\cdot,0) + \bar{\rho}(t)\right) + |S_\epsilon|\left(\bar{\rho}(t) - \epsilon\right) \\
&= |\mathbb{T}^d|\bar{\rho}(t) + |\mathbb{T}^d \setminus S_\epsilon|\mathrm{e}^{-\int_0^t \omega(s)\mathrm{d}s}\max_{\mathbb{T}^d}\rho(\cdot,0) - \epsilon|S_\epsilon(t)|.
\end{aligned}$$

Rearranging the first line and the last line, using $|\mathbb{T}^d| = 1$, and recalling there is an inequality produces

$$\epsilon|S_\epsilon(t)| \leq \mathrm{e}^{-\int_0^t \omega(s)\mathrm{d}s}\left(\max_{\mathbb{T}^d}\rho(\cdot,0)|\mathbb{T}^d \setminus S_\epsilon| - ||\rho||_{L^1(\mathbb{T}^d)}(0)\right).$$

Since we assume here that ω is strictly bounded below by 0 so that $\int_0^\infty \omega(s)\mathrm{d}s$ diverges, the right hand side decays to zero and thus

$$\epsilon|S_\epsilon(t)| \downarrow 0$$

as $t \to \infty$ for all $\epsilon > 0$. □

In one dimension, we can prove a stronger result.

Theorem 3.3 (Spatially Homogeneous Forcing Induces Uniform Decay of Gradient in One Dimension). *Let $\rho(x,t) \in \mathcal{C}^3_x \cap \mathcal{C}^1_t(\mathbb{T} \times [0,\infty) \to \mathbb{R}_{\geq 0})$, having ρ_{xt} and ρ_{tx} continuous, be a solution to* (3.1)-(3.4) *with $\Omega = \mathbb{T}$, $\rho(\cdot,0) = \rho_0 > 0$ with $\rho_0 \in \mathcal{C}^3$, $\theta = \theta(t)$ (making κ constant in space), $\eta = \eta(t) \geq 0$, $\omega = \omega(t) \geq 0$, $\gamma = \gamma(t) \geq 0$, and $\tau = 1$. We also assume that one of ω or γ is bounded below by a positive constant, m. Then for all $\rho(\cdot,0) \in \mathcal{C}^3$, $\rho(\cdot,t) \to \bar{\rho}(t)$ uniformly where $\bar{\rho}(t) = \mathrm{e}^{-\int_0^t \omega(s)\mathrm{d}s}\int_0^t \eta(r)\mathrm{e}^{\int_0^r \omega(s)\mathrm{d}s}\mathrm{d}r$.*

Remark 3.4. Note that we require 3 continuous derivatives in space. Also, we potentially allow $\omega = 0$ here.

Proof. [**Proof of theorem 3.3**] We will prove that the gradient ρ_x tends to zero uniformly. We can rewrite (2.13) in one spatial dimension and with the imposed hypotheses as

$$\rho_t = \delta(\rho u)_x + \eta - \omega\rho + \gamma\int_{\mathbb{T}} \rho u \mathrm{d}y - \gamma\rho u.$$

We begin by taking an $x-$partial-derivative using (3.15) with κ, ω, and γ constant in space to obtain:

$$\rho_{tx} = \delta((u^+ - \kappa M'(\rho))\rho_{xxx} - 3\kappa M''(\rho)\rho_x\rho_{xx} - \kappa M'''(\rho)\rho_x^3) \\ - \omega\rho_x - \gamma(u^+ - \kappa M'(\rho))\rho_x$$

since the nonlocal operator results in a constant in space. By interchanging $x-$ and $t-$ derivatives since the mixed partial derivatives are assumed continuous (and hence equal), we rewrite the equation with $\psi = \rho_x$:

$$\psi_t = \delta((u^+ - \kappa M'(\rho))\psi_{xx} - \frac{3}{2}\kappa M''(\rho)(\psi^2)_x - \kappa M'''(\rho)\psi_x^2) - \omega\psi - \gamma(u^+ - \kappa M'(\rho))\psi.$$

Now we multiply by ψ so that

$$\frac{1}{2}(\psi^2)_t = \delta((u^+ - \kappa M'(\rho))\psi\psi_{xx} - \frac{3}{2}\kappa M''(\rho)\psi(\psi^2)_x - \kappa M'''(\rho)\psi\psi_x^2) \\ - \big(\omega + \gamma(u^+ - \kappa M'(\rho))\big)\psi^2$$

Let $q = \frac{1}{2}\psi^2$. If we can prove $q \to 0$ uniformly over $\mathbb{T}$ we are done. Note that we have

$$q_t = \delta((u^+ - \kappa M'(\rho))(q_{xx} - \psi_x^2) - 6\kappa M''(\rho)\psi_x q - \kappa M'''(\rho)\psi\psi_x^2) \\ - 2\big(\omega + \gamma(u^+ - \kappa M'(\rho))\big)q$$

Note that $q \geq 0$ and if $q(x,t) = r(x,t)\mathrm{e}^{\xi t}$ then

$$r_t - \delta(u^+ - \kappa M'(\rho))r_{xx} + \delta(u^+ - \kappa M'(\rho))\psi_x^2 r\mathrm{e}^{-\xi t} + 6\kappa M''(\rho)\psi_x r \\ - \kappa M'''(\rho)\psi\mathrm{e}^{-\xi t}\psi_x^2 = -\Big(\xi + 2\big(\omega + \gamma(u^+ - \kappa M'(\rho))\big)\Big)r$$

If the supremum of r is zero, there is nothing more to prove. Otherwise, at any nonzero local maximum for r, $\psi\psi_x = 0$ and since $\psi \neq 0$ in such a case, we have $\psi_x = 0$. If ξ is chosen so that $\xi + 2\big(\omega + u^+ - \kappa M'(\rho)\big) \geq 0$ then by lemma 3.1, the global maximum for r is achieved at $t = 0$. Many ξ can be chosen but to establish decay, we choose $\xi = -2m$ if $\omega \geq m > 0$ with γ reaching zero and otherwise we pick $\xi = -2m(u^+ - u^-)$. This establishes $\frac{1}{2}\psi^2 \leq \mathrm{e}^{-mt}\max_{\mathbb{T}}\frac{1}{2}\psi(\cdot,0)^2$ so the gradient of ρ tends to zero uniformly. Combining this fact with theorem 3.2 proves the result□

We examine the the consequences of theorem 3.3 by plotting a solution subject to time-dependent, but spatially constant, forcing along with the decay of its gradient.

We now turn to the problem of parameters and solutions that are time-independent and their perturbations.

Proposition 3.4 (With constant γ and symmetric τ, steady state solutions are stable to L^2-perturbations).

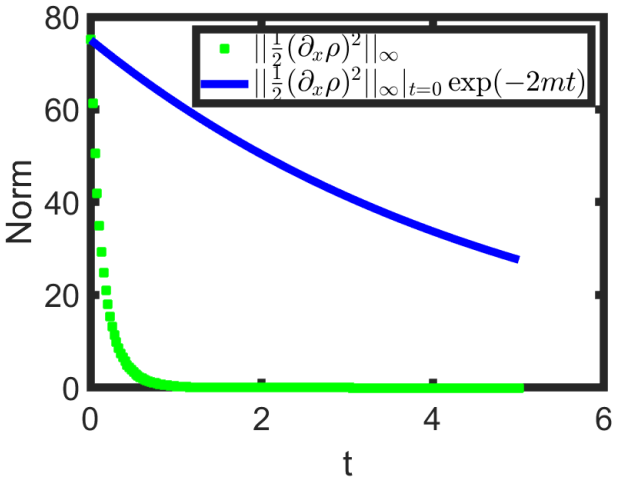

Fig. 5. Maximum of solution gradient with initial condition $\rho(x,0) = 2\sin^2(2\pi x)$ with $\eta = 2+t$, $\kappa = \frac{1}{2}$, $\omega = 0.1 + 0.09\sin^2 t$, $\gamma = 4\mathrm{e}^{-3t}$, $\tau = 1$, $u^+ = 0.98$, $u^- = 0.96$, $\bar{\rho} = 20$, $\delta = 0.02$.Here $m = 0.1$ since $\omega \geq 0.1$. The diffusion also helps to level out the solution, even more than m accounts for.

We consider equations (3.1)-(3.4) *with* $\Omega = \mathbb{T}^d$*;* $\gamma = \gamma(x)$*;* τ *time-independent and symmetric with* $\tau(x,y) = \tau(y,x)$*; let* $\eta(x,t) = \eta(x) \geq 0$*,* $\omega(x,t) = \omega(x) > 0$ *constants in time. We let* $\gamma \geq 0$ *be a constant. Suppose that* $\rho_0 \in \mathcal{C}^2(\mathbb{T}^d)$*,* $\rho_0 > 0$ *obeys the steady-state equation:*

$$0 = \delta\Delta(\rho u) + \eta - \omega\rho + (\int_{\mathbb{T}^d} \tau(y,x)\gamma(y)\rho(y)u(y)\mathrm{d}y - \gamma\rho u) \tag{3.26}$$

alongside (2.14) *and* (2.15)*. Then, for sufficiently small, periodic* $\bar{\rho}_0 \in \mathbb{T}^{\lceil 2}(\Omega)$*, the solution to* (2.13) *with* $\rho(\cdot,0) = \rho_0 + \bar{\rho}_0$ *upholds* $\rho \to \rho_0$ *in* $L^2(\mathbb{T}^d)$ *as* $t \to \infty$*.*

Proof. We define $R(x) = u^+ - \kappa M'(\rho_0)$ and $u_0 = u(\kappa, \rho_0)$ so that for small $\bar{\rho}$, up to first order, $(\rho_0 + \bar{\rho})u|_{\rho_0+\bar{\rho}} = \rho_0 u_0 + R(\rho_0)\bar{\rho}$. We remark that $R(x) > 0$ for all $x \in \mathbb{T}^d$. Now, we linearize (3.1)-(3.4) by $\rho = \rho_0 + \bar{\rho}$ to furnish:

$$\bar{\rho}_t = \Delta(R\bar{\rho}) - \omega\bar{\rho} + \gamma\left(\int_{\mathbb{T}^d} \tau(y,x)R\bar{\rho}|_y \mathrm{d}y - R\bar{\rho}\right)$$

Now, we multiply the equation by $R\bar{\rho}$ and integrate over $\mathbb{T}^d$:

$$\begin{aligned}\int_{\mathbb{T}^d} R\bar{\rho}\bar{\rho}_t \mathrm{d}x &= \int_{\mathbb{T}^d} R\bar{\rho}\Delta(R\bar{\rho})\mathrm{d}x - \int_{\mathbb{T}^d} \omega R\bar{\rho}^2 \mathrm{d}x \\ &\quad + \gamma \int_{\mathbb{T}^d} R\bar{\rho}|_x \left(\int_{\mathbb{T}^d} \tau(y,x) R\bar{\rho}|_y \mathrm{d}y - R\bar{\rho}|_x \right) \mathrm{d}x \\ \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{T}^d} R\bar{\rho}^2 \mathrm{d}x\right) &= -\int_{\mathbb{T}^d} |\nabla(R\bar{\rho})|^2 \mathrm{d}x - \int_{\mathbb{T}^d} R\omega\bar{\rho}^2 \mathrm{d}x \\ &\quad + \gamma \int_{\mathbb{T}^d} R\bar{\rho}|_x \left(\int_{\mathbb{T}^d} \tau(y,x) R\bar{\rho}|_y \mathrm{d}y - R\bar{\rho}|_x \right) \mathrm{d}x \\ \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{T}^d} R\bar{\rho}^2 \mathrm{d}x\right) &+ \int_{\mathbb{T}^d} R\omega\bar{\rho}^2 \mathrm{d}x \leq \\ &\gamma \left(\int_{\mathbb{T}^d} R\bar{\rho}|_x \int_{\mathbb{T}^d} \tau(y,x) R\bar{\rho}|_y \mathrm{d}y\mathrm{d}x - \int_{\mathbb{T}^d} R^2\bar{\rho}^2 \mathrm{d}x \right)\end{aligned}$$

We denote $\underline{R} = \min_{\mathbb{T}^d} R > 0$, and $\underline{\omega} = \inf_{\mathbb{T}^d} \omega > 0$ and with the above, we apply Cauchy-Schwarz:

$$\begin{aligned}&\frac{\underline{R}}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\bar{\rho}||^2_{L^2(\mathbb{T}^d)} + \underline{R\omega}||\bar{\rho}||^2_{L^2(\mathbb{T}^d)} \leq \\ &\qquad \gamma\left(\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} |R(x)\bar{\rho}(x,t)R(y)\bar{\rho}(y,t)\tau(y,x)|\mathrm{d}y\mathrm{d}x - \int_{\mathbb{T}^d} R^2\bar{\rho}^2\mathrm{d}x \right) \\ &\leq \gamma\left(\int_{\mathbb{T}^d} |R(y)\bar{\rho}(y,t)| \left(\int_{\mathbb{T}^d} \tau(y,x)\mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{T}^d} R(x)^2\bar{\rho}(x,t)^2\tau(y,x)\mathrm{d}x\right)^{1/2}\mathrm{d}y - \int_{\mathbb{T}^d} R^2\bar{\rho}^2\mathrm{d}x \right) \\ &\quad = \gamma\left(\int_{\mathbb{T}^d} |R(y)\bar{\rho}(y,t)| \left(\int_{\mathbb{T}^d} R(x)^2\bar{\rho}(x,t)^2\tau(y,x)\mathrm{d}x\right)^{1/2}\mathrm{d}y - \int_{\mathbb{T}^d} R^2\bar{\rho}^2\mathrm{d}x \right) \\ &\quad \leq \gamma\left(\int_{\mathbb{T}^d} R^2\bar{\rho}^2\mathrm{d}y\right)^{1/2}\left(\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} R(x)^2\bar{\rho}(x,t)^2\tau(y,x)\mathrm{d}x\mathrm{d}y\right)^{1/2} - \gamma\int_{\mathbb{T}^d} R^2\bar{\rho}^2\mathrm{d}x \\ &\quad = \gamma\left(\int_{\mathbb{T}^d} R^2\bar{\rho}^2\mathrm{d}y\right)^{1/2}\left(\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} R(x)^2\bar{\rho}(x,t)^2\tau(x,y)\mathrm{d}y\mathrm{d}x\right)^{1/2} - \gamma\int_{\mathbb{T}^d} R^2\bar{\rho}^2\mathrm{d}x \\ &\qquad = \gamma\left(\int_{\mathbb{T}^d} R^2\bar{\rho}^2\mathrm{d}y\right)^{1/2}\left(\int_{\mathbb{T}^d} R(x)^2\bar{\rho}(x,t)^2\mathrm{d}x\right)^{1/2} - \gamma\int_{\mathbb{T}^d} R^2\bar{\rho}^2\mathrm{d}x = 0\end{aligned}$$

Thus,

$$||\bar{\rho}(\cdot,t)||^2_{L^2(\mathbb{T}^d)} + 2\underline{\omega}\int_0^t ||\bar{\rho}(\cdot,s)||^2_{L^2(\mathbb{T}^d)}\mathrm{d}s \leq ||\bar{\rho}(\cdot,0)||^2_{L^2(\mathbb{T}^d)}$$

We claim this proves $||\bar{\rho}(\cdot,t)||_{L^2(\mathbb{T}^d)} \downarrow 0$ as $t \uparrow \infty$. We have that $||\bar{\rho}(\cdot,t)||_{L^2(\mathbb{T}^d)}$ is monotonically non-increasing. Were it to not tend to zero then there must exist $\epsilon > 0$ so that $\lim_{t\to\infty}||\bar{\rho}(\cdot,t)||_{L^2(\mathbb{T}^d)} \geq \epsilon$. But this would force the integral to diverge to $+\infty$ and we would have a contradiction to the boundedness of $||\bar{\rho}(\cdot,t)||_{L^2(\mathbb{T}^d)} + 2\underline{\omega}\int_0^t ||\bar{\rho}(\cdot,s)||_{L^2(\mathbb{T}^d)}\mathrm{d}s$. □

3.3. *Generalizations*

Some of the proofs provided here could readily be generalized to include other features. For example, the proof of positivity, and the bounds on the L^1 and L^∞ norms of solutions depended upon the positivity, normalization, and boundedness of τ. But more complicated τ choices could also be used, such as those of the form $\tau(y, x, t, \rho/||\rho||_{L^1(\Omega)})$ to furnish the same results, allowing effects of aggregation due to localized peaks in the population density. Other proofs may require more control over τ and further properties of solutions ρ.

4. Qualitative Features of Real Data

After nondimensionalization and smoothing, the homeless population densities for four consecutive years are plotted in figure 6. Within the real data, we remark that while some areas have high homeless population densities consistently from year to year, encampments can form over the course of a year and a new bump in the density appears.

The PDE model has many parameters, most of which can vary over space and time. As such, there is a danger of overfitting. In order to illustrate qualitatively consistent behavior in our model with the real data of encampment formation, we restrict ourselves to describing the formation of a new encampment (a new local maximum). To achieve this, we keep all of η, ω, γ constant in space and time and only allow κ to vary with time. We select a travel term of

$$\tau(y, x, t) = \frac{\kappa(x, t)}{\int_\Omega \kappa(x', t)\mathrm{d}x'} \tag{4.1}$$

and allow a localized region in the domain to increase its κ value over dimensionless time interval of 1. Over the domain we impose no-flux boundary conditions. This choice induces a change in homeless population density consistent with an encampment formation. See figure 7.

5. Conclusions and Future Work

Motivated by observations that homeless populations are influenced by local geographic features, we formulated a continuum PDE model to describe the evolution of the homeless population density. We have proven that smooth periodic solutions to the model equations enjoy a maximum principle, a positive population density, uniqueness, a flattening phenomena with spatially uniform forcing, and L^2-stability for constant transfer rates and symmetric travel kernels. The model is well-behaved and for suitable choices of parameters, it can produce population changes that are qualitatively consistent with real homeless population data.

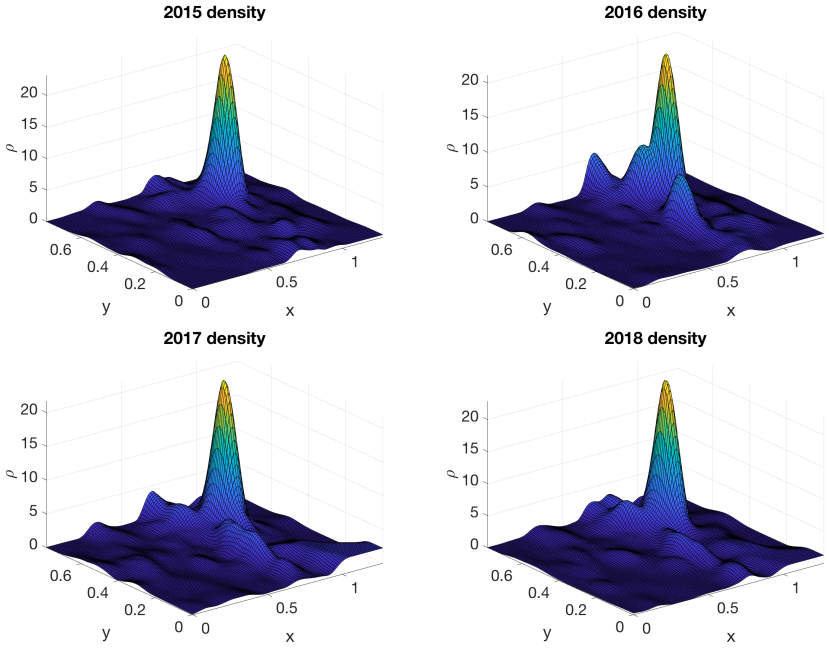

Fig. 6. Plots of the interpolated and smoothed homeless population densities for a subset of Los Angeles. The population has been nondimensionalized by a scale of 68 mi^{-2} and the length scale is 12 mi. The data have been smoothed: population densities were interpolated onto a regular mesh from LAHSA data and then locally averaged over a radius of 1 mi.

From this preliminary model, many new avenues of research open up. It would be worth understanding how to empirically model the various parameters/functions in (3.1)-(3.4) so that the model can be used in a quantitative capacity and in identifying what can be done to combat homelessness. From the viewpoint of mathematical analysis, a rigorous proof of existence of solutions is worth pursuing. It would also be interesting to study this model or its natural variants as applied to other contexts.

6. Appendix

6.1. *Numerical Methods*

In one spatial dimension on the torus, we use semi-implicit stepping. To solve (3.1)-(3.4), assume we have a numerical solution on a regular spatial grid at time t_k, ρ^k. We write $u(\rho)$ to indicate the u-function evaluated at a given place with argument ρ. A method-of-lines implementation with time step Δt is to find ρ^{k+1} that solves:

$$\frac{\rho^{k+1}-\rho^k}{\Delta t} = \delta\Delta(\rho^{k+1}u(\rho^k))+\eta^{k+1}-\omega^{k+1}\rho^{k+1}+I[\gamma^{k+1}\rho^{k+1}u(\rho^k)]-\gamma^{k+1}\rho^{k+1}u(\rho^k).$$

This requires inverting a dense matrix, which is not overly costly in one dimen-

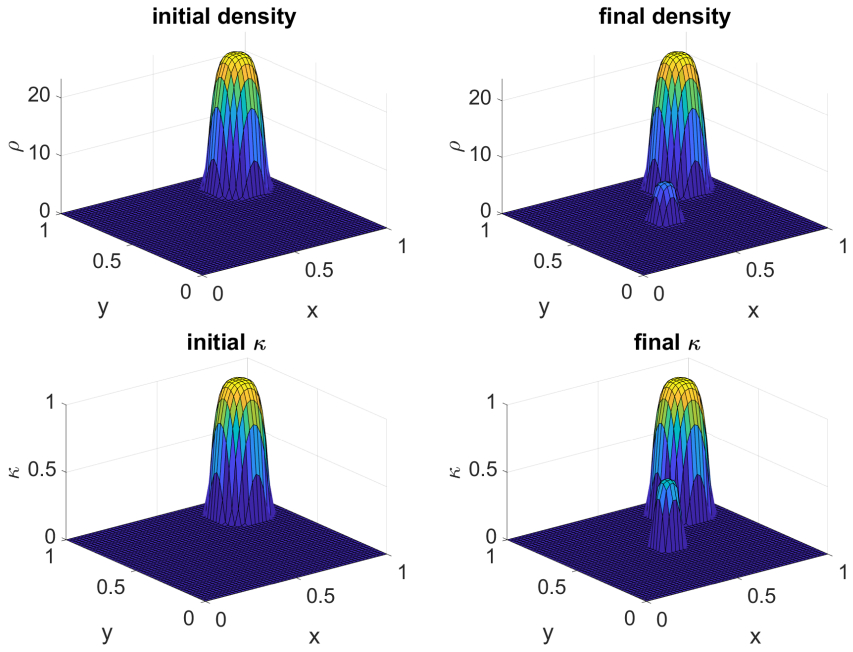

Fig. 7. Top row: initial homeless population density at $t = 0$ (left) and final homeless population density at $t = 1$ (right). Bottom row: initial relative capacity κ at $t = 0$ and final relative capacity at $t = 1$. Between $t = 0$ and $t = 1$, the relative capacity was a linear interpolation between the initial and final κ plots. Parameters are $\delta = 10^{-4}$, $u^+ = 0.999$, $u^- = 0.998$, $\tilde{\rho} = 1/20$, $\eta = 1/10$, $\omega = 1/100$, $\gamma = 5$, with τ given by (4.1).

sion. And standard second-order quadrature (trapezoid rule) is sufficient. In two dimensions with Neumann boundary conditions, we implement a split-step finite difference scheme using semi-implicit stepping for the Laplacian, explicit stepping for the travel terms, and implicit stepping for the sink terms. We adopt the same notation as above but also allow for fractional superscripts so that $\eta^{k+1/3}$ means η evaluated at $t_k + \frac{\Delta t}{3}$. Our method is to

(1) solve $\frac{\varrho^{(1)} - \rho^k}{\Delta t/3} = \delta\Delta(\varrho^{(1)} u^k)$ for $\varrho^{(1)}$ also with semi-implicit boundary conditions;
(2) solve $\frac{\varrho^{(2)} - \varrho^{(1)}}{\Delta t} = \eta^{k+1/3} + I[\gamma^{k+1/3}\varrho^{(1)} u(\varrho^{(1)}) - \gamma^{k+1/3}\varrho^{(1)} u(\varrho^{(1)})$ for $\varrho^{(2)}$ with the quadrature described below;
(3) solve $\frac{\varrho^{(3)} - \varrho^{(2)}}{\Delta t/3} = \delta\Delta(\varrho^{(3)} u(\varrho^{(2)}))$ for $\varrho^{(3)}$ also with semi-implicit boundary conditions;
(4) solve $\frac{\varrho^{(4)} - \varrho^{(3)}}{\Delta t} = -\omega^{k+2/3}\varrho^{(4)}$;
(5) solve $\frac{\rho^{k+1} - \varrho^{(4)}}{\Delta t/3} = \delta\Delta(\rho^{k+1} u(\varrho^{(4)}))$ for ρ^{k+1} also with semi-implicit boundary conditions.

The Laplacian term solves the equation with the appropriate boundary condi-

tions so we split it into three so that the nonlocal stepping and stepping forward with the sink will be done with the correct boundary conditions and so that the output also has the proper boundary conditions. The nonlocal operator is done explicitly to avoid inverting a large dense matrix in the case of an implicit scheme. It is combined with the positive source term η and positivity can be preserved when done explicitly. The sink term $-\omega\rho$ is dealt with implicitly to preserve positivity.

The quadrature for the nonlocal movement is delicate. In second-order quadrature, values of $\gamma\rho u$ in the integral are weighted by either $1/2$ or $1/4$ at the boundary. Proper mass balance is achieved by weighting the corresponding sinks of $-\gamma\rho u$ by either $1/2$ or $1/4$.

6.2. *Further Proofs*

Proof. [**Proof of lemma 3.1**] We prove the sup-case as the inf-case can be done mutatis mutandis. Let $\epsilon > 0$ and write $q = q_\epsilon + \epsilon t$. Then we have

$$q_{\epsilon,t} + \epsilon + \Phi(\nabla q_\epsilon, x, t) + \lambda q - \alpha\Delta F(q,t) = \Theta.$$

At a local maximum for q_ϵ within S, $\Delta F(q,t) \leq 0$, $\Phi(\nabla q_\epsilon, x, t) = 0$, $q_{\epsilon,t} \geq 0$. And at the global maximum we know $\Theta \leq 0$ because the addition of the constant ϵt will preserve $q_\epsilon + \epsilon t$ having a global maximum at the same point in $\mathbb{T}^d$. So

$$\overbrace{\epsilon + \lambda q - \alpha\Delta F(q)}^{>0} = \overbrace{\Theta}^{\leq 0},$$

a contradiction. Thus,

$$\sup_S q = \sup_S(q_\epsilon + \epsilon t) \leq \sup_S q_\epsilon + \epsilon T = \max_{\partial^* S} q_\epsilon + \epsilon T \leq \max_{\partial^* S} q + \epsilon T \leq \sup_S q + \epsilon T$$

Taking the limit as $\epsilon \downarrow 0$ furnishes that $\sup_S q = \max_{\partial^* S} q$ and the global maximum for q cannot be attained within S and must occur in $\mathbb{T}^d \times \{t_0\}$. □

Proof. [**Proof of proposition 3.1**] Let $\rho(x,t) = \mathrm{e}^{\xi t} q(x,t)$ for a $\xi \in \mathbb{R}$ to be chosen judiciously. Then from (3.1)-(3.4) and expressing things in terms of q where appropriate:

$$\begin{aligned} q_t + \xi q = {} & \delta(u^+ - \kappa M'(\rho))\Delta q - \delta\kappa M''(\rho)e^{\xi t}|\nabla q|^2 - 2\delta M'(\rho)\nabla\kappa\cdot\nabla q - \delta M(\rho)\Delta\kappa \\ & + \eta\mathrm{e}^{-\xi t} - \omega q + I[\gamma q u] - \gamma q u \end{aligned}$$

so that upon rearranging:

$$\begin{aligned} q_t + (\delta\kappa M''(\rho)\mathrm{e}^{\xi t}|\nabla q|^2 + 2\delta M'(\rho)\nabla\kappa\cdot\nabla q) + [(\xi + \gamma u + \omega)q - \delta\Delta\kappa M(\rho)] & \\ - \delta(u^+ - \kappa M'(\rho))\Delta q = \overbrace{\eta\mathrm{e}^{-\xi t} + I[\gamma q u]}^{\geq 0} & \end{aligned}$$

Note by (3.9) and hypothesis 2 we have that $|\delta\Delta\kappa M(\rho)| \leq \delta\Xi u^- \rho$ and so we choose ξ so that

$$\xi + \gamma u + \omega + \delta\Xi u^- \mathrm{e}^{\xi t} \leq 0,$$

which can be done by choosing $\xi = -2\Xi - \delta\Xi u^-$. The solution q must attain its global minimum at $t = 0$ by lemma 3.1. Then $\rho = \mathrm{e}^{\xi t} q \geq \mathrm{e}^{\xi t} \min_{\mathbb{T}^d} q(\cdot, 0) = \rho_m \mathrm{e}^{-(2\Xi + \delta\Xi u^-)t} > 0$ since $\rho(\cdot, 0) = q(\cdot, 0)$. □

Proof. [**Proof of proposition: 3.2**] We integrate (2.13) over Ω:

$$\begin{aligned}
\int_{\mathbb{T}^d} \rho_t \mathrm{d}x &= \int_{\mathbb{T}^d} \left(\delta\Delta(\rho u) + \eta - \omega\rho + I[\gamma\rho u] - \gamma\rho u\right) \mathrm{d}x \\
&= \int_{\mathbb{T}^d} (\eta - \omega\rho)\mathrm{d}x + \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \tau(y, x, t)\gamma\rho u|_y \mathrm{d}y\mathrm{d}x - \int_{\mathbb{T}^d} \gamma\rho u \mathrm{d}x \\
&= \int_{\mathbb{T}^d} (\eta - \omega\rho)\mathrm{d}x + \int_{\mathbb{T}^d} (\int_{\mathbb{T}^d} \tau(y, x, t)\mathrm{d}x)\gamma\rho u|_y \mathrm{d}y - \int_{\mathbb{T}^d} \gamma\rho u \mathrm{d}x \\
&= \int_{\mathbb{T}^d} (\eta - \omega\rho)\mathrm{d}x
\end{aligned}$$

where in getting from line 2 to 3 the boundary term vanished (they cancel due to the periodicity) and we interchanged the order of integration, and in getting from line 3 to 4 we used that $\int_{\mathbb{T}^d} \tau(\cdot, x, \cdot)\mathrm{d}x = 1$ so that there is a cancellation of the integrals with γ. We find therefore that, since $\rho > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}||\rho||_{L^1(\mathbb{T}^d)} = \int_{\mathbb{T}^d} \eta \mathrm{d}x - \int_{\mathbb{T}^d} \omega\rho \mathrm{d}x \leq \int_{\mathbb{T}^d} \eta \mathrm{d}x$$

and thus

$$||\rho||_{L^1(\mathbb{T}^d)} \leq ||\rho(x, 0)||_{L^1(\mathbb{T}^d)} + \int_0^t (\int_{\mathbb{T}^d} \eta \mathrm{d}x)\mathrm{d}t \leq ||\rho(x, 0)||_{L^1(\mathbb{T}^d)} + \Xi t. \tag{6.1}$$

since $|\mathbb{T}^d| = 1$. □

Proof. [**Proof of proposition 3.3**] This is similar to the proof of positivity. Let $\rho(x, t) = \xi(t) + q(x, t)$ for smooth $\xi : \mathbb{R} \to \mathbb{R}$ to be chosen judiciously. Then in terms of q where appropriate:

$$\begin{aligned}
q_t + (\delta\kappa M''(\rho)|\nabla q|^2 + 2\delta M'(\rho)\nabla\kappa \cdot \nabla q) + (\gamma u + \omega)q \\
- \delta(u^+ - \kappa M'(\rho))\Delta q = \eta + I[\gamma q u] + \delta M(\rho)\Delta\kappa - \xi'(t)
\end{aligned}$$

Here we choose ξ so that $\eta + I + \delta M(\rho)\Delta\kappa - \xi'(t) \leq 0$. Note that $I[\gamma\rho u] \leq I[\gamma\rho] \leq \Xi^2 ||\rho(\cdot, t)||_{L^1(\mathbb{T}^d)}$. Therefore we let $\xi(t)$ be defined by

$$\begin{aligned}
\xi(t) = \int_0^t (\Xi^2 ||\rho(\cdot, s)||_{L^1(\mathbb{T}^d)} + \Xi + \delta\tilde{\rho}u^-\Xi)\mathrm{d}s \leq \\
(\Xi^2 ||\rho(\cdot, 0)||_{L^1(\mathbb{T}^d)} + \Xi + \delta\tilde{\rho}u^-\Xi)t + \frac{\Xi^3}{2}t^2 < \infty
\end{aligned}$$

owing to the bound of $L^1(\mathbb{T}^d)$ from proposition 3.2. This forces the maximum of q to occur at $t = 0$ and since $||\rho(\cdot, 0)||_{L^\infty(\mathbb{T}^d)} = ||q(\cdot, 0)||_{L^\infty(\mathbb{T}^d)}$, we have

$$||\rho(\cdot, t)||_{L^\infty(\mathbb{T}^d)} \leq ||\rho(\cdot, 0)||_{L^\infty(\mathbb{T}^d)} + (\Xi^2||\rho(\cdot, 0)||_{L^1(\mathbb{T}^d)} + \Xi + \delta\tilde{\rho}u^-\Xi)t + \frac{\Xi^3}{2}t^2. \quad \square$$

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