Hybrid Inverse Problems for Nonlinear Elasticity

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January 21, 2020

Abstract

We consider the problem of nonlinear elasticity in two and three dimensions. Under the hypothesis the fluid is incompressible, we recover the displaced field and the Lame parameter μ from power density measurements in two dimensions. A stability estimate is shown to hold for small displacement fields, under some natural hypotheses on the direction of the displacement, with the background pressure fixed. We also prove in dimensions two and three a stability result for the (nonlinear) Saint Venant model in the case of displaced solution measurements. The techniques introduced show the difficulties of using hybrid imaging techniques for non-linear inverse problems.

Mathematics Subject Classification(MSC2000): 35B30.

Keywords: stability analysis, shear modulus reconstruction, magnetic resonance elastography, biological tissues, optimal control.

The content of this chapter corresponds to the article H. Carrillo and A. Waters. "Nonlinear hybrid inverse problems in elasticity". In preparation.

1 Introduction

We consider an isotropic nonlinear elastic wave equation in a bounded domain Ω. The stress the material is under going is described by the Lamé parameters, λ, μ, and ρ. We study the following problem: is it possible to determine the Lamé parameters λ, μ and ρ from the knowledge of Neumann data of the solution on the boundary? We are interested in the global recovery problem of the displacement field.

Our main motivation is the structure of hyper-elastic materials, many of which are not accurately described by linear elastic models. A hyperelastic model is one for an ideally elastic material in which the stress-strain relationship is derived from the strain energy density function. This type of model is often known as Green's model which was made rigorous by Ogden [26]. Hyper-elastic models accurately describe the stress-strain behaviour of materials such as rubber, [24]. Unfilled vulcanized elastomers almost always conform to the hyperelastic ideal. Filled elastomers and biological tissues are also modelled via the hyper-elastic idealisation, [15]. In the linear case, for reconstruction of the Lamé coefficients concerning biological tissues, one can see [2] for example. Our focus is on a non-linear model, and the reduction of the amount of required data to recover the coefficients uniquely. Of the three parameters required to recover the material structure, it is often the most natural to recover the parameter μ which encodes more about possible disease in patients than the other parameters. Several diseases involve changes in the mechanical properties of tissue and normal function of tissue, for example in skeletal muscle, heart, lungs and gut [17, 22, 16].

From power density measurements we are able to to prove a stability estimate for both the solution and the parameter μ. Even in the linear case for elasticity this has not been shown before in the literature. For the linear problem, the closest works in 2 and 3 dimensions are for the anisotropic conductivity problem [9] and for full solution measurements in [8, 6, 32]. However, this list is not exhaustive there are numerous results on recovering the parameters μ and λ from knowledge of the solution in a domain for the linear problem [30, 28, 25, 14, 13]. As such, the significant contribution of this article is the extension to the nonlinear problem. The difficult symbol computations used to find stability estimates for the nonlinear problem can also be used to extend known results on the linear problem.

In Section 4, we give precise stability estimates for the linearized incompressible model of elasticity in 2 dimensions with the background pressure held fixed, see Theorem 5. These stability estimates have no kernel (they are injective) for all ω sufficiently large on the entirety of the domain with two measurements. In Section 5, we can extend these estimates to include some generic nonlinear forcing terms. This is the first time global injectivity with a single fixed ω has been shown under any conditions.

For the later part of the article, in Section 7, we consider the Saint-Venant model of elasticity. Because the (nonlinear) Saint-Venant model depends on the parameter λ and this in practice is large, we also prove convergence of the linearized Saint Venant model in 2 and 3 dimensions using a differential operator (the curl) which removes the parameter λ. In the process of doing so, in Section 6, we correct an earlier computational error in the stability estimates for the linearized compressible problem in two dimensions in [6] which affects the 2d stability estimates. The size of the parameter λ aversely affects the size of the class of solutions which can be considered in the linearized Saint-Venant model, unless we apply the curl. This means our convergence results are sharper than in [18]. Indeed, the main assumption on their nonlinear models does not give any convergence result for the Saint-Venant model when λ is very large. We use solution measurements in the linearized Saint-Venant model, since power density measurements do not work well when using the annihilation (curl) operator.

Iterative algorithms for the recovery of μ and convergence results are presented for each model in Sections 4.5, 5.3 and 7.3. Main tools in this article come from the theory of over-determined elliptic boundary-value problems. In Section 3 we present necessary preliminaries.

2 Notation

In this paper we use the Einstein summation convention. For two vectors a and b, the exterior product is denoted by

$$a \otimes b = ab^{\intercal},$$

i.e., $a \otimes b$ is a matrix with entries

$$(a \otimes b)_{ij} = a_i b_j.$$

More generally, the exterior product between a tensor A of order m and B of order n is a new tensor $A \otimes B$ of order $m + n$ with entries

$$(A \otimes B)_{i_1 \dots i_m j_1 \dots j_n} = A_{i_1 \dots i_m} B_{j_1 \dots j_n}.$$

For two matrices A and B of the same size, the inner product is denoted by

$$A : B = a_{ij} b_{ji},$$

and we write $|A|^2 = A : A$. In addition, we consider the product between a tensor A of order $(n+1)$ and other B of order n as the vector AB with entries

$$(AB)_{i_0} = A_{i_0 i_1 \dots i_n} B_{i_1 \dots i_n}.$$

Let $\Omega \subset \mathbb{R}^d$ be a simply-connected smooth bounded domain in $\mathbb{R}^d$. For vector–valued functions

$$f(x) = (f_1(x), f_2(x), \dots, f_d(x)) : \Omega \to \mathbb{R}^d \ ,$$

the Hilbert space $H_m(\Omega)^d$, $m \in \mathbb{N}$ is defined as the completion of the space $\mathcal{C}_c^\infty(\Omega)^d$ with respect to the norm

$$\|f\|_m^2 = \|f\|_{m,\Omega}^2 = \sum_{|i|=1}^{m} \int_\Omega |\nabla^i f(x)|^2 + |f(x)|^2 \, dx,$$

where we write $\nabla^i = \partial^{i_1} \dots \partial^{i_d}$ for $i = (i_1, \dots, i_d)$ for the higher-order derivative. Let E be the symmetric gradient acting on $u \in H_0^1(\Omega)^d$ as

$$Eu = \frac{1}{2}(\nabla u + (\nabla u)^{\intercal}) = \nabla^S u. \tag{1}$$

In general, we assume the Lamé coefficients are $C^3(\overline{\Omega})$ where $\overline{\Omega}$ denotes the closure of Ω and that they satisfy the following conditions

$$\begin{aligned}\lambda(x) &\geq \lambda_{\min} = \min\{\lambda(x) : x \in \overline{\Omega}\} > 0, \\ \mu(x) &\geq \mu_{\min} = \min\{\mu(x) : x \in \overline{\Omega}\} > 0, \\ \rho(x) &\geq \rho_{\min} = \min\{\rho(x) : x \in \overline{\Omega}\} > 0.\end{aligned} \tag{2}$$

We consider the density $\rho(x)$ to be fixed for this article, and as such we remove it from the symbol computations. We remind the definition of the divergence for a matrix function: if $T : \bar{\Omega} \to \mathbb{M}^n$ (square matrices of order n) is differentiable, then

$$\operatorname{div}(T)(x) = \partial_j T_{ij}(x)\hat{e}_i \ \in \mathbb{R}^d.$$

Also we remind the definition of the curl of a function $f : \Omega \to \mathbb{R}^d$:

$$\nabla \times f = \partial_1 f_2 - \partial_2 f_1$$

in dimension $d = 2$, and

$$\nabla \times f = (\partial_2 f_3 - \partial_3 f_2)\boldsymbol{e}_1 - (\partial_1 f_3 - \partial_3 f_1)\boldsymbol{e}_2 + (\partial_1 f_2 - \partial_2 f_1)\boldsymbol{e}_3$$

in dimension $d = 3$.
And finally we remind the reader of a useful integration by parts identity. If $S : \bar{\Omega} \to \mathbb{S}^d$ (symmetric matrices) and $v : \bar{\Omega} \to \mathbb{R}^d$, then

$$\int_\Omega \operatorname{div}(S) \cdot v \, dx = \int_{\partial\Omega} (S\nu) \cdot v \, da - \int_\Omega S : \nabla^S v \, dx,$$

where ν denotes the outward unit normal on $\partial\Omega$.

We will also need the following lemma.

Lemma 1. *[Korn's inequality] Let Ω be as above. Let $u \in H_0^1(\Omega)^d$ then*

$$\int_\Omega |\nabla u|^2 \, dx \leq 2 \int_\Omega |\nabla^S u|^2 \, dx,$$

c.f., for instance, [5].

We now review the existence and uniqueness results for the elasticity system. We consider the following boundary value problem for the elasticity equations

$$\begin{cases} \nabla(\lambda(x)\nabla \cdot u_\lambda) + \omega^2 u_\lambda(x) + 2\nabla \cdot \mu(x)\nabla^S u_\lambda(x) = 0 \quad \text{in } \Omega, \\ u_\lambda(x) = g(x) \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

with $\mu(x), \lambda(x) \in \mathcal{C}^1(\bar{\Omega})$ the Lamé coefficients.

The solution $u_\lambda(x)$ is such that

$$u_\lambda(x) : \Omega \to \mathbb{R}^d.$$

It is known that the solution $u_\lambda(x)$ exists and is unique. In particular, $\nabla^S u_\lambda(x) \in L^2(\Omega)^d$ if $g(x) \in H^{1/2}(\partial\Omega)$, $\lambda, \mu \in L^\infty(\Omega)$ and satisfy (2) and $\nabla^S u_\lambda(x) \in H^4(\Omega)^d$ under the additional assumptions that $\mu(x), \lambda(x) \in \mathcal{C}^4(\bar{\Omega})$, $g \in H^{9/2}(\partial\Omega)^d$. We need the latter regularity assumption for later stability estimates.

The Poisson ratio σ of the anomaly is given in terms of the Lamé coefficients by

$$\sigma = \frac{\lambda/\mu}{1 + 2\lambda/\mu}.$$

It is known in soft tissues $\sigma \approx 1/2$ or equivalently $\lambda >> \mu$. This makes it difficult to reconstruct both parameters μ and λ simultaneously [21],[20]. Therefore we first construct asymptotic solutions to the problem (3) when $\lambda_{\min} \to \infty$. The following theorem loosely follows [3] and [4] which consider piecewise constant Lamé coefficients. We recall that in the limit, the elasticity equations (3) reduces to the following Stokes system

$$\begin{cases} \omega^2 u(x) + 2\nabla \cdot \mu(x)\nabla^S u(x) + \nabla p(x) = 0 \quad \text{in } \Omega, \\ \nabla \cdot u(x) = 0 \quad \text{in} \quad \Omega, \\ u(x) = g(x) \quad \text{on} \quad \partial\Omega, \\ \int\limits_{\Omega} p(x)\, dx = 0. \end{cases} \tag{4}$$

Theorem 1 ([6] '14)**.** *Suppose that ω^2 is not an eigenvalue of the problem (4) with $g(x) = 0$, then there exists a positive constant C which is independent of λ such that the following error estimates hold for λ_{min} large enough*

$$||u_\lambda - u||_{H^1(\Omega)^d} \leq \frac{C}{\sqrt{\lambda_{min}}}. \tag{5}$$

Remark 2.1. *The relation between the pressure p in* (4) *and u_λ in* (3) *is that p is the limit of $\lambda\nabla \cdot u_\lambda$ as $\lambda_{min} \to \infty$.*

3 Preliminaries on Over-determined Elliptic Boundary-Value Problems

In this section, we present some basic properties about over-determined elliptic boundary-value problems which plays a key role in our stability estimates in the next sections. The presentation follows closely to the ones in [27, 32]. We present it here for the convenience of the reader.

We first recall the definition of ellipticity in the sense of Douglis-Nirenberg. Consider the (possibly) redundant system of linear partial differential equations

$$\begin{aligned} \mathcal{L}(x, \frac{\partial}{\partial x})y &= \mathcal{S}, \\ \mathcal{B}(x, \frac{\partial}{\partial x})y &= \phi, \end{aligned} \tag{6}$$

for m unknown functions $y = (y_1, \dots, y_m)$ comprising in total of M equations. Here $\mathcal{L}(x, \frac{\partial}{\partial x})$ is a matrix differential operator of dimension $M \times m$ with entries $L_{ij}(x, \frac{\partial}{\partial x})$. For each $1 \leq i \leq M$, $1 \leq j \leq m$ and for each point x, the entry $L_{ij}(x, \frac{\partial}{\partial x})$ is a polynomial in $\frac{\partial}{\partial x_i}$ $i = 1, \dots, d$. If the system is redundant, then there are possibly more equations than unknowns, $M \geq m$. The matrix $\mathcal{B}(x, \frac{\partial}{\partial x})$ has entries $B_{ij}(x, \frac{\partial}{\partial x})$ for $1 \leq k \leq Q, 1 \leq j \leq m$ consisting of Q equations at the boundary. The operators are also polynomial in the partials of x. Naturally, the vector $\mathcal{S}$ is a vector of length M, and ϕ is a vector of length Q.

Definition 1. *[c.f.[1],[12]] Let integers $s_i, t_j \in \mathbb{Z}$ be given for each row $1 \leq i \leq M$ and column $1 \leq j \leq m$ with the following property: for $s_i + t_j \geq 0$ the order of L_{ij} does not exceed $s_i + t_j$. For $s_i + t_j < 0$, one has $L_{ij} = 0$. Furthermore, the numbers are normalized so that for all i one has $s_i \leq 0$. The numbers s_i, t_j are known as Douglis-Nirenberg numbers.*

The principal part of $\mathcal{L}$ for this choice of numbers s_i, t_j is defined as the matrix operator $\mathcal{L}^0$ whose entries are composed of those terms in L_{ij} which are exactly of order $s_i + t_j$.

The principal part $\mathcal{B}^0$ of $\mathcal{B}$ is composed of the entries which are composed of those terms in B_{kj} which are exactly of order $\sigma_k + t_j$. The numbers σ_k, $1 \leq k \leq Q$ are computed as

$$\sigma_k = \max_{1 \leq j \leq m} (b_{kj} - t_j)$$

with b_{kj} denoting the order of B_{kj}. Real directions with $\xi \neq 0$ and

$$\operatorname{rank} \mathcal{L}^0(x, i\xi) < m$$

are called characteristic directions of $\mathcal{L}$ at x. The operator $\mathcal{L}$ is said to be (possibly) over-determined elliptic in Ω if $\forall x \in \overline{\Omega}$ and for all real nonzero vectors ξ one has

$$\operatorname{rank} \mathcal{L}^0(x, i\xi) = m.$$

We next recall the following Lopatinskii boundary condition.

Definition 2. *Fix $x \in \partial\Omega$ and let ν be the inward unit normal vector at x. Let ζ be any non-zero tangential vector to Ω at x. We consider the line $\{x + z\nu, z > 0\}$ in the upper half plane and the following system of ODE's*

$$\mathcal{L}^0(x, i\zeta + \nu\frac{d}{dz})\tilde{y}(z) = 0 \qquad z > 0, \tag{7}$$

$$\mathcal{B}^0(x, i\zeta + \nu\frac{d}{dz})\tilde{y}(z) = 0 \qquad z = 0. \tag{8}$$

We define the vector space V of all solutions to the system (7)-(8) which are such that $\tilde{y}(z) \to 0$ as $z \to \infty$. If $V = \{0\}$, then we say that the Lopatinskii condition is fulfilled for the pair $(\mathcal{L}, \mathcal{B})$ at x.

Now, let $\mathcal{A}$ be the operator defined by

$$\mathcal{A} = (\mathcal{L}, \mathcal{B}).$$

Then the equations (6) read as $\mathcal{A}y = (\mathcal{S}, \phi)$.

Let $\mathcal{A}$ act on the space

$$D(p,l) = W_p^{l+t_1}(\Omega) \times \ldots \times W_p^{l+t_m}(\Omega)$$

with $l \geq 0$, $p > 1$. Here W_p^α denotes the standard Sobolev space with α's order partial derivatives in the L^p space. With some regularity assumptions on the coefficients of $\mathcal{L}$ and $\mathcal{B}$, $\mathcal{A}$ is bounded with range in the space

$$R(p,l) = W_p^{l-s_1}(\Omega) \times \ldots \times W_p^{l-s_m}(\Omega) \times W_p^{l-\sigma_1-\frac{1}{p}} \times \ldots \times W_p^{l-\sigma_q-\frac{1}{p}}(\partial\Omega).$$

We have the following result, see [32, Theorem 1].

Theorem 2. *Let the integers $l \geq 0, p > 1$ be given. Let $(\mathcal{S}, \phi) \in \mathcal{R}(p,l)$. Let the Douglis-Nirenberg numbers s_i and t_j be given for $\mathcal{L}$ and σ_k be as in Definition 1. Let Ω be a bounded domain with boundary in $\mathcal{C}^{l+\max t_j}$. Also assume that $p(l - s_i) > d$ and $p(l - \sigma_k) > d$ for all i and k. Let the coefficients L_{ij} be in $W_p^{l-s_i}(\Omega)$ and the coefficients of B_{kj} be in $W^{l-\sigma_k-\frac{1}{p}}$. The following statements are equivalent:*

1. *$\mathcal{L}$ is over-determined elliptic and the Lopatinskii condition is fulfilled for $(\mathcal{L}, \mathcal{B})$ on $\partial\Omega$.*

2. *There exists a left regularizer $\mathcal{R}$ for the operator $\mathcal{A} = \mathcal{L} \times \mathcal{B}$ such that*

$$\mathcal{R}\mathcal{A} = \mathcal{I} - \mathcal{T}$$

with $\mathcal{T}$ compact from $R(p,l)$ to $D(p,l)$.

3. *The following a priori estimate holds*

$$\sum_{j=1}^{m} ||y_j||_{W_p^{l+t_j}(\Omega)} \leq C_1 \left(\sum_{i=1}^{M} ||\mathcal{S}_i||_{W_p^{l-s_i}(\Omega)} + \sum_{k=1}^{Q} ||\phi_k||_{W_p^{l-\sigma_j-\frac{1}{p}}(\partial\Omega)} \right) + C_2 \sum_{t_j>0} ||y_j||_{L^p(\Omega)},$$

where y_j is the j-th component of the solution y.

4 Linear elasticity with elastic energy density measurements

Given Theorem 1 for the system (4), we chose to consider the system

$$\left\{\begin{array}{rcll} \omega^2 u_j + 2\nabla\cdot\mu\nabla^S u_j &=& -\nabla p_j & \text{in } \Omega, \\ \nabla\cdot u_j &=& 0 & \text{in } \Omega, \\ u_j &=& g_j & \text{on } \partial\Omega, \end{array}\right. \tag{9}$$

for $j = 1, \ldots, J$. We add the power density measurements:

$$\frac{\mu}{2}|\nabla^S u_j|^2 = H_j \quad \text{in } \Omega, \tag{10}$$

for $j = 1, \ldots, J$. Power density measurements are essentially a measure of the local energy of the solution, as a result of the Lebesgue differentiation theorem. An example of imaging technique that measures power densities, but under an similar scalar model, is ultrasound modulated electrical impedance tomography (UMEIT) [30].

Let $v = (\mu, \{u_j\}_{j=1}^J)$. Then the system (9)-(10) may be recast as

$$\left\{\begin{array}{rcll} \mathcal{F}v &=& \mathcal{H} & \text{in } \Omega, \\ \mathcal{B}v &=& g & \text{on } \partial\Omega. \end{array}\right. \tag{11}$$

where $\mathcal{F}$ and $\mathcal{B}$ are the differential operators defining the system (9)-(10). We know that the underlying unperturbed equations are well posed, and the main result of section 4.5 will be to provide an existence and uniqueness result for the linearisation of this equation (11). We consider the background pressure ∇p to be fixed. The stability estimates given here then would allow us to go back and solve for p as soon as u and μ are known, since by applying divergence we can determine Δp and then obtain an elliptic equation in p. We do not perform this calculation here, but it is the motivation behind our choice of model in this section.

4.1 Ellipticity arguments in dimension 2

In dimension $d = 2$, notice that $\xi \in \mathbb{R}^2$ can be written as

$$\xi = |\xi|\begin{pmatrix}\cos(\theta) \\ \sin(\theta)\end{pmatrix}$$

for some $\theta \in]-\pi, \pi]$. Moreover, the symmetric gradient of a incompressible vector valued function u satisfies

$$\nabla^S u = (\nabla^S u)^\intercal, \quad \mathrm{tr}(\nabla^S u) = 0,$$

then $\nabla^S u$ can be written as

$$\nabla^S u(x) = \frac{|\nabla^S u(x)|}{\sqrt{2}}\begin{bmatrix}\cos(\alpha(x)) & \sin(\alpha(x)) \\ \sin(\alpha(x)) & -\cos(\alpha(x))\end{bmatrix}$$

for some $\alpha(x) \in]-\pi, \pi]$. We will use these structures along the section. We also use the following notation where F is a vector or a matrix:

$$\hat{F} = \frac{F}{|F|}.$$

4.1.1 One measurement, lack of invertibility

We consider the case of dimension $d = 2$ only in this section. Consider the case $J = 1$, that is, only one measurement. Let us define $F_j = \nabla^S u_j$ and assume that $|F_j| > 0$ for all $x \in \Omega$. From equation (10) we obtain

$$\mu = \frac{2H_j}{|F_j|^2} \tag{12}$$

and then we can replace μ in equation (9) to obtain:

Lemma 2.

$$\frac{\omega^2|F_j|^2}{2H_j}u_j + \nabla^S u_j \nabla\ln(H_j) + (\mathbb{I} - 2\hat{F}_j\otimes\hat{F}_j)\nabla\otimes\nabla^S u_j = -\frac{|F_j|^2}{2H_j}\nabla p_j \tag{13}$$

where $\mathbb{I}$ *is a fourth order tensor whose entries are defined as*

$$\mathbb{I}_{ijkl} = \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}.$$

Proof. We have (dropping the sub-index j):

$$2\nabla \cdot \mu \nabla^S u + \omega^2 u + \nabla p = 0$$

where μ is given by (12). We analyze the first term in the left side of the equation, considering additionally that $\nabla \cdot u = 0$:

$$2\nabla \cdot \mu \nabla^S u = 2\mu \Delta u + 2\nabla^S u \nabla \mu = \frac{4H}{|F|^2}\Delta u - 4\nabla^S u \nabla\Big(\frac{H}{|F|^2}\Big).$$

Then we compute $\nabla\Big(\frac{H}{|F|^2}\Big)$:

$$\nabla\Big(\frac{H}{|F|^2}\Big) = \frac{1}{|F|^4}\Big(|F|^2 \nabla H - H\nabla |F|^2\Big)$$

where:

$$\nabla |F|^2 = \frac{\partial |F|^2}{\partial x_k}\hat{e}_k = 2F_{ij}\frac{\partial F_{ij}}{\partial x_k}\hat{e}_k = 2(\nabla \otimes F)F.$$

Therefore

$$2\nabla \cdot \mu \nabla^S u = \frac{2H}{|F|^2}\Delta u + \frac{2}{|F|^2}\nabla H - \frac{4H}{|F|^4}(\nabla \otimes F)F$$

and then

$$\frac{2H}{|F|^2}\Delta u + \frac{2}{|F|^2}\nabla^S u \nabla H - \frac{4H\nabla^S u}{|F|^4}(\nabla \otimes F)F + \omega^2 u + \nabla p = 0.$$

Multipyling both sides of the equation by $\frac{|F|^2}{2H}$ we obtain:

$$\Delta u + \nabla^S u \nabla \ln(H) - 2\hat{F}(\nabla \otimes F)\hat{F} + \omega^2 \frac{|F|^2}{2H} u + \frac{|F|^2}{2H}\nabla p = 0$$

Finally, we notice that

$$\Delta u - 2\hat{F}(\nabla \otimes F)\hat{F} = (\mathbb{I} - 2\hat{F} \otimes \hat{F})\nabla \otimes \nabla^S u.$$

□

Now, identifying the leading term of (13), we define the operator:

$$P_j(x, D) = \Big(\mathbb{I} - 2\hat{F}_j \otimes \hat{F}_j\Big)\nabla \otimes \nabla^S$$

and it has the symbol:

$$q_j(x,\xi) = 2(\hat{F}_j\xi) \otimes (\hat{F}_j\xi) - \frac{1}{2}\Big(|\xi|^2 I_d + (\xi \otimes \xi)\Big). \tag{14}$$

Lemma 3. *In dimension* $d = 2$*, let*

$$\xi = |\xi|\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}, \qquad \hat{F}_j(x) = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos(\alpha(x)) & \sin(\alpha(x))\\ \sin(\alpha(x)) & -\cos(\alpha(x))\end{bmatrix}. \tag{15}$$

Computing we have that

$$det(q_j(x,\xi)) = -\frac{|\xi|^4}{2}\sin^2\Big(2\theta - \alpha(x)\Big). \tag{16}$$

The conclusion is the operator is not elliptic for only one set of measurements given by (10) *with* $J = 1$*.*

Proof. In this case, we have

$$
\begin{aligned}
q_j(x,\xi) &= -\frac{1}{2}\Big(\begin{bmatrix}|\xi|^2 & 0\\ 0 & |\xi|^2\end{bmatrix} + \begin{bmatrix}\xi_1^2 & \xi_1\xi_2\\ \xi_1\xi_2 & \xi_2^2\end{bmatrix}\Big) + 2\begin{bmatrix}A^2 & AB\\ AB & B^2\end{bmatrix}\\
&= \begin{bmatrix}-\frac{1}{2}(|\xi|^2+\xi_1^2)+2A^2 & -\frac{\xi_1\xi_2}{2}+2AB\\ -\frac{\xi_1\xi_2}{2}+2AB & -\frac{1}{2}(|\xi|^2+\xi_2^2)+2B^2\end{bmatrix}
\end{aligned}
$$

where

$$A = (\hat{F}_j\xi)_1, \qquad B = (\hat{F}_j\xi)_2.$$

Then,

$$
\begin{aligned}
\det(q_j(x,\xi)) &= \frac{1}{4}(|\xi|^2+\xi_1^2)(|\xi|^2+\xi_2^2) - B^2(|\xi|^2+\xi_1^2) - A^2(|\xi|^2+\xi_2^2)\\
&\quad +4A^2B^2 - \Big(2AB - \frac{\xi_1\xi_2}{2}\Big)^2\\
&= \frac{|\xi|^4}{2} - |\xi|^2(A^2+B^2) - B^2\xi_1^2 - A^2\xi_2^2 + 2AB\xi_1\xi_2\\
&= \frac{|\xi|^4}{2} - |\xi|^2(A^2+B^2) - (A\xi_2 - B\xi_1)^2
\end{aligned}
$$

In addition, notice that using the representation (15), we have

$$A = (F_{11}\xi_1 + F_{12}\xi_2) = \frac{|\xi|}{\sqrt{2}}(\cos(\alpha)\cos(\theta)+\sin(\alpha)\sin(\theta)) = \frac{|\xi|}{\sqrt{2}}\cos(\alpha-\theta),$$

$$B = (F_{21}\xi_1 + F_{22}\xi_2) = \frac{|\xi|}{\sqrt{2}}(\sin(\alpha)\cos(\theta)+\cos(\alpha)\sin(\theta)) = \frac{|\xi|}{\sqrt{2}}\sin(\alpha-\theta).$$

So, the determinant of $q_j(x,\xi)$ can be written as

$$
\begin{aligned}
\det(q_j(x,\xi)) &= \frac{|\xi|^4}{2} - |\xi|^2\Big(\frac{|\xi|^2}{2}\cos^2(\alpha-\theta) - \frac{|\xi|^2}{2}\sin^2(\alpha-\theta)\Big) - (A\xi_2 - B\xi_1)^2\\
&= -(A\xi_2 - B\xi_1)^2\\
&= -\Big(\frac{|\xi|^2}{\sqrt{2}}\cos(\alpha-\theta)\sin(\theta) - \frac{|\xi|^2}{\sqrt{2}}\sin(\alpha-\theta)\cos(\theta)\Big)^2\\
&= -\frac{|\xi|^4}{2}\sin^2(2\theta-\alpha)
\end{aligned}
$$

and we conclude the proof of the estimate on the principal symbol. Notice that for all $\hat{F}_j(x)$ with the structure given in equation (15), the operator $P_j(x,D)$ is not elliptic, since for all $x\in\Omega$ and for all $\hat{F}_j(x)$ it is possible to find $\xi = (\cos(\alpha(x)/2), \sin(\alpha(x)/2)) \in \mathbb{S}^1$ such that $\det(q_j(x,\xi)) = 0$, i.e., $q_j(x,\xi)$ is not of full rank. ☐

Remark 4.1. *Although this result gives us an idea about the ellipticity for the equation, this is a result of the ellipticity for the operator $P_j(x,D)$. Similar problems have been studied in [30, 7], where a result says that an analogue system (in scalar case) is in fact hyperbolic. It seems natural to linearize in nonlinear models, since the problem is reduced to a linear one, and better mathematical results are known to hold. In the remaining of the article, we show results concerning to linearization of the models in study.*

4.1.2 Linearisation of the model problem for J measurements

We consider the background pressure to be fixed, and let d be the dimension which is arbitrary for this system. The linearized problem for $j \in \{1, \dots, J\}$ is then given by

$$\begin{cases} 2\nabla \cdot \delta\mu \nabla^S u_j + 2\nabla \cdot \mu \nabla^S \delta u_j + \omega^2 \delta u_j &= 0 & \text{in } \Omega, \\ \dfrac{\delta\mu}{2} |\nabla^S u_j|^2 + \mu \nabla^S u_j : \nabla^S \delta u_j &= \delta H_j & \text{in } \Omega, \\ \nabla \cdot \delta u_j &= 0 & \text{in } \Omega, \\ \delta u_j &= 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

We make the definition $w = (\delta\mu, \{\delta u_j\}_{j=1}^J)$ which allows us to re-write the system as:

$$\begin{cases} \mathcal{L}w &= \mathcal{S} & \text{in } \Omega, \\ \mathcal{B}w &= g & \text{on } \partial\Omega. \end{cases} \tag{18}$$

The principal symbol associated to (17) is, rearranging rows, the following:

$$\mathcal{P}_J(x,\xi) =$$

$$\begin{bmatrix} \frac{|F_1|^2}{2} & i\mu(F_1\xi)^\intercal & 0 & \cdots & 0 \\ 2iF_1\xi & -\mu\Big(|\xi|^2 I_d + (\xi\otimes\xi)\Big) & 0 & \cdots & 0 \\ 0 & i\xi^\intercal & 0 & \cdots & 0 \\ \frac{|F_2|^2}{2} & 0 & i\mu(F_2\xi)^\intercal & \cdots & 0 \\ 2iF_2\xi & 0 & -\mu\Big(|\xi|^2 I_d + (\xi\otimes\xi)\Big) & \cdots & 0 \\ 0 & 0 & i\xi^\intercal & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{|F_J|^2}{2} & 0 & 0 & \cdots & i\mu(F_J\xi)^\intercal \\ 2iF_J\xi & 0 & 0 & \cdots & -\mu\Big(|\xi|^2 I_d + (\xi\otimes\xi)\Big) \\ 0 & 0 & 0 & \cdots & i\xi^\intercal \end{bmatrix}$$

which is a matrix of size $J(d+2) \times (Jd+1)$. We can recognize the following family of submatrices

$$\rho_j(x,\xi) = \begin{bmatrix} \frac{|F_j|^2}{2} & i\mu(F_j\xi)^\intercal \\ 2iF_j\xi & -\mu\Big(|\xi|^2 I_d + (\xi\otimes\xi)\Big) \end{bmatrix} \tag{19}$$

and we have from the formulas for the determinant of block matrices that (see, for example, Section 6.2 in [23]):

$$\det\big(\rho_j(x,\xi)\big) = 2^{d-1}\mu^d |F_j|^2 \det\big(q_j(x,\xi)\big), \tag{20}$$

where q_j is defined in (14). Note that Lemma 3 now says that the linearized operator $\mathcal{L}$ is not elliptic. On the other hand, if we take determinant for the submatrices with the rows containing the highest power of ξ in $\mathcal{P}_j$, we obtain, by applying properties for determinant of block matrices, the following:

$$(-1)^{(J-1)d} \frac{\mu^{Jd}}{2^{(J-1)d}} |F_j|^2 \det\Big(|\xi|^2 I_d + \xi\otimes\xi\Big)^{J-1} \det\big(q_j(x,\xi)\big).$$

Definition 3. *We say that a family $\{Op(\rho_j(x,\xi))\}_{j=1}^J$ of operators is elliptic if $\rho_j(x,\xi)$ is invertible for all $x \in \Omega$ and all $j = 1, \dots, J$ implies $\xi = 0$.*

This definition is inspired by the one in [10], Definition 2.1.

Lemma 4. *If $\{\rho_j\}$ forms an elliptic family and $|F_j| > 0$ for all $x \in \Omega$ and $j = 1, ...J$, then the full linearized operator $\mathcal{L}(x,\xi)$ is elliptic.*

Proof. Let C_0 and $\{C_j\}_{j=1}^J$ be the submatrices of $\mathcal{P}_J$ defined by

$$C_0 = \begin{pmatrix} |F_1|^2 \\ 2iF_1\xi \\ 0 \\ |F_2|^2 \\ 2iF_2\xi \\ 0 \\ \vdots \\ |F_J|^2 \\ 2iF_J\xi \\ 0 \end{pmatrix}, \qquad C_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 2i\mu(F_j\xi)^\intercal \\ -\mu(|\xi|^2 I_d + \xi\otimes\xi) \\ i\xi^\intercal \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{row } ((j-1)(d+2)+1)$$

where $C_0 \in \mathcal{M}_{J(d+2)\times 1}(\mathbb{C})$ and $C_j \in \mathcal{M}_{J(d+2)\times d}(\mathbb{C})$ for $j = 1, \ldots, J$.
Let $\xi \neq 0$. Then we can see easily that $-\mu(|\xi|^2 + \xi\otimes\xi)$ is invertible, hence C_j has complete column rank. In addition, if $j_1 \neq j_2$, then C_{j_1} and C_{j_2} do not have the same nonzero rows.
If $\mathcal{L}(x,\xi)$ is not full rank, then it is clear that there exists j_0 and $\alpha_{j_0} \in \mathbb{R}^d\backslash\{0\}$ such that in the nonzero rows of C_{j_0} we have

$$\begin{pmatrix} |F_{j_0}|^2 \\ 2i\mu F_{j_0}\xi \\ 0 \end{pmatrix} = \begin{pmatrix} 2i\mu(F_{j_0}\xi)^\intercal \\ -\mu(|\xi|^2 I_d + \xi\otimes\xi) \\ i\xi^\intercal \end{pmatrix} \begin{pmatrix} \alpha_{j_0 1} \\ \vdots \\ \alpha_{j_0 d} \end{pmatrix}$$

and then we have that $\xi^\intercal \alpha_{j_0} = 0$ and

$$\begin{pmatrix} |F_{j_0}|^2 & 2i\mu(F_{j_0}\xi)^\intercal \\ 2i\mu F_{j_0}\xi & -\mu(|\xi|^2 I_d + \xi\otimes\xi) \end{pmatrix} \begin{pmatrix} -1 \\ \alpha_{j_0 1} \\ \vdots \\ \alpha_{j_0 d} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

That is, $\rho_{j_0}(x,\xi)$ is not invertible. □

Theorem 3. *For $J = 2$, $d = 2$, if $\alpha_2(x) \neq \alpha_1(x) + k\pi$ for all $k \in \mathbb{Z}$ and for all $x \in \Omega$, then the differential operator corresponding with the system* (17) *is elliptic.*

Proof. We have to prove that

$$\det(q_j(x,\xi)) = 0 \;\; \forall j \;\Rightarrow \xi = 0$$

since equation (20) establishes that $\rho_j(x,\xi)$ is invertible if and only if $q_j(x,\xi)$ is invertible.
If $\det(q_j(x,\xi)) = 0$ for $j = 1, 2$, then we have

$$\sin(2\theta - \alpha_1(x)) = 0 \quad \wedge \quad \sin(2\theta - \alpha_2(x)) = 0 \tag{21}$$

or

$$\xi = 0$$

but (21) implies

$$\alpha_2 = \alpha_1 + k\pi \quad \text{for some } k \in \mathbb{Z}$$

which is false by hypothesis. So we conclude that $\xi = 0$. That is, (q_1, q_2) forms an elliptic family. We conclude the proof using Lemma 4. □

4.2 Lopatinskii condition

We prove now the following in dimension $d = 2$.

Lemma 5. *Consider $v = (\mu, \{u_j\}_{j=1,\ldots,J})$. Let $x \in \partial\Omega$, ν the outward unit normal to Ω at x, and $\zeta \in \mathbb{S}^{d-1}$ satisfying $\zeta\cdot\nu = 0$. Define $\tilde{v}(z) = v(x - \nu z)$. Then the only solution of the system of ODEs*

$$\begin{cases} \mathcal{P}_J(x, i\zeta + \nu\partial_z)\tilde{v} &= 0, \quad z > 0, \\ \mathcal{B}\tilde{v} &= 0, \quad z = 0, \end{cases} \tag{22}$$

such that $\tilde{v}(z) \to 0$ *as* $z \to \infty$ *is* $\tilde{v} \equiv 0$.

Proof. The system can be seen as the following

$$\begin{cases} |F_j|^2\tilde{\mu} + 2\mu\Big(F_j[i\zeta + \nu\partial_z]\Big)^{\intercal}\tilde{u}_j = 0, \quad z > 0, \\ F_j[i\zeta + \nu\partial_z]\tilde{\mu} - \dfrac{\mu}{2}\Big((i\zeta + \nu\partial_z)^2 I_d + (i\zeta + \nu\partial_z) \otimes (i\zeta + \nu\partial_z)\Big)\tilde{u}_j = 0, \quad z > 0, \\ i(i\eta + \nu\partial_z)^{\intercal}\tilde{u}_j = 0, \quad z > 0, \\ \tilde{u} = 0, \quad z = 0, \end{cases} \tag{23}$$

for all $j = 1, \dots J$.
We can eliminate $\tilde{\mu}$ using the first equation

$$\tilde{\mu} = -\frac{2\mu}{|F_j|^2}\Big(F_j[i\zeta + \nu\partial_z]\Big)^{\intercal}\tilde{u}_j. \tag{24}$$

Replacing it in the second equation, after some calculations we have

$$q_j(x,\nu)\partial_z^2\tilde{u}_j + ir_j(x,\nu,\zeta)\partial_z\tilde{u}_j + s_j(x,\zeta)\tilde{u}_j = 0 \tag{25}$$

for all $j = 1, \dots, J$, where q_j is the same matrix of previous sections, and r_j, s_j are real matrices given by

$$r_j(x,\nu,\zeta) = 2(\hat{F}_j\nu \otimes \hat{F}_j\zeta + \hat{F}_j\zeta \otimes \hat{F}_j\nu) - \frac{1}{2}(\nu \otimes \zeta + \zeta \otimes \nu), \quad s_j(x,\zeta) = -q_j(x,\zeta).$$

We look the imaginary part of (25):

$$r_j\partial_z\tilde{u}_j = 0, \quad z > 0.$$

After some calculations (see Lemma 6), we have

$$\det(r_j) \neq 0$$

so we have

$$\partial_z\tilde{u}_j = 0$$

and this implies $\tilde{u}_j \equiv 0$ since $\tilde{u}(0) = 0$. Then using (38) we obtain $\tilde{\mu} \equiv 0$. Therefore we conclude $\tilde{v} \equiv 0$. □

Lemma 6. *In dimension* $d = 2$, *we have* $det(r_j(x,\nu,\zeta)) \neq 0$.

Proof. We have

$$r_j(x,\nu,\zeta) \quad = \quad M + N$$

where

$$M \quad = \quad \begin{bmatrix} 2AC & AD + BC \\ AD + BC & 2BD \end{bmatrix},$$

$$N \quad = \quad -\frac{1}{2}\begin{bmatrix} 2\nu_1\zeta_1 & \nu_1\zeta_2 + \zeta_1\nu_2 \\ \nu_1\zeta_2 + \zeta_1\nu_2 & 2\nu_2\zeta_2 \end{bmatrix}$$

and

$$A = (\hat{F}\nu)_1, \quad B = (\hat{F}\nu)_2, \quad C = (\hat{F}\zeta)_1, \quad D = (\hat{F}\zeta)_2.$$

Since $\nu \cdot \zeta = 0$, without loss of generality we can take $\zeta_1 = -\nu_2$ and $\zeta_2 = \nu_1$, and using the properties of $\hat{F}_j$ we have

$$C = (\hat{F}_j)_{11}\zeta_1 + (\hat{F}_j)_{12}\zeta_2 = -(\hat{F}_j)_{11}\nu_2 + (\hat{F}_j)_{12}\nu_1 = B,$$

$$D = (\hat{F}_j)_{21}\zeta_1 + (\hat{F}_j)_{22}\zeta_2 = -(\hat{F}_j)_{21}\nu_2 + (\hat{F}_j)_{22}\nu_1 = -A.$$

Then

$$r_j = \begin{bmatrix} 4AB + \nu_1\nu_2 & 2(B^2 - A^2) - \frac{1}{2}(\nu_1^2 - \nu_2^2) \\ 2(B^2 - A^2) - \frac{1}{2}(\nu_1^2 - \nu_2^2) & -(4AB + \nu_1\nu_2) \end{bmatrix}$$

and we can compute the determinant

$$-\det(r_j) = \Big(4AB + \nu_1\nu_2\Big)^2 + \Big(2(B^2 - A^2) - \frac{1}{2}(\nu_1^2 - \nu_2^2)\Big)^2. \tag{26}$$

Using the fact that $\nabla^S u_j$ are divergence free, we have

$$A = \frac{1}{\sqrt{2}}\cos(\alpha_j - \theta), \quad B = \frac{1}{\sqrt{2}}\sin(\alpha_j - \theta)$$

where $\theta = \arg(\nu)$, so that $\nu = (\cos(\theta), \sin(\theta))$. Then

$$\begin{aligned}
-\det(r_j) &= \Big(2\cos(\alpha_j - \theta)\sin(\alpha_j - \theta) + \cos(\theta)\sin(\theta)\Big)^2 \\
&\qquad + \Big((\cos^2(\alpha_j - \theta) - \sin^2(\alpha_j - \theta)) + \frac{\cos^2(\theta) - \sin^2(\theta)}{2}\Big)^2 \\
&= \Big(\sin(2(\alpha_j - \theta)) + \frac{\sin(2\theta)}{2}\Big)^2 + \Big(\cos(2(\alpha_j - \theta)) + \frac{\cos(2\theta)}{2}\Big)^2 \\
&= \frac{5}{4} + \cos(2\alpha_j - 3\theta) \\
&\neq 0 \qquad \forall x, \nu, \zeta.
\end{aligned}$$

□

Remark 4.2. *It should be possible to show the theorem holds under weaker assumptions given the form of the determinant* (26)*.*

4.3 Stability estimates

In any dimension d with J measurements, we can see the problem (17) in the framework of Section 3. The Douglis-Niernberg numbers are

$$s_i = \begin{cases} -1 & \text{if } i = k' \cdot (d+2) + k'',\ k' = 0, 1, \dots, J,\ k'' = 0, 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$t_j = \begin{cases} 1 & \text{if } j = 1, \\ 2 & \text{otherwise ,} \end{cases}$$

$$\sigma_k = -1,\ k = 1, \dots, Jd.$$

where $i = 1, \dots, J(d+2)$ and $j = 1, \dots, Jd+1$. The operator $\mathcal{A} = (\mathcal{L}, \mathcal{B})$ is defined from

$$\mathcal{X} = \prod_{j=1}^{Jd+1} H^{l+t_j}(\Omega)$$

to

$$\mathcal{Y} = \prod_{i=1}^{J(d+2)} H^{l-s_i}(\Omega) \times \prod_{j=1}^{Jd} H^{l-\sigma_j-1/2}(\partial\Omega)$$

where we choose l such that $2(l - s_i) > d$, $2(l - \sigma_k) > d$. In dimension $d = 2$, we can choose $l = 2$. Moreover, if $d = 2$ and $J = 2$, the we have

$$\mathcal{X} = H^3(\Omega) \times \Big(H^4(\Omega)^2\Big)^2 \tag{27}$$

with norm

$$||(\delta\mu, \{\delta u_j\}_{j=1}^J)||_{\mathcal{X}} = ||\delta\mu||_{H^3(\Omega)} + \sum_{j=1}^{J} ||\delta u_j||_{H^4(\Omega)^2}$$

and

$$\mathcal{Y} = \Big(H^3(\Omega) \times H^2(\Omega)^2 \times H^3(\Omega)\Big)^2 \times \Big(H^{5/2}(\partial\Omega)^2\Big)^2 \tag{28}$$

with norm

$$||(\{\delta f_j^{pd}\}_{j=1}^J, \{\delta f_j^{ec}\}_{j=1}^J, \{\delta f_j^{div}\}_{j=1}^J, \{\delta g_j\}_{j=1}^J)||_{\mathcal{Y}}$$

$$= \sum_{j=1}^J ||\delta f_j^{pd}||_{H^3(\Omega)} + ||\delta f_j^{ec}||_{H^2(\Omega)^2} + ||\delta f_j^{div}||_{H^3(\Omega)} + ||\delta g_j||_{H^{5/2}(\partial\Omega)^2}.$$

Theorem 4. *Let $d = 2, J = 2$, we have the estimate for $w = (\delta\mu, \{\delta u_j\}_{j=1}^J)$ a solution to* (17)

$$\begin{aligned} ||\delta\mu||_{H^3(\Omega)} + \sum_{j=1}^2 ||\delta u_j||_{H^4(\Omega)^2} \le C \sum_{j=1}^J \Big(||\mathcal{L}_j^{ec}(\delta\mu, \delta u_j)||_{H^2(\Omega)^2} + ||\mathcal{L}_j^{pd}(\delta\mu, \delta u_j)||_{H^3(\Omega)} \\ + ||\mathcal{L}^{div}(\delta\mu, \delta u_j)||_{H^3(\Omega)} + ||\mathcal{B}\delta u_j||_{H^{5/2}(\Omega)^2}\Big) \\ + C_2\Big(||\delta\mu||_{L^2(\Omega)^2} + \sum_{j=1}^J ||\delta u_j||_{L^2(\Omega)^2}\Big) \end{aligned} \tag{29}$$

where $\mathcal{L}_j^{ec}, \mathcal{L}_j^{pd}, \mathcal{L}^{div}$ are the parts of $\mathcal{L}$ coming from the elasticity equations, the power density measurements and the divergence condition, respectively. If $C_2 = 0$, then the inverse operator is locally well-defined.

Proof. Since $(\mathcal{L}, \mathcal{B})$ satisfies the Lopatinskii condition, by Theorem 2 we have the estimate

$$||w||_{\mathcal{X}} \le C||(\mathcal{S}, g)||_{\mathcal{Y}} + C_2||w||_{L^2(\Omega)^{d\cdot J}}. \tag{30}$$

If $C_2 = 0$, then the inverse operator is locally well-defined. We remark that in dimension 2, we can choose $l = 2$. □

4.4 Injectivity

Lemma 7. *Let the dimension be $d = 2$. The boundary value problem given by*

$$\begin{cases} \tilde{\mathcal{L}}_j\delta u_j := -2\nabla\cdot\Big(\frac{2\mu}{|F_j|^2}(F_j : \nabla^S\delta u_j)F_j\Big) + 2\nabla\cdot\mu\nabla^S u_j + \omega^2\delta u_j = \tilde{f} \quad \text{in } \Omega, \\ \tilde{\mathcal{B}}_j\delta u_j := \delta u_j = \tilde{g}_j \quad \text{on } \partial\Omega, \end{cases} \tag{31}$$

is elliptic. In addition, we have:

$$\sum_{j=1}^2 ||\delta u_j||_{H^4(\Omega)^2} \le \tilde{C}\sum_{j=1}^J \Big(||\tilde{\mathcal{L}}_j\delta u_j||_{H^2(\Omega)^2} + ||\tilde{\mathcal{B}}_j\delta u_j||_{H^{5/2}(\Omega)^2}\Big) + \tilde{C}_2\sum_{j=1}^J ||\delta u_j||_{L^2(\Omega)^2}. \tag{32}$$

Proof. In fact, the principal symbol of the operator corresponding with this equation is $c \cdot q_j(x, \xi)$ where c is a constant, hence the ellipticity of the operator is given by Theorem 3. The Lopatinskii condition is given by the proof of Lemma 5. Therefore we have the proposed estimate by Theorem 2. □

Lemma 8. *Let $\mathcal{A}$ be the operator corresponding to the equation given in the previous lemma. Let the dimension be 2. If $\delta u_j \in ker(\tilde{\mathcal{L}}_j, \tilde{\mathcal{B}}_j)$, then*

$$\int_\Omega |\delta u_j|^2 \le \frac{2||\mu||_{L^\infty}^2}{\omega^2}\int_\Omega |\nabla(\delta u_j)|^2. \tag{33}$$

Proof. Multiplying the equation (31) by δu_j, and integrating by parts we have

$$\int_\Omega 2\mu|\nabla^S \delta u_j : \hat{F}_j|^2 dx - \int_\Omega \mu|\nabla^S \delta u_j|^2 + \omega^2 \int_\Omega |\delta u_j|^2 dx = 0. \tag{34}$$

On the other hand, let $F_j^\perp$ be such that $F_j : F_j^\perp = 0$ and $|F_j^\perp| = |F_j|$. Then $\nabla^S \delta u_j$ can be expressed as

$$\nabla^S \delta u_j = (\nabla^S \delta u_j : \hat{F}_j)\hat{F}_j + (\nabla^S \delta u_j : \hat{F}_j^\perp)\hat{F}_j^\perp \tag{35}$$

and then

$$\int_\Omega \mu|\nabla^S \delta u_j|^2 = \int_\Omega \mu\Big(|\nabla^S \delta u_j : \hat{F}_j|^2 + |\nabla^S \delta u_j : \hat{F}_j^\perp|^2\Big)dx. \tag{36}$$

Summing (34) and (36) we have

$$\begin{aligned}
\omega^2 \int_\Omega |\delta u_j|^2 dx &= \int_\Omega \mu\Big(|\nabla^S \delta u_j : \hat{F}_j^\perp|^2 - |\nabla^S \delta u_j : \hat{F}_j|^2\Big)dx \\
&\leq \int_\Omega \mu\Big(|\nabla^S \delta u_j : \hat{F}_j|^2 + |\nabla^S \delta u_j : \hat{F}_j^\perp|^2\Big)dx \\
&= \int_\Omega \mu|\nabla^S \delta u_j|^2 dx
\end{aligned}$$

and we conclude noticing that

$$\begin{aligned}
|\nabla^S \delta u_j|^2 &= \frac{1}{4}\Big|\nabla \delta u_j + \nabla \delta u_j^\intercal\Big|^2 \\
&= \frac{1}{4}\Big(|\nabla \delta u_j|^2 + |\nabla \delta u_j^\intercal|^2 + 2\nabla \delta u_j : \nabla \delta u_j^\intercal\Big) \\
&\leq \frac{1}{4}\Big(2|\nabla \delta u_j|^2 + 2|\nabla \delta u_j^\intercal|^2\Big) \\
&= |\nabla \delta u_j|^2.
\end{aligned}$$

□

Lemma 9. *In dimension $d = 2$, there exists $\omega_0 > 0$ such that $\forall \omega \geq \omega_0$ we have $ker(\tilde{\mathcal{L}}_j, \tilde{\mathcal{B}}_j) = \{0\}$ for all j. In other words, the operator $(\tilde{\mathcal{L}}, \tilde{\mathcal{B}})$ is injective, where $\tilde{\mathcal{L}} = \{\tilde{\mathcal{L}}_j\}_{j=1}^J$ and $\tilde{\mathcal{B}} = \{\tilde{\mathcal{B}}_j\}_{j=1}^J$.*

Proof. From Lemma 7 with $(\tilde{\mathcal{L}}_j w, \tilde{\mathcal{B}}_j \delta u_j) = (0, 0)$ for all j and from Lemma 8, we have

$$\sum_{j=1}^J ||\delta u_j||_{H^4(\Omega)^2} \leq \tilde{C}_2 \sum_{j=1}^J ||\delta u_j||_{L^2(\Omega)^2} \leq \frac{\tilde{C}_2||\mu||_{L^\infty(\Omega)}}{\omega} \sum_{j=1}^J ||\nabla \delta u_j||_{L^2(\Omega)^2}.$$

If we take ω large enough such that $\tilde{C}_2||\mu||_{L^\infty} < \omega$, we can absorb the right side of the estimate into the left hand side. So we conclude $\delta u_j = 0$. □

As a result we have the following result.

Theorem 5. *Let $d = 2, J = 2$, $\omega \geq \omega_0$ as in the previous lemma and the hypothesis of Theorem 4. Then we have the estimate for $(\delta\mu, \delta u_j)$ a solution to* (17)

$$\begin{aligned}
||\delta\mu||_{H^3(\Omega)} + \sum_{j=1}^2 ||\delta u_j||_{H^4(\Omega)^2} \leq C \sum_{j=1}^2 \Big(&||\mathcal{L}_j^{ec}(\delta\mu, \delta u_j)||_{H^2(\Omega)^2} + ||\mathcal{L}_j^{pd}(\delta\mu, \delta u_j)||_{H^3(\Omega)} \\
&+ ||\mathcal{L}^{div}(\delta\mu, \delta u_j)||_{H^3(\Omega)} + ||\mathcal{B}\delta u_j||_{H^{5/2}(\Omega)^2}\Big).
\end{aligned} \tag{37}$$

Proof. Considering equation (17) with the right hand side equal to zero, we can take the second equation and obtain

$$\delta\mu = -\frac{2\mu}{|F_j|^2} F_j : \nabla^S \delta u_k. \tag{38}$$

Then we replace $\delta\mu$ in the first equation, so we obtain the equation (31). By Lemma 9 we obtain $\delta u_j = 0$ and using equation (38) we conclude $\delta\mu = 0$. Hence, we can eliminate the terms multiplying C_2 in equation (29), which is valid because we have the hypothesis of Theorem 4. □

4.5 Fixed-point algorithm

We introduce the general fixed point Lemmas which are needed to solve nonlinear PDE with small data. Let J be a linear operator, and N a power nonlinearity. We view the nonlinear PDE as

$$\begin{aligned} &J(w) = N(w) \quad \text{in}\, \Omega, \\ &w = f \quad \text{in}\ \Omega, \\ &w = 0 \quad \text{on}\ \partial\Omega. \end{aligned}$$

The solution then looks like

$$w = w_{lin} + J^{-1} N(w). \tag{39}$$

We also have the following abstract iteration result:

Lemma 10. *[[29] Prop 1.38] Let $\mathcal{N}, \mathcal{S}$ be two Banach spaces and suppose we are given an invertible linear operator $J : \mathcal{N} \to \mathcal{S}$ with the bound*

$$||J^{-1}F||_{\mathcal{S}} \le C_0 ||F||_{\mathcal{N}} \tag{40}$$

for all $F \in \mathcal{N}$ and some $C_0 > 0$. Suppose that we are given a nonlinear operator $N : \mathcal{S} \to \mathcal{N}$ which is a sum of a u dependent part and a u independent part. Assume the u dependent part N_u is such that $N_u(0) = 0$ and obeys the following Lipschitz bounds

$$||N(u) - N(v)||_{\mathcal{N}} \le \frac{1}{2C_0} ||u - v||_{\mathcal{S}} \tag{41}$$

for all $u, v \in B_\epsilon = \{u \in \mathcal{S} : ||u||_S \le \epsilon\}$ for some $\epsilon > 0$. In other words we have that $||N||_{\dot{C}^{0,1}(B_\epsilon \to \mathcal{N})} \le \frac{1}{2C_0}$. Then, for all $u_{lin} \in B_{\epsilon/2}$ there exists a unique solution $u \in B_\epsilon$ with the map $u_{lin} \mapsto u$ Lipschitz with constant at most 2. In particular we have that

$$||u||_{\mathcal{S}} \le 2||u_{lin}||_{\mathcal{S}}. \tag{42}$$

Remark 4.3. *The proof of Lemma 10 consists in establishing the convergence of the following iterative sequence:*

$$u^{(n)} = \begin{cases} u_{lin} & \text{if } n = 0, \\ u_{lin} + J^{-1} N(u^{(n-1)}) & \text{if } n \ge 1. \end{cases}$$

Therefore, the Lemma 10 also establishes the convergence of this kind of sequences.

Given the abstract convergence Lemma above, we want to apply this to the linearised elasticity problem to give a direct proof of existence and uniqueness to the system (17).

We set the following notation:

- $v_j = (\mu, \{u_j\}_j)$ and $v = \{v_j\}_{j=1}^J$.
- Also, $v = v_0 + \delta v$, where $v_0 = (\mu_0, \{u_{0,j}\}_{j=1}^J) = \{v_j\}_{j=1}^J$.
- $\delta v = (\delta\mu, \{\delta u_j\}_j) = \{w_j\}_{j=1}^J = w$.

- $\mathcal{F}(v_j) = \begin{pmatrix} \frac{\mu}{2}|\nabla^S u_j|^2 \\ \\ 2\nabla\cdot\mu\nabla^S u_j + \omega^2 u_j \\ \\ \nabla\cdot u_j \end{pmatrix}$, $\mathcal{H}_j = \begin{pmatrix} H_j \\ G_j \\ 0 \end{pmatrix}$, $\mathcal{B}v_j = g_j$.
- $\mathcal{F}v = \{\mathcal{F}v_j\}_{j=1}^J$, $\mathcal{H} = \{\mathcal{H}_j\}_{j=1}^J$, $\mathcal{B}v = \{\mathcal{B}v_j\}_{j=1}^J$.
- $\mathcal{L}_j = \mathcal{F}'(v_{0j})$, that is,

$$\mathcal{L}_j w_j = \mathcal{F}'(v_{0j})w_j = \begin{pmatrix} \frac{\delta\mu}{2}|\nabla^S u_{0j}|^2 + \mu\nabla^S u_{0j} : \nabla^S\delta u_{0j} \\ \\ 2\nabla\cdot\delta\mu\nabla^S u_{0j} + 2\nabla\cdot\mu\nabla^S\delta u_{0j} + \omega^2\delta u_{0j} \\ \\ \nabla\cdot\delta u_{0j} \end{pmatrix}.$$

- $\mathcal{S}_j = \begin{pmatrix} \delta H_j \\ \delta G_j \\ 0 \end{pmatrix}$.
- $\mathcal{L}w = \{\mathcal{L}_j w_j\}_{j=1}^J$, $\mathcal{S} = \{\mathcal{S}_j\}_{j=1}^J$.
- $\mathcal{H}_0 := \mathcal{F}(v_{0j})$, $g_0 = \mathcal{B}v_0$.

And consider the following nonlinear problem:

$$\left\{ \begin{array}{rcll} \mathcal{F}(v_0 + w) & = & \mathcal{H} & \text{in } \Omega, \\ \mathcal{B}w & = & g - g_0 & \text{on } \partial\Omega, \end{array} \right. \tag{43}$$

and the linear problem

$$\left\{ \begin{array}{rcll} \mathcal{L}w & = & \mathcal{S} & \text{in } \Omega, \\ \mathcal{B}w & = & g - g_0 & \text{on } \partial\Omega. \end{array} \right. \tag{44}$$

The system (44) can be written as

$$\mathcal{A}w = \begin{pmatrix} \mathcal{S} \\ g - g_0 \end{pmatrix}. \tag{45}$$

Note that

$$\mathcal{F}(v_0 + w) = \mathcal{F}(v_0) + \mathcal{F}'(v_0)w + \mathcal{G}(w; v_0)$$

where $\mathcal{G}(w; v_0)$ is given by

$$\mathcal{G}_j(w; v_0) = \begin{pmatrix} \delta\mu\nabla^S u_{0j} : \nabla^S\delta u_j + \frac{(\mu_0 + \delta\mu)}{2}|\nabla^S\delta u_j|^2 \\ \\ 2\nabla\cdot\delta\mu\nabla^S\delta u_j \\ \\ 0 \end{pmatrix} \tag{46}$$

is such that

$$||(\mathcal{G}(w; v_0)||_{\mathcal{Y}} \leq C||w||_{\mathcal{X}}^2 \tag{47}$$

where the constant C depends only on the $L^\infty(\Omega)$ norm of $|\nabla^S u_j|$ and μ for $j = 1, 2$ so that we can write the problem as

$$\left\{ \begin{array}{rcll} \mathcal{L}w & = & \mathcal{H} - \mathcal{H}_0 - \mathcal{G}(w; v_0) & \text{in } \Omega, \\ \mathcal{B}w & = & g - g_0 & \text{on } \partial\Omega. \end{array} \right. \tag{48}$$

We define the following fixed point Algorithm:

4.5.1 Algorithm 1:

Input.

- A function $v_0 = (\mu_0, \{u_{0j}\})$, where μ_0 is given and then $u_{0,j}$ is the solution of the system:

$$\begin{cases} 2\nabla \cdot \mu_0 \nabla^S u_j + \omega^2 u_j &= -\nabla p_j \quad \text{in } \Omega, \\ \nabla \cdot u_j &= 0 \quad \text{in } \Omega, \\ u_j &= g_j \quad \text{on } \partial\Omega. \end{cases} \tag{49}$$

- Observations $\mathcal{H}$ in Ω and boundary information g on $\partial\Omega$, i.e., $\mathcal{H} = \mathcal{F}(v_0 + w_{true})$ and $g = g_0 + \mathcal{B}w_{true}$.
- A tolerance $\varepsilon > 0$.

Steps.

- Compute $\mathcal{H}_0$ via the formula $\mathcal{H}_0 = \mathcal{F}(v_0)$.
- Define $w^0 = 0$.
- Iterations, from k to $k+1$:
  - $w^{k+1} = \mathcal{I}(w^k) := \mathcal{A}^{-1}(\mathcal{H} - \mathcal{H}_0 - \mathcal{G}(w^k; v_0), g - g_0)$,
  - Stop if $||w^{k+1} - w^k|| < \varepsilon$.
- Define $v = v_0 + w^{k+1}$

Return v

4.5.2

Lemma 11. *There exist a constant $c_1 = c_1(\varepsilon) > 0$ such that*

$$||\mathcal{G}(w; v_0) - \mathcal{G}(\tilde{w}; v_0)||_{\mathcal{Y}} \leq c_1 \Big(||\delta\mu - \delta\tilde{\mu}||_{H^3(\Omega)} + \sum_j ||\delta u_j - \delta\tilde{u}_j||_{(H^4(\Omega))^2} \Big) \tag{50}$$

provided $||\delta\mu||_{H^3(\Omega)}, ||\delta u_j||_{H^4(\Omega)^2} \leq \varepsilon$, for some $\varepsilon > 0$. Such a constant satisfies $c_1(\varepsilon) \to 0$ whenever $\varepsilon \to 0$.

Proof. The definition of $\mathcal{G}_j(w, v_0)$ in (46), implies $\mathcal{G}_j(w, v_0)$ is a differentiable function of w. The mean value theorem gives the result. Alternatively, using that $H^2(\Omega)^d$ and $H^3(\Omega)^d$ are Banach algebras gives a bound for c_1:

$$c_1 \leq C_{BA}\varepsilon \Big(JC_{BA} \max_j ||u_{0j}||_{H^4(\Omega)^d} + J||\mu_0||_{H^3(\Omega)} + 5\varepsilon \Big)$$

with $C_{BA} > 0$ the constant from the bound given by the fact $H^2(\Omega)$ and $H^3(\Omega)$ are Banach algebras, c.f. [11] Theorem 6.1-4. □

Theorem 6. *If $\varepsilon > 0$ is sufficiently small so that*

$$c_1(\varepsilon)||\mathcal{A}^{-1}||_{\mathcal{L}(\mathcal{Y},\mathcal{X})} < \frac{1}{2}$$

where $c_1(\varepsilon)$ is given by the previous Lemma. Then the algorithm converges if in addition we have

$$||(\mathcal{H} - \mathcal{H}_0, g - g_0)||_{\mathcal{X}} \leq \frac{\varepsilon}{2},$$

and we obtain

$$||w||_{\mathcal{X}} < \varepsilon. \tag{51}$$

Proof. We take

$$J = \mathcal{A}, \quad N(w) = (\mathcal{G}(w; v_0), 0), \quad w_{lin} = (\mathcal{H} - \mathcal{H}_0, g - g_0).$$

Because the nonlinearity satisfies the conditions for the fixed point iteration by Lemma 11 application of the previous convergence Lemma 10 gives the desired result. □

Remark 4.4. *Note that the bound on $\mathcal{A}^{-1}\tilde{w}$ can be made precise by taking the constant from* (37)*, with $w = \mathcal{A}^{-1}\tilde{w}$, but it depends on the constant C appearing in Theorem 2.*

5 Model with generic forcing term $f(u)$

Let $f \in \mathcal{C}^3(H^3(\mathbb{R}^d)^d, L^2(\mathbb{R}^d)^d)$ be a differentiable function whose symbol is a polynomial with degree at most 1. The model studied in this section is

$$\left\{\begin{array}{rcll} 2\nabla\cdot\mu\nabla^S u_j + \omega^2 u_j - f(u_j) & = & -\nabla p_j & \text{in } \Omega, \\ \dfrac{\mu}{2}|\nabla^S u_j|^2 - f(u_j)\cdot u_j & = & H_j & \text{in } \Omega, \\ \nabla\cdot u_j & = & 0 & \text{in } \Omega, \\ u_j & = & g_j & \text{on } \partial\Omega, \end{array}\right. \tag{52}$$

where $j = 1,\ldots,J$. The motivation for considering the term $f(u_j)$ is to have a first intuition on more general nonlinear elasticity models in dimension $d = 2$. In [31], a simplified nonlinear elasticity model is studied in dimension $d = 3$ with scalar valued functions.

The system (52) can be written as

$$\left\{\begin{array}{rcll} \mathcal{F}_{FT} v & = & \mathcal{H} & \text{in } \Omega, \\ \mathcal{B} v & = & g & \text{on } \partial\Omega, \end{array}\right. \tag{53}$$

where $v = (\mu, \{u_j\}_{j=1}^J)$. The linearized problem for $j \in \{1,\ldots,J\}$ is then given by

$$\left\{\begin{array}{rcll} 2\nabla\cdot\delta\mu\nabla^S u_j + 2\nabla\cdot\mu\nabla^S\delta u_j + \omega^2\delta u_j & = & Df(u_j)\delta u_j & \text{in } \Omega, \\ W_j[\delta\mu,\delta u_j] & = & \delta H_j & \text{in } \Omega, \\ \nabla\cdot\delta u_j & = & 0 & \text{in } \Omega, \\ \delta u_j & = & 0 & \text{on } \partial\Omega, \end{array}\right. \tag{54}$$

where

$$W_j[\delta\mu,\delta u_j] = \frac{\delta\mu}{2}|\nabla^S u_j|^2 + \mu\nabla^S u_j : \nabla^S\delta u_j - (Df(u_j)\delta u_j)\cdot u_j - f(u_j)\cdot\delta u_j$$

and if we take $w = (\delta\mu, \{\delta u_j\}_{j=1}^J)$ it can be re-written as

$$\left\{\begin{array}{rcll} \mathcal{L}_{FT} w & = & \mathcal{S} & \text{in } \Omega, \\ \mathcal{B} w & = & g & \text{on } \partial\Omega, \end{array}\right. \tag{55}$$

and it can be seen as the equation

$$\mathcal{A}_{FT} w = \begin{pmatrix}\mathcal{S}\\ g\end{pmatrix}.$$

5.1 Ellipticity and Lopatinskii condition

The principal symbol associated to (54) measurements is exactly $\mathcal{P}_J(x,\xi)$ given in section (4.1.2). That is, for $J = 2$ measurements:

$$\mathcal{P}_J(x,\xi) = \begin{bmatrix} \dfrac{|F_1|^2}{\mu} & i\mu(F_1\xi)^\intercal & 0 \\ 2iF_1\xi & -\mu\Big(|\xi|^2 + (\xi\otimes\xi)\Big) & 0 \\ 0 & i\xi^\intercal & 0 \\ \dfrac{|F_2|^2}{2} & 0 & i\mu(F_2\xi)^\intercal \\ 2iF_2\xi & 0 & -\mu\Big(|\xi|^2 + (\xi\otimes\xi)\Big) \\ 0 & 0 & i\xi^\intercal \end{bmatrix}$$

which is a matrix of size $J(d+2)\times(Jd+1)$.

Corollary 1. *Let $d = 2, J = 2$. Then the operator $\mathcal{L}_{FT}$ is elliptic and $\mathcal{B}$ covers $\mathcal{L}_{FT}$. Moreover we have* *the estimate for $w = (\delta\mu, \{\delta u_j\}_{j=1}^2)$ a solution to (54)*

$$||\delta\mu||_{H^3(\Omega)} + \sum_{j=1}^2 ||\delta u_j||_{H^4(\Omega)^2} \leq C \sum_{j=1}^2 \Big(||\mathcal{L}_{FT,j}^{ec}(\delta\mu, \delta u_j)||_{H^2(\Omega)^2} + ||\mathcal{L}_{FT,j}^{pd}(\delta\mu, \delta u_j)||_{H^3(\Omega)}$$

$$+ ||\mathcal{L}_{FT}^{div}(\delta\mu, \delta u_j)||_{H^3(\Omega)} + ||\mathcal{B}\delta u_j||_{H^{5/2}(\Omega)^2} \Big) \tag{56}$$

$$+ C_2 \Big(||\delta\mu||_{L^2(\Omega)^2} + \sum_{j=1}^2 ||\delta u_j||_{L^2(\Omega)^2} \Big)$$

where $\mathcal{L}_{FT,j}^{ec}, \mathcal{L}_{FT,j}^{pd}, \mathcal{L}_{FT}^{div}$ are the parts of $\mathcal{L}_{FT}$ coming from the elasticity equations, the power density measurements and the divergence condition, respectively.

Proof. Since the ellipticity and Lopatinskii condition depend only on the principal symbol, then we have the result inmediatly from Theorem 4. □

5.2 Injectivity

Lemma 12. *The following boundary problem is elliptic:*

$$\begin{cases} L_{j,FT}[\delta u_j] = 0 & \text{in } \Omega, \\ \nabla \cdot \delta u_j = 0 & \text{in } \Omega, \\ \delta u_j = 0 & \text{on } \partial\Omega, \end{cases} \tag{57}$$

for $j = 1, 2$, $d = 2$, where

$$L_{j,FT}[\delta u_j] = 2\nabla\Big(\Big[-\frac{2\mu}{|F|^2}(F : \nabla^S \delta u_j) + h(u_j)\delta u_j \Big] F_j \Big) + 2\nabla \cdot \mu \nabla^S \delta u_j$$

$$+\omega^2 \delta u_j - Df(u_j)\delta u_j$$

and

$$h(u_j) = \frac{2}{|F_j|^2}\Big(u_j^\intercal Df(u_j) - f(u_j)^\intercal \Big)$$

is elliptic. Therefore we have

$$\sum_{j=1}^2 ||\delta u_j||_{H^4(\Omega)^2} \leq C \sum_{j=1}^2 \Big(||\mathcal{L}_{FT}\delta u_j||_{H^2(\Omega)^2} + ||\mathcal{B}_{FT}\delta u_j||_{H^{5/2}(\Omega)^2} \Big) + C_2 \sum_{j=1}^2 ||\delta u_j||_{L^2(\Omega)^2}.$$

Proof. In fact, since the symbol of f is a polynomial with degree at most 1, we notice that the principal symbol for the system (57) is given by the principal symbol associated to (17). The Lopatinskii condition is satisfied because it depends only on the principal symbol. Therefore we conclude the ellipticity and the estimate by considering Theorem 4. □

Lemma 13. *Let $\tilde{\mathcal{A}}_{FT}$ be the operator corresponding to the equation given in the previous lemma. In dimension 2, if $\{\delta u_j\} \in ker(\tilde{\mathcal{A}}_{FT})$, then*

$$\int_\Omega |\delta u_j|^2 \leq \tilde{C}(\omega^2) \int_\Omega |D\delta u_j|^2 \tag{58}$$

where $\tilde{C}(\omega^2) = \dfrac{1 + 2||\mu||_{L^\infty}}{\omega^2 - (||Df(u_j)||_{L(H^1,L^2)} + ||h(u_j)||_{L(H^1,L^2)})}$.

Proof. If $\delta u_j \in \ker(\tilde{\mathcal{A}}_{FT})$, then:

$$\begin{cases} L_{j,FT}[\delta\mu, \delta u_j] = 0 & \text{in } \Omega, \\ \nabla \cdot \delta u_j = 0 & \text{in } \Omega, \\ \delta u_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{59}$$

Note that

$$\frac{1}{|F_j|^2}(Df(u_j)\delta u_j)\cdot u_j = \frac{1}{|F_j|^2}(u_j^\intercal Df(u_j))\delta u_j.$$

From the second equation in (59) we obtain:

$$\delta\mu = -\frac{2\mu}{|F_j|^2}(F_j : \nabla^S \delta u_j) + h(u_j)\delta u_j.$$

On the other hand, multiplying the first equation of (59) by δu_j and integrating, we obtain:

$$\begin{aligned}\omega^2\int_\Omega |\delta u_j|^2 &= \int_\Omega (Df(u_j)\delta u_j)\cdot\delta u_j - 4\int_\Omega \mu|\hat{F}_j : \nabla^S\delta u_j|^2 \\ &\quad +2\int_\Omega (h(u_j)\delta u_j)(\hat{F}_j : \nabla^S \delta u_j) + 2\int_\Omega \mu|\nabla^S\delta u_j|^2\end{aligned}$$

and considering the identity (36):

$$\begin{aligned}\omega^2\int_\Omega |\delta u_j|^2 &= \int_\Omega (Df(u_j)\delta u_j)\cdot\delta u_j + 2\int_\Omega (h(u_j)\delta u_j)(\hat{F}_j : \nabla^S\delta u_j) \\ &\quad +2\int_\Omega \mu|\hat{F}_j^\perp : \nabla^S\delta u_j|^2 - 2\int_\Omega \mu|\hat{F}_j : \nabla^S\delta u_j|^2 \\ &\leq \Big(||Df(u_j)||_{L(H^1,L^2)} + ||h(u_j)||_{L(H^1,L^2)}\Big)||\delta u_j||^2_{L^2} \\ &\quad +(1+2||\mu||_{L^\infty})\int_\Omega |\nabla^S\delta u_j|^2.\end{aligned}$$

Therefore we obtain the desired result

$$\int_\Omega |\delta u_j|^2 \leq \frac{1+2||\mu||_{L^\infty}}{\omega^2 - (||Df(u_j)||_{\mathcal{L}(H^3,L^2)} + ||h(u_j)||_{\mathcal{L}(H^3,L^2)})}\int_\Omega |\nabla u_j|^2.$$

□

Lemma 14. *In dim 2, there exists $\omega_0 > 0$ such that $\forall\omega \geq \omega_0$ we have $ker(\tilde{\mathcal{A}}_{FT}) = \{0\}$. In other words, the operator is injective.*

Proof. From Corollary 1 taking $\tilde{\mathcal{A}}_{FT}w = (0,0)$, we have, using the previous lemma:

$$\sum_j ||\delta u_j||_{H^4(\Omega)^2} \leq C_2 \sum_j ||\delta u_j||_{L^2(\Omega)^2} \leq C_2\tilde{C}(\omega)\sum_j ||\nabla\delta u_j||_{L^2}$$

where $\tilde{C}(\omega^2)$ is given in (58). If we take ω large enough such that $C_2\tilde{C}(\omega^2) < 1$, we can absorb the right side of the estimate. So we conclude that $\delta u_j = 0$. □

As a result we have the following corollary.

Corollary 2. *Let $d = 2, J = 2$, and $\omega \geq \omega_0$ as in the previous lemma. Then we have the estimate for $(\delta\mu, \delta u_j)$ a solution to* (54)

$$||\delta\mu||_{H^3(\Omega)} + \sum_{j=1}^{2}||\delta u_j||_{H^4(\Omega)^2} \leq C\sum_{j=1}^{2}\Big(||\mathcal{S}_j||_{H^3(\Omega)\times H^2(\Omega)^2} + ||g_j||_{H^{5/2}(\Omega)^2}\Big). \tag{60}$$

Proof. Considering equation (54) with the terms not depending on u_j equal to zero, we can take the second equation and obtain

$$\delta\mu = \frac{1}{|F_j|^2}\Big[\Big(f(u_j) + u_j^\intercal Df(u_j)\Big)\cdot\delta u_j - 2\mu\nabla^S u_j : \nabla^S\delta u_j\Big]. \tag{61}$$

Then we replace $\delta\mu$ in the first equation, so we obtain the equation (57). By Lemma 14 we obtain $\delta u_j = 0$ and using equation (61) we conclude $\delta\mu = 0$. Hence, we can eliminate the terms multiplying C_2 in equation (56), which is valid because we have the hypothesis of Corollary 1. □

5.3 Fixed point algorithm

We note that $\mathcal{F}_{FT} = \mathcal{F} + \mathcal{F}_{add}$ and $\mathcal{L}_{FT} = \mathcal{L} + \mathcal{L}_{add}$ with $\mathcal{F}$ and $\mathcal{L}$ given in the previous case and

$$\mathcal{F}_{add} v_j = \begin{pmatrix} -f(u_j)\cdot u_j \\ -f(u_j) \\ 0 \end{pmatrix}, \quad \mathcal{L}_{j,add} v_j = \begin{pmatrix} -(Df(u_j)\delta u_j)\cdot u_j - f(u_j)\cdot \delta u_j \\ -Df(u_j)\delta u_j \\ 0 \end{pmatrix}.$$

In addition we define $\mathcal{G}_{FT}(w;v) = \mathcal{F}(v+w) - \mathcal{F}v - \mathcal{L}w$. It is clear that $\mathcal{G}_{FT}(w;v) = \mathcal{G}(w;v) + \mathcal{G}_{add}(w;v)$ with $\mathcal{G}$ defined as before and

$$\mathcal{G}_{j,add}(w;v) = \begin{pmatrix} o(\delta u_j)\cdot(u_j + \delta u_j) - (Df(u_j)\delta u_j)\cdot u_j \\ o(\delta u_j) \\ 0 \end{pmatrix}$$

where

$$o(\delta u_j) = \int_0^1 (1-t) D^2 f(u + t\delta u_j)[\delta u_j, \delta u_j] dt$$

comes from Taylor's formula

$$f(u_j + \delta u_j) = f(u_j) + Df(u_j)\delta u_j + \int_0^1 (1-t) D^2 f(u + t\delta u_j)[\delta u_j, \delta u_j] dt.$$

The Fixed Point Algorithm for this case is the same as Algorithm 1, with the following changes:

- Instead of $\mathcal{F}, \mathcal{L}, \mathcal{G}, \mathcal{A}$, we use $\mathcal{F}_{FT}, \mathcal{L}_{FT}, \mathcal{G}_{FT}, \mathcal{A}_{FT}$
- In the step of solving equation (49), we solve

$$\begin{cases} 2\nabla\cdot\mu_0\nabla^S u_j + \omega^2 u_j - f(u_j) &= -\nabla p_j \quad \text{in } \Omega, \\ \nabla\cdot u_j &= 0 \quad \text{in } \Omega, \\ u_j &= g_j \quad \text{on } \partial\Omega. \end{cases} \tag{62}$$

Lemma 15. *There exists a constant $c_2 = c_2(\varepsilon) > 0$ such that*

$$||\mathcal{G}_{FT}(w;v_0) - \mathcal{G}_{FT}(\tilde{w};v_0)||_{\mathcal{Y}} \le c_2\Big(||\delta\mu - \delta\tilde{\mu}||_{H^3(\Omega)} + \sum_j ||\delta u_j - \delta\tilde{u}_j||_{(H^4(\Omega))^2}\Big), \tag{63}$$

provided $||\delta\mu||_{H^3(\Omega)}, ||\delta u_j||_{H^4(\Omega)^2} \le \varepsilon$, for some $\varepsilon > 0$.

Proof. Let

$$\begin{aligned}\psi(\delta u_j, \delta\tilde{u}_j) &= D^2 f(u + t\delta u_j)[\delta u_j, \delta u_j] - D^2 f(u + t\delta\tilde{u}_j)[\delta\tilde{u}_j, \delta\tilde{u}_j] \\ &= D^2 f(u_j + t\delta u_j)[\delta u_j - \delta\tilde{u}_j, \delta u_j] + D^2 f(u_j + t\delta u_j)[\delta\tilde{u}_j, \delta u_j - \delta\tilde{u}_j] \\ &\quad + \Big(D^2 f(u_j + t\delta u_j) - D^2 f(u_j + t\delta\tilde{u}_j)\Big)[\delta\tilde{u}_j, \delta\tilde{u}_j],\end{aligned}$$

hence

$$||\psi(\delta u_j, \delta\tilde{u}_j)||_{L^2(\Omega)^2} \le c_3\varepsilon ||\delta u_j - \delta\tilde{u}_j||_{H^1(\Omega)^2},$$

with c_3 being the maximum between

$$2\sup\{||D^2 f(h)||_{\mathcal{L}(H^4(\Omega)^2, \mathcal{L}(H^4(\Omega)^2, L^2(\Omega)^2))};\ ||u_j - h||_{H^3(\Omega)^2} \le \varepsilon\}$$

and

$$2\varepsilon^2 \sup\{||D^3 f(h)||_{\mathcal{L}(H^4(\Omega)^2, \mathcal{L}(H^4(\Omega)^2, \mathcal{L}(H^4(\Omega)^2, L^2(\Omega)^2)))}; ||u_j - h||_{H^4(\Omega)^2} \le \varepsilon\}$$

given by the mean value theorem over $D^2 f$. Then

$$\begin{aligned} ||o(\delta u_j) - o(\delta \tilde{u}_j)||_{L^2(\Omega)^2} &= \int_0^1 |1-t| c_3 \varepsilon ||\delta u_j - \delta \tilde{u}_j||_{H^4(\Omega)^2} dt \\ &\leq c_3 \varepsilon ||\delta u_j - \delta \tilde{u}_j||_{H^4(\Omega)^2}. \end{aligned}$$

Then the conclusion is direct from Lemma 11 and the definition of $\mathcal{G}_{add}$. □

Then we have the following analogue to Theorem 6:

Corollary 3. *If $\varepsilon > 0$ is sufficiently small so that*

$$c_2(\varepsilon) ||\mathcal{A}_{FT}^{-1}||_{\mathcal{L}(\mathcal{Y},\mathcal{X})} < \frac{1}{2}$$

then the algorithm of this case converges if in addition we have

$$||(\mathcal{H} - \mathcal{H}_0, g - g_0)||_{\mathcal{X}} \leq \frac{\varepsilon}{2},$$

and we obtain

$$||w||_{\mathcal{X}} < \varepsilon. \tag{64}$$

6 Linear Elasticity with internal measurements, incompressible case

The model considered in this section is given by the system (9), but with internal measurements of u_j, i.e.,

$$u_j = H_j \quad \text{in } \Omega. \tag{65}$$

In [32], Proposition 1 c), they proved that there is no ellipticity for the joint recovery of μ and p. Therefore we must either apply the curl to the operator to remove ∇p or we must hold ∇p fixed. This last case is studied in [32], establishing the ellipticity and Lopatinskii condition with at least one measurement, but null kernel with two measurements. If we are to use the model with ∇p fixed, then we know that λ is large. This causes serious convergence problems when considering the Saint-Venant model of non-linear elasticity, for example with results like 10, where we need to have a contraction map, so we chose to apply the curl operator, which eliminates the λ terms.

Hence, we consider the model

$$\left\{ \begin{array}{rcll} \omega^2 \nabla \times u_j + 2\nabla \times \nabla \cdot \mu \nabla^S u_j &=& 0 & \text{in } \Omega, \\ u_j &=& H_j & \text{in } \Omega, \\ \nabla \cdot u_j &=& 0 & \text{in } \Omega, \\ u_j &=& g_j & \text{on } \partial\Omega. \end{array} \right. \tag{66}$$

The linearization of (66) gives the following system:

$$\left\{ \begin{array}{rcll} \omega^2 \nabla \times \delta u_j + 2\nabla \times \nabla \cdot \mu \nabla^S \delta u_j + 2\nabla \times \nabla \cdot \delta\mu \nabla^S u_j &=& 0 & \text{in } \Omega, \\ \delta u_j &=& \delta H_j & \text{in } \Omega, \\ \nabla \cdot \delta u_j &=& 0 & \text{in } \Omega, \\ \delta u_j &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{67}$$

6.1 Ellipticity

Let $\Sigma_{\text{curl}}(\xi)$ be the symbol of the curl operator, that is

$$\Sigma_{\text{curl}}(\xi) = i \begin{pmatrix} -\xi_2 & \xi_1 \end{pmatrix}$$

in dimension 2, and

$$\Sigma_{\mathrm{curl}}(\xi) = i \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix}$$

in dimension 3. Note that if $b \in \mathbb{R}^d$, then $\Sigma_{\mathrm{curl}}(\xi)\, b = i\, b \times \xi$.

The linearized system then has the following principal symbol:

$$\mathcal{P}(x,\xi) = \begin{bmatrix} 2(\nabla^S u\ \xi) \times \xi & -2\mu\ \Sigma_{\mathrm{curl}}(\xi)\Big(|\xi|^2 I_d + \xi \otimes \xi\Big) \\ 0 & I_d \\ 0 & i\xi^\intercal \end{bmatrix}$$

with is a matrix with size $(2d+1) \times (d+1)$. Let $\xi \neq 0$ and $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_{d+1}$ the columns of that matrix. Let $\alpha_1, \ldots, \alpha_{d+1} \in \mathbb{C}$ such that

$$\sum_{i=1}^{d+1} \alpha_i \boldsymbol{C}_i = \boldsymbol{0}.$$

We see that, because of the identity matrix, necessarily $\alpha_2 = \cdots = \alpha_{d+1} = 0$, so we have to analyse the equation $\alpha_1 \boldsymbol{C}_1 = \boldsymbol{0}$. This last equation can be reduced to the case studied in [6], giving the nonellipticity for 1 measurement. If we consider the augmented system for 2 measurements, we obtain the ellipticity as in [6] for 3 dimensions. Notice that this computation in 2 dimensions corrects a mistake in the original computations presented there.

The symbol for the augmented system is

$$\mathcal{P}_2(x,\xi) = \begin{bmatrix} 2(\nabla^S u_1\ \xi) \times \xi & P(\xi) & 0 \\ 0 & I_d & 0 \\ 0 & i\xi^\intercal & 0 \\ 2(\nabla^S u_2\ \xi) \times \xi & 0 & P(\xi) \\ 0 & 0 & I_d \\ 0 & 0 & i\xi^\intercal \end{bmatrix}$$

where

$$P(\xi) = -2\mu \Sigma_{\mathrm{curl}}(\xi)\Big(|\xi|^2 I_d + \xi \otimes \xi\Big).$$

In order to have ellipticity, that is, in order to $\mathcal{P}_2(x,\xi)$ being column rank, we need that he following condition holds:

$$|(\nabla^S u_1 \xi) \times \xi| + |(\nabla^S u_2 \xi) \times \xi| \neq 0 \quad \forall |\xi| \neq 0. \tag{68}$$

This is slightly different to the case in [6] where the following is considered:

$$|(\nabla^S u_1 \xi) \times \xi| + |(\nabla^S u_2 \xi) \times \xi| \geq |\xi|^2. \tag{69}$$

It is unclear to the authors which condition is more natural.

Condition (68) is equivalent to the following: let $A^{(j)} = \nabla^S u_j$, and the matrices $B^{(j)}$ defined in two dimensions by

$$B^{(j)} = \begin{pmatrix} a_{11}^{(j)} - a_{22}^{(j)} & 2(a_{12}^{(j)} + a_{21}^{(j)}) \end{pmatrix} \tag{70}$$

and in three dimensions by

$$B^{(j)} = \begin{pmatrix} a_{23}^{(j)} & 0 & 0 & a_{22}^{(j)} - a_{33}^{(j)} & a_{12}^{(j)} & -a_{13}^{(j)} \\ 0 & -a_{13}^{(j)} & 0 & -a_{12}^{(j)} & a_{33}^{(j)} - a_{11}^{(j)} & a_{23}^{(j)} \\ 0 & 0 & a_{12}^{(j)} & a_{13}^{(j)} & -a_{23}^{(j)} & a_{11}^{(j)} - a_{22}^{(j)} \end{pmatrix}. \tag{71}$$

A condition in dimension $d = 2, 3$ for having ellipticity is that the $d \times d$ matrix

$$\begin{pmatrix} B^{(1)} \\ B^{(2)} \end{pmatrix} \text{ must be invertible.} \tag{72}$$

The equivalence between (68) and (72) comes from the equality

$$\begin{pmatrix} ((A^{(1)}\xi)\times\xi)^\intercal \\ ((A^{(1)}\xi)\times\xi)^\intercal \end{pmatrix} = \begin{pmatrix} B^{(1)} \\ B^{(2)} \end{pmatrix} \begin{pmatrix} \xi_2^2-\xi_1^2 \\ \xi_1\xi_2 \end{pmatrix}$$

in dimension 2, and

$$\begin{pmatrix} ((A^{(1)}\xi)\times\xi)^\intercal \\ ((A^{(1)}\xi)\times\xi)^\intercal \end{pmatrix} = \begin{pmatrix} B^{(1)} \\ B^{(2)} \end{pmatrix} \begin{pmatrix} \xi_3^2-\xi_2^2 \\ \xi_3^2-\xi_1^2 \\ \xi_2^2-\xi_1^2 \\ \xi_2\xi_3 \\ \xi_1\xi_3 \\ \xi_1\xi_2 \end{pmatrix}$$

in dimension 3. Note that condition (72) is a sufficient condition for $\nabla^S u_1 \neq \alpha\nabla^S u_2\ \forall\alpha\in\mathbb{R}$.

6.2 Lopatinskii condition

The Lopatinskii condition we show is based in [6]. The analysis is the same, but in certain step we consider the condition (68) instead of (69).
If

$$\mathcal{P}_2(x, i\eta+\nu\partial_z)(\tilde\mu,\tilde u) = 0,$$

then we easily see that $\tilde u \equiv 0$, due to the identity blocks. Then, consider $A^{(j)} = \nabla^S u_j$. Then we have the equation

$$(A^{(j)}\nu\times\nu)\partial_z^2\tilde\mu + i(A^{(j)}\eta\times\nu + A^{(j)}\nu\times\eta)\partial_z\tilde\mu - (A^{(j)}\eta\times\eta)\tilde\mu = 0, \quad j=1,2.$$

If in each equation we apply the dot product with $A^{(j)}\nu\times\nu$, and then we sum both equations, we obtain:

$$a\partial_z^2\tilde\mu + b\partial_z\tilde\mu + c\tilde\mu = 0 \tag{73}$$

with

$$a = \sum_j |A^{(j)}\nu\times\nu|^2$$

which is nonzero by (68). Then, let $\lambda_{1,2} = \dfrac{-ib\pm\sqrt{-b^2-4ac}}{2a}$ the roots of the characteristic polynomial related to equation (73). The solutions have the structure

$$\tilde\mu(z) = \alpha(\exp(\lambda_1 z) - \exp(\lambda_2 z))$$

since $\tilde\mu(0)=0$. If $\lambda_{1,2}$ is purely imaginary, the only option for $\tilde\mu$ going to 0 when $z\to\infty$ is when $\alpha=0$. If $\lambda_{1,2}$ has a real part, then one of the exponentials goes to infinity and the other goes to zero when $z\to\infty$, so the only option we have is $\alpha=0$. That is, we have the Lopatinskii condition.

The Douglis numbers are:

$$s_i = \begin{cases} 0 & \text{if } i\in\{1,\dots,d+1, 2d+2,\dots,3d+1\}, \\ -2 & \text{otherwise,} \end{cases}$$

$$t_j = \begin{cases} 2 & \text{if } i=1, \\ 3 & \text{otherwise,} \end{cases}$$

$$\sigma_k = -1, \quad k=1,\dots,2d.$$

Then the operator over $(\delta\mu, \{\delta u_j\}_{j=1}^J)$ given by equation (67) with 2 measurements is defined from

$$\mathcal{X} = H^{l+2}(\Omega)\times H^{l+3}(\Omega)^d\times H^{l+3}(\Omega)^d$$

to

$$\mathcal{Y} = \left(H^l(\Omega)^d \times H^{l+2}(\Omega)^d \times H^{l+2}(\Omega) \times H^{l+1/2}(\Omega)^d\right)^2$$

where we can take $l = 2$ in dimension 2 and dimension 3. Then we have the following estimate:

$$||\delta\mu||_{H^{l+2}(\Omega)} + \sum_{j=1}^{2} ||\delta u_j||_{H^{l+3}(\Omega)^d} \leq C \sum_{j=1}^{2} \Big(||\mathcal{L}_j^{ec}(\delta\mu, \delta u_j)||_{H^l(\Omega)^d}$$

$$+||\mathcal{L}_j^{int}(\delta\mu, \delta u_j)||_{H^{l+2}(\Omega)^d}$$

$$+||\mathcal{L}_j^{div}\delta u_j||_{H^{l+2}(\Omega)} + ||\delta u_j||_{H^{l+1/2}(\partial\Omega)^d}\Big)$$

$$+C_2\Big(||\delta\mu||_{L^2(\Omega)} + \sum_{j=1}^{2} ||\delta u_j||_{L^2(\Omega)^d}\Big).$$

6.3 Local injectivity

The results in [19] prove local injectivity and the convergence of an algorithm for the recovery of μ. They use unique continuation properties assuming $\delta\mu|_{\partial\Omega} = 0$ (in our notation). In this section we show another injectivity argument, based on [10].

If we consider the right hand side of (66) being 0, then we have

$$\nabla \times \nabla \cdot \left(\delta\mu A^{(j)}\right) = 0, \quad j = 1, 2.$$

Let $\rho(x, \xi)$ be the principal symbol for this last equation. Then

$$\rho(x, \xi) = \begin{pmatrix} (A^{(1)}\xi) \times \xi \\ (A^{(2)}\xi) \times \xi \end{pmatrix}.$$

In dimension 2, we need to assume that $A_{12}^{(j)} \neq 0$ to obtain that $(0, 1)$ is non characteristic at the origin, since

$$A^{(j)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A_{12}^{(j)}.$$

In dimension 3, we need to assume that $a_{13}^{(j)}, a_{23}^{(j)} \neq 0$ to obtain that $(0, 0, 1)$ is non characteristic at the origin, since

$$A^{(j)}(0, 0, 1) \times (0, 0, 1) = (a_{23}^{(j)}, -a_{13}^{(j)}, 0).$$

The condition (68) provides the hypothesis for Theorem 3.6 in [10], since there are not real roots, and then, due to the fundamental algebra theorem, we have two different complex roots. Therefore, we have a unique continuation principle for μ and we can take $C_2 = 0$ in the last estimate above.

7 Nonlinear Elasticity (Saint-Venant model) with internal measurements

Saint-Venant model is the first nonlinear model in elasticity that is studied in the literature. It is a generalization of the linear model studied before, and it comes from the simplification of the Green strain tensor

$$Eu = \nabla^S u + \frac{1}{2}\nabla u^\intercal \nabla u. \tag{74}$$

In linear elasticity, it is assumed that the displacements are sufficiently small for neglecting the term $\nabla u^\intercal \nabla u$, considering the *small strain tensor*,

$$\epsilon u = \nabla^S u, \tag{75}$$

c.f. [26] for the constant coefficient calculations. The Saint Venant-Kirchhoff model considers (74) instead of (75), since it is assumed that the deformations are not so small, and Eu plays the role of ϵu in the

constitutive equations of linear elasticity.
In this section, we consider the Saint-Venant's model under a periodic force with frequency ω, which can be written as a "steady state" equation by:

$$\left\{\begin{array}{rcll}(L_{\mu,\lambda}+N_{\mu,\lambda})u+\omega^2 u &=& 0 & \text{in } \Omega,\\ u &=& g & \text{on } \partial\Omega.\end{array}\right. \tag{76}$$

where

$$\begin{array}{rcl}L_{\mu,\lambda}u &=& 2\nabla\cdot\mu\nabla^S u+\nabla(\lambda\nabla\cdot u),\\ N_{\mu,\lambda}u &=& 2c_\tau\nabla\cdot(\mu\nabla u^\intercal\nabla u)+\nabla(\lambda|\nabla u|^2).\end{array}$$

and c_τ is a constant in x coming from the fact that we cannot obtain a time independent equation by applying a periodic force in time, as in the previous cases, since they are linear in u. So, our model is considered for a fixed time τ.
The measurements are

$$u=H \quad \text{in } \Omega. \tag{77}$$

Applying the curl operator in (76), we obtain

$$\left\{\begin{array}{rcll}(\tilde{L}_\mu+\tilde{N}_\mu)u+\omega^2\nabla\times u &=& 0 & \text{in } \Omega,\\ u &=& 0 & \text{on } \partial\Omega.\end{array}\right. \tag{78}$$

where

$$\tilde{L}_\mu u=2\nabla\times\nabla\cdot\mu\nabla^S u, \qquad \tilde{N}_\mu u=2c_\tau\nabla\times\nabla\cdot(\mu\nabla u^\intercal\nabla u).$$

The linearized system from (78) with internal measurements is

$$\left\{\begin{array}{rcll}D\tilde{L}(\mu,u)[\delta\mu,\delta u]+D\tilde{N}(\mu,u)[\delta\mu,\delta u]+\omega^2\nabla\times\delta u &=& 0 & \text{in } \Omega,\\ \delta u &=& \delta H & \text{in } \Omega,\\ \delta u &=& 0 & \text{on } \partial\Omega.\end{array}\right. \tag{79}$$

where $D\tilde{L}$ and $D\tilde{N}$ are the Fréchet derivatives of $\tilde{L}$ and $\tilde{N}$, respectively, given by:

$$\begin{array}{rcl}D\tilde{L}(\mu,u)[\delta\mu,\delta u] &=& 2\nabla\times\nabla\cdot\delta\mu\nabla^S u+2\nabla\times\nabla\cdot\mu\nabla^S\delta u\\ D\tilde{N}(\mu,u)[\delta\mu,\delta u] &=& 2c_\tau\nabla\times\nabla\cdot(\delta\mu\nabla u^\intercal\nabla u)+2\nabla\times\nabla\cdot(\mu\nabla\delta u^\intercal\nabla u),\\ && +2\nabla\times\nabla\cdot(\mu\nabla u^\intercal\nabla\delta u).\end{array}$$

7.1 Ellipticity

The symbol of the linearized operator is

$$\mathcal{P}(x,\xi)=\left[\begin{array}{cc}2\Big((\nabla^S u+c_\tau\nabla u^\intercal\nabla u)\xi\Big)\times\xi & P(\xi)\\ 0 & I_d\end{array}\right]$$

where

$$P(\xi)=-2\mu\Sigma_{\text{curl}}(\xi)\Big(|\xi|^2 I_d+\xi\otimes\xi\Big)(I_d+\nabla u^\intercal).$$

We see that $Op(\mathcal{P}(x,\xi))$ is not elliptic. If we add a measurement, we will have the symbol

$$\mathcal{P}_2(x,\xi)=\left[\begin{array}{ccc}2\Big((\nabla^S u_1+c_\tau\nabla u_1^\intercal\nabla u_1)\xi\Big)\times\xi & P(\xi) & 0\\ 0 & I_d & 0\\ 2\Big((\nabla^S u_2+c_\tau\nabla u_2^\intercal\nabla u_2)\xi\Big)\times\xi & 0 & P(\xi)\\ 0 & 0 & I_d\end{array}\right]$$

and we see that the linearized operator is elliptic if

$$\left|\Big((\nabla^S u_1+c_\tau\nabla u_1^\intercal\nabla u_1)\xi\Big)\times\xi\right|+\left|\Big((\nabla^S u_2+\nabla c_\tau u_2^\intercal\nabla u_2)\xi\Big)\times\xi\right|\neq 0 \quad \forall\xi\neq 0.$$

Let $A^{(j)}=\nabla^S u_j+c_\tau\nabla u_j^\intercal\nabla u_j$, and and the matrices $B^{(j)}$ defined as in (70)-(71). Then a condition for having ellipticity is (72).

7.2 Lopatinskii condition and local injectivity

The deduction of the Lopatinskii condition and local injectivity are the same as the presented in section 6, with the change

$$A^{(j)} = \nabla^S u_j + c_\tau \nabla u_j^\intercal \nabla u_j, \quad j = 1, 2.$$

The Douglis numbers are:

$$s_i = \begin{cases} 0 & \text{if } i \in \{1, \dots, d, 2d+1, \dots, 3d\}, \\ -2 & \text{otherwise,} \end{cases}$$

$$t_j = \begin{cases} 2 & \text{if } i = 1, \\ 3 & \text{otherwise} \end{cases}$$

$$\sigma_k = -1, \quad k = 1, \dots, 2d.$$

Then the operator over $(\delta\mu, \{\delta u_j\}_{j=1}^J)$ given by equation (79) with 2 measurements is defined from

$$\mathcal{X} = H^{l+2}(\Omega) \times H^{l+3}(\Omega)^d \times H^{l+3}(\Omega)^d$$

to

$$\mathcal{Y} = \Big(H^l(\Omega)^d \times H^{l+2}(\Omega)^d \times H^{l+1/2}(\Omega)^d\Big)^2$$

with $l = 2$ in dimension 2 and 3. Then we have the following estimate:

$$\begin{aligned} ||\delta\mu||_{H^{l+2}(\Omega)} + \sum_{j=1}^{2} ||\delta u_j||_{H^{l+3}(\Omega)^d} \leq \quad & C \sum_{j=1}^{2} \Big(||\mathcal{L}_j^{ec}(\delta\mu, \delta u_j)||_{H^l(\Omega)^d} \\ & + ||\mathcal{L}_j^{int}(\delta\mu, \delta u_j)||_{H^{l+2}(\Omega)^d} + ||\delta u_j||_{H^{l+\frac{1}{2}}(\partial\Omega)^d} \Big) \\ & + C_2 \Big(||\delta\mu||_{L^2(\Omega)} + \sum_{j=1}^{2} ||\delta u_j||_{L^2(\Omega)^d} \Big). \end{aligned}$$

Since we have local injectivity, we can take $C_2 = 0$. That is, we have

$$\begin{aligned} ||\delta\mu||_{H^{l+2}(\Omega)} + \sum_{j=1}^{2} ||\delta u_j||_{H^{l+3}(\Omega)^d} \leq \quad & C \sum_{j=1}^{2} \Big(||\mathcal{L}_j^{ec}(\delta\mu, \delta u_j)||_{H^l(\Omega)^d} \\ & + ||\mathcal{L}_j^{int}(\delta\mu, \delta u_j)||_{H^{l+2}(\Omega)^d} + ||\delta u_j||_{H^{l+\frac{1}{2}}(\partial\Omega)^d} \Big). \end{aligned} \tag{80}$$

7.3 Algorithm

We define μ_0 as our first guess for μ and u_0 a solution of the equation (78) corresponding to $\mu = \mu_0$. Let $\delta\hat{u}$ be the displacement satisfying $H = u_0 + \delta\hat{u}$, and $\delta\mu_{tr}$ such that $\mu_{tr} = \mu_0 + \delta\mu_{tr}$.

In order to reconstruct μ_{tr} from the measurements H, we define the discrepancy functional:

$$\mathcal{J}[\delta\mu] = \mathcal{J}_1[\delta\mu] + \mathcal{J}_2[\delta\mu] = \frac{1}{2}\Big(||P_1[\delta\mu] - \delta\hat{u}_1||_{X_u}^2 + ||P_2[\delta\mu] - \delta\hat{u}_2||_{X_u}^2 \Big)$$

where P_j is the operator given by $P_j : \delta\mu \mapsto \delta u_j$ defined by equation (79) and $X_u = H^{l+3}(\Omega)^d$. Our strategy for determining μ from $\{H_1, H_2\}$ is to solve the problem

$$\min_{\delta\mu \in X_\mu} \mathcal{J}[\delta\mu]$$

where $X_\mu = H^{l+2}(\Omega)$.

Following [6], we consider the following iterations

$$\begin{cases} \delta\mu_{n+1} &= \delta\mu_n - \eta D\mathcal{J}[\delta\mu_n], \\ \delta\mu_0 &= 0. \end{cases} \tag{81}$$

For each α multi-index of dimension d satisfying $|\alpha| \leq l+3$, let $\psi_j^\alpha = \dfrac{\partial^{|\alpha|} P_j[\delta\mu]}{\partial x^\alpha}$, $\psi_{j,tr}^\alpha = \dfrac{\partial^{|\alpha|}\delta\hat{u}_j}{\partial x^\alpha}$ and the functionals:

$$\mathcal{J}_j^\alpha[\delta\mu] = \frac{1}{2}||\psi_j^\alpha - \psi_{j,tr}^\alpha||^2_{L^2(\Omega)^d},$$

The derivative of $\mathcal{J}_j^\alpha$ can be written

$$D\mathcal{J}_j^\alpha[\delta\mu] = \Big(\frac{\partial^{|\alpha|}}{\partial x^\alpha} DP_j[\delta\mu]\Big)^* \Big(\frac{\partial^{|\alpha|}}{\partial x^\alpha} P_j[\delta\mu] - \frac{\partial^{|\alpha|}}{\partial x^\alpha} P_j[\delta\mu_{tr}]\Big). \tag{82}$$

This expression allows to conclude that the algorithm given by (81) converges (see [6] Theorem 6) when (80) holds and the initial guess μ_0 is close enough to μ.

On the other hand we need to determine each $D\mathcal{J}_j^\alpha[\delta\mu]$ in a more explicit form.

Lemma 16. *Let ϕ_j^α be a solution of the following adjoint system:*

$$\begin{cases} 2\nabla\cdot\Big[\Big(\delta\mu(I_d + 2c_\tau\nabla u_j) + 2c_\tau\mu\nabla\delta u_j\Big)\nabla^S(\nabla\times\phi_j^\alpha)\Big] = & \psi_j^\alpha - \psi_{j,tr}^\alpha \quad \text{in } \Omega, \\ \nabla^S(\nabla\times\phi_j^\alpha) = & 0 \quad \text{on } \partial\Omega, \\ \nabla\times\phi_j^\alpha = & 0 \quad \text{on } \partial\Omega, \\ \phi_j^\alpha = & 0 \quad \text{on } \partial\Omega, \end{cases} \tag{83}$$

Then, the Fréchet-derivative of $\mathcal{J}_j^\alpha$ is given by:

$$D\mathcal{J}_j^\alpha[\delta\mu] = 2\Big[\nabla^S\delta u_j + c_\tau\Big((\nabla\delta u_j^\intercal\nabla u_j) + (\nabla\delta u_j^\intercal\nabla u_j)^\intercal\Big)\Big] : \nabla^S(\nabla\times\phi_j^\alpha).$$

Proof. We compute the second order derivatives of $\tilde{L}$ and $\tilde{N}$:

$$\begin{aligned} D^2\tilde{L}(\mu,u_j)[\delta\mu,\delta u_j;\gamma,w] &= 2\nabla\times\nabla\cdot(\delta\mu\nabla^S w) + \omega^2\nabla\times w \\ &\quad +2\nabla\times\nabla\cdot\gamma\nabla^S\delta u_j, \\ D^2\tilde{N}(\mu,u_j)[\delta\mu,\delta u_j;\gamma,w] &= 2c_\tau\nabla\times\nabla\cdot\Big(\delta\mu(\nabla w^\intercal\nabla u_j + \nabla u_j^\intercal\nabla w) \\ &\quad +\mu(\nabla\delta u_j^\intercal\nabla w + \nabla w^\intercal\nabla\delta u_j)\Big) \\ &\quad +2c_\tau\nabla\times\nabla\cdot\Big(\gamma(\nabla\delta u_j^\intercal\nabla u_j + \nabla u_j^\intercal\nabla\delta u_j)\Big). \end{aligned}$$

We have, using the adjoint equation and integrating by parts:

$$\begin{aligned} &\int_\Omega(\psi_j^\alpha - \psi_{j,tr}^\alpha)\cdot w dx \\ &= \int_\Omega 2\nabla\cdot\Big[\Big(\delta\mu(I_d + 2c_\tau\nabla u_j) + 2c_\tau\mu\nabla\delta u_j\Big)\nabla^S(\nabla\times\phi_j^\alpha)\Big]\cdot w\ dx \\ &= \int_\Omega \nabla\times\Big[\Big(2\nabla\cdot\delta\mu\nabla^S w + \omega^2 w\Big) + 2c_\tau\Big(\delta\mu(\nabla w^\intercal\nabla u_j + \nabla u_j^\intercal\nabla w) \\ &\qquad +\mu(\nabla\delta u_j^\intercal\nabla w + \nabla w^\intercal\nabla\delta u_j)\Big)\Big]\cdot\phi_j^\alpha dx \\ &= -\int_\Omega\Big[2\nabla\times\nabla\cdot\gamma\Big(\nabla^S\delta u_j + c_\tau(\nabla\delta u_j^\intercal\nabla u_j + \nabla u_j^\intercal\nabla\delta u_j)\Big)\Big]\cdot\phi_j^\alpha dx \end{aligned}$$

where in the last step we used the computation of the second order derivative of $\tilde{L}+\tilde{N}$ and that

$$D^2(\tilde{L}+\tilde{N})(\mu,u_j)[\delta\mu,\delta u_j;\gamma,w]=0.$$

Finally, using (82), integrating by parts and considering $w=\gamma\dfrac{\partial^\alpha}{\partial x^\alpha}DP_j[\delta\mu]$, we have:

$$\begin{aligned}D\mathcal{J}_j^\alpha[\delta\mu]\gamma &= \int_\Omega(\psi_j^\alpha-\psi_{j,tr}^\alpha)\cdot w\;dx\\ &= \int_\Omega\gamma\Big[2\Big(\nabla^S\delta u+c_\tau(\nabla\delta u^\intercal\nabla u+\nabla u^\intercal\nabla\delta u)\Big):\nabla^S(\nabla\times\phi_j^\alpha)\Big]dx.\end{aligned}$$

Therefore,

$$D\mathcal{J}_j^\alpha[\delta\mu]=2\Big(\nabla^S\delta u+c_\tau(\nabla\delta u^\intercal\nabla u+\nabla u^\intercal\nabla\delta u)\Big):\nabla^S(\nabla\times\phi_j^\alpha).$$

□

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