

A Rotating-Grid Upwind Fast Sweeping Scheme for a Class of Hamilton-Jacobi Equations

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Abstract

We present a fast sweeping method for a class of Hamilton-Jacobi equations that arise from time-independent problems in optimal control theory. The basic method in two dimensions uses a four point stencil and is extremely simple to implement. We test our basic method against Eikonal equations in different norms, and then suggest a general method for rotating the grid and using additional approximations to the derivatives in different directions in order to more accurately capture characteristic flow. We display the utility of our method by applying it to relevant problems from engineering.

1 Introduction

The general Hamilton-Jacobi (HJ) equation in d -dimensions is given by

$$H(x, \nabla \phi(x)) = 0, \quad x \in \Omega \quad (1)$$

where $\Omega \subset \mathbb{R}^d$ and $H : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the *Hamiltonian* function. Along with equation (1), one is often supplied boundary data $\phi(x) = g(x)$ on a set $\Gamma \subset \mathbb{R}^d$, which typically has dimension smaller than d . Common scenarios are $\Gamma = \partial\Omega$ or $\Gamma = \{x_0\}$, a single point. These equations have diverse application in fields including traffic modeling [29], medical imaging [31], path-planning [39], and dynamic visibility [26, 33, 54] to name a few.

The fast sweeping method is a type of finite difference scheme used to approximate (1). The basic strategy involves discretizing the domain and devising update rules

$$u_i = F_i(u_j|_{j \in N(i)}) \quad (2)$$

that locally approximate the equation at grid nodes i , where $N(i)$ is comprised of the nodes in some neighborhood of node i . Using these update rules, one sweeps through the domain in the Gauss-Seidel manner, iteratively updating the solution values at grid nodes until a steady state is reached. As far as this author can discern, the fast sweeping method was first used by Boué and Dupuis [9] and Zhao et al. [60]. Shortly afterwards, there was much work toward developing fast sweeping methods for different types of Hamiltonians, and using different

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strategies for numerical approximation [23, 24, 25, 55, 61]. Subsequent work was devoted to adapting fast sweeping methods to irregular grids [41, 42], improving the accuracy [28, 30], and extending them to other equations, such as conservation laws [18, 19]. Luo and Zhao [32] provide a nice overview of fast sweeping methods, which we will refer to in section 3.1.

Besides fast sweeping schemes, other grid-based methods used to approximate steady-state HJ equations can be largely divided into two categories. The first category is fast marching methods for monotonically advancing fronts, pioneered by Tsitsiklis [56]. These rely on a single-pass, Dijkstra-type algorithm to update that solution value at grid nodes as characteristics flow outward from boundary data [1, 2, 46, 47, 48]. The second category is time-dependent methods. Osher [35] showed that in many cases one can recast the steady-state HJ equation in a time-dependent manner, and there are very general methods which can approximate time-dependent HJ equations at high accuracy, and also allow for non-monotonic flow of information [22, 37, 49]. More recently, there has been increased interest in algorithms for numerical solutions of HJ equations which break the curse of dimensionality. These typically rely on Hopf-Lax or Lax-Oleinik type formulas for time-dependent HJ equations, and use optimization routines to approximate the solution at individual points [11, 16, 27]. However, due to the wide applicability and relative ease of both implementation and analysis, fast sweeping methods have remained a popular option for approximating solutions of steady-state HJ equations.

We present an exceedingly simple fast sweeping scheme for a class of Hamilton-Jacobi equations arising from optimal control theory. For ease of exposition, we develop our method in two spatial dimensions. The method applies in higher dimensions, though for dimensions $d > 3$, one will encounter the curse of dimensionality. In two dimensions, our most basic method includes a four-point stencil on a rectangular grid, using only the ordinary forward and backward difference operators. We then describe a general method for using rotated coordinates to improve the accuracy of the scheme. We implement our method with special application toward Eikonal equations in different norms, and also mention a few other applications.

2 Hamilton-Jacobi Equations in Optimal Control Theory

We will address a specific class of Hamilton-Jacobi equations arising from deterministic optimal control theory. A basic problem in optimal control theory is to choose the best control plan $\mathbf{a} : [0, T] \rightarrow A$ to steer a trajectory \mathbf{x} obeying

$$\begin{aligned}\dot{\mathbf{x}}(t) &= f(\mathbf{x}(t), \mathbf{a}(t), t), \quad 0 < t \leq T, \\ \mathbf{x}(0) &= x_0,\end{aligned}\tag{3}$$

to an optimal destination $\mathbf{x}(T)$. Here $A \subset \mathbb{R}^m$ is the set of admissible control actions and $f : \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \rightarrow \mathbb{R}^d$ is a function describing the dynamics along the trajectory. The “optimal destination” is determined in view of a cost functional

$$C[\mathbf{x}(\cdot), \mathbf{a}(\cdot)] = g(\mathbf{x}(T)) + \int_0^T r(\mathbf{x}(t), \mathbf{a}(t), t) dt\tag{4}$$

that one wishes to minimize. The function $r : \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \rightarrow \mathbb{R}$ accounts for a running cost along the trajectory, and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is an exit cost. While it is not necessary in all cases, we will assume that $r, g \geq 0$ which is common in many applications where cost cannot be negative. To analyze this problem using dynamic programming [7, 8], one defines the value function $\phi : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}$ by

$$\phi(x, t) := \inf_{\mathbf{x}(\cdot), \mathbf{a}(\cdot)} C_{x,t}[\mathbf{x}(\cdot), \mathbf{a}(\cdot)] \quad (5)$$

where $C_{x,t}[\mathbf{x}(\cdot), \mathbf{a}(\cdot)]$ is the remaining cost functional, restricted to trajectories \mathbf{x} on the time interval $(t, T]$ and satisfying $\mathbf{x}(t) = x$. Thus ϕ is the optimal remaining cost for a trajectory that is at position x at time t . Under mild conditions on the data, this value function is the unique viscosity solution [14] of the terminal value Hamilton-Jacobi-Bellman equation [3, 5]

$$\begin{aligned} \phi_t(x, t) + \inf_{a \in A} \left\{ \langle f(x, a, t), \nabla \phi(x, t) \rangle + r(x, a, t) \right\} &= 0, \\ \phi(x, T) &= g(x). \end{aligned} \quad (6)$$

Note that the viscosity solution of (6) should remain non-negative: by (5), ϕ is non-negative whenever r and g are non-negative.

We observe that (6) is of the form (1) if we consider generalized coordinates $\tilde{x} = (t, x)$ and $\nabla_{\tilde{x}} = (\partial_t, \nabla_x)$. In this case $\Omega = \mathbb{R}^d \times [0, T)$ and $\Gamma = \mathbb{R}^d \times \{T\}$. Thus this can be analyzed in the framework of the more general equation (1), but time-dependent equations like (6) are so ubiquitous in application that they are often analyzed independently. Indeed, in their two original papers, Crandall and Lions established the notion of viscosity solutions specifically for time-dependent Hamilton-Jacobi equations [13, 14], and later the theory was extended to more general equations; see, for example, [12].

2.1 Our Class of Equations

We restrict our focus to a special class of optimal control problems. We consider the case that the dynamic function f does not depend explicitly on t , and the running cost function r does not depend explicitly on either t or $\mathbf{a}(\cdot)$. The removal of the explicit dependence on t is not a particularly stringent condition; this is very natural many applications. Removing the dependence of r on $\mathbf{a}(\cdot)$ is a more serious restriction. For example, this will exclude essentially any problem from mathematical finance where the control variable could represent the fraction of capital one wishes to invest, or the amount of goods a company would like to produce [40]. In this case, the cost and profit very explicitly depend on the value of the control variable. However, control problems of this our type still have diverse application. Minimal-time path-planning [39] and reach avoid games [62] are two classical problems in applied optimal control theory that fit into this framework. Otherwise, four of the five examples given by Evans [20, chap. 1] fall into this category. This includes the moon lander problem, optimally stopping a pendulum, and a model for growth of ant colonies originally proposed by Oster and Wilson [38].

When neither f nor r depend on t , one can neglect the time horizon T and formulate a steady-state Hamilton-Jacobi-Bellman equation for the value function. Given that r does

not depend on $\mathbf{a}(\cdot)$, this takes the form

$$-r(x) = \inf_{a \in A} \{ \langle f(x, a), \nabla \phi(x) \rangle \}, \quad (7)$$

or alternately

$$-r(x) = \inf_{a \in A} \left\{ \sum_{\ell=1}^d f_{\ell}(x, a) \phi_{x_{\ell}}(x) \right\} \quad (8)$$

where $x = (x_1, \dots, x_d)$ and $f(x, a) = (f_1(x, a), \dots, f_d(x, a))$. We focus on numerical solutions for this equation with boundary data $\phi(x) = g(x)$ on a set $\Gamma \subset \mathbb{R}^d$. For example, in the case of optimal-time path-planning, we will take $\Gamma = \{x_f\}$, where $x_f \in \mathbb{R}^d$ is the desired ending point, and let $\phi(x_f) = 0$. This signifies that paths ending at the desired location incur no exit cost, while other paths are not admissible (i.e., they incur infinite cost).

Many classical Hamilton-Jacobi equations can be expressed in this form. Notably, the Eikonal equation

$$1 = v(x) |\nabla \phi(x)| \quad (9)$$

is of this form. The travel-time function for isotropic motion $\dot{\mathbf{x}}(t) = v(\mathbf{x}(t))\mathbf{a}(t)$, where $\mathbf{a}(\cdot)$ is a unit vector, is the viscosity solution of this equation, and in the case that $v(x) \equiv 1$, this yields a signed distance function [36]. Assuming $v > 0$, equation (9) can be re-written

$$-1/v(x) = \inf_{a \in \mathbb{S}^{d-1}} \{ a \cdot \nabla \phi \} \quad (10)$$

whereupon casting the equation in the form (8) is accomplished by parameterizing the unit sphere \mathbb{S}^{d-1} . For example in dimension $d = 2$, we have

$$-1/v(x, y) = \inf_{a \in [0, 2\pi)} \{ \phi_x \cos(a) + \phi_y \sin(a) \}, \quad (11)$$

or in dimension $d = 3$,

$$-1/v(x, y, z) = \inf_{a, b} \{ \phi_x \cos(a) \cos(b) + \phi_y \sin(a) \cos(b) + \phi_z \sin(b) \}, \quad (12)$$

where $(a, b) \in [0, 2\pi) \times [-\pi/2, \pi/2]$ represent the xy -planar angle and the angle of inclination from the xy -plane, respectively. We return to Eikonal equations when testing our method in section 3.2 and section 4.1.

3 A Basic Fast Sweeping Scheme for (8)

As stated in section 1, for simplicity of exposition, we will describe our fast sweeping scheme in dimension $d = 2$. We consider a rectangular domain $[x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}]$ and a uniform grid discretization with $I + 1$ points in the x -direction, and $J + 1$ points in the y -direction. Thus the grid is given by

$$\begin{aligned} x_i &:= x_{\min} + i\Delta x, & \Delta x &= \frac{x_{\max} - x_{\min}}{I}, & i &= 0, 1, \dots, I, \\ y_j &:= y_{\min} + j\Delta y, & \Delta y &= \frac{y_{\max} - y_{\min}}{J}, & j &= 0, 1, \dots, J. \end{aligned} \quad (13)$$

In two-dimensions, the equation of interest is

$$-r(x, y) = \inf_{a \in A} \left\{ f_1(x, y, a) \phi_x(x, y) + f_2(x, y, a) \phi_y(x, y) \right\}. \quad (14)$$

Let ϕ_{ij} be the numerical approximation to $\phi(x_i, y_j)$, and for a fixed $a \in A$, let $f_{\ell,ij}(a) = f_{\ell}(x_i, y_j, a)$ for $\ell = 1, 2$. Further let

$$\xi_{\ell,ij}(a) = \text{sign}(f_{\ell}(x_i, y_j, a)), \quad \ell = 1, 2. \quad (15)$$

Then the upwind approximations to the derivatives are given by

$$\begin{aligned} \left(f_1(x, y, a) \phi_x(x, y) \right)_{ij} &= |f_{1,ij}(a)| \frac{\phi_{i+\xi_{1,ij}(a),j} - \phi_{ij}}{\Delta x}, \\ \left(f_2(x, y, a) \phi_y(x, y) \right)_{ij} &= |f_{2,ij}(a)| \frac{\phi_{i,j+\xi_{2,ij}(a)} - \phi_{ij}}{\Delta y}. \end{aligned} \quad (16)$$

Supposing that a is the correct control value at the node (i, j) , we can insert these approximations into (14) to arrive at

$$-r_{ij} = |f_{1,ij}(a)| \frac{\phi_{i+\xi_{1,ij}(a),j} - \phi_{ij}}{\Delta x} + |f_{2,ij}(a)| \frac{\phi_{i,j+\xi_{2,ij}(a)} - \phi_{ij}}{\Delta y}, \quad (17)$$

where $r_{ij} = r(x_i, y_j)$. Isolating ϕ_{ij} , we see that

$$\phi_{ij}^*(a) = \frac{r_{ij} + \frac{|f_{1,ij}(a)|}{\Delta x} \phi_{i+\xi_{1,ij}(a),j} + \frac{|f_{2,ij}(a)|}{\Delta y} \phi_{i,j+\xi_{2,ij}(a)}}{\frac{|f_{1,ij}(a)|}{\Delta x} + \frac{|f_{2,ij}(a)|}{\Delta y}} \quad (18)$$

is a first-order upwind approximation to equation (14), when a is the correct control value at node (i, j) . This suggests the fast sweeping scheme detailed in algorithm 1.

We include some comments regarding the algorithm. First, at each iteration, we sweep through the indices in alternating directions until all combinations of sweeping directions have been performed. Thus each iteration consists of four sweeps; in MATLAB notation:

- (1) $i = 1 : I - 1, \quad j = 1 : J - 1,$
- (2) $i = 1 : I - 1, \quad j = J - 1 : -1 : 1,$
- (3) $i = I - 1 : -1 : 1, \quad j = J - 1 : -1 : 1,$
- (4) $i = I - 1 : -1 : 1, \quad j = 1 : J - 1.$

Generally, in dimension d , there will be 2^d sweeps in each iteration. Second, it is important that we assign $\phi_{ij}^n \leftarrow \phi_{ij}^{n-1}$ at the beginning of each iteration and then operate only with ϕ_{ij}^n . This ensures that sweeping is carried out in the Gauss-Seidel sense: updating values, and then using the most recently updated values to resolve the ensuing values. Third, for the convergence criterion, we continue the iteration until $\|\phi^n - \phi^{n-1}\| = \max_{ij} |\phi_{ij}^n - \phi_{ij}^{n-1}| \leq 10^{-8}$, though other criteria could be used. Fourth, the scheme is fully upwind meaning that numerical characteristics flow away from the boundary set Γ . If Γ corresponds to the computational

boundary, then information flows into the domain. If Γ is contained in the computational domain, then characteristics will flow out of the computational boundary. In this case, no special considerations are necessary at the computational boundaries. The values at the boundary nodes will remain large, but will not affect the solution at interior nodes. In this way, our scheme is similar to Godunov-inspired methods such as [55]. In a different approach, Kao et al. [24] devise a sweeping method with a Lax-Friedrichs Hamiltonian, wherein added numerical diffusion will cause boundary data to seep into the domain, requiring special consideration.

Perhaps the most important notes regard the minimization over $a \in A$, which takes place at each grid point in each sweep. Thus a single iteration requires this minimization to be resolved roughly $4IJ$ times. Because of this, the shape of A is somewhat crucial to the algorithm. For example, in the Eikonal equation, we have $A = \mathbb{S}^1$, meaning this optimization

Algorithm 1 A fast sweeping scheme to solve (14)

Initialization: Input boundary data (a function g and set Γ), a grid discretization as in (13), and a small error tolerance $\varepsilon > 0$. Initialize $\phi_{ij}^0 = g(x_i, y_j)$ for the grid nodes corresponding to Γ and $\phi_{ijk}^0 = +\infty$ (or some large number) for all other grid nodes. Initialize $\phi_{ij}^1 = 0$ at all grid points, and $n = 1$.

while $\|\phi^n - \phi^{n-1}\| > \varepsilon$ **do**

Assign $\phi_{ij}^n \leftarrow \phi_{ij}^{n-1}$ for all (i, j) .

for $i = 1$ **to** $I - 1$ **do**

for $j = 1$ **to** $J - 1$ **do**

For each $a \in A$, compute

$$\phi_{ij}^*(a) \leftarrow \frac{r_{ij} + \frac{|f_{1,ij}(a)|}{\Delta x} \phi_{i+\xi_{1,ij}(a),j}^n + \frac{|f_{2,ij}(a)|}{\Delta y} \phi_{i,j+\xi_{2,ij}(a)}^n}{\frac{|f_{1,ij}(a)|}{\Delta x} + \frac{|f_{2,ij}(a)|}{\Delta y}}.$$

Assign $\phi_{ij}^n \leftarrow \min\{\min_a \phi_{ij}^*(a), \phi_{ij}^{n-1}\}$

end for

end for

Repeat the above **for** loops, sweeping in alternating directions until all combinations of sweeping directions have been completed (a total of 4 sweeps).

Assign $n \leftarrow n + 1$

end while

return the values ϕ_{ij}^{end} for all (i, j)

is performed over a continuous set. One can either discretize the set choose from finitely many values, or introduce an optimization routine of their choosing. Either way, this is likely to represent the largest computational burden. The algorithm performs extraordinarily well when A is finite. For example, this occurs in bang-bang control problems, where the optimal controls switch between finitely many control values [50]. One application of this is in kinematic models for simple self-driving cars [17, 44]. Takei and Tsai were the first to analyze this problem in the Hamilton-Jacobi setting [52, 53], and they used a sweeping scheme just like ours. We will return to the example of self-driving cars in section 5, where we present a slight generalization of [52, 53].

3.1 Upwinding, Monotonicity & Convergence

Luo and Zhao [32] discuss and analyze fast sweeping methods in reasonable generality. In particular, they consider (1) with a Hamiltonian H that is

- (i) continuous on $\Omega \times \mathbb{R}^n$,
- (ii) convex and coercive in $\nabla\phi$,
- (iii) compatible, in that $H(x, 0) \leq 0$ for $x \in \Omega$.

Under these conditions and some mild conditions on the boundary data g , they prove that if a fast sweeping scheme is consistent, monotone, and obeys a causality condition, then the approximate solution produced by the scheme will converge to the viscosity solution of the Hamilton-Jacobi equation under grid refinement.

An annoying but necessary facet of the theory of viscosity solutions is that orientation matters. Formally, the viscosity solution of $H(x, \nabla\phi(x)) = 0$ is the negative of the viscosity solution of $-H(x, \nabla\phi(x)) = 0$. Our orientation is reversed from that in [32] but modulo some sign changes and inequality flips, the analysis is the same. Our scheme is consistent to first order, as can be shown by a simple Taylor expansion. In our case, the monotonicity requirement is trivially satisfied since the update rule (18) is clearly non-decreasing in the values at the surrounding grid nodes. The causality condition states in essence that the characteristic flowing into grid node (i, j) is contained in the polygon formed by the nodes used for the finite difference approximations at (i, j) . This is illustrated in fig. 1, where the characteristic curve (blue) enters from the positive- x and positive- y direction, specifying that one should use nodes (i, j) , $(i + 1, j)$, $(i, j + 1)$ to approximate $(\nabla\phi)_{ij}$. For us, the causality condition corresponds exactly to the upwind approximations (16). Note that because of the negative sign in the equation, the characteristic direction at (x, y) is $-f(x, y, a)$ when a is the correct control value at (x, y) . Thus our scheme fits into their framework, and we have convergence to the viscosity solution of (7) as the grid parameters go to zero.

Determining the order of convergence is subtle. Classical proofs of convergence for numerical solutions of Hamilton-Jacobi equations depend not only on the order of local truncation error, but also on the regularity of the viscosity solution [4, 6, 51]. Typically one can guarantee convergence at order no less than $1/2$ when the scheme is consistent at order 1. However, one often sees full first-order convergence in regions where the solution is smooth [32], and one can achieve higher order accuracy in some cases using techniques such as ENO or WENO

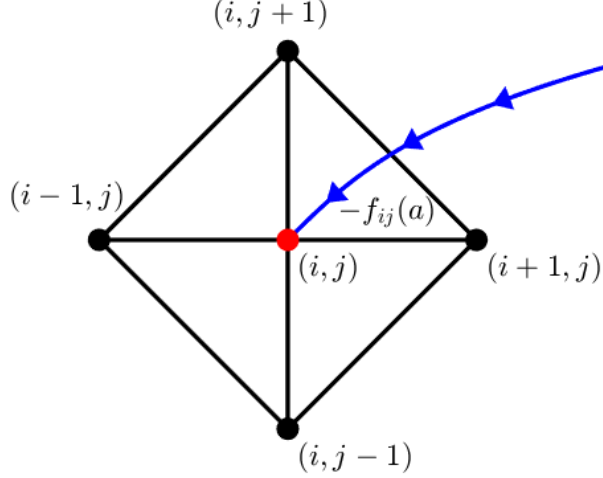


Figure 1: The causality condition specifies that the nodes used to approximate $(\nabla\phi)_{ij}$ form a polygon containing the characteristic (blue) flowing into (i, j) . Here, one would use nodes (i, j) , $(i+1, j)$, $(i, j+1)$. The characteristic direction is given by $-f_{ij}(a)$ if a is the correct control value at the grid node.

schemes [22, 37, 49].

3.2 Application of the Basic Method to Eikonal Equations

To empirically study error and convergence, we test our method on three different Eikonal equations:

$$1 = \|\nabla\phi(x)\|_p \quad (19)$$

where $p = 1, 2, \infty$. Given the boundary data $\phi(0) = 0$, we see that the unique (positive) viscosity solution of (19) is $\phi_p(x) = \|x\|_{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. This fact is somewhat trivial to intuit from the ensuing optimal control problem, and essentially follows from the dual definition of the norm:

$$\|z\|_p = \sup_{\|a\|_{p'} \leq 1} \langle z, a \rangle. \quad (20)$$

However, proving this in full generality is surprisingly intricate. A discussion of such equations is included in [34], and a full analysis is given in [10].

Each of these equations is solved by travel time function for a minimal-time path-planning problem of the form above. Indeed, consider the equation of motion

$$\dot{\mathbf{x}}(t) = \mathbf{a}(t), \quad \mathbf{a}(\cdot) \in B_1^{(p')}, \quad (21)$$

where $B_1^{(p')}$ is the unit ball in the p' -norm (centered at the origin). If we pair this equation with the cost functional

$$C[\mathbf{x}(\cdot), \mathbf{a}(\cdot)] = \iota_0(\mathbf{x}(T)) + \int_0^T 1 \, dt \quad (22)$$

where ι_0 is the convex indicator of the origin (0 at the origin; $+\infty$ elsewhere) and allow for infinite horizon time, then the Hamilton-Jacobi-Bellman equation for the value function

is the p -norm Eikonal equation (19), and the optimal control plan steers the trajectory to the origin in the minimal possible time, where distance from the origin is computed in the p' -norm. In particular, since the unit ball has finitely many extreme points in the case that $p' = \infty$ or $p' = 1$, this leads to a bang-bang control problem for $p = 1$ or $p = \infty$.

In two-dimensions, equation (11) shows that the 2-norm Eikonal equation can be written in the form (14). We can write the other equations in this form as well. For $p = 1$, we have

$$-1 = \inf_{a_1, a_2 \in \{\pm 1\}} \left\{ a_1 \phi_x(x, y) + a_2 \phi_y(x, y) \right\} \quad (23)$$

and for $p = \infty$, we have

$$-1 = \inf_{a \in \{\pm e_1, \pm e_2\}} \left\{ a_1 \phi_x(x, y) + a_2 \phi_y(x, y) \right\}, \quad (24)$$

where in the latter equation, e_1, e_2 are the standard basis vectors, and $a = (a_1, a_2)$.

We would like derive the specific update formula (18) for each of these cases. For the ordinary Eikonal equation in the 2-norm, we find

$$\phi_{ij}^{*,2}(a) = \frac{1 + \frac{|\cos(a)|}{\Delta x} \phi_{i+\text{sign}(\cos(a)),j}^{n,2} + \frac{|\sin(a)|}{\Delta y} \phi_{i,j+\text{sign}(\sin(a))}^{n,2}}{\frac{|\cos(a)|}{\Delta x} + \frac{|\sin(a)|}{\Delta y}}. \quad (25)$$

and use the update $\phi_{ij}^{n,2} = \min\{\min_{a \in [0, 2\pi)} \phi_{ij}^{*,2}(a), \phi_{ij}^{n-1,2}\}$. To use this update, we will need to resolve the minimization over $a \in [0, 2\pi)$. To do so, we simply sample $a = 2\pi k/K$ for $k = 0, \dots, K-1$ and choose the minimum from these finitely many points. In our tests, we fix $K = 400$. This will incur some minor error, but we found empirically that the overall error in the approximation is much less sensitive to changes in K than it is to changes in the grid parameters.

For the 1-norm and ∞ -norm equations, we can explicitly write the update rule, by considering all possible combinations of control variables. For the case $p = 1$, we have

$$\phi_{ij}^{n,1} = \min \left\{ \phi_{ij}^{n-1,1}, \frac{1 + \frac{1}{\Delta x} \phi_{i+1,j}^{n,1} + \frac{1}{\Delta y} \phi_{i,j+1}^{n,1}}{\frac{1}{\Delta x} + \frac{1}{\Delta y}}, \frac{1 + \frac{1}{\Delta x} \phi_{i-1,j}^{n,1} + \frac{1}{\Delta y} \phi_{i,j+1}^{n,1}}{\frac{1}{\Delta x} + \frac{1}{\Delta y}}, \right. \\ \left. \frac{1 + \frac{1}{\Delta x} \phi_{i+1,j}^{n,1} + \frac{1}{\Delta y} \phi_{i,j-1}^{n,1}}{\frac{1}{\Delta x} + \frac{1}{\Delta y}}, \frac{1 + \frac{1}{\Delta x} \phi_{i-1,j}^{n,1} + \frac{1}{\Delta y} \phi_{i,j-1}^{n,1}}{\frac{1}{\Delta x} + \frac{1}{\Delta y}} \right\}. \quad (26)$$

In the $p = \infty$ case, the update is even simpler since one of a_1, a_2 in (24) is zero. Plugging the values into the general update formula (18) and clearing the denominator yields

$$\phi_{ij}^{n,\infty} = \min \left\{ \phi_{ij}^{n-1,\infty}, \Delta x + \phi_{i+1,j}^{n,\infty}, \Delta x + \phi_{i-1,j}^{n,\infty}, \Delta y + \phi_{i,j+1}^{n,\infty}, \Delta y + \phi_{i,j-1}^{n,\infty} \right\}. \quad (27)$$

We note that (27) is perfectly satisfied by the exact solution $\phi_\infty(x, y) = \|(x, y)\|_1 = |x| + |y|$, and thus when $p = \infty$, our scheme will solve the equation exactly, so long as the origin is a grid node. Otherwise, the error in the approximation will only depend on the distance from the origin to the nearest grid node in each direction.

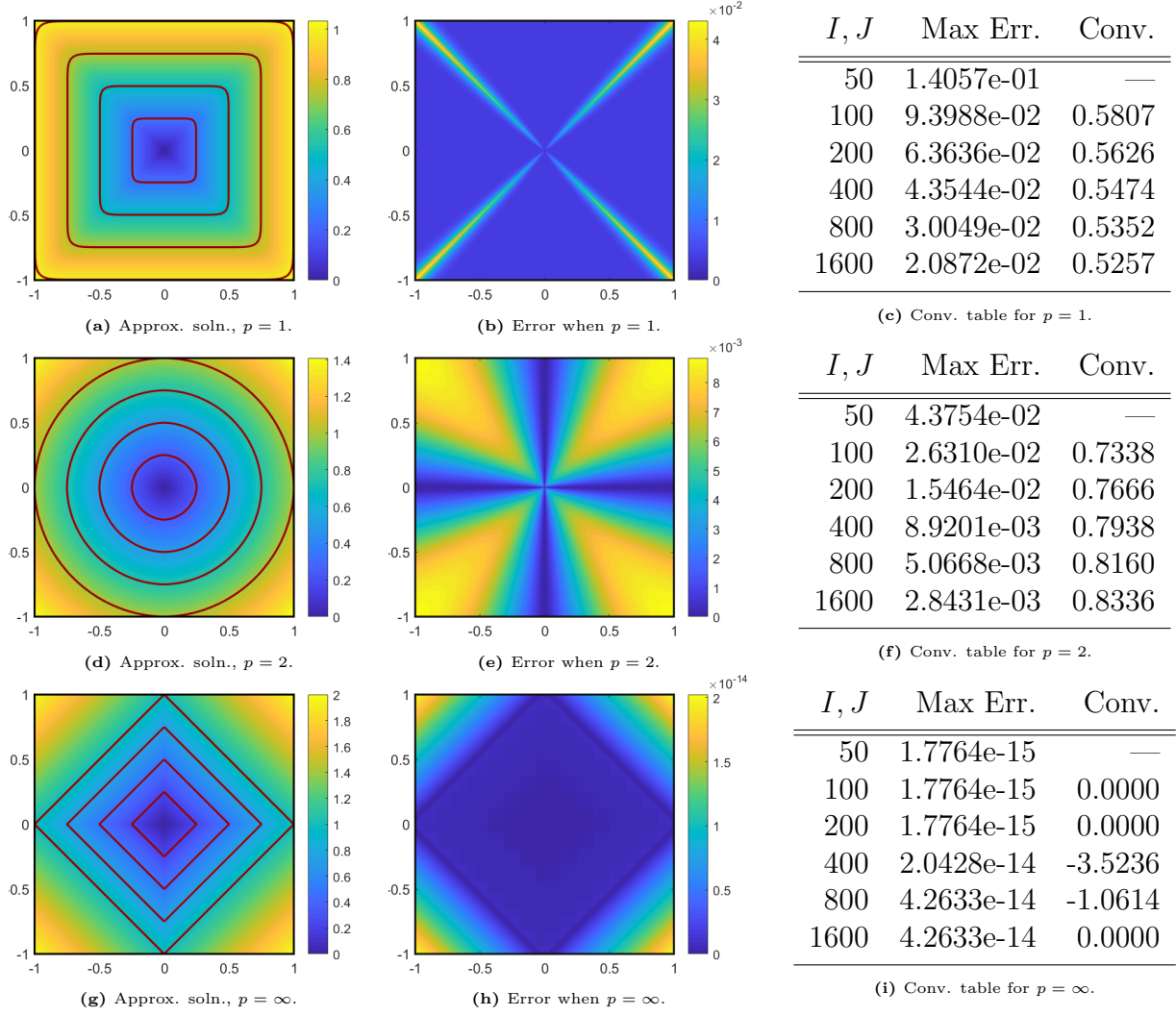


Figure 2: Approximation of $\|\nabla\phi\|_p = 1$ using our fast sweeping method. Plots display results from the 401×401 grid. Red lines are level sets of the solution.

Using these update rules, and the boundary condition $\phi(0, 0) = 0$, we simulated equation (19) for $p = 1, 2, \infty$. The results are included in fig. 2; specifically, results for $p = 1$ are included in figures 2a, 2b, 2c; $p = 2$ in figures 2d, 2e, 2f; and $p = \infty$ in figures 2g, 2h, 2i. Recall again the exact solution $\phi_p(x, y) = \|(x, y)\|_{p'}$. The left most figure in each column show contour plots of the approximations to these solutions on $[-1, 1] \times [-1, 1]$ with a 401×401 grid, along with level sets of the approximations. The middle figure in each column shows a contour plot of the error in the approximation. The right most figure includes the convergence table in each case. We note that there is a different scale in each plot.

When $p = 1$, the level sets should be perfect squares since these are balls in the ∞ -norm. At the corners of those squares, the ordinary forward and backward difference operators cannot capture the sharp edges, which leads to some rounding off. Because of this, the error is large along the lines $y = \pm x$, and the order of convergence is roughly $1/2$; the minimal convergence rate guaranteed by the classical theory [4, 51].

When $p = 2$, the maximum error is less than in the $p = 1$ case, and the error itself

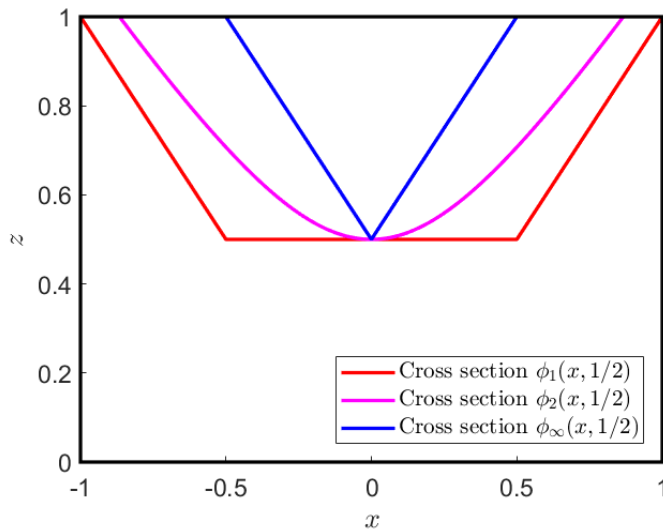


Figure 3: Horizontal cross section of $\phi_1, \phi_2, \phi_\infty$ at $y = 1/2$.

is more evenly spread throughout the entirety of each quadrant, rather than being focused along specific lines. The convergence rate here is roughly $3/4$, showing improved convergence behavior compared with the $p = 1$ case. An interesting note here is that along the lines $x = 0$ and $y = 0$, the error is effectively zero. This is because the finite difference approximations are focused in those directions, and the cross sections of the exact solution in those directions are linear rays increasing outward from the origin. Thus, for example, when $x > 0$, the exact solution satisfies $\phi_2(x + \Delta x, 0) = \Delta x + \phi_2(x, 0)$, and our discretization captures this relationship with no error. We will return to this line of thought momentarily.

When $p = \infty$, we noted earlier that our scheme should be exact. Indeed, we see that the level sets of the approximate solution are sharp-edged diamonds, exactly mirroring the level sets of $\phi_\infty(x, y) = |x| + |y|$. In this case, the error is near machine- ε , and thus the convergence table is not informative.

We remarked about the low error along the lines $x = 0$ and $y = 0$ in the $p = 2$ case, and the relationship between this low error and the cross sections of the exact solution along those lines. This remark very closely relates to the improved order of convergence for larger p . As p increases (and thus p' decreases), the cross sections of the exact solution $\phi_p(x) = \|x\|_{p'}$ in the vertical or horizontal directions more closely resemble the absolute value function, and thus can be captured more accurately by the finite difference approximations. This is seen in fig. 3, where we have plotted horizontal cross sections of ϕ_1, ϕ_2 and ϕ_∞ at level $y = 1/2$. For $\phi_\infty(x, y) = |x| + |y|$, this cross section is exactly $|x| + 1/2$. For $\phi_2(x, y) = \sqrt{x^2 + y^2}$, the cross section is a smooth curve, which cannot be captured perfectly by our discretization, but is better approximated than the cross section of $\phi_1(x, y) = \max\{|x|, |y|\}$, which has two kinks. The accuracy of the method depends on how well these cross sections can be approximated, since any error in these approximations will propagate to other regions.

With this in mind, we note that for $\phi_1(x, y) = \max\{|x|, |y|\}$, while the cross sections in the horizontal and vertical direction have these two kinks, the cross sections in the diagonal directions $y = x_0 \pm x$ will look like shifted absolute value functions. If we used first order ap-

proximations to $\nabla\phi_1$ along these diagonals, we would perfectly capture these cross sections, and thus reconstruct the solution exactly. This suggests that we should rotate the grid and consider alternative approximations to $\nabla\phi_1$.

4 A Rotating-Grid Fast Sweeping Scheme

We would like to append the basic algorithm with additional approximations to the gradient $\nabla\phi$ in directions that are not vertical and horizontal (with respect to the rectangular domain). To do so, we must first recast equation (14) in new coordinates (\bar{x}, \bar{y}) , rotated versions of the standard Cartesian coordinates. Again, we describe this procedure in two dimensions. Here the extension to higher dimensions is not quite as straightforward but can still be accomplished in a relatively predictable, if very tedious, manner.

Suppose that (\bar{x}, \bar{y}) are the typical Cartesian coordinates, rotated counterclockwise by an angle $\beta \in (0, \pi/2)$, as pictured in fig. 4. Note that it is sufficient to consider this range of angles; rotations by larger angles results in the same transform up to renaming coordinates and flipping positive and negative directions. One easily verifies the relationship

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \longleftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}. \quad (28)$$

Thus the derivatives in the (x, y) directions can be expressed

$$\begin{aligned} \phi_x &= \frac{\partial \bar{x}}{\partial x} \phi_{\bar{x}} + \frac{\partial \bar{y}}{\partial x} \phi_{\bar{y}} = \cos(\beta) \phi_{\bar{x}} - \sin(\beta) \phi_{\bar{y}}, \\ \phi_y &= \frac{\partial \bar{x}}{\partial y} \phi_{\bar{x}} + \frac{\partial \bar{y}}{\partial y} \phi_{\bar{y}} = \sin(\beta) \phi_{\bar{x}} + \cos(\beta) \phi_{\bar{y}}. \end{aligned} \quad (29)$$

Inserting these representations into (14) yields

$$\begin{aligned} -r(\bar{x}, \bar{y}) &= \inf_{a \in A} \left\{ [\cos(\beta) f_1(\bar{x}, \bar{y}, a) + \sin(\beta) f_2(\bar{x}, \bar{y}, a)] \phi_{\bar{x}}(\bar{x}, \bar{y}) \right. \\ &\quad \left. + [\cos(\beta) f_2(\bar{x}, \bar{y}, a) - \sin(\beta) f_1(\bar{x}, \bar{y}, a)] \phi_{\bar{y}}(\bar{x}, \bar{y}) \right\}. \end{aligned} \quad (30)$$

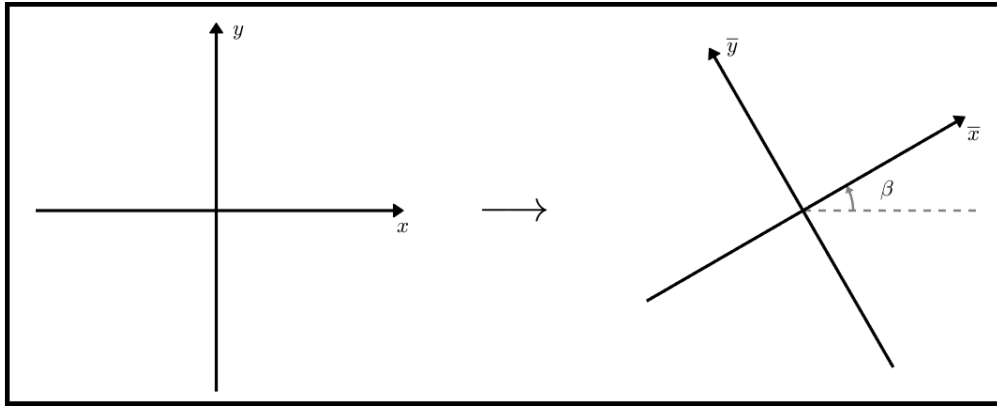


Figure 4: Cartesian coordinates rotated by $\beta \in (0, \pi/2)$ in the counterclockwise direction.

Defining

$$\begin{aligned}\bar{f}_1(\bar{x}, \bar{y}, a) &= \cos(\beta) f_1(\bar{x}, \bar{y}, a) + \sin(\beta) f_2(\bar{x}, \bar{y}, a), \\ \bar{f}_2(\bar{x}, \bar{y}, a) &= \cos(\beta) f_2(\bar{x}, \bar{y}, a) - \sin(\beta) f_1(\bar{x}, \bar{y}, a),\end{aligned}\tag{31}$$

we arrive at

$$-r(\bar{x}, \bar{y}) = \inf_{a \in A} \{ \bar{f}_1(\bar{x}, \bar{y}, a) \phi_{\bar{x}}(\bar{x}, \bar{y}) + \bar{f}_2(\bar{x}, \bar{y}, a) \phi_{\bar{y}}(\bar{x}, \bar{y}) \}.\tag{32}$$

The idea is now to write the upwind finite difference approximations in the directions of (\bar{x}, \bar{y}) . Doing so shows that

$$\phi(\bar{x}, \bar{y}) = \frac{r(\bar{x}, \bar{y}) + \frac{|\bar{f}_1(\bar{x}, \bar{y}, a)|}{\Delta \bar{x}} \phi(\bar{x} + \bar{\xi}_1 \Delta \bar{x}, \bar{y}) + \frac{|\bar{f}_2(\bar{x}, \bar{y}, a)|}{\Delta \bar{y}} \phi(\bar{x}, \bar{y} + \bar{\xi}_2 \Delta \bar{y})}{\frac{|\bar{f}_1(\bar{x}, \bar{y}, a)|}{\Delta \bar{x}} + \frac{|\bar{f}_2(\bar{x}, \bar{y}, a)|}{\Delta \bar{y}}}\tag{33}$$

is a first order upwind approximation to (32) at the point (\bar{x}, \bar{y}) when a is the correct control value, and $\bar{\xi}_\ell = \text{sign}(\bar{f}_\ell(\bar{x}, \bar{y}, a))$. Thus one could add this approximation into the sweeping scheme and use the update rule

$$\phi_{ij}^n = \min \left\{ \phi_{ij}^{n-1}, \min_{a \in A} \phi_{ij}^*(a), \min_{a \in A} \bar{\phi}_{ij}^*(a) \right\},\tag{34}$$

where $\bar{\phi}_{ij}^*(a)$ is computed from (33). However, this raises the question of how to evaluate (33) on the grid, since for example, $(\bar{x} \pm \Delta \bar{x}, \bar{y})$ may not be grid nodes.

Rotated finite differences are extensively used in computational wave mechanics. So-called rotated-staggered-grid methods were introduced by Saenger et al. [45], and are still being developed and improved today [15, 21, 43, 57, 59]. The philosophy of these methods is the same: using finite differences in multiple orientations will more accurately capture the upwind direction. Their strategy is to define a new grid corresponding to the points (\bar{x}, \bar{y}) and keep track of solution values ϕ_{ij} and $\bar{\phi}_{ij}$ separately, while using both sets of values to approximate the derivatives on both grids. To this author's knowledge, the idea of fixing a square grid and computing approximations to $\nabla \phi$ in different directions has not been widely used in the context of fast sweeping methods. Takei et al. [52] suggest using approximations along different directions. However, in their case, the upwind direction is fixed (in analogy to our setup, they have f_1, f_2 independent of a) which significantly simplifies the matter. In a follow up article [53], when the upwind direction is no longer fixed, they abandon the idea.

We would like to maintain a single grid (x_i, y_j) . To do so, one can interpolate values of ϕ_{ij} to off grid values, and compute the upwind approximation in any direction β . This will be computationally expensive since, in order to maintain the Gauss-Seidel sweeping, this interpolation will need to be performed separately for every (i, j) using the newest updated values. Alternatively, we can choose particular values of β and $\Delta \bar{x}, \Delta \bar{y}$ such that the points $(\bar{x} \pm \Delta \bar{x}, \bar{y})$, $(\bar{x}, \bar{y} \pm \Delta \bar{y})$ fall on the grid.

Explicitly, rather than choosing β and the rotated grid parameters $(\Delta \bar{x}, \Delta \bar{y})$, we choose natural numbers (\hat{i}, \hat{j}) , and define $\beta = \arctan(\hat{j}/\hat{i})$. We then let this β determine the grid rotation. This is pictured in fig. 5. Here we have used $(\hat{i}, \hat{j}) = (2, 1)$. As pictured, the nodes used to approximate $\phi_{\bar{x}}$ at (i, j) will be $\{(i, j), (i+2, j+1)\}$ for the forward approximation,

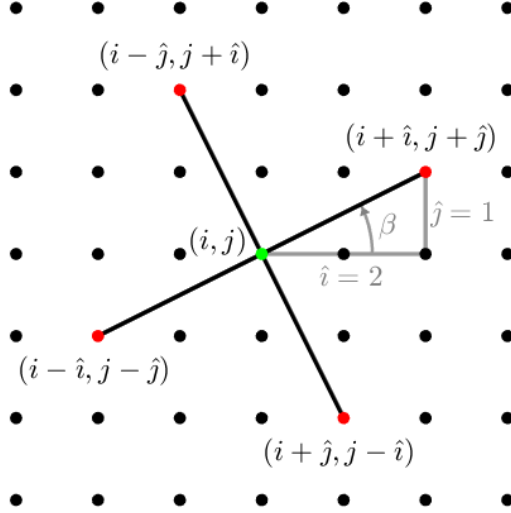


Figure 5: Rotated stencil at (i, j) using the rotation determined by $(\hat{i}, \hat{j}) = (2, 1)$.

and $\{(i - 2, j - 1), (i, j)\}$ for the backward approximation. Similarly, the nodes used to approximate $\phi_{\bar{y}}$ at (i, j) will be $\{(i, j), (i - 1, j + 2)\}$ for the forward approximation and $\{(i + 1, j - 2), (i, j)\}$ for the backward approximation.

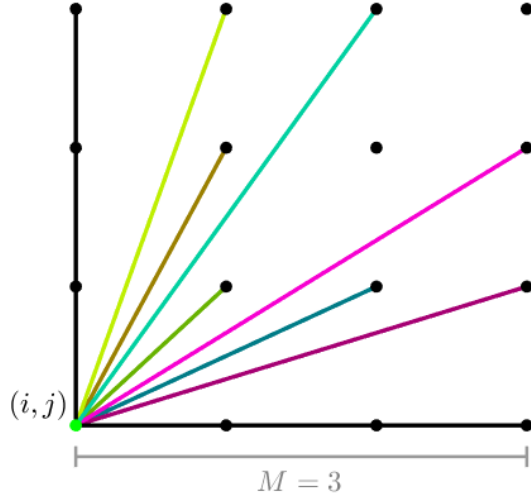
We note that as described, this will only work on a square grid ($\Delta x = \Delta y$). The extension to a non-square grid is a bit more complicated. In that case, there would be two rotation angles that rotate the x -axis and y -axis differently, and thus the resulting coordinate system would no longer be orthogonal. Again, this could be implemented in a straightforward, but tedious, manner. For the remainder of this document, we will assume that $\Delta x = \Delta y$ so that the rotation method works as described.

With these parameters (\hat{i}, \hat{j}) determining the rotation, we define the new grid discretization parameter $\Delta s = \sqrt{(\hat{i}\Delta x)^2 + (\hat{j}\Delta y)^2}$. Note that this Δs will take the place of $\Delta \bar{x}, \Delta \bar{y}$ in the case of a square grid. Thus we can translate equation (33) onto the grid:

$$\bar{\phi}_{ij}^*(a) = \frac{r_{ij}\Delta s + |\bar{f}_{1,ij}(a)|\phi_{i+\bar{\xi}_{1,ij}(a)\hat{i}, j+\bar{\xi}_{1,ij}(a)\hat{j}} + |\bar{f}_{2,ij}(a)|\phi_{i-\bar{\xi}_{2,ij}(a)\hat{i}, j+\bar{\xi}_{2,ij}(a)\hat{j}}}{|\bar{f}_{1,ij}(a)| + |\bar{f}_{2,ij}(a)|}, \quad (35)$$

which, one sees, is exactly analogous to (18), except that the coordinates are rotated and the grid parameters are equal. Inserting this approximation into (34) provides a new update rule that can be used in algorithm 1. Of course, it is not necessary to limit oneself to a single rotation (\hat{i}, \hat{j}) . To further improve the scheme, one can choose as many pairs as desired, compute the rotated derivative approximations in each of these directions, and take the minimum over all such approximations. Since the stencil at each grid node will be larger, the scheme will require a larger layer of ghost nodes padding the computational boundary; otherwise, algorithm 1 will operate in the exact same fashion, but with extra approximations added into the update rule. In general, if one imposes $1 \leq \hat{i}, \hat{j} \leq M$, one should buffer the computational domain with M layers of grid nodes, and there will be some finite number $C(M)$ of distinct angles β created by different pairs (\hat{i}, \hat{j}) .¹ This is pictured in fig. 6, where

¹In fact, one has $C(M) = 2\left(\sum_{m=1}^M \varphi(m)\right) - 1$ where φ is the Euler totient function, as detailed in the



(a) The possible rotation angles if $1 \leq \hat{i}, \hat{j} \leq 3$.

M	$C(M)$
1	1
2	3
3	7
4	11
5	19
6	23
7	35
8	43
9	55
10	63

(b) Number of possible angles if $1 \leq \hat{i}, \hat{j} \leq M$.

Figure 6: If we restrict $1 \leq \hat{i}, \hat{j} \leq M$ there will be some finite number $C(M)$ of distinct rotation angles $\beta = \arctan(\hat{j}/\hat{i})$, each represented by a colored line.

each colored line represents a distinct rotation angle β when $M = 3$. Fixing M , we propose two strategies for choosing different rotation angles: first, one could simply use every possible rotation angle. This may be computationally expensive since, for example, when $M = 5$, there are $C(M) = 19$ angles to consider. Accordingly, our second strategy will be to choose some fixed size subcollection at random. This will not be able to guarantee the same level of accuracy, but will be significantly cheaper computationally. It may also be better than choosing a fixed subcollection of angles since, in application, one may not be able to intuit the “principal” directions that need to be captured as we can for the Eikonal equations.

Note that we will always use the ordinary forward and backward approximations in the (x, y) directions, and include approximations in other directions as desired. This is to establish a baseline. In this manner, the approximation provided by the algorithm when additional directions are used can be no worse than that provided by the basic method presented in algorithm 1.

4.1 Application of the Rotating-Grid Method to Eikonal Equations

We apply the sweeping scheme with rotated derivative approximations to the Eikonal equation in the $p = 1$ and $p = 2$ norms. We remarked earlier that cross sections of the solution $\phi_1(x, y) = \max\{|x|, |y|\}$ along the diagonal lines $y = x_0 \pm x$ could be captured exactly by our scheme if we use the rotation $\beta = \pi/4$, which is the same as $(\hat{i}, \hat{j}) = (1, 1)$. In this case, the rotated coefficients are $\bar{f}_1 = \frac{1}{\sqrt{2}}(a_1 + a_2)$ and $\bar{f}_2 = \frac{1}{\sqrt{2}}(a_2 - a_1)$, where $a_1, a_2 \in \{\pm 1\}$.

Online Encyclopedia of Integer Sequences: <http://oeis.org/A018805>

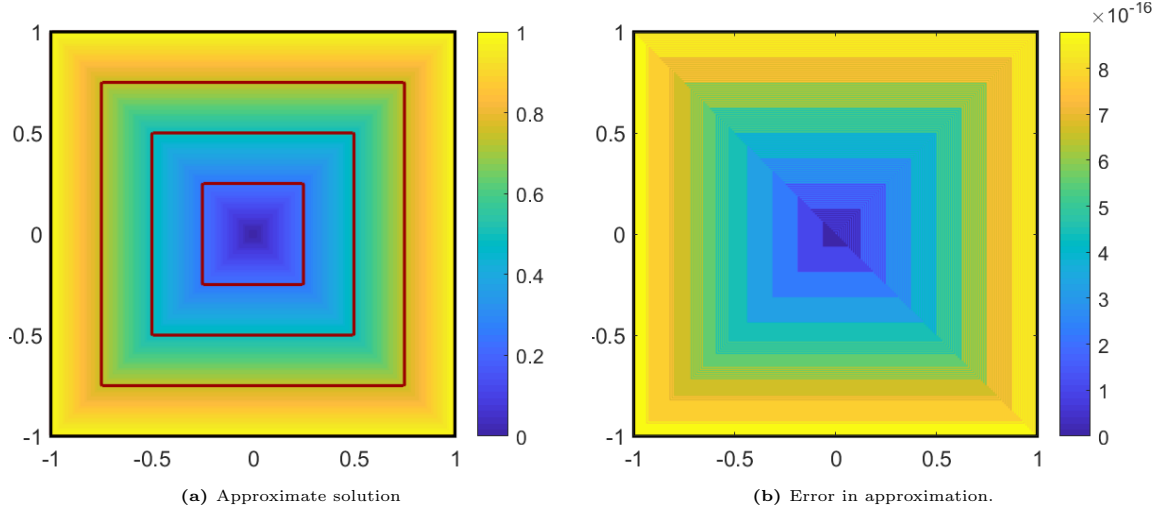


Figure 7: Numerical solution of $\|\nabla\phi\|_1 = 1$ using with additional approximations to $\nabla\phi$ in the direction of $\beta = \pi/4$. Compare with figures 2a, 2b.

Since one of these is zero, the update rule is

$$\begin{aligned} \phi_{ij}^{n,1} = \min \left\{ \phi_{ij}^{n-1,1}, \frac{1 + \frac{1}{\Delta x}\phi_{i+1,j}^{n,1} + \frac{1}{\Delta y}\phi_{i,j+1}^{n,1}}{\frac{1}{\Delta x} + \frac{1}{\Delta y}}, \frac{1 + \frac{1}{\Delta x}\phi_{i-1,j}^{n,1} + \frac{1}{\Delta y}\phi_{i,j+1}^{n,1}}{\frac{1}{\Delta x} + \frac{1}{\Delta y}}, \right. \\ \left. \frac{1 + \frac{1}{\Delta x}\phi_{i+1,j}^{n,1} + \frac{1}{\Delta y}\phi_{i,j-1}^{n,1}}{\frac{1}{\Delta x} + \frac{1}{\Delta y}}, \frac{1 + \frac{1}{\Delta x}\phi_{i-1,j}^{n,1} + \frac{1}{\Delta y}\phi_{i,j-1}^{n,1}}{\frac{1}{\Delta x} + \frac{1}{\Delta y}}, \right. \\ \left. \phi_{i+1,j+1}^{n,1} + \frac{\Delta s}{\sqrt{2}}, \phi_{i-1,j-1}^{n,1} + \frac{\Delta s}{\sqrt{2}}, \right. \\ \left. \phi_{i-1,j+1}^{n,1} + \frac{\Delta s}{\sqrt{2}}, \phi_{i+1,j-1}^{n,1} + \frac{\Delta s}{\sqrt{2}} \right\}. \end{aligned} \quad (36)$$

We use this update rule in algorithm 1 to solve $\|\nabla\phi\|_1 = 1$. The results are seen fig. 7. We note that the level sets of the solution have sharp edges, as opposed to fig. 2a, where they were rounded off. In this case, the error in the solution is on the order of machine- ε .

Next we solve $\|\nabla\phi\|_2 = 1$. Here, in contrast with $\|\nabla\phi\|_1 = 1$ or $\|\nabla\phi\|_\infty = 1$, we will never be able to solve the equation exactly with finitely many grid rotations. The solution will be resolved exactly along any line through the origin if we consider the derivatives in the direction along that line. We saw this in fig. 2e; the error is approximately zero along the x -axis and y -axis. We see it further in fig. 8. In that figure, we first solve $\|\nabla\phi\|_2 = 1$ using the basic method (subfigures 8a, 8b, 8c). We then compare this to results when using approximations to the derivatives in one additional direction (subfigures 8d, 8e, 8f), and three additional directions (subfigures 8g, 8h, 8i). As expected, we see that for a fixed I, J , the error only decreases as we incorporate additional approximations to $\nabla\phi$ in different directions. Interestingly, the convergence seems to slow down slightly when additional directions are included. However, we also note that when using three additional directions one only needs 51 grid points in each direction to achieve the same approximation error as the basic method with 401 points in each direction.

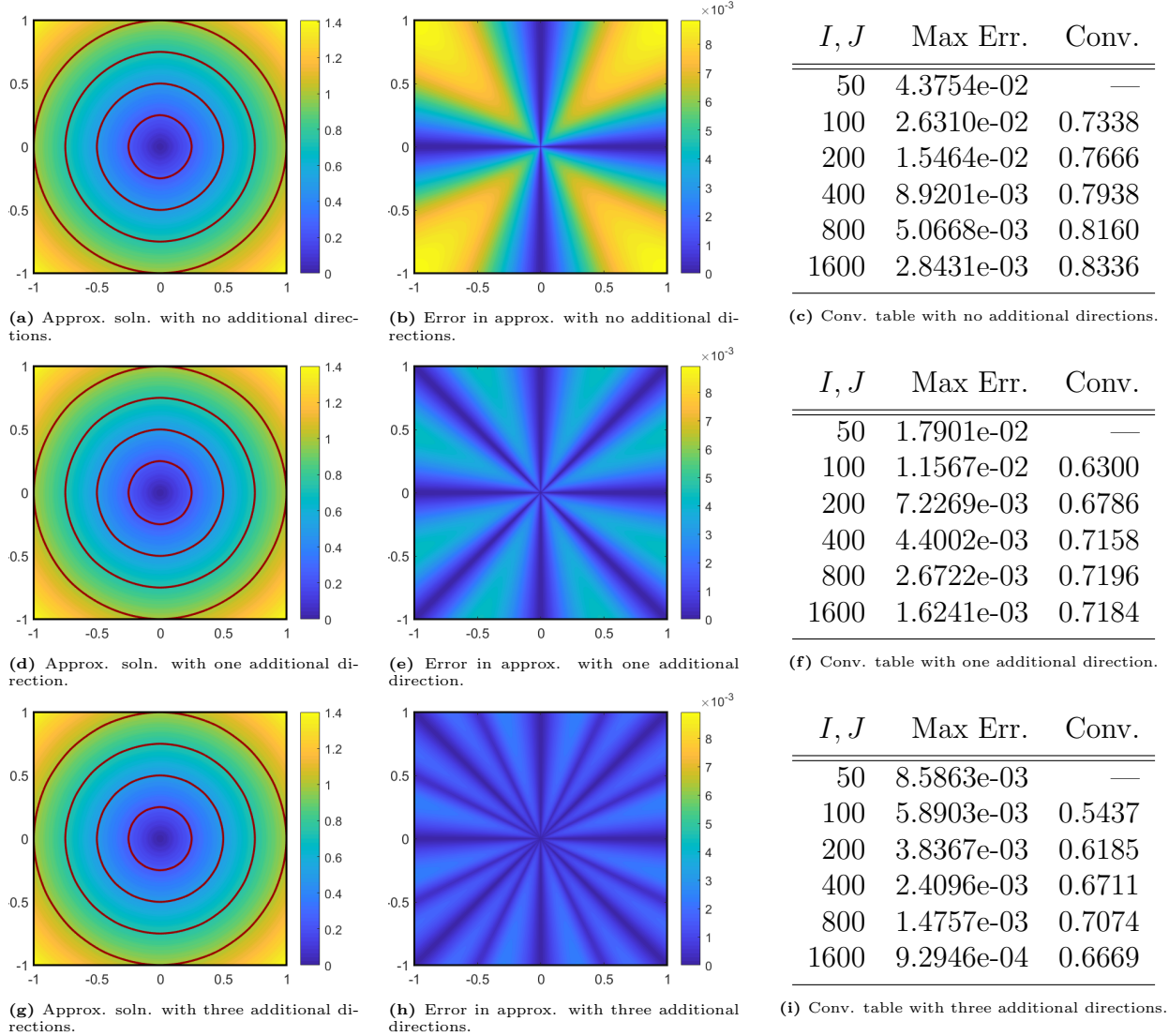


Figure 8: Numerical solution of $\|\nabla\phi\|_2 = 1$ using our fast sweeping method with additional approximations to $\nabla\phi$ in different directions. Scale on error plots is fixed. Error is approximately zero in the directions of the derivative approximations.

Finally, we solve the same equation using a 401×401 grid and all 19 grid rotations $\beta = \arctan(\hat{j}/\hat{i})$ corresponding to $1 \leq \hat{i}, \hat{j} \leq 5$. In fig. 9a, we see that when using all 19 rotations, we achieve an approximation error of 8.7914×10^{-4} . In this case, the algorithm required 12 iterations to terminate, and each iteration requires 20 times the computation as in the basic method (since there are 20 total approximations to $\nabla\phi$ being computed). In fig. 9b, we use the same 19 possible grid rotations, but for each iteration we choose only two rotations to use at random. We achieve similar approximation error: 8.7941×10^{-4} . The algorithm required 40 iterations to converge, but each iteration is 3 times as costly as in the basic method. Thus while there are roughly 3 times as many iterations, each iteration requires only 15% of the computation, meaning one can achieve similar approximation error with roughly half the computation. It should be mentioned that these results have some randomness, but the numbers presented are quite typical.

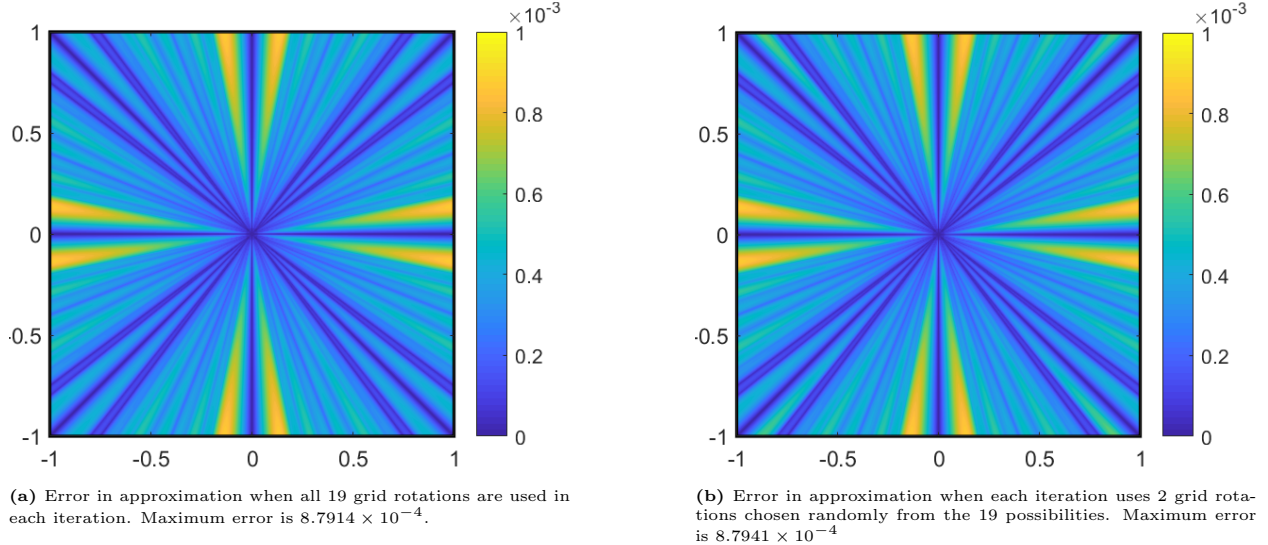


Figure 9: Error in approximation using rotations $\beta = \arctan(j/i)$ where $1 \leq i, j \leq 5$.

We note that Darbon and Osher [16] solve similar Eikonal equations using a variational method based on the Hopf-Lax formula. Their method is applicable in high dimensions and can resolve the solution with essentially no error. However, the method only applies to Hamiltonians which are state-independent: $H = H(\nabla\phi)$. Fast sweeping methods are more general, but suffer from the curse of dimensionality. We have included Eikonal equations as an example because they are the prototypical steady-state Hamilton-Jacobi equations.

5 Other Applications

Lastly, we present two applications of our method to problems arising in engineering. First we consider the visibility problem. Here one could imagine placing cameras at fixed points in a domain. The cameras have omnidirectional view, but the view is occluded by obstacles. The problem is to find the region that is visible to the cameras.

This problem was first formulated using partial differential equations and the level set method by Tsai et al. [54]. However, that formulation involves a nonlocal equation. More recently, Oberman and Salvador were able to recast the problem in terms of a simple, local equation [33]. Specifically, supposing that $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is the signed distance function to the obstacles (positive inside the obstacles) and $x^* \in \mathbb{R}^d$ is the vantage point, the visibility function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies

$$0 = \min\{\phi(x) - g(x), \langle x - x^*, \nabla\phi(x) \rangle\} \quad (37)$$

with the boundary condition $\phi(x^*) = g(x^*)$. The visibility set is then given by $\{\phi \leq 0\}$. To include multiple vantage points, one solves (37) individually for each point, and combines the solution via minima and maxima to account for different scenarios (for example, the minimum of all such solutions will provide the set of points visible from at least one vantage point, while the maximum of all such solutions provides the set of points that are visible from all vantage points simultaneously).

Note that while the equation doesn't directly follow from an optimal control problem, (37) does fall into our framework. If one sets $\phi_{ij}^0 = g_{ij}$ for the nodes closest to the vantage point (x^*, y^*) and $\phi_{ij}^0 = -\infty$ at other nodes, one can use the update rule

$$\phi_{ij}^* = \frac{\frac{|x_i - x^*|}{\Delta x} \phi_{i - \text{sign}(x_i - x^*), j} + \frac{|y_j - y^*|}{\Delta y} \phi_{i, j - \text{sign}(y_j - y^*)}}{\frac{|x_i - x^*|}{\Delta x} + \frac{|y_j - y^*|}{\Delta y}}, \quad (38)$$

and iterate $\phi_{ij}^n = \max\{\phi_{ij}^{n-1}, g_{ij}, \phi_{ij}^*\}$. [Note that the upwind direction is reversed, which explains the slight deviations between these formulas and those above.] One can then use additional approximations to $\nabla\phi$ as desired. We used this update rule, and applied algorithm 1 with a 401×401 grid and with approximations to $\nabla\phi$ along the x -axis and y -axis as well as the $\beta = \pi/4$ direction. The results are seen in fig. 10, where the yellow set represents the visible set, the black shapes are obstacles and the green dots are the vantage points. In this case, because there is no control variable, the upwind direction is fixed and characteristics are straight lines flowing away from the vantage points. Because of this simple geometry, the scheme requires only one iteration and values at grid nodes are resolved during one of directional sweeps depending on where they lie relative to the vantage point. For example, if the vantage point is at grid node (i^*, j^*) , then the forward-forward sweep will resolve all values ϕ_{ij} with $i > i^*$ and $j > j^*$. It should be noted that Oberman and Salvador also devised an upwind sweeping scheme that also approximates (37) with one sweep in each direction by using interpolation to explicitly capture the exact upwind direction. Our method is not an improvement of theirs; we include this example only to demonstrate the diverse applicability of our method. For a full discussion of the visibility problem including rigorous analysis of (37), see [33].

Our final application is in time-optimal path planning for simple self-driving cars. This problem was first analyzed by Dubins [17] and later Reeds and Shepp [44] in a purely geometric sense, and later analyzed in the Hamilton-Jacobi formulation by Takei, Tsai and others [52, 53]. Let (x, y) denote the location of the center of mass of the vehicle and θ denote the orientation. If W is the maximum angular velocity of the car (which enforces a minimum turning radius) and d is the distance from the rear wheels—which drive the car—to the center of mass, then the kinematics are

$$\begin{aligned} \dot{x} &= v \cos(\theta) - \omega W d \sin(\theta), \\ \dot{y} &= v \sin(\theta) + \omega W d \cos(\theta), \\ \dot{\theta} &= W \omega, \end{aligned} \quad (39)$$

where $v, \omega \in [-1, 1]$ are normalized control variables representing tangential and angular velocity respectively [58].

With these kinematics, the optimal travel time function solves the Hamilton-Jacobi equation

$$-1 = \inf_{v, \omega} \left\{ [v \cos(\theta) - \omega W d \sin(\theta)] \phi_x + [v \sin(\theta) + \omega W d \cos(\theta)] \phi_y + \omega W \phi_\theta \right\}. \quad (40)$$

For a full derivation of this equation, we direct the reader to [53]; they consider the case that $d = 0$ so the car is simplified to a point mass, but otherwise the derivation is the same.

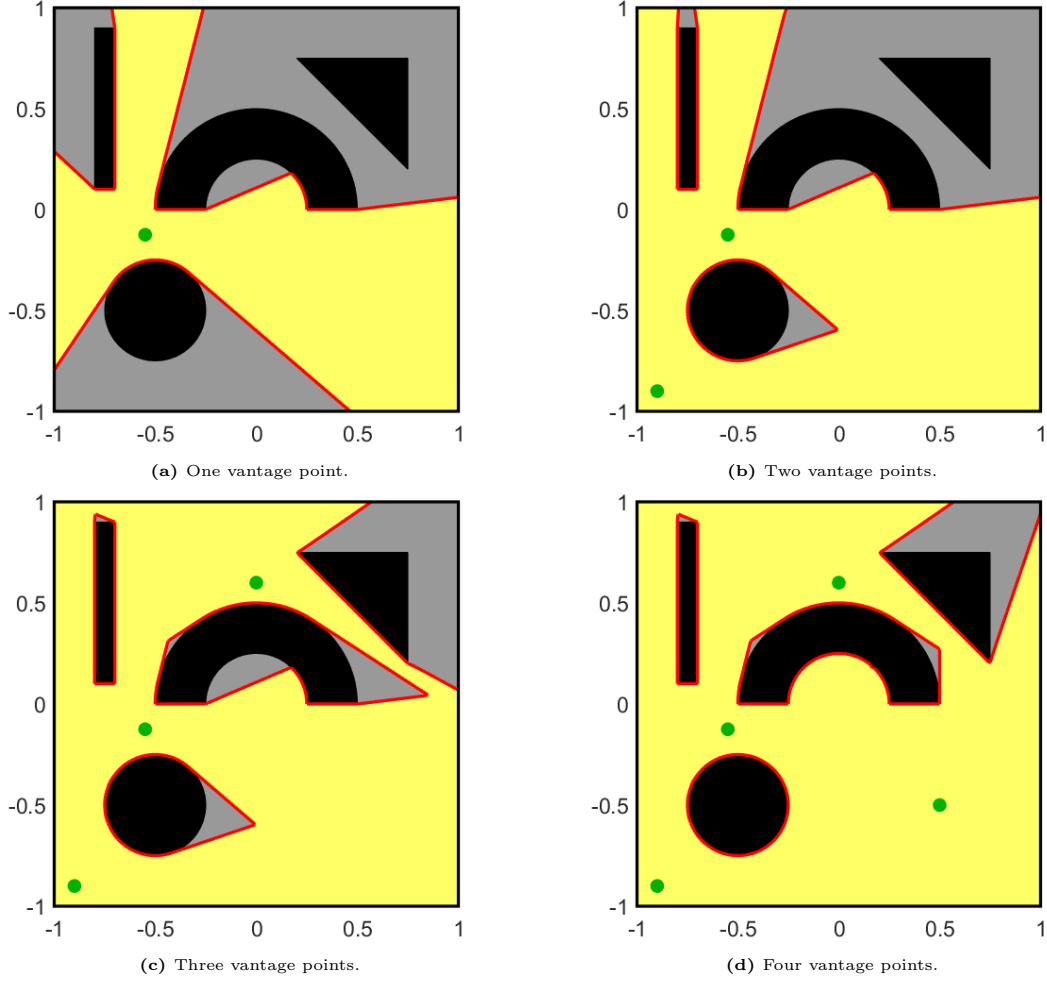


Figure 10: Computing the visibility set using (37) with different vantage points. The green dots represent the vantage points. The black shapes are obstacles. The yellow set is the visible set. The grey set is the unobserved set.

One notes that the minimization is linear in (v, ω) , and thus, since the minimization set $[-1, 1] \times [-1, 1]$ has finitely many extreme points, there are finitely many values that the pair (v, ω) will take. For technical reasons, one should allow $v \in \{-1, 1\}$ and $\omega \in \{-1, 0, 1\}$ [53].

Equation (40) fits directly into our framework. Discretizing (x_i, y_j, θ_k) , (40) is approximated by the update rule

$$\begin{aligned}
 \phi_{ijk}^*(v, \omega) = & \left\{ 1 + \frac{|A_k(v, \omega)|}{\Delta x} \phi_{i+a_k(u,v),j,k} \right. \\
 & + \frac{|B_k(v, \omega)|}{\Delta y} \phi_{i,j+b_k(v,\omega),k} \\
 & \left. + \frac{|\omega| W}{\Delta \theta} \phi_{i,j,k+\text{sign}(\omega)} \right\} \\
 & / \left\{ \frac{|A_k(v, \omega)|}{\Delta x} + \frac{|B_k(v, \omega)|}{\Delta y} + \frac{|\omega| W}{\Delta \theta} \right\},
 \end{aligned} \tag{41}$$

where

$$\begin{aligned}
A_k(v, \omega) &= v \cos(\theta_k) - \omega W d \sin(\theta_k), \\
B_k(v, \omega) &= v \sin(\theta_k) + \omega W d \cos(\theta_k), \\
a_k(v, \omega) &= \text{sign}(v \cos(\theta_k) - \omega W d \sin(\theta_k)), \\
b_k(v, \omega) &= \text{sign}(v \sin(\theta_k) + \omega W d \cos(\theta_k)).
\end{aligned} \tag{42}$$

One can use this update rule in algorithm 1 (accounting for three dimensions by performing 8 sweeps per iteration) with the boundary condition $\phi_{i^*, j^*, k^*}^0 = 0$ for the desired ending configuration and $\phi_{ijk}^0 = +\infty$ otherwise. Then ϕ_{ijk} will represent the approximate time needed to travel from grid node (i, j, k) to grid node (i^*, j^*, k^*) while obeying (39).

To incorporate grid rotation, one needs to specify a rotation direction $u \in \mathbb{S}^2$. This will become quite involved to write down especially when the grid is not square. However, if we restrict to rotations in the xy -plane, we can again trade (x, y) for (\bar{x}, \bar{y}) exactly as in the two-dimensional case. Here if $\beta = \arctan(\hat{j}/\hat{i})$, the new update scheme is

$$\begin{aligned}
\bar{\phi}_{ijk}^*(v, \omega) &= \left\{ \Delta s + |\bar{A}_k(v, \omega)| \phi_{i+\bar{a}_k(u, v), j+\bar{a}_k(u, v), k} \right. \\
&\quad + |\bar{B}_k(v, \omega)| \phi_{i-\bar{b}_k(v, \omega), j+\bar{b}_k(v, \omega), k} \\
&\quad \left. + \frac{\Delta s |\omega| W}{\Delta \theta} \phi_{i, j, k+\text{sign}(\omega)} \right\} \\
&\quad / \left\{ |A_k(v, \omega)| + |B_k(v, \omega)| + \frac{\Delta s |\omega| W}{\Delta \theta} \right\},
\end{aligned} \tag{43}$$

where $\Delta s = \sqrt{(\hat{i}\Delta x)^2 + (\hat{j}\Delta y)^2}$ as before, and

$$\begin{aligned}
\bar{A}_k(v, \omega) &= v \cos(\theta_k + \beta) - \omega W d \sin(\theta_k + \beta), \\
\bar{B}_k(v, \omega) &= v \sin(\theta_k + \beta) + \omega W d \cos(\theta_k + \beta), \\
\bar{a}_k(v, \omega) &= \text{sign}(v \cos(\theta_k + \beta) - \omega W d \sin(\theta_k + \beta)), \\
\bar{b}_k(v, \omega) &= \text{sign}(v \sin(\theta_k + \beta) + \omega W d \cos(\theta_k + \beta)).
\end{aligned} \tag{44}$$

We used these formulas on a $201 \times 201 \times 201$ discretization of $[-1, 1] \times [-1, 1] \times [0, 2\pi]$ to compute the travel-time function for this control problem when the ending configuration is $(\frac{1}{2}, \frac{1}{2}, 0)$ meaning the car should end at $(x_f, y_f) = (\frac{1}{2}, \frac{1}{2})$ facing in the positive x -direction. Specifically, we used three additional directions to approximate ϕ_x, ϕ_y : the directions of $\beta = \arctan(1/2), \arctan(1), \arctan(2/1)$. Results are included in fig. 11 and fig. 12. One way to evaluate the results is to compare against known values of the travel-time function. For example, anywhere along the line $(x, \frac{1}{2}, 0)$, the optimal travel time is $|x - \frac{1}{2}|$ since the optimal path simply requires pulling forward or reversing into the final configuration. Accordingly, on the level set plots in fig. 11, we plot the point $(-\frac{1}{2}, \frac{1}{2}, 0)$ in red. This point should satisfy $\phi(-\frac{1}{2}, \frac{1}{2}, 0) = 1$ and indeed, it seems to approximately lie in the level set $\phi(x, y, \theta) = 1$ [fig. 11d]. Likewise, in fig. 12, we display the contours of $\phi(x, y, 0)$ which show the values of the travel-time function given that the car is facing in the positive x -direction. Using these, we can directly compare values of $\phi(x, \frac{1}{2}, 0)$ and $|x - \frac{1}{2}|$ and the results line up very well.

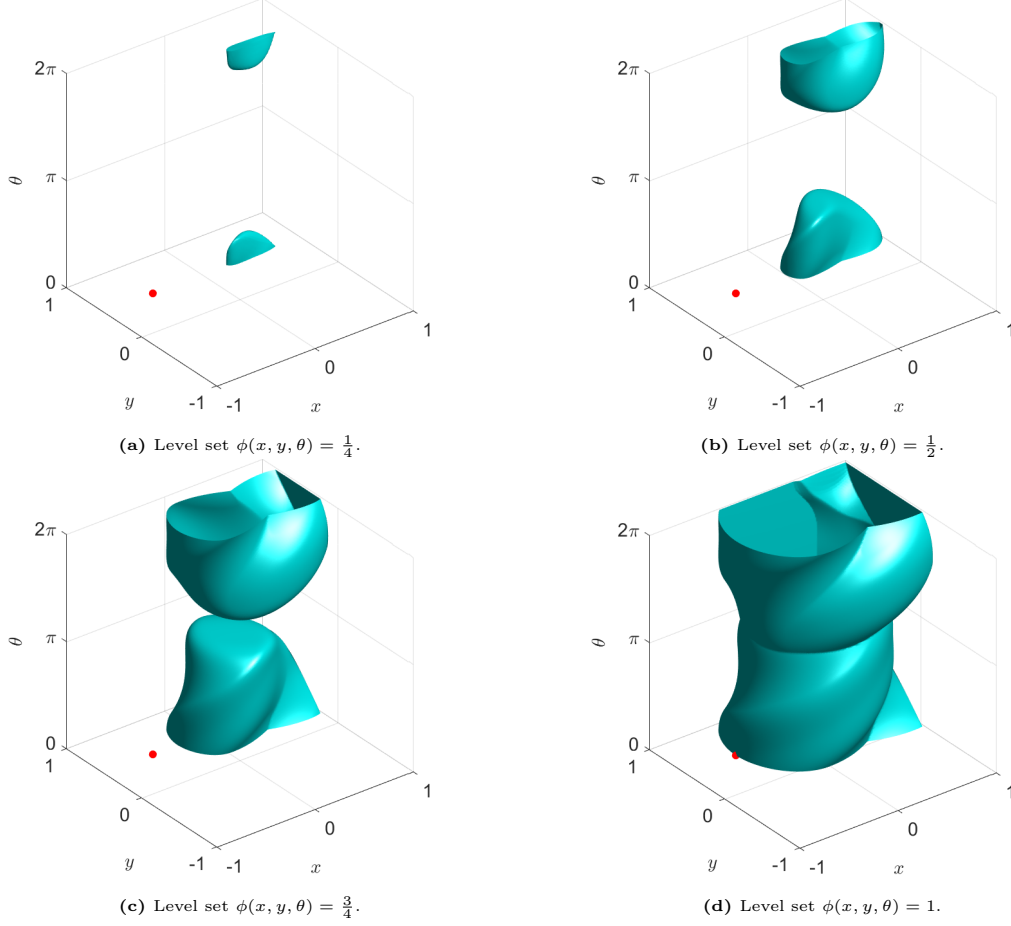


Figure 11: Level sets (cyan) of the travel-time function $\phi(x, y, \theta)$ with ending point $(\frac{1}{2}, \frac{1}{2}, 0)$. Plotted in red is the point $(-\frac{1}{2}, \frac{1}{2}, 0)$. This point should have a travel time of 1, and indeed the level set $\phi(x, y, \theta) = 1$ includes the point.

Another way one can verify results is to compute the actual paths given by the control problem. Having computed the travel-time function ϕ one can determine optimal trajectories by integrating (39) using control values

$$\begin{aligned} v &= -\text{sign}(\phi_x \cos \theta + \phi_y \sin \theta), \\ \omega &= -\text{sign}(-d\phi_x \sin \theta + d\phi_y \cos \theta + \phi_\theta). \end{aligned} \tag{45}$$

This is seen in fig. 13. In those plots, the final location is marked by the red star, and the initial locations are marked by colored dots. The positions of the vehicles are displayed at several points along their respective optimal trajectories. Note, these optimal paths were computed independently and are simply plotted on top of each other; the paths will require different amounts of time to traverse and there is no interaction between the cars.

6 Conclusion

Fast sweeping methods provide a simple and robust framework for numerical solutions of steady state Hamilton-Jacobi equations. We have developed a fast sweeping scheme for

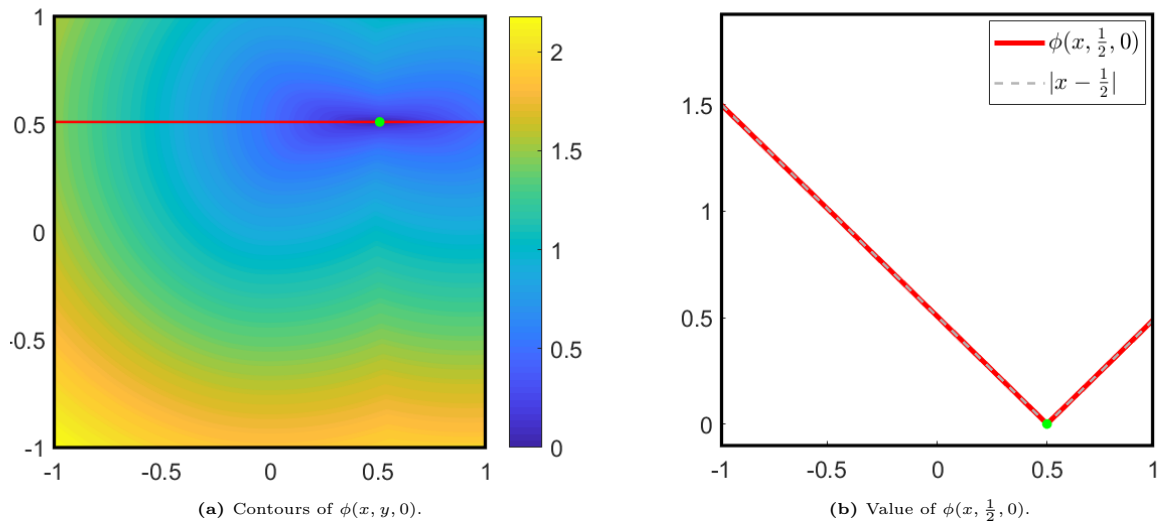


Figure 12: Contour plot of the travel-time function $\phi(x, y, 0)$ with ending point $(\frac{1}{2}, \frac{1}{2}, 0)$ [green]. Along the line $(x, \frac{1}{2}, 0)$ [red] the solution is $|x - 1/2|$.

a class of Hamilton-Jacobi equations arising from steady-state optimal control problems wherein the running cost is independent of the control variables. Our method is exceedingly simple to implement and applies to a wide range of problems. We tested our method against Eikonal equations in different norms. We then suggested a general method for maintaining a square grid, but using approximations to derivatives in rotated directions, so as to more accurately capture the information flow along characteristics. Finally, we demonstrated the utility of our method by applying it to two problems arising from engineering applications.

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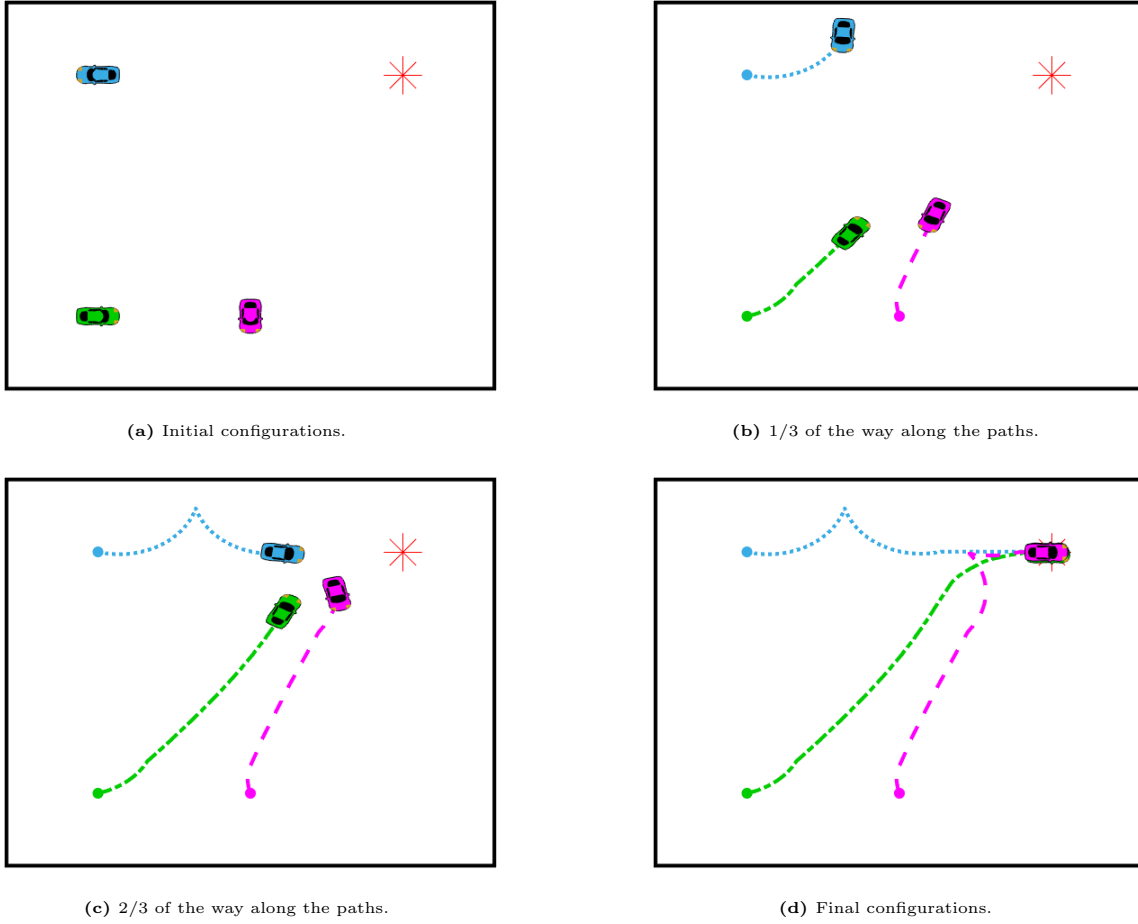


Figure 13: Optimal paths for cars with initial configurations $(-\frac{1}{2}, \frac{1}{2}, \pi)$ [blue], $(-\frac{1}{2}, -\frac{1}{2}, 0)$ [green], and $(0, -\frac{1}{2}, \frac{5\pi}{4})$ [pink]. Final configuration is $(\frac{1}{2}, \frac{1}{2}, 0)$ [red star].

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