Optimal Transport of Nonlinear Control-Affine Systems

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***Abstract*— In this paper, we consider the optimal transport problem for general nonlinear control systems that are of control-affine form. When the system is driftless, this corresponds to the sub-Riemannian optimal transport problem. We consider the Benamou-Brenier formulation of this problem. We first establish a controllability result that controllability of the underlying control system implies controllability of the continuity equation using Borel measurable feedback controls from a given initial measure to a terminal measure. Then we consider the problem of numerically computing the feedback control laws that generate optimal transport. We propose fast algorithms to calculate the sub-Riemannian Wasserstein-p distance (W_p), $p = 1, 2$ on the discretized domain with the rate of convergence independent of grid size, which is important for large scale problems. For sub-Riemannian W_1 cost, we formalize the optimization problem to be independent of time-variable which reduces the dimensionality of the problem significantly. We validate our numerical approach on a 2-dimensional system and a 3-dimensional system, the Grushin plane system, and the unicycle model, respectively.**

I. INTRODUCTION

The optimal transport problem [25] of transporting one probability distribution to another using transport maps, in some optimal manner, has found a large number of applications in a wide range of fields such as the theory of nonlinear PDEs [3], machine learning [23] and image processing [24]. One promising direction of investigation is the application of the Benamou-Brenier fluid dynamical formulation [5] of optimal transport, which frames the optimal transport problem as an optimal control problem for a continuity equation, to multi-agent control problems [12]. The several applications of optimal transport have also lead to many works on developing fast algorithms to compute optimal transport costs and their extensions. The constrained optimization problem arising from an optimal transport problem can be turned into an unconstrained saddle point problem that can be solved with the first-order primal-dual method of Chambolle and Pock [10] or the alternating direction method of multipliers (ADMM) [6]. Various types of optimal transport costs and its generalizations have been solved using these approaches [22], [20].

There has been an increasing interest in considering optimal transport problems associated with costs arising from optimal control problems. There has been some recent effort to extend the existing theory to costs arising from optimal control of general linear systems with some success, e.g., [19], [11]. However, a large number of models, such as those arising in robotics [21], are nonlinear. This motivates our investigation of optimal transport problems associated with optimal control of nonlinear systems.

There has been some work on the *Monge formulation* [25] of the optimal transport problem for costs arising from optimal control of nonlinear control-affine systems in the setting of sub-Riemannian optimal transport [1], [17]. For driftless control-affine systems when the corresponding sub-Riemannian distance is semi-concave [17], it has been shown that the results are very close to the Riemannian case [18]. In [13], the authors considered the fluid-dynamical formulation of optimal transport formulation of the optimal transport problem by discretizing the space and hence, reducing the sub-Riemannian optimal transport problem to an optimal transport problem on a graph [13]. More recently, in [9] the authors considered the Benamou-Brenier formulation of optimal for *feedback-linearizable* systems. In addition, the geometry calculations for sub-Riemannian structure in density space have been considered in [16].

Contributions: In this paper, we make theoretical and numerical contributions towards optimal transport of control-affine systems. First, we consider the issue of controllability of the resulting continuity equation arising from the Benamou-Brenier formulation of the optimal transport problem for control-affine systems. The second main contribution is on developing efficient numerical methods for the optimal transport control affine systems. We first formally derive the optimality conditions for the case of L_2 cost and demonstrate that the Benamou-Brenier version of the problem has a very similar structure as in the linear case, and can be similarly posed as a convex optimization problem using a suitable change of variables. The optimality conditions can be framed as a convex-concave problem involving the continuity equation and a Hamilton-Jacobi-Bellman (HJB) equation. Then we consider the case of homogeneous of degree one cost, in which case the optimal transport problem can be transformed into a static, time-independent version of the problem . We also apply the primal-dual hybrid gradient (PDHG) method to implement these formulations and obtain a very fast method whose rate of convergence is independent of grid size.

We organize this paper as follows. In section II, we formulate the variational problem for optimal transport of nonlinear control-affine systems. We discuss the controllability problems in section III. In section IV, we design primal-dual hybrid algorithms to solve the proposed variational formulations. Numerical examples, including the Grushin

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plane and the unicycle model, are presented.

II. PROBLEM FORMULATION

Suppose $g_i : \mathbb{R}^d \to \mathbb{R}^d$ are smooth vector fields for $i = 1, ..., m$. Consider the following finite-dimensional control system on $\mathbb{R}^d$,

$$\dot{x}(t) = g_0(x(t)) + \sum_{i=1}^{n} v_i(t) g_i(x(t)) \tag{II.1}$$

where $x(t)$ represents the state and $v_i(t)$ are the control inputs of the system. Control systems of this type are said to be in control-affine form and are well-studied in control theory literature [2]. Such systems commonly arise in models in robotics [21].

Let $T > 0$. Given $x_0, x_T \in \mathbb{R}^d$, a standard instance of the optimal control problem for this system is to solve the following optimization problem,

$$\inf_{v_i, x} \int_0^T \frac{1}{p} \sum_{i=1}^{n} |v_i(t)|^p dt \tag{II.2}$$

subject to (II.1) and the constraints

$$x(0) = x_0 \quad x(T) = x_T \tag{II.3}$$

The optimal transport problem that we are interested in is a variation of the above problem where the initial and final condition of the state $x(t)$ are represented by probability densities $\rho_0 : \mathbb{R}^d \to \mathbb{R}$ and $\rho_T : \mathbb{R}^d \to \mathbb{R}$. These densities, for example, could either represent the uncertainty in the initial and final condition of the control system (II.1), or they could also represent a density of agents such as animals, crowds, robots. etc, each of which can be modeled using (II.1).

Suppose that the controls $v_i(t)$ are given in feedback form, using functions $u_i(x, t)$, as $v_i(t) = u_i(x(t), t)$. The probability density $\rho(t, x)$ of the variable $x(t)$ is then given by the following system of equations,

$$\begin{cases} \partial_t \rho + \nabla \cdot (g_0 \rho) + \sum_{i=1}^{n} \nabla \cdot (u_i g_i \rho) = 0, & \text{in } \mathbb{R}^d \times [0, T] \\ \rho(0, \cdot) = \rho_0, \quad \rho(T, \cdot) = \rho_T & \text{in } \mathbb{R}^d. \end{cases} \tag{II.4}$$

Subject to these constraints, the optimal transport problem that we wish is to solve is the following optimization problem,

$$\inf_{u_i, \rho} \int_0^T \int_{\mathbb{R}^d} \frac{1}{p} \sum_{i=1}^{n} \rho(t, x) |u_i(t, x)|^p dx dt \tag{II.5}$$

From a probabilistic point of view, the cost functional in (II.5) can be interpreted as the *expectation* (with respect to density $\rho(t, x)$) of the control cost considered in the original optimal control problem (II.2).

Remark II.1. When $g_0 \equiv \mathbf{0}$, $m = d$ and g_i are coordinate vector fields for each $i = 1, ..., m$, and p=2, this problem is the Benamou-Brenier fluid dynamical formulation of optimal transport [5] for the squared-Euclidean distance. In this case, the continuity equation simplifies to,

$$\partial_t \rho + \nabla \cdot (\mathbf{u} \rho) = 0, \text{ in } \mathbb{R}^d \times [0, T] \tag{II.6}$$

and the optimization problem can be alternatively expressed as,

$$\inf_{u_i, \rho} \int_0^T \int_{\mathbb{R}^d} \frac{1}{p} \rho(t, x) \|\mathbf{u}(t, x)\|_p^p dx dt \tag{II.7}$$

where $\|\mathbf{u}\|_p$ denotes is the usual p-norm. For the case of $p = 2$, and when the underlying control system is *linear*, i.e., there exists a time-varying matrix $A(t) \in \mathbb{R}^{d \times d}$ and vectors $b_i(t) \in \mathbb{R}^d$ are such that $g_0(t, x) = A(t)x$ and $g_i(x) = b_i(t)$, this problem has been addressed in [11]. For $n = d$ and $\text{span}\{g_i(x); i \in \{1, ..., n\}\} = \mathbb{R}^d$, this problem can be shown to coincide with the Riemannian optimal transport problem.

Remark II.2. It is worth mentioning that in the classical Riemannian setting, the optimal transport problem (II.5) also exhibits a mapping and linear programming based formulations, known as Monge problem and Kantorovich problem [25], respectively. One can also consider these formulations for the transport problem corresponding to costs arising from optimal control of control-affine systems. See [19], [1], [17] for these formulations of the optimal transport problem for such costs.

In this paper, we are particularly interested in the case when it is possible that $n < d$. In this case, $\text{span}\{g_i(x); i \in \{1, ..., n\}\} \neq \mathbb{R}^d$, and hence it might be possible that the constraints (II.4) are not feasible. The feasibility of constraints (II.4), and hence the finiteness of the cost (II.5), is closely related to properties of the control system (II.1). For linear systems, it has been shown than the Kalman rank condition for controllability of (II.1) implies well-posedness of the optimal transport problem [11]. Our goal will be to consider theoretical and numerical aspects of this optimization problem for the general class of vector fields $\{g_i\}$.

III. CONTROLLABILITY

In this section, we will consider the issue of feasibility of the constraints (II.4). Particularly, before attempting to solve the optimization problem, we are interested in establishing if, given probability densities ρ_0 and ρ_T, there exist feedback control laws $\mathbf{u}(t, x) = [u_1(t, x) ... u_m(t, x)]^T$ such that the solution of the continuity equation in (II.4) satisfies the initial and terminal constraints. In the linear case, this for the continuity equation follows from existing results on optimal transport using a suitable coordinate transformation [11]. Since such a technique does not seem feasible in the general nonlinear case, we will address this issue of feasibility by a different approach using results from [7].

To address this controllability problem, instead of densities $\rho(t, x)$, we will work with measures that are not necessarily absolutely continuous with respect to the Lebesgue measure. Moreover, to avoid issues regarding compactness, we will frame the problem on a compact smooth manifold M without a boundary. Let $TM = \cup_{x \in M} T_x M$ be the tangent bundle, where $T_x M$ is the tangent space of M at x. Accordingly, the vector-fields $g_i : M \to TM$ will be smooth vector fields defined on M, such that $g_i(x) \in T_x M$ for each $x \in M$. Moreover, we will also assume that the control set U is a compact subset of $\mathbb{R}^d$. For notational convenience we also define the control-dependent vector-field $f : M \times U \to TM$, given by,

$$f(x, \mathbf{v}) = g_0(x) + \sum_{i=1}^{n} v_i g_i(x) \qquad \text{(III.1)}$$

for each $(x, \mathbf{v}) := (x, [v_1 ... v_n]^T) \in M \times U$.

We will need an appropriate notion of the solution of the continuity equation (II.4). Towards this end, given a topological space X, we will denote by $\mathcal{P}(X)$ the set of Borel probability measures on X. The solutions of the PDE will be considered in the following sense. Let $T > 0$. We will say that a narrowly (or weakly) continuous [8] curve $\mu : [0, T] \to \mathcal{P}(M)$ solves the PDE

$$\partial_t \mu + \nabla \cdot (f(\cdot, \mathbf{u}))\mu) = 0, \text{ in } M \times [0, T] \qquad \text{(III.2)}$$

in a weak sense, with initial and terminal conditions, $\mu_0 \in \mathcal{P}(M)$ and $\mu_T \in \mathcal{P}(M)$ respectively, if the following holds,

$$\begin{aligned} &\int_I \int_M [\partial_t g + \partial_x g \cdot f(\cdot, \mathbf{u}(t, \cdot))] d\mu_t(x) dt \\ &= \int_M g(T, x) d\mu_T(x) - \int_M g(0, x) d\mu_0(x) \qquad \text{(III.3)} \end{aligned}$$

for all smooth functions $g \in C^\infty([0, T] \times M)$, where, $\partial_x g$ denotes the differential of g.

In order to address the controllability problem, we will first consider a *relaxed* version of the problem, where instead of looking for control laws that assign to each (t, x) a fixed control in U, we search instead for *Young measure* that assigns to each $t \in [0, T]$ a Borel probability measure on $M \times U$. Towards this end, let X be a topological space. We denote by $\mathcal{Y}(I; X)$ the set of measurable maps $I := [0, T] \ni t \mapsto K_t(\cdot) \in \mathcal{P}(X)$. By measurable, we mean that the function $t \mapsto K_t(A)$ is measurable for each Borel set $A \subseteq X$. We will first establish that, for given $\mu_0 \in \mathcal{P}(X)$ and $\mu_T \in \mathcal{P}(X)$, there exists a $K \in \mathcal{Y}(I; M \times U)$ such that

$$\begin{aligned} &\int_I \int_{M \times U} [\partial_t g + \partial_x g \cdot f(\cdot, \mathbf{v})] dK_t(x, \mathbf{v}) \\ &= \int_M g(T, x) d\mu_T(x) - \int_M g(0, x) d\mu_0(x) \qquad \text{(III.4)} \end{aligned}$$

for all smooth functions $g \in C^\infty([0, T] \times M)$.

Definition III.1. *We will say that a point $x_0 \in M$ is **reachable** from $x_T \in M$ within time $T \in (0, \infty)$ if there exists a measurable function, or control, $\mathbf{v} : [0, T] \to U$ such that solution of the following equation*

$$\dot{x}(t) = f(x(t), \mathbf{v}(t)) \qquad \text{(III.5)}$$

satisfies $x(0) = x_0$ and $x(T) = x_T$.

Given a measurable control $\mathbf{v} : [0, T] \to U$, and trajectory $x : [0, T] \to M$ that is a solution of the differential equation (III.5), $\delta_{x(t)}$ satisfies equation (III.3) for $\mathbf{u}(t, \cdot) = \mathbf{v}(t)$, with $\mu_0 = \delta_{x(0)}$ and $\mu_T = \delta_{x(T)}$. Similarly, $K_t = \delta_{x(t), \mathbf{v}(t)}$ satisfies equation (III.4). From this observation, the following result follows.

Theorem III.2. *Suppose that $x_T \in M$ is reachable from $x_0 \in M$. Suppose $\mu_0 = \delta_{x_0}$ and $\mu_T = \delta_{x_T}$, for some points $x_0, x_T \in M$ such that x_T is reachable from x_0 within time $T > 0$. Then there exists a Young measure $K \in \mathcal{Y}(I; M \times U)$ such that* (III.4) *is satisfied.*

The above result, that relates the reachability properties of the system (III.5) to the reachability properties of the continuity equation (III.2) for the special case of Dirac measures will be fundamental to the controllability result established in this section. The **idea behind the proof** is quite straightforward and is the following. To generalize the controllability result in Theorem III.2 to the controllability of the continuity equation for general initial and target measures, we approximate the measures using sums of Dirac measures and then take the limit. A similar approach was used in [14] to establish controllability for nonlinear discrete-time systems using *stochastic* or measure-valued feedback laws. In contrast, for the continuous-time control-affine case considered in this paper, using results from [7], we will be able to prove a more computationally practical result that there exists a deterministic, albeit possibly very irregular, feedback control law $\mathbf{u}(t, x)$ and a narrowly continuous curve μ_t such that (III.4) holds.

Theorem III.3. *For each $i \in \{1, .., N\}$ and each $j \in \{1, .., L\}$, let $x_0^i, x_T^j \in M$ be such that and x_T^j is reachable from $x_0^i \in M$ within time $T > 0$. Then there exists a $K \in \mathcal{Y}(I; M \times U)$ such that* (III.4) *is satisfied with $\mu_0 = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_0^i}$ and $\mu_T = \frac{1}{L} \sum_{j=1}^{L} \delta_{x_T^j}$.*

Proof. The proof follows from a superposition principle. We know from Theorem III.2 that, For each $i \in \{1, .., N\}$ and each $j \in \{1, .., L\}$, there exists $K_j^i \in \mathcal{Y}(I; M \times U)$ such that the equation (III.4) holds for $\mu_0 = \delta_{x_0^i}$ and $\mu_T = \delta_{x_T^j}$. Then setting $K = \frac{1}{L} \sum_{j=1}^{L} \frac{1}{N} \sum_{i=1}^{N} K_j^i$, we have our result. ☐

In the following theorem and henceforth, let $\text{supp}\ \mu$ denote the support of a measure μ. The statement of the theorem is a natural generalization of the previous Theorem to general initial and target measures.

Proposition III.4. *Let $\mu_0, \mu_T \in \mathcal{P}(M)$ be such that each point in $\text{supp}\ \mu_T$ is reachable by each point in $\text{supp}\ \mu_0$. Then there exists $K \in \mathcal{Y}(I; M \times U)$ such that* (III.4) *is satisfied.*

Proof. There exist sequences of measures of the form $\mu_{0N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_0^i}$, $\mu_{TN} = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_T^i}$ such that $\lim_{N \to \infty} \mu_{0N} = \mu_0$ and $\lim_{N \to \infty} \mu_{TN} = \mu_T$ in the narrow topology [8]. From Proposition III.4, there exists a sequence

K^N such that (III.4) is satisfied with initial and terminal conditions, μ_{0N} and μ_{TN}, respectively, for each $N \in \mathbb{Z}_+$. Since $M \times U$ is compact, we can extract a sub-sequence, again denoted by K^N, that narrowly converges to an element $K \in \mathcal{Y}(I; M \times U)$ [15][Theorem 12.5.9]. The function $h : [0,T] \times M \times U \to \mathbb{R}$, defined by $h(t,x,u) = \partial_x g(t,x) \cdot f(x,u)$, is continuous, for each $g \in C^\infty([0,T] \times M)$. This implies that (III.4) holds with K, initial and terminal conditions, μ_0 and μ_T, respectively. □

In the next result, we establish that we can use the last proposition on controllability for the relaxed problem using Young measures to conclude that the solution can, in fact, be realized using a vector-field that takes values in the set of admissible velocities.

Theorem III.5. *Let $\mu_0, \mu_T \in \mathcal{P}(M)$ be such that each point in* supp μ_T *is reachable by each point in* supp μ_0. *Suppose, additionally that $f(x,U) := \{f(x,r); r \in U\}$ is convex for each $x \in M$. Then there exists a Borel vector-field $V : [0,T] \times M \to TM$ such that* (III.4) *is satisfied and $V(t,x) \in f(x,U)$ for μ_t almost every $x \in M$.*

$$\begin{aligned}\int_I \int_M &[\partial_t g + \partial_x g(t,x) \cdot V(t,x)] d\mu_t(x) dt \\ &= \int_M g(T,x) d\mu_T(x) - \int_M g(0,x) d\mu_0(x) \qquad \text{(III.6)}\end{aligned}$$

Proof. Given $\mu_0, \mu_T \in \mathcal{P}(M)$, let $K \in \mathcal{Y}(I; M \times U)$ be such that (III.4) holds. Let $\eta \in \mathcal{Y}(I;TM)$ be the Young measure obtained by pushing forward the Young measure K using the vector-field f. That is, for almost every $t \in [0,T]$, $\eta_t(A) = K_t(f^{-1}(A))$ for each Borel measurable set $A \subseteq TM$. Since K satisfies the equation (III.4), we can infer that η satisfies

$$\begin{aligned}\int_I \int_{TM} &[\partial_t g + \partial_x g \cdot v] d\eta_t(v) dt \\ &= \int_M g_1(x) d\mu_T(x) - \int_M g(0,x) d\mu_0(x) \qquad \text{(III.7)}\end{aligned}$$

for all smooth functions $g \in C^\infty([0,T] \times M)$. Let $\pi : TM \to M$ be the projection defined by $\pi(v) = x$ for each $v \in TM$ whenever $v \in T_x M$. From [7][Lemma 4.5], there exists a Borel vector-field $V : I \times M \to TM$ such that the following holds,

$$\begin{aligned}\int_I \int_M &\partial_t g + \partial_x g(t,x) \cdot V(t,x) d\mu_t(x) dt \\ &= \int_M g(T,x) d\mu_T(x) - \int_M g(0,x) d\mu_0(x) \qquad \text{(III.8)}\end{aligned}$$

where $\mu_t(A) = \eta_t(\pi^{-1}(A))$ for each Borel measurable set $A \subseteq M$, and $V(t,x) = \int_{T_x M} v d\eta_{t,x}(v)$ is the *barycenter* of $\eta_{t,x}$, a measurable family of of probability measures on $T_x M$, obtained by *disintegration* of η such that $\eta_t(A) = \int_A \eta_{t,x}(A) d\mu_t(x)$. Moreover, for each $t \in [0,T]$, we have that supp $\eta_{t,x} \subseteq$ supp $\eta_t \cap T_x M$ for μ_t almost every $x \in M$. This implies for each $t \in [0,T]$, we have that supp $\eta_{t,x} \subseteq f(M \times U) \cap T_x M$ for μ_t almost every $x \in M$, since η_t is the measure pushforward of K_t under the action of the map f. From this it follows that for each $t \in [0,T]$, we have that supp $\eta_{t,x} \subseteq f(x,U)$. for μ_t almost every $x \in M$. By assumption U is compact and $f(x,U)$ is convex for every $x \in M$. This implies that, for each $t \in [0,T]$, $V(t,x) = \int_{T_x M} v d\eta_{t,x}(v) \in f(x,U)$ for μ_t almost every $x \in M$. □

From the above result we can conclude that, in fact, there exist measurable controls that transport the system from a given measure to a terminal measure, provided that points in the support of the terminal measure and reachable by points in the support of the target measure. This is summarized in the next theorem.

Theorem III.6. *Let $\mu_0, \mu_T \in \mathcal{P}(M)$ be such that each point in* supp μ_T *is reachable by each point in* supp μ_0. *Let U be convex and compact. Then there exists a Borel measurable control $\mathbf{u} : [0,T] \times M \to U$ such that $\mu : [0,T] \to \mathcal{P}(M)$ satisfies* (III.3).

Proof. From Theorem III.5 there exists a Borel measurable vector-field $V : [0,T] \to TM$ and $\mu : [0,T] \to \mathcal{P}(M)$ such that

$$\begin{aligned}\int_I \int_M &[\partial_t g + \partial_x g(t,x) \cdot V(t,x)] d\mu_t(x) dt \\ &= \int_M g_1(x) d\mu_T(x) - \int_M g_0(x) d\mu_0(x) \qquad \text{(III.9)}\end{aligned}$$

and $V(t,x) \in f(x,U)$, for μ_t almost every $x \in M$, since convexity of U implies convexity of $f(x,U)$. Then, it follows from [4][Theorem 8.2.10], that there exists a Borel measurable (feedback) control $\mathbf{u} : [0,T] \times M \to U$ such that $f(x, \mathbf{u}(t,x)) = V(t,x)$ for each $t \in [0,T]$ and each $x \in M$. This concludes the proof. □

IV. Primal Dual Formulations and Numerical Algorithms

In this section, we study the primal dual formulation of problem (II.5) and provide fast numerical algorithms to calculate the control $u_i(t,x)$ and its sub-Riemannian optimal transport cost $W_p(\rho_0, \rho_T)$ for $p = 1, 2$, where

$$W_p(\rho_0, \rho_T)^p = \inf_{u,\rho} \Big\{ \int_0^T \int_\Omega \tfrac{1}{p} \sum_{i=1}^n \rho(t,x) |u_i(t,x)|^p dx dt \quad \text{s.t. (II.4) holds.} \Big\}$$

Henceforth, for computational purposes, we will consider the optimization problem over control laws $\mathbf{u}$ that are unconstrained and can take values in $\mathbb{R}^d$.

A. Sub-Riemannian Wasserstein-2 Distance (W_2)

For $p = 2$, we introduce Lagrange multiplier $\phi(t,x)$ to handle the transport equation constraint and define a Hamiltonian as

$$H(x,p) = \sup_v f(x,v) \cdot p - L(v)$$

Since $L(v) = \frac{1}{2}\|v\|^2$, we can explicitly compute H as follows:

$$H(x,p) = \frac{1}{2} p \cdot \left(g_0(t,x) + G(x)^T G(x) p\right) \tag{IV.1}$$

where $G(x) := [g_1(x), g_2(x), ..., g_n(x)]^T$ is the vector formation of $g_i(x)$. Using Lagrange multipliers, and via integration by parts, we get the following:

$$W_2\left(\rho_0, \rho_T\right)^2 = \inf_\rho \sup_\phi \Big\{ \textstyle\int_0^T \int_{\mathbb{R}^d} -\rho\left(\phi_t + H\left(x, \nabla\phi\right)\right) dxdt \\ + \textstyle\int_\Omega \rho_0(x)\phi(0,x) - \rho_T(x)\phi(T,x) dx \Big\}.$$

Hence, solving the original problem (II.5) is equivalent as solving the following dual problem:

$$W_2\left(\rho_0, \rho_T\right)^2 = \sup_\phi \Big\{ \int_{\mathbb{R}^d} \rho_T(x)\phi(T,x) - \rho_0(x)\phi(0,x) dx \\ \text{s.t.} \quad \phi_t(t,x) + H(x, \nabla\phi(t,x)) \le 0 \Big\}. \tag{IV.2}$$

To turn the previous problem into a convex-concave optimization problem, we introduce

$$m_i(t,x) = \rho(t,x) u_i(t,x) \quad i = 1, ..., n, \\ m = [m_1, ..., m_n]^T.$$

Furthermore, we denote by

$$\mathcal{L}(m,\rho,\phi) = \int_{\mathbb{R}^d} \rho_T(x)\phi(T,x) - \rho_0(x)\phi(0,x) dx \\ + \int_0^T \int_{\mathbb{R}^d} -\rho\left(\phi_t + \nabla\phi \cdot g_0(t,x)\right) \\ -\nabla\phi \cdot \left(\sum_{i=1}^n m_i(t,x) g_i(x)\right) + \sum_{i=1}^n \frac{|m_i(t,x)|^2}{2\rho} dxdt.$$

Thus, the unconstrained min-max problem can be written as

$$\inf_{m,\rho} \sup_\phi \mathcal{L}(m,\rho,\phi). \tag{IV.3}$$

From the KKT conditions, the minimizer satisfies

$$\begin{cases} \partial_t \rho + \nabla \cdot (g_0 \rho) + \sum_{i=1}^n \nabla \cdot (m_i g_i) = 0, \\ \dfrac{m(t,x)}{\rho(t,x)} = G(x)^T \nabla\phi(t,x) \\ \phi_t(t,x) + H(x, \nabla\phi(t,x)) \le 0 \end{cases}$$

Note that when $\rho(t,x) > 0$, the last inequality becomes equality, which is a HJB equation of ϕ.

B. G-prox PDHG for sub-Riemannian W_2

We apply G-prox[20] version of Primal-Dual Hybrid Gradient Algorithm (PDHG) to solve the saddle point problem (IV.3). Specifically, when we solve for proximal step of the dual variable, we use $\|\cdot\|^2_{H^1_{t,x}}$ instead of $\|\cdot\|^2_{L^2}$, where

$$\|w(t,x)\|^2_{H^1_{t,x}} = \|\partial_t w(t,x)\|^2_{L^2} + \|\partial_x w(t,x)\|^2_{L^2}$$

This is the crucial step that makes the convergence rate of our numerical algorithm to be independent of grid size. If $L^2_{t,x}$-type of proximal step is implemented, the convergence rate scales linearly with the number of grid points, which will slow down the convergence as refining the mesh gird. In fact, $H^1_{t,x}$ type of proximal step gives an insight of the regularity of primal variables [20].

With $\tau_\phi, \tau_\rho, \tau_m$ are our choice of stepsizes, at the k-th iteration, the PDHG update reads as follows

$$\begin{cases} (\rho^{k+1}, m^{k+1}) = \text{argmin}_{\rho,m} \mathcal{L}(\bar{\phi}^k, \rho, m) + \frac{1}{2\tau_\rho}\|\rho - \rho^k\|^2_{L^2_{t,x}} \\ \qquad\qquad + \frac{1}{2\tau_m}\|m - m^k\|^2_{L^2_{t,x}} \\ \phi^{k+1} = \text{argmax}_\phi \mathcal{L}(\phi, \rho^{k+1}, m^{k+1}) - \frac{1}{2\tau_\phi}\|\phi - \phi^k\|^2_{H^1_{x,t}} \\ \bar{\phi}^{k+1} = 2\phi^{k+1} - \phi^k. \end{cases}$$

Step 1. Equation for ρ:

$$\bar{\phi}^k_t + \nabla\bar{\phi}^k \cdot g_0(t,x) + \frac{\|m\|^2}{2\rho^2} - \frac{\rho - \rho^k}{\tau_\rho} = 0.$$

Equation for m:

$$-G(x)\nabla\bar{\phi}^k + \frac{m}{\rho} + \frac{m - m^k}{\tau_m} = 0.$$

Step 2. Equation for ϕ:

$$\begin{cases} \rho^{k+1}_t + \text{div}(\rho^{k+1} g_0(t,x) + G(x)^T m^{k+1}) \\ \quad + \frac{\phi_{tt} - \phi^k_{tt}}{\tau_\phi} + \frac{\Delta_x(\phi - \phi^k)}{\tau_\phi} = 0, \\ \rho^{k+1}(0,x) - \rho_0(x) + \frac{\phi_t(0,x) - \phi^k_t(0,x)}{\tau_\phi} = 0, \\ \rho^{k+1}(T,x) - \rho_T(x) + \frac{\phi_t(T,x) - \phi^k_t(T,x)}{\tau_\phi} = 0, \end{cases}$$

Note here, the updates for $\phi(x)$ corresponds to solving a Poisson equation with Neumann boundary (in time), which has infinite many solutions that are different by a constant. To get a unique solution, we impose another boundary condition:

$$\int_{\mathbb{R}^d} \phi(T,x)\ dx = 0.$$

We will implement this algorithm in numerical examples presented in section V.

C. sub-Riemannian Wasserstein-1 Distance (W_1)

When $p = 1$, we are solving for sub-Riemannian optimal transport cost for the so-called Manhattan distance metric.

$$W_1\left(\rho_0, \rho_T\right) = \inf_{u,\rho} \Big\{ \int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^n \rho(t,x)\|u(t,x)\|_1\ dxdt \\ \text{s.t. (II.4) holds} \Big\} \tag{IV.4}$$

where $\|u\|_1 = \sum_i |u_i|$. We introduce flux m

$$m(x) = \int_0^T \rho(t,x) u(t,x)\ dt.$$

According to Jensen's inequality,

$$\int_0^T \int_{\mathbb{R}^d} \rho(t,x)\|u(t,x)\|_1\ dxdt \ge \int_{\mathbb{R}^d} \|m(x)\|_1\ dx$$

By Integrating with respect t on $[0,T]$ for (II.4), we have the following:

$$\rho_T(x) - \rho_0(x) + \nabla \cdot \left(G(x)^T m(x)\right) = 0, \text{ for all } x \in \mathbb{R}^d. \tag{IV.5}$$

It is easy to verify that solving (IV.4) is equivalent to solving the following

$$\inf_m \left\{ \int_{\mathbb{R}^d} \|m(x)\|_1 \, dx \text{ s.t. (IV.5) holds} \right\} \tag{IV.6}$$

via adapting the proof in [22]. However, (IV.6) can have multiple minimizers as the objective function is not strictly convex. To remedy this issue, we add quadratic regularization with a small ϵ.

$$\inf_m \left\{ \int_{\mathbb{R}^d} \|m(x)\|_1 \; + \epsilon \|m(x)\|_2^2 dx \text{ s.t. (IV.5) holds} \right\} \tag{IV.7}$$

Now as our objective function is strictly convex, we introduce Lagrangian multiplier $\phi(x)$ and define $\mathcal{L}(m,\phi)$ as follows:

$$\mathcal{L}(m,\phi) = \int_{\mathbb{R}^d} \|m(x)\|_1 + \epsilon \|m(x)\|_2^2 dx$$
$$+ \int_{\mathbb{R}^d} \phi(x) \left(\rho_T(x) - \rho_0(x) + \nabla \cdot \left(G(x)^T m(x)\right)\right) \, dx,$$

Hence, we can solve (IV.4) by solving the min-max problem

$$\inf_m \sup_\phi \mathcal{L}(m,\phi). \tag{IV.8}$$

D. G-prox PDHG for sub-Riemannian W_1

To solve (IV.8), we adapted both G-prox version of PDHG and shrink operator techniques from [22], [20], with τ_ϕ, τ_m are our choice of stepsizes. At the k-th iteration,

$$\begin{cases} m^{k+1} = \text{argmin}_m \; \mathcal{L}(\bar{\phi}^k, m) + \frac{1}{2\tau_m}\|m - m^k\|_{L_x^2}^2 \\ \phi^{k+1} = \text{argmax}_\phi \; \mathcal{L}(\phi, m^{k+1}) - \frac{1}{2\tau_\phi}\|\phi - \phi^k\|_{H_x^1}^2 \\ \bar{\phi}^{k+1} = 2\phi^{k+1} - \phi^k, \end{cases}$$

where $\|w(t,x)\|_{H_x^1}^2 = \|\partial_x w(x)\|_{L^2}^2$. The detail of updates are as follows:

Step 1. Equation for m:

$$m_i(x)^{k+1} = \text{shrink}_1\left(m_i(x)^k + \tau_m \left(G(x)\nabla\phi(x)\right)_i, \tau_m\right).$$

The shrink operator shrink_1 is defined as

$$\text{shrink}_1(v,\mu) = \begin{cases} \left(1 - \frac{\mu}{|v|}\right) v & \text{for } |v| \geq \mu \\ 0 & \text{for } |v| < \mu. \end{cases}$$

Step 2. Equation for ϕ:

$$\rho_1(x) - \rho_0(x) + \text{div}(G(x)^T m^{k+1}) + \frac{\Delta_x(\phi - \phi^k)}{\tau_\phi} = 0.$$

Similar to the sub-Riemannian W_2 case, to get a unique solution, we impose another condition:

$$\int_{\mathbb{R}^d} \phi(x) \, dx = 0.$$

E. Reconstruct control u from m for W_1

Given an $m(x)$ feasible for (IV.6), denote

$$\rho(t,x) = \frac{t}{T}\rho_1(x) + \frac{T-t}{T}\rho_0(x)$$
$$u(t,x) = \frac{m(x)}{\rho(t,x)}$$

Then $u(t,x)$ is feasible for (IV.4) and has the same objective value as $m(x)$ does for (IV.6).

In the case that $\rho(t,x)$ vanishes while $m(x) \neq 0$ at some $x \in \mathbb{R}^d$, for instance, the support of ρ_0, ρ_T do not overlap, we modify the initial-terminal density with

$$\hat{\rho}_0 = \rho_0 + \delta\rho_{\text{unif}}(x),$$
$$\hat{\rho}_T = \rho_T + \delta\rho_{\text{unif}}(x)$$

where ρ_{unif} is the uniform distribution in Ω with some small $\delta > 0$. To recover the control, we calculate the following:

$$\hat{u}(t,x) = \frac{m(x)}{\frac{t}{T}\hat{\rho}_T(x) + \frac{T-t}{T}\hat{\rho}_0(x)} = \frac{m(x)}{\frac{t}{T}\rho_T(x) + \frac{T-t}{T}\rho_0(x) + \delta}.$$

Note that if (ρ_0, ρ_T, m) solves (IV.5), $(\hat{\rho}_0, \hat{\rho}_T, m)$ also satisfies (IV.5).

V. Numerical Examples

In this section, we demonstrate two sub-Riemannian optimal transport examples solved using finite difference method for $p = 1, 2$ on a bounded domain $\Omega \subset \mathbb{R}^d$.

A. Discretization and Optimization Parameters

To illustrate the sub-Riemannian W_2 model on graph, we first consider the discretization in one spatial dimension $[0,1]$ with uniform spatial mesh size Δx and temporal mesh size Δt. For $x_j = j\Delta x, t_l = l\Delta t$, define

$$\rho_j^l = \rho(t_l, x_j) \quad 1 \leq j \leq M_x, 1 \leq l \leq N_t$$
$$m_{j+\frac{1}{2}}^l = m(t_l, x_{j+\frac{1}{2}}) \quad 1 \leq j \leq M_x, 1 \leq l \leq N_t$$
$$\phi_j^l = \phi(t_l, x_j) \quad 1 \leq j \leq M_x, 1 \leq l \leq N_t$$
$$g_{i,j}^l = g_i(x_j) \quad 1 \leq j \leq M_x, 1 \leq i \leq d$$

For simplicity, we consider the case that $g_0 = 0$ and $d = 1$. The Fokker-Planck equation discretized with forward difference in time as follows:

$$\frac{1}{\Delta t}\left(\rho_j^{l+1} - \rho_j^l\right) + \frac{1}{\Delta x}\left(g_{1,j+\frac{1}{2}} m_{j+\frac{1}{2}}^l - g_{1,j-\frac{1}{2}} m_{j-\frac{1}{2}}^l\right) = 0,$$

The coupling HJB equation is discretized with backward difference in time as follows:

$$\frac{1}{\Delta t}\left(\phi_j^l - \phi_j^{l-1}\right) + H\left(x_j, \frac{\phi_{j+1}^l - \phi_j^l}{\Delta x}\right) = 0,$$

The extension to multi-spatial dimension is straight forward. As for W_1, we choose $\epsilon = 10^{-3}$. We have the discretized variables $\rho_{0,j}, \rho_{T,j}, m_j$ defined on spatial domain:

$$(\rho_{0,j} - \rho_{T,j}) + \frac{1}{\Delta x}\left(g_{1,j+\frac{1}{2}} m_{j+\frac{1}{2}} - g_{1,j-\frac{1}{2}} m_{j-\frac{1}{2}}\right) = 0.$$

As for the choice of optimization step sizes, in order to guarantee the convergence, we usually have to refine the optimization stepsize as we refine the mesh grid on the graph. This leads to deceleration of the convergence. Thanks to [20], we only need $\tau_m \tau_\phi < \lambda_{\max}(G(x))$, $\tau_\rho \tau_\phi < \lambda_{\max}(G(x))$ to guarantee convergence of PDHG. $\lambda_{\max}(\cdot)$ denotes the largest eigenvalues of the operator in the discrete setting.

In the following numerical experiments, we have $\lambda_{max}(G(x)) \leq 1$. Thus, we set $\tau_m = \tau_\rho = \tau_\phi = 0.99$. As for stopping criteria, we track the residuals for the HJB equation and the Fokker-Planck equation:

$$R_{\text{HJB}} = \| (\phi_t + H(x, \nabla\phi))_+ \|_{L^2([0,T]\times\Omega)}$$
$$R_{\text{FP}} = \|\partial_t \rho + \nabla \cdot (g_0 \rho) + \sum_{i=1}^{n} \nabla \cdot (m_i g_i) \|_{L^2([0,T]\times\Omega)}$$

We run the algorithm for total number of iterations $n_{\text{itr}} = 10^4$ or stop when $R_{\text{HJB}}, R_{\text{FP}} < 10^{-3}$

B. Grushin Plane Model

Grushin plane is a 2-D driftless system with control vector fields $g_0 = [0,0]^T$, $g_1(x_1,x_2) = [1,0]^T$, $g_2(x_1,x_2) = [0, \sin(2\pi x_1)]^T$ on $[0,T]\times\Omega = [0,1]\times[0,1]^2$, with non-flux boundary condition:

$$\begin{cases} \dot{x}_1 = u_1, \\ \dot{x}_2 = u_2 x_1. \end{cases}$$

We have $d = n = 2$, specifically, the dimension of control is the same as the dimension of system, except there exists a ‘singularity’ at $x_1 = 0.5$ along x_2-direction.

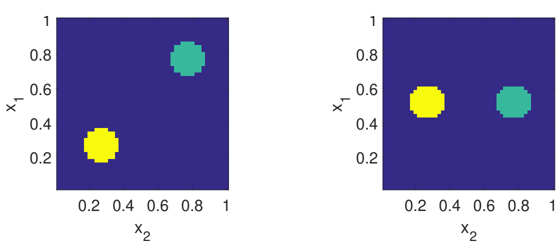

Fig. 1. Initial (green) and terminal (yellow) measure of the optimal transport on Grushin plane for case 1 (left) and case 2 (right)

We solve the discretized system with mesh grid $64^2 \times 64$ in space-time for two sets of initial and terminal densities shown in Figure1:

- For case 1, ρ_0 is uniformly supported in the disk $d_0 = \{(x_1,x_2)|(x_1-0.75)^2 + (x_2-0.75)^2 \leq 0.1^2$, while ρ_1 is uniformly supported in the disk $d_1 = \{(x_1,x_2)|(x_1 - 0.25)^2 + (x_2 - 0.25)^2 \leq 0.1^2$.
- For case 2, ρ_0 ρ_1 are uniformly supported in a disk with radius $r = 0.1$, centered at $(0.5, 0.75)$ and $(0.5, 0.25)$ respectively.

For sub-Riemannian W_2, see Figure 2, 3 for change of ρ with respect to time.

For sub-Riemannian W_1, we plot $(G^T m)(x)$ as red arrows in Figure 4. The densities travels along different trajectories as to avoid the ‘singularity’ at $x_1 = 0.5$ along x_2-direction.

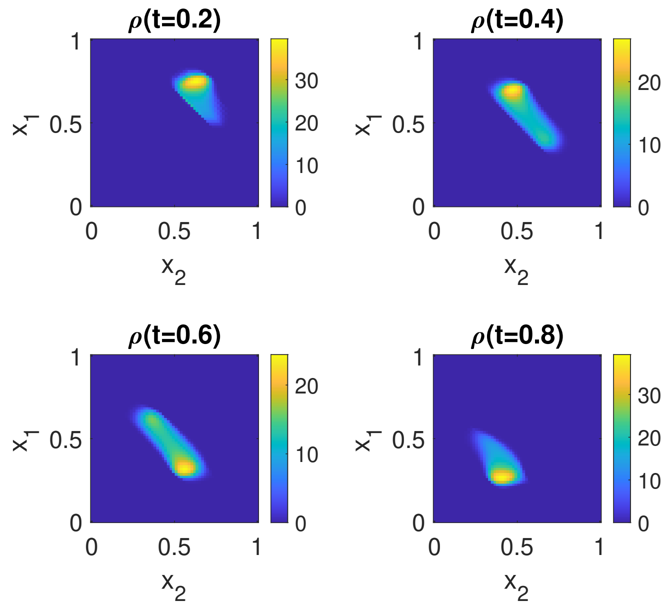

Fig. 2. Grushin plane W_2 case 1

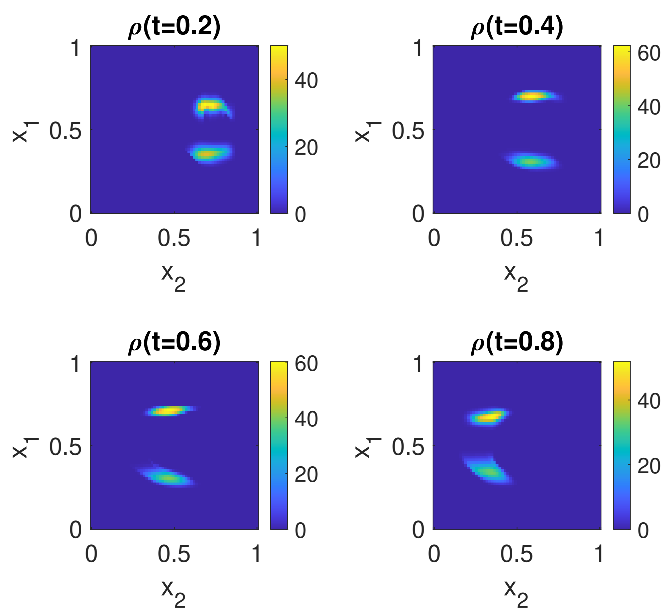

Fig. 3. Grushin plane W_2 case 2

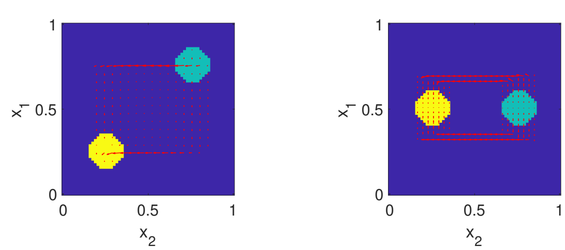

Fig. 4. Grushin plane W_1 case 1 (left) and case 2 (right)

C. Unicycle Model

We consider the sub-Riemannian optimal transport in a three-dimensional unicycle model, which is the classic dubins car that describes vehicle kinematics. The states are cartesian coordinates $(x_1, x_2) \in [0,1]^2$, and orientation $\theta \in \mathcal{S}^1$. The system equations on $\Omega = \mathcal{S}^1 \times [0,1]^2$ are given by

$$\begin{cases} \dot{\theta} = u_1, \\ \dot{x_1} = u_2 \cos\theta, \\ \dot{x_2} = u_2 \sin\theta, \end{cases}$$

where the change of x_1, x_2 depends on the translation speed u_2 and angular of the car θ, while the angle θ depends on the steering speed u_1. The above system is a driftless system with control vector fields

$$g_1(\theta, x_1, x_2) = [1, 0, 0]^T$$
$$g_2(\theta, x_1, x_2) = [0, \cos\theta, \sin\theta]^T.$$

We discretize the domain with mesh grid $32^3 \times 32$ in space-time and compute optimal transport solutions for two scenarios for $[0,T] = [0,1]$ as shown in Figure 5:

- In the first case, ρ_0 is supported in a ball centering at $(\theta, x_1, x_2) = (0, 0.1, 0.5)$ with radius $r = 0.1$, and ρ_1 is uniformly supported in balls centering at $(\theta, x_1, x_2) = (0, 0.9, 0.1)$ and $(\theta, x_1, x_2) = (0, 0.9, 0.9)$ with radius $r = 0.1$.
- In the second case, ρ_0 is supported in a ball centering at $(\theta, x_1, x_2) = (0, 0.1, 0.5)$ with radius $r = 0.1$, and ρ_1 is uniformly supported in balls centering at $(\theta, x_1, x_2) = (\frac{3\pi}{2}, 0.9, 0.1)$ and $(\theta, x_1, x_2) = (\frac{\pi}{2}, 0.9, 0.9)$ with radius $r = 0.1$.

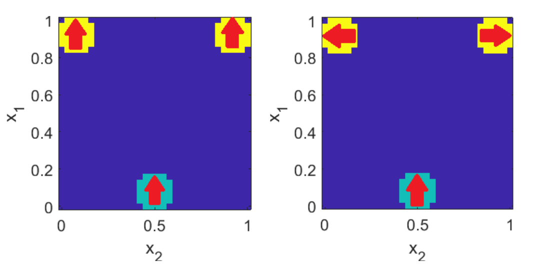

Fig. 5. The initial and terminal distributions of unicycle model for two scenarios: case 1(left) and case 2 (right). The red arrows indicate the first coordinate θ

Different final density orientations lead to two qualitatively different controls. For case 1 in W_2 solution shown in 6, the densities steer and then split to their final positions. The optimal transport solution for the second case in W_2 is shown in Fig.7. They reach x_1-direction position first, and then split to move horizontally towards final positions.

We see that in W_1, the flux (red arrows) recovered by $\left(G^T m\right)(x)$ in Figure 8 follows similar trajectories as the W_2 type.

VI. CONCLUSION

In this paper, we studied the fluid dynamic formulation for optimal transport of nonlinear control-affine systems.

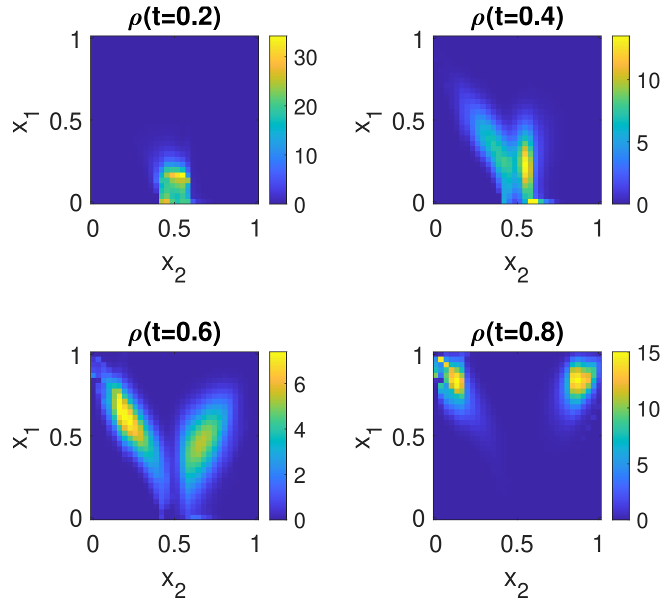

Fig. 6. Unicycle model W_2 case 1

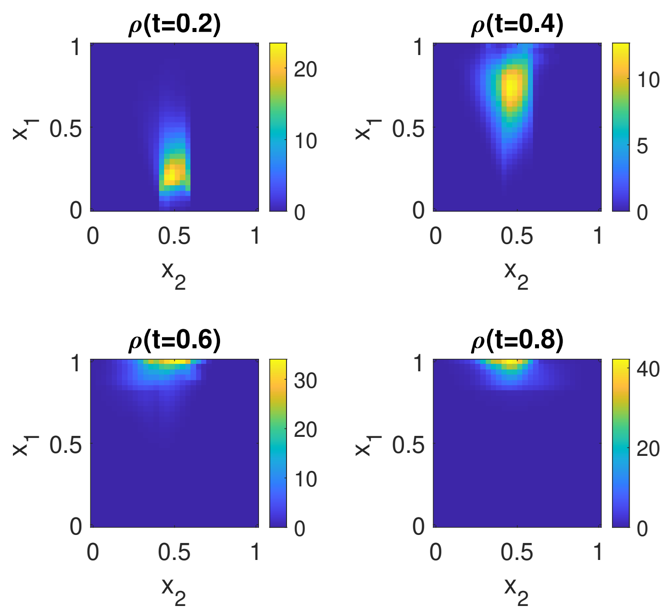

Fig. 7. Unicycle model W_2 case 2

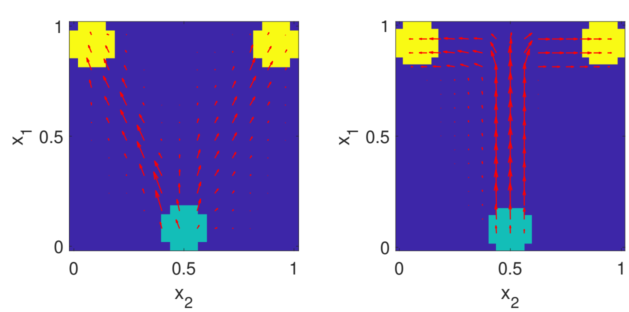

Fig. 8. Unicycle W1 case 1 (left) and case 2 (right)

We establish controllability of the continuity equation for control-affine systems. We also implemented fast algorithms for computing the control laws for the corresponding optimal transport problems, with convergence rate independent of grid size.

There are many natural directions in future work. One possible direction is to formulate mean-field game problems for control-affine systems. Another possibility is to use this formulation to study degenerate partial differential equations as gradient flows on metric spaces corresponding to control affine systems. We expect that it would be useful in understanding the convergence property of gradient flows studied in [16].

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