

A Hamilton–Jacobi Formulation for Optimal Coordination of Heterogeneous Multiple Vehicle Systems*

Matthew R. Kirchner¹, Mark J. Debord², and João P. Hespanha¹

Abstract— We present a method for optimal coordination of multiple vehicle teams when multiple endpoint configurations are equally desirable, such as seen in the autonomous assembly of formation flight. The individual vehicles’ positions in the formation are not assigned a priori and a key challenge is to find the optimal configuration assignment along with the optimal control and trajectory. Commonly, assignment and trajectory planning problems are solved separately. We introduce a new multi-vehicle coordination paradigm, where the optimal goal assignment and optimal vehicle trajectories are found simultaneously from a viscosity solution of a single Hamilton–Jacobi (HJ) partial differential equation (PDE), which provides a necessary and sufficient condition for global optimality. Intrinsic in this approach is that individual vehicle dynamic models need not be the same, and therefore can be applied to heterogeneous systems. Numerical methods to solve the HJ equation have historically relied on a discrete grid of the solution space and exhibits exponential scaling with system dimension, preventing their applicability to multiple vehicle systems. By utilizing a generalization of the Hopf formula, we avoid the use of grids and present a method that exhibits polynomial scaling in the number of vehicles.

I. INTRODUCTION

Multi-robot motion planning presents several challenges, and key among them are trajectory planning and formation control, as well as the assignment of vehicles to goal states. When the assignment of vehicles to goal states is made beforehand, many methods have been proposed. These include control theory approaches [1], [2], graph-based techniques [3], and optimization methods [4], [5]. Approximations to assist in dimensionality reductions have also been proposed, such as sequential methods [6] and hybrid approaches [7], [8].

When the goal states of multi-vehicle teams are not set a priori, it is necessary to determine which vehicle to allocate to which terminal goal. There is a large body of literature developing allocation and assignment algorithms [9], and applying these methods to engineering problems has appeared in diverse forms including sensor coverage [10], weapon-target assignment [11], or network routing [12]. Dynamics are not considered in this body of work, which

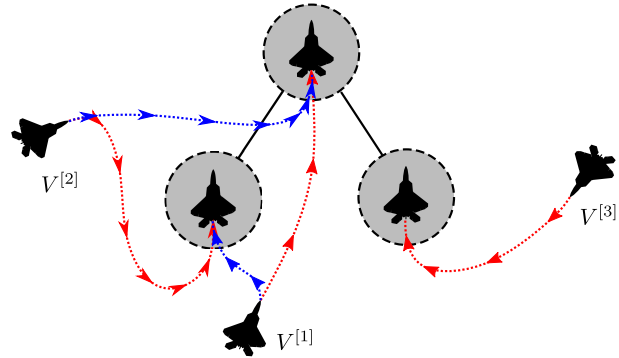


Fig. 1: An illustration of the vehicle coordination problem. If a vehicle’s position in the formation is not assigned a priori, then many different configurations are possible to achieve the desired formation, each with a differing system objective cost. Two such possibilities are shown, one in red and an alternate assignment in blue.

commonly study worst-case system performance under equilibrium conditions using a game-theoretical framework [13], [14], [15]. Without consideration of dynamics, these methods have limited applicability to vehicle coordination problems and there exists a technical gap relating vehicle-goal allocation and motion planning, as most prior work assumes each is done independently. The critical dependency between assignment and trajectory planning is illustrated in Figure 1.

There have been recent attempts to close this gap as in [16]. But this work only evaluates discrete spatial waypoints on a rectangular grid without considering vehicle dynamics, and therefore the paths generated are not feasible for most physical systems. Morgan et al. [17] generalizes this concept to linear, discrete-time dynamics and constructs a non-linear programming problem that is solved with sequential convex programming in conjunction with an auction algorithm [9]. However, both [17] and [16] restrict the cost function to one of additive individual vehicle weights (similar to the cost function presented in [9]), which excludes many vehicle optimization problems of interest, including assembling the desired formation shape in minimal time.

We present how the optimal assignment and trajectory can be found simultaneously from the viscosity solution to a single Hamilton–Jacobi PDE, a necessary and sufficient condition for optimality. We assume the vehicles can each have unique linear dynamics. Remarkably, we show that the global viscosity solution can be found as the solution to a linear bottleneck assignment problem (LBAP) [18]. This

*This research was supported in part by the Office of Naval Research under Grant N00014-18-WX01382 and Grant N00014-19-WX00546 and in part by the CogDeCon program under contract FA8750-18-C-0014 and contract 88ABW-2019-4739.

¹M. Kirchner and J. P. Hespanha are with Center for Control, Dynamical-systems and Computation (CCDC), University of California, Santa Barbara, CA, USA. {kirchner, hespanha}@ece.ucsb.edu

²M. Debord is with the Image and Signal Processing Branch, Research and Intelligence Department, Code D5J1, Naval Air Warfare Center Weapons Division, China Lake, CA, USA. mark.debord@navy.mil

fact can be utilized to construct a level set method that has polynomial computational scaling with the number of vehicles and still ensures a global optimum.

Our formulation has a close relation to reachability analysis which can be useful for the design and analysis of heterogeneous systems. When the dynamics of some vehicles differ greatly from the rest, some formations may not be feasible (e.g. a slow-moving vehicle attempting to join a formation with a much faster vehicle). The zero sub-level set of the viscosity solution of a related HJ PDE defines an implicit surface representation of the backwards reachable set [19], and from this one determines whether the formation can be achieved given the unique collection of vehicle dynamics.

Traditionally, numerical solutions to HJ equations require a dense, discrete grid of the solution space [20], [21]. Computing the elements of this grid scales poorly with dimension and has limited use for problems with dimension greater than four. The exponential dimensional scaling in optimization is sometimes referred to as the ‘‘curse of dimensionality’’ [22], [23]. A new result in [24] discovered a numerical solution based on the Hopf formula [25] that does not require a grid and can be used to efficiently compute solutions of a certain class of Hamilton–Jacobi PDEs. However, that only applies to systems with time-independent Hamiltonians of the form $\dot{x} = f(u(t))$, and has limited use for general linear control problems. Recently, the classes of systems were expanded upon and generalizations of the Hopf formula are used to solve optimal linear control problems in high-dimensions [26] and differential games as applied to multi-vehicle, collaborative pursuit-evasion problems [27].

The key contributions of this paper are to formulate the multi-vehicle coordination problem as a single Hamilton–Jacobi equation, show that the solution of which can be found from an LBAP, and use the generalized Hopf formula to simultaneously solve for optimal trajectory, vehicle control, and goal assignment. By utilizing the generalized Hopf formula, we create a level-set method to find the viscosity solution of the HJ in a computationally efficient manner.

We introduce a multi-vehicle dynamical system and formulate the coordination problem in Section II. Section III formulates the optimal vehicle coordination problem as the viscosity solution to a single Hamilton–Jacobi PDE. Section IV constructs a level-set method for the multi-vehicle coordination problem and utilizes the generalized Hopf formula as a computational tool to find the viscosity solution to the HJ PDE of the joint, multi-vehicle system. Section V shows the method as applied to several examples including a case of planar motion planning with 4 vehicles. This example shows how assignment and path planning cannot be decoupled and even if a single vehicle has a different initial condition, it may result in a different assignment for optimal coordination.

II. PROBLEM FORMULATION

We consider a system that consists of N vehicles and each vehicle, $i \in \mathcal{V} = \{1, \dots, N\}$, has linear dynamics

$$\frac{d}{ds}x_i(s) := A_i x_i(s) + B_i \alpha_i(s), \quad (1)$$

for $s \in [0, t]$, where $x_i \in \mathbb{R}^{n_i}$ is the system state and $\alpha_i(s) \in \mathcal{A}_i \subset \mathbb{R}^{m_i}$ is the control input, constrained to a convex admissible control set \mathcal{A}_i . We add the assumption that no eigenvalue of matrix A_i has a strictly positive real part. We let $[0, t] \ni s \mapsto \gamma_i(s; x_i, \alpha_i(\cdot)) \in \mathbb{R}^{n_i}$ denote a state trajectory for vehicle i that evolves in time with measurable control sequence $\alpha_i(\cdot) \in \mathcal{A}_i$, according to (1) starting from initial state x_i at $s = 0$. The trajectory γ_i is a solution of (1) in that it satisfies (1) almost everywhere:

$$\begin{aligned} \frac{d}{ds}\gamma_i(s; x_i, \alpha_i(\cdot)) &= A_i \gamma_i(s; x_i, \alpha_i(\cdot)) + B_i \alpha_i(s), \\ \gamma_i(0; x_i, \alpha_i(\cdot)) &= x_i. \end{aligned}$$

A. Multi-Vehicle Model

For the set of N vehicles, we construct a joint state space with state vector $x = (x_1, \dots, x_i, \dots, x_N) \in \mathbb{R}^n$ and control $\alpha = (\alpha_1, \dots, \alpha_i, \dots, \alpha_N) \in \mathcal{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_i \times \dots \times \mathcal{A}_N \subset \mathbb{R}^m$ which is written as follows

$$\begin{aligned} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_i \\ \vdots \\ \dot{x}_N \end{bmatrix} &= \begin{bmatrix} A_1 & \cdots & & 0 \\ & \ddots & & \\ \vdots & & A_i & \\ & & & \ddots \\ 0 & \cdots & & A_N \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_N \end{bmatrix} \\ &+ \begin{bmatrix} B_1 & \cdots & & 0 \\ & \ddots & & \\ \vdots & & B_i & \\ & & & \ddots \\ 0 & \cdots & & B_N \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_N \end{bmatrix} \\ \implies \dot{x} &= Ax + B\alpha, \end{aligned} \quad (2)$$

We reiterate the above definition of the joint state space with the fact that when quantities x , A , B , γ , and α appear without subscripts, they refer to the joint system in (2), and when subscripts are used, they refer to a specific individual vehicle as defined in (1).

B. Vehicle Coordination

We assume there exists a set of N goals, and for each vehicle $i \in \mathcal{V}$ we associate closed convex sets $\Omega_{i,j} \subset \mathbb{R}^{n_i}$, $j \in \mathcal{V}$ with the understanding that $x_i \in \Omega_{i,j}$ means the vehicle i is assigned to goal j . Our goal is to make sure that we have one vehicle at each goal, but it does not matter which vehicle is at each goal. This goal can be expressed by the requirements that the multi-vehicle state, x , belongs to the following goal set

$$\Theta := \{x \in \mathbb{R}^n : \exists \sigma \in \mathcal{S}_N \text{ s.t. } \forall i \in \mathcal{V}, x_i \in \Omega_{i,\sigma(i)}\}, \quad (3)$$

where \mathcal{S}_N denotes the set of all permutations of \mathcal{V} . We represent Θ implicitly with the function $J : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\Theta = \{x \in \mathbb{R}^n | J(x) \leq 0\}, \quad (4)$$

and use it to construct a cost functional for the system trajectory $\gamma(s; x, \alpha(\cdot))$, given terminal time t as

$$R(t, x, \alpha(\cdot)) = \int_0^t C(\alpha(s)) ds + J(\gamma(t; x, \alpha(\cdot))), \quad (5)$$

where the function $C : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ represents the rate that cost is accrued over time. The value function $\varphi : \mathbb{R}^n \times (0, +\infty) \rightarrow \mathbb{R}$ is defined as the minimum cost, R , among all admissible controls for a given initial state x with

$$\varphi(x, t) = \inf_{\alpha(\cdot) \in \mathcal{A}} R(t, x, \alpha(\cdot)). \quad (6)$$

III. HAMILTON–JACOBI EQUATIONS FOR OPTIMAL COORDINATION

The value function in (6) satisfies the dynamic programming principle [28], [29] and also satisfies the following initial value Hamilton–Jacobi (HJ) equation with φ being the viscosity solution of

$$\begin{aligned} \frac{\partial \varphi}{\partial s}(x, s) + H(s, x, \nabla_x \varphi(x, s)) &= 0, \\ \varphi(x, 0) &= J(x), \end{aligned} \quad (7)$$

for $s \in [0, t]$, where the Hamiltonian $H : (0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$H(s, x, p) = -x^\top A^\top p + \sup_{\alpha \in \mathbb{R}^m} \{-B\alpha, p\} - C(\alpha), \quad (8)$$

where $p := \nabla_x \varphi(x, s)$ in (8) denotes the *costate*. We denote by $[0, t] \ni s \mapsto \lambda(s; x, \alpha(\cdot)) \in \mathbb{R}^n$ the costate trajectory that can be shown to satisfy almost everywhere:

$$\begin{aligned} \frac{d}{ds} \lambda(s; x, \alpha(\cdot)) &= \nabla_x f(\gamma(s; x, \alpha(\cdot)), s)^\top \lambda(s; x, \alpha(\cdot)) \\ \lambda(t; x, \alpha(\cdot)) &= \nabla_x \varphi(\gamma(t; x, \alpha(\cdot))), \end{aligned}$$

with initial costate denoted by $\lambda(0; x, \alpha(\cdot)) = p_0$. With a slight abuse of notation, we will hereafter use $\lambda(s)$ to denote $\lambda(s; x, \alpha(\cdot))$, when the initial state and control sequence can be inferred through context with the corresponding state trajectory, $\gamma(s; x, \alpha(\cdot))$.

A. System Hamiltonian

We consider a time-optimal formulation with

$$C(\alpha) = \mathcal{I}_{\mathcal{A}}(\alpha),$$

where $\mathcal{I}_{\mathcal{A}} : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ is the characteristic function of the set of admissible controls and is defined by

$$\mathcal{I}_{\mathcal{A}}(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \mathcal{A} \\ +\infty & \text{otherwise.} \end{cases}$$

In this case, the integral term in (5) disappears (as long as α remains in \mathcal{A}) and $R(t, x, \alpha(\cdot))$ is simply the value of $J(\gamma(t; x, \alpha(\cdot)))$ at the terminal time, t . For this problem

$$\varphi(x, t) = \inf_{\alpha(\cdot) \in \mathcal{A}} J(\gamma(t; x, \alpha(\cdot))) \leq 0$$

means that there exists a feasible trajectory for the vehicles the ends at the state $x(t) \in \Theta$; whereas $\varphi(x, t) > 0$ means that such trajectories do not exist under the system dynamics

and initial conditions. Since each vehicle has independent dynamics, the Hamiltonian in (8) is of the form

$$H(s, x, p) = \sum_i H_i(s, x_i, p_i), \quad (9)$$

where each vehicle's Hamiltonian is given by

$$\begin{aligned} H_i(s, x_i, p_i) &= -x_i^\top A_i^\top p_i \\ &+ \sup_{\alpha_i \in \mathbb{R}^{m_i}} \{-B_i \alpha_i, p_i\} - \mathcal{I}_{\mathcal{A}_i}(\alpha_i). \end{aligned} \quad (10)$$

B. Linear Bottleneck Assignment

Our goal is solving the Hamilton–Jacobi PDE in (7), and any $J(x)$ that satisfies (4) is, in general, non-convex and presents numerical challenges. We show that we can overcome this challenge by alternatively solving for the global value function with the following linear bottleneck assignment problem [18]:

$$\varphi(x, t) = \min_{\sigma \in \mathcal{S}_N} \max_{i \in \mathcal{V}} \phi_{i, \sigma(i)}(x_i, t), \quad (11)$$

where $\phi_{i,j}(x_i, t)$ is the viscosity solution to

$$\begin{aligned} \frac{\partial \phi_{i,j}}{\partial s}(x_i, s) + H_i(s, x, \nabla_x \phi_{i,j}(x_i, s)) &= 0, \\ \phi_{i,j}(x_i, 0) &= J_{i,j}(x_i), \end{aligned} \quad (12)$$

with Hamiltonian defined by (27). The function $J_{i,j} : \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ is an implicit surface representation of $\Omega_{i,j}$ such that

$$\Omega_{i,j} = \{x_i \in \mathbb{R}^{n_i} | J_{i,j}(x_i) \leq 0\}. \quad (13)$$

The solution to (11) can be found from $\phi_{i,j}$ using the appropriate linear bottleneck assignment algorithms (for example [30, Section 6.2]), requiring only N^2 evaluations of lower dimensional viscosity solutions and avoiding computation involving all $|\mathcal{S}_N| = N!$ permutations of vehicle-goal pairs. Also of note is that if each HJ equation (12) has convex initial data, it enables the use of computationally efficient techniques that can guarantee convergence to the appropriate viscosity solution. Additionally, since each of the solutions are independent, the N^2 solutions can be computed in parallel.

We introduce a set of mild regularity assumptions that guarantee Hamilton–Jacobi equations have a unique viscosity solution [31, Chapter 7, p. 63]:

1) Each Hamiltonian

$[0, t] \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} \ni (s, x_i, p_i) \mapsto H_i(s, x_i, p_i) \in \mathbb{R}, \forall i$ is continuous.

2) There exists a constant $c_i > 0$ such that for all $(s, x_i) \in [0, t] \times \mathbb{R}^{n_i}$ and for all $p', p'' \in \mathbb{R}^{n_i}$, the following inequalities hold

$$|H_i(s, x_i, p') - H_i(s, x_i, p'')| \leq \rho_i(x_i) \|p' - p''\|, \forall i$$

and

$$|H_i(s, x_i, 0)| \leq \rho_i(x_i), \forall i$$

with $\rho_i(x_i) = c_i(1 + \|x_i\|)$.

- 3) For any compact set $M \subset \mathbb{R}^{n_i}$ there exists a constant $\kappa_i(M) > 0$ such that for all $x', x'' \in M$ and for all $(s, p_i) \in [0, t] \times \mathbb{R}^{n_i}$ the inequality holds, $\forall i$

$$\|H_i(s, x', p_i) - H_i(s, x'', p_i)\| \leq \mu_i(p_i) \|x' - x''\|,$$

with $\mu_i(p_i) = \kappa_i(M) (1 + \|p_i\|)$.

- 4) Each terminal cost function

$$\mathbb{R}^{n_i} \ni x_i \mapsto J_{i,j}(x_i) \in \mathbb{R}, \forall i, j$$

is continuous.

Lemma 1: If each vehicle Hamiltonian, $H_i(s, x_i, p_i)$, meets assumptions 1-3, then the Hamiltonian defined in (9) also meets assumptions 1-3.

The proof is given in the appendix.

Theorem 2: Under assumptions 1-4, $\phi_{i,j}$ is a unique viscosity solution to (12) for all i, j , and there exists a $J : \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies (4) such that with the Hamiltonian given by (9), (11) is a viscosity solution to (7).

Proof: We will prove the theorem constructively by proposing a particular $J : \mathbb{R}^n \rightarrow \mathbb{R}$ given as

$$J(x) = \min_{\sigma \in \mathcal{S}_N} J_\sigma(x), \quad (14)$$

with

$$J_\sigma(x) := \max_{i \in \mathcal{V}} J_{i,\sigma(i)}(x_i), \quad (15)$$

where $J_{i,\sigma(i)}(x_i)$ is the implicit representation of $\Omega_{i,j}$, defined in (13), with $i, j = i, \sigma(i)$. We see that for any $\sigma \in \mathcal{S}_N$ when $x_i \in \Omega_{i,\sigma(i)}$, $\forall i \in \mathcal{V} \implies J_{i,\sigma(i)}(x_i) \leq 0$, $\forall i$ which implies $J_\sigma(x) \leq 0$. We also see if there exists an i such that $x_i \notin \Omega_{i,\sigma(i)}$, then $J_{i,\sigma(i)} > 0$ and implies $J_\sigma(x) > 0$. Therefore the J proposed in (14) satisfies (4) and is an implicit surface representation of the set Θ . Furthermore, since by assumption each $J_{i,j}(x_i)$ is continuous, then (15) is convex as the max of a finite number of continuous functions is also continuous. This implies (14) is continuous as the minimum of a finite number of continuous functions is continuous. Therefore (14) satisfies assumption 4.

From Lemma 1, the system Hamiltonian defined in (9) meets assumptions 1-3 and implies that (7) has a unique viscosity solution, denoted as $\varphi_{\text{HJ}}(x, t)$, when the initial cost function is given by (14) [31, Theorem 8.1, p. 70]. The uniqueness of the solution $\varphi_{\text{HJ}}(x, t)$ implies that the viscosity solution is equivalent to the value function in (6). It follows that

$$\varphi_{\text{HJ}}(x, t) = \inf_{\alpha(\cdot) \in \mathcal{A}} J(\gamma(t; x, \alpha(\cdot))), \quad (16)$$

since $C(\alpha) = 0$ when $\alpha \in \mathcal{A}$. Denoting by α^* as the control that optimizes (16), we have

$$\varphi_{\text{HJ}}(x, t) = J(\gamma(t; x, \alpha^*(\cdot))).$$

Substituting (15) and (14) we have

$$\varphi_{\text{HJ}}(x, t) = \min_{\sigma \in \mathcal{S}_N} \max_{i \in \mathcal{V}} J_{i,\sigma(i)}([\gamma(t; x, \alpha^*(\cdot))]_i), \quad (17)$$

where we use the notation $[\gamma(t; x, \alpha^*(\cdot))]_i$ to represent the i -th block of the vector $\gamma(t; x, \alpha^*(\cdot))$. Likewise (12) has a

unique viscosity solution, $\phi_{i,j}$, for each i, j with initial cost given (13), and as such

$$\begin{aligned} \phi_{i,\sigma(i)}(x_i, t) &= \inf_{\alpha_i(\cdot) \in \mathcal{A}_i} J_{i,\sigma(i)}(\gamma_i(t; x_i, \alpha_i(\cdot))) \\ &= J_{i,\sigma(i)}(\gamma_i(t; x_i, \alpha_i^*(\cdot))), \end{aligned} \quad (18)$$

for each i and with α_i^* denoting the control that optimizes (18). Note that if $J_{i,\sigma(i)}([\gamma(t; x, \alpha^*(\cdot))]_i) < J_{i,\sigma(i)}(\gamma_i(t; x_i, \alpha_i^*(\cdot)))$ it would contradict (18) that α_i^* is the optimal control. Also if $J_{i,\sigma(i)}([\gamma(t; x, \alpha^*(\cdot))]_i) > J_{i,\sigma(i)}(\gamma_i(t; x_i, \alpha_i^*(\cdot)))$, then it would contradict (16) that α^* is the optimal control of the entire system. Therefore, $J_{i,\sigma(i)}([\gamma(t; x, \alpha^*(\cdot))]_i) = J_{i,\sigma(i)}(\gamma_i(t; x_i, \alpha_i^*(\cdot))) = \phi_{i,\sigma(i)}(x_i, t)$ and our value function in (16) becomes

$$\varphi_{\text{HJ}}(x, t) = \min_{\sigma \in \mathcal{S}_N} \max_{i \in \mathcal{V}} \phi_{i,\sigma(i)}(x_i, t),$$

and we arrive at our result. \blacksquare

IV. A LEVEL SET METHOD WITH THE GENERALIZED HOPF FORMULA

First, we introduce the Fenchel–Legendre transform of a convex, proper, lower semicontinuous function $g : \mathbb{R}^\ell \rightarrow \mathbb{R} \cup \{+\infty\}$, denoted as $g^* : \mathbb{R}^\ell \rightarrow \mathbb{R} \cup \{+\infty\}$, and is defined as [32]

$$g^*(p) = \sup_{y \in \mathbb{R}^\ell} \{ \langle p, y \rangle - g(y) \}. \quad (19)$$

We propose to find the viscosity solutions of (12) using the generalized Hopf formula.

Theorem 3: Each viscosity solution of (12) can be found from the formula

$$\begin{aligned} \phi_{i,j}(x_i, t) &= - \min_{p_i \in \mathbb{R}^{n_i}} \left\{ J_{i,j}^* \left(e^{-tA_i^\top} p_i \right), \right. \\ &\quad \left. + \int_0^t \widehat{H}_i(s, p_i) ds - \langle x_i, p_i \rangle \right\} \end{aligned} \quad (20)$$

with

$$\widehat{H}_i(s, p_i) = \sup_{\alpha_i \in \mathcal{A}_i} \{ \langle -e^{-sA_i} B_i \alpha_i, p_i \rangle \}. \quad (21)$$

Proof: Proceeding similar to [27], we apply a change of variables to (1) with

$$z_i(s) = e^{-sA_i} x_i(s), \quad (22)$$

which results in the following system

$$\frac{d}{ds} z_i(s) = e^{-sA_i} B_i \alpha_i(s). \quad (23)$$

By utilizing this change of variables, we construct an equivalent HJ equation

$$\begin{aligned} \frac{\partial \tilde{\phi}_{i,j}}{\partial s}(z_i, s) + \widehat{H}_i \left(s, \nabla_z \tilde{\phi}_{i,j}(z_i, s) \right) &= 0, \\ \tilde{\phi}_{i,j}(z_i, 0) &= J_{i,j}(e^{tA_i} z_i). \end{aligned} \quad (24)$$

From Lemma 5 (given in the appendix), (21) meets assumption 1-3, and by composition rule $J_{i,j}(e^{tA_i} z_i)$ is continuous [33, Theorem 4.7] and meets assumption 4. Therefore (24)

has a unique viscosity solution that is equivalent to the cost functional

$$\tilde{\phi}_{i,j}(z_i, t) = \inf_{\alpha_i(\cdot) \in \mathcal{A}_i} J_{i,j}(e^{tA}\xi_i(t; z_i, \alpha_i(\cdot))),$$

where $\xi_i(s; z_i, \alpha(\cdot))$ is a solution trajectory that satisfies (23) almost everywhere. Since

$$z_i(t) = e^{-tA_i}x_i(t),$$

we have

$$\begin{aligned} \tilde{\phi}_{i,j}(z_i, t) &= \inf_{\alpha_i(\cdot) \in \mathcal{A}_i} J_{i,j}(e^{tA}\xi_i(t; z_i, \alpha_i(\cdot))) \\ &= \inf_{\alpha_i(\cdot) \in \mathcal{A}_i} J_{i,j}(\gamma_i(t; x_i, \alpha_i(\cdot))) \\ &= \phi_{i,j}(x_i, t), \end{aligned}$$

noting that since $x_i = \gamma_i(0; x_i, \alpha_i(\cdot))$ by the transform (22) implies $z_i = x_i$ at $t = 0$. Thus, $\tilde{\phi}(z_i, t) = \phi(x_i, t)$ and we can find $\phi(x_i, t)$ by finding the viscosity solution to (24).

Since $\Omega_{i,j}$ is assumed closed and convex, this implies $J_{i,j}$ is convex and by assumption 4 is continuous in z_i . By assumption 1, $H(p_i)$ is continuous in p_i , therefore (20) gives an exact, point-wise viscosity solution for (24) [34, Section 5.3.2, p. 215]. ■

Formula (20) shows that we can compute a viscosity solution to (24) by solving a finite dimensional optimization problem. This avoids constructing a discrete spatial grid, and is numerically efficient to compute even when the dimension of the state space is large. Additionally, no spatial derivative approximations are needed with Hopf formula-based methods, and this eliminates the numeric dissipation introduced with the Lax–Friedrichs scheme [35], which is necessary to maintain numeric stability in grid-based methods.

A. Numerical Optimization of the Hopf Formula

We transcribe the Hopf formula into a non-linear programming problem by approximating the integral in (20) with an N -point quadrature rule sampled on the time grid

$$\pi^N = \{s_k : k = 0, \dots, N\},$$

with $s_k \in [0, t]$ and corresponding weights w_k . Additionally, we make a simple, invertible change of variable $\tilde{p}_i = e^{-tA_i^\top} p_i$ and substituting into (20) results in the following unconstrained optimization problem that solves (20):

$$\left\{ \min_{\tilde{p}_i} J_{i,j}^*(\tilde{p}_i) + \sum_{j=0}^N w_j \hat{H}_i(s_k, \tilde{p}_i) - \langle e^{tA_i} x_i, \tilde{p}_i \rangle, \right. \quad (25)$$

with \hat{H}_i now being defined as

$$\hat{H}_i(s, \tilde{p}_i) = \sup_{\alpha_i \in \mathcal{A}_i} \left\{ \left\langle -B_i \alpha_i, e^{sA_i^\top} \tilde{p}_i \right\rangle \right\}. \quad (26)$$

The variable transformation is done so that when the matrix A has at least one eigenvalue with strictly negative real part, we avoid computation of e^{-sA^\top} which would have exponentially unstable poles since we are evaluating the matrix exponential of $-A$. This divergence would cause sensitivity in the evaluation of the Hopf formula in (20)

with respect to small changes of p . By utilizing the variable transformation and optimizing (25), we avoid this sensitivity.

Remark 4: Often the expression in (26) is known in closed form and we present one such common case. Recall that $(\cdot)^*$ denotes the Fenchel–Legendre transform of a function defined in (19), and suppose \mathcal{A}_i is the closed convex set defined as

$$\mathcal{A}_i := \{u : \|u\| \leq 1\},$$

where $\|(\cdot)\|$ is any norm. Then $(\mathcal{I}_{\mathcal{A}_i})^*$ defines a norm, which we denote with $\|(\cdot)\|_*$, which is the dual norm to $\|(\cdot)\|$ [32]. From this we write (26) as

$$\hat{H}_i(s, \tilde{p}_i) = \left\| -B_i^\top e^{sA_i^\top} \tilde{p}_i \right\|_*. \quad (27)$$

B. Time-Optimal Control to a Goal Set

The task of determining the control that drives the system into Θ in minimal time can be determined by finding the smallest t such that

$$\varphi(x, t) \leq 0. \quad (28)$$

When the system (1) is constrained controllable, then the set of times such that x is reachable with respect to Θ contains the open interval $[t', \infty)$ [36]. This insures that if x is outside the set Θ at time $t = 0$, then $\varphi(x, 0) > 0$ and there exists a time t' such that $\varphi(x, \tau) < 0$ for all $\tau > t'$. This implies standard root-finding algorithms can be used to find (28). As noted in [26], we solve for the minimum time to reach the set Θ by constructing a newton iteration, starting from an initial guess, t_0 , with

$$t_{k+1} = t_k - \frac{\varphi(x, t_k)}{\frac{\partial \varphi}{\partial t}(x, t_k)}, \quad (29)$$

where $\varphi(x, t_k)$ is the solution to (7) at time t_k . The value function must satisfy the HJ equation

$$\frac{\partial \varphi}{\partial t}(x, t_k) = -H(s, x, \nabla_x \varphi(x, t_k)),$$

where $\nabla_x \varphi(x, t_k) = (e^{t_k A_1^\top} \tilde{p}_1^*, \dots, e^{t_k A_N^\top} \tilde{p}_N^*)$ and each \tilde{p}^* is the argument of the minimizer in (25). We iterate (29) until convergence at the optimal time to reach, which we denote as t^* . This process is summarized in Algorithm 1.

The optimal control and trajectory for each vehicle is found directly from the necessary conditions of optimality established by Pontryagin's principal [37] by noting the optimal trajectory must satisfy

$$\begin{aligned} \frac{d}{ds} \gamma_i^*(s; x_i, \alpha_i^*(\cdot)) &= -\nabla_p H(s, x_i(s), \lambda_i^*(s)) \\ &= A_i \gamma_i(s; x_i, \alpha_i^*(\cdot)) \\ &\quad + B_i \nabla_p \left\| -B_i^\top \lambda_i^*(s) \right\|_*, \end{aligned}$$

where λ_i^* is the optimal costate trajectory and is given by

$$\lambda_i^*(s) = e^{-(t^*-s)A_i^\top} \tilde{p}_i^*.$$

This implies our optimal control is

$$\alpha_i^*(s) = \nabla_p \left\| -B_i^\top e^{-(t^*-s)A_i^\top} \tilde{p}_i^* \right\|_*, \quad (30)$$

for all time $s \in [0, t^*]$, provided the gradient exists.

Algorithm 1 Algorithm to compute the viscosity solution at the minimum time to reach of the set Θ .

1: **Inputs:**
 $x_i, J_{i,j}, \forall i, j$

2: **Initialize:**
 $t = t_0, \varphi = \infty, \epsilon = 10^{-5}$

3: **while** $|\varphi| \geq \epsilon$ **do**

4: **for all** i, j **do**

5: $Q_{i,j} = \min_{\tilde{p}_i} J_{i,j}^*(\tilde{p}_i) + \sum_{j=0}^N w_j \widehat{H}_i(s_k, \tilde{p}_i) - \langle e^{tA_i} x_i, \tilde{p}_i \rangle$

6: **end for**

7: $\varphi = \text{LBAP}(Q)$

8: $p = (e^{tA_1^\top} \tilde{p}_1^*, \dots, e^{tA_i^\top} \tilde{p}_i^*, \dots, e^{tA_N^\top} \tilde{p}_N^*)$

9: $t = t + \frac{\varphi}{H(x,p)}$

10: **end while**

11: **Return:**
 p, t

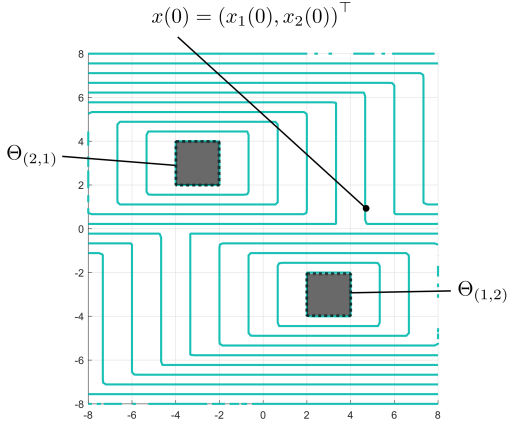


Fig. 2: The two vehicle example of Section V-A. The zero level set evolution solved at various times shown in green.

V. RESULTS

We present results for two examples. For the first we choose a toy example of two vehicles, each with differing single integrator dynamics. The purpose of this example is that it allows easy visualization of the level set propagation and the solution can be verified by comparing with methods such as [38], which cannot be applied to the higher dimensional second example to follow. The LBAP was solved using [30, Algorithm 6.1] and the optimization (25) used sequential quadratic programming [39, Chapter 18] with initial conditions $\tilde{p}_i = e^{tA_i} x_i(0)$ for each vehicle. To prevent a singular control condition, a slight smoothing was applied to the Hamiltonian in (30) and (26) using [40] with parameter $\mu = 10^{-6}$.

A. Toy Problem

We present a two vehicle problem where the dynamics are $\dot{x}_1 = 3\alpha_1$ for the first vehicle and $\dot{x}_2 = \alpha_2$ for the second vehicle. The states x_1, x_2 are the linear position of

each respective vehicle. The control is bounded by $|\alpha_i| \leq 1$. The goal states are $|x_i - 3| \leq 1$ for $j = 1$ and $|x_i + 3| \leq 1$ for $j = 2$. The level set contours of the joint space are shown in Figure V-A, for ten time samples equally spaced on $t \in [0, 4]$. Notice this seemingly simple heterogeneous system can give rise to counterexamples for the assignment formulations given in [17] where the sum of the vehicle distances are used as the assignment metric and [16] where the time of arrival of vehicles was used. Take the initial state $x(0) = (4.667, 0.5)^\top$, which is shown on Figure 2. For the time metric, the the assignment (1, 2) gives $0.222 + 2.5 = 2.7223$, while the assignment (2, 1) gives $2.222 + 1.5 = 3.722$, indicating (1, 2) is the clear choice. However, (2, 1) is the global optimum in this example, as both vehicles reach their desired goal states in a time of 2.222, as the zero level set of the global value function intersects the point $x(0)$ at that time, giving the minimum time-to-reach. For the distance metric, the misassignment is more pronounced, with the assignment (1, 2) giving $0.667 + 2.5 = 3.1670$ as compared to the assignment (2, 1) gives $4.667 + 1.5 = 8.167$.

B. Planar Motion

We choose for dynamics (1) with state $x \in [r, \dot{r}]^\top$, where $r \in \mathbb{R}^2$ is spatial position of a robot and $\dot{r} \in \mathbb{R}^2$ is the velocity and

$$A_i = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, B_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

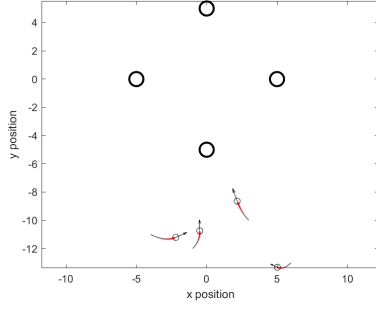
for each vehicle $i \in \mathcal{V}$. The control $\alpha_i \in \mathbb{R}^2$ is constrained to lie in the set $\|\alpha_i\|_2 \leq 1$. The robots are tasked with reaching the goal formation and coming to rest, in minimum time. The goal sets each have radius of 0.5 and the centers of the goals are located spatially at $(0, 5)^\top, (-5, 0)^\top, (5, 0)^\top$, and $(0, -5)^\top$. Since the 2-norm is self-dual, the Hamiltonian (27) for each vehicle is

$$\widehat{H}_i(s, \tilde{p}_i) = \left\| -B_i^\top e^{sA_i^\top} \tilde{p}_i \right\|_2.$$

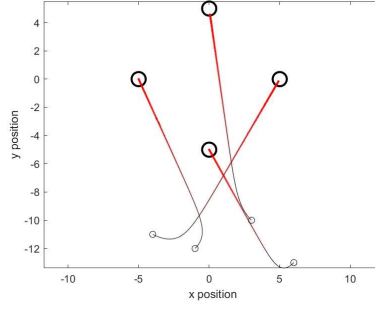
Since A_i contains eigenvalues with negative real part, we optimize using the variable transformation of Section IV-A. The initial conditions of the vehicles were

$$\begin{aligned} x_1(0) &= (3, -10, -1, 1)^\top, \\ x_2(0) &= (-1, -12, 1, 1)^\top, \\ x_3(0) &= (-4, -11, 2, -1)^\top, \\ x_4(0) &= (6, -13, -1, -1)^\top. \end{aligned}$$

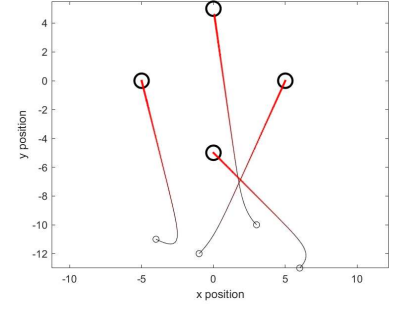
Only 11 iterations of (29) were needed to solve for the minimum time to reach the goal formation, which was found to be $t^* = 15.015$. Figures 3a and 3b show the optimal paths found from (30). Figure 3c shows the optimal paths when the initial condition for vehicle 4 was changed to $x_4(0) = (6, -13, 1, 1)^\top$ and a different optimal assignment results.



(a) Optimal trajectories at $t = 1.50$ (10% of path).



(b) Complete optimal trajectories.



(c) Complete optimal trajectories for an alternate initial condition.

Fig. 3: The example presented in Section V-B with four vehicles guiding to four possible goal sets. The optimal formation is achieved in $t^* = 15.015$. The trajectories are shown for two different times as they traverse the optimal path in the left two figures. The figure on the right is the trajectories when the initial condition for vehicle 4 is different.

VI. CONCLUSIONS AND FUTURE WORK

We presented how to formulate vehicle coordination problems with unknown goal assignments as the viscosity solution to a single Hamilton–Jacobi equation. We show the solution of this single HJ PDE is equivalent to decomposing the problem and performing a linear bottleneck assignment using the viscosity solutions of independent single-vehicle problems. This allows quadratic computational scaling in the number of vehicles. Finally, a level set method based on the Hopf formula was presented for efficient computation of the vehicle value functions, in which each can be computed in parallel. The Hamilton–Jacobi formulation presented has other advantages for multi-robot systems, such as the compensation of time delays which can be induced in several ways including computation, sensing, and inter-robot communication. See for example [41].

Future work includes to expand the class of allowable vehicle dynamics from general linear models to certain types of nonlinear dynamics and to allow for dependencies between vehicles both in the dynamics models and in the cost functional.

APPENDIX

Lemma 5: Given that (10) satisfies assumptions 1-3, then (21) also satisfies assumption 1-3.

Proof: From assumption 1, \widehat{H}_i is continuous by composition rule [33, Theorem 4.7] and since (24) does not depend on state, it therefore trivially meets assumption 3. It remains to show that (21) meets assumption 2. We know assumption 2 holds for all $x_i \in \mathbb{R}^{n_i}$ and for all $p', p'' \in \mathbb{R}^{n_i}$, therefore the following holds

$$\begin{aligned} & \left| H_i \left(s, x_i, e^{-sA_i^\top} p' \right) - H_i \left(s, x_i, e^{-sA_i^\top} p'' \right) \right| \\ & \leq c_i (1 + \|x_i\|) \left\| e^{-sA_i^\top} p' - e^{-sA_i^\top} p'' \right\| \\ & \leq c_i (1 + \|x_i\|) \left| \lambda_{\max} \left(e^{-sA_i^\top} \right) \right| \|p' - p''\| \\ & = c (1 + \|x_i\|) \|p' - p''\|. \end{aligned}$$

When $x_i = 0$ we have

$$\begin{aligned} & \left| H_i \left(s, 0, e^{-sA_i^\top} p' \right) - H_i \left(s, 0, e^{-sA_i^\top} p'' \right) \right| \\ & = \left| \widehat{H}_i \left(s, p' \right) - \widehat{H}_i \left(s, p'' \right) \right| \\ & \leq c \|p' - p''\| \leq c (1 + \|x_i\|) \|p' - p''\|, \end{aligned}$$

for any x_i . And since $\left| \widehat{H}_i \left(s, x_i, 0 \right) \right| = 0$ for all x_i , it follows that assumption 2 is met. \blacksquare

Proof of Lemma 1. *Proof:* The sum of continuous functions is also continuous [33, Theorem 4.9], therefore assumption 1 is met. To prove assumption 2, we first write, for all $\bar{p}', \bar{p}'' \in \mathbb{R}^n$,

$$\begin{aligned} & |H(s, x, \bar{p}') - H(s, x, \bar{p}'')| \\ & = \left| \sum_i H_i(s, x_i, \bar{p}'_i) - \sum_i H_i(s, x_i, \bar{p}''_i) \right| \\ & \leq \sum_i |H_i(s, x_i, \bar{p}'_i) - H_i(s, x_i, \bar{p}''_i)|, \end{aligned} \quad (31)$$

$$\begin{aligned} & \leq \sum_i c_i (1 + \|x_i\|) \|\bar{p}'_i - \bar{p}''_i\| \\ & \leq \sum_i c_i (1 + \|x\|) \|\bar{p}' - \bar{p}''\| \end{aligned} \quad (32)$$

$$c (1 + \|x\|) \|\bar{p}' - \bar{p}''\|,$$

where line (31) comes from the triangular inequality and line (32) comes by noting $\forall i, \|x_i\| \leq \|x\|$ and $\|\bar{p}'_i - \bar{p}''_i\| \leq \|\bar{p}' - \bar{p}''\|$. Therefore, there exists a $c = \sum_i c_i$ such that the inequality holds and we arrive at our result for part 1 of assumption 2. The second part of assumption follows from part 1,

$$\begin{aligned} |H(s, x, 0)| & = \left| \sum_i H(s, x_i, 0) \right| \\ & \leq \sum_i |H(s, x_i, 0)| \leq \sum_i c_i (1 + \|x_i\|) \\ & \leq \sum_i c_i (1 + \|x\|) = c (1 + \|x\|). \end{aligned}$$

And we have shown part 2. The proof of assumption 3 follows that of above. For any compact set $M \subset \mathbb{R}^n$ and for all $\bar{x}', \bar{x}'' \in M$

$$\begin{aligned}
& |H(s, \bar{x}', p) - H(s, \bar{x}'', p)| \\
&= \left| \sum_i H(s, \bar{x}'_i, p_i) - \sum_i H_i(s, \bar{x}''_i, p_i) \right| \\
&\leq \sum_i |H(s, \bar{x}'_i, p_i) - H_i(s, \bar{x}''_i, p_i)|, \\
&\leq \sum_i \kappa_i(M) (1 + \|p_i\|) \|\bar{x}'_i - \bar{x}''_i\| \\
&\leq \sum_i \kappa_i(M) (1 + \|p\|) \|\bar{x}' - \bar{x}''\| \\
&= \kappa(M) (1 + \|p\|) \|\bar{x}' - \bar{x}''\|.
\end{aligned}$$

Therefore, there exists a $\kappa(M) = \sum_i \kappa_i(M)$ such that the inequality holds. ■

REFERENCES

- [1] D. M. Stipanović, G. Inalhan, R. Teo, and C. J. Tomlin, “Decentralized overlapping control of a formation of unmanned aerial vehicles,” *Automatica*, vol. 40, no. 8, pp. 1285–1296, 2004.
- [2] X. Dong, B. Yu, Z. Shi, and Y. Zhong, “Time-varying formation control for unmanned aerial vehicles: Theories and applications,” *IEEE Transactions on Control Systems Technology*, vol. 23, no. 1, pp. 340–348, 2015.
- [3] A. Felner, R. Stern, S. E. Shimony, E. Boyarski, M. Goldenberg, G. Sharon, N. Sturtevant, G. Wagner, and P. Surynek, “Search-based optimal solvers for the multi-agent pathfinding problem: Summary and challenges,” in *Tenth Annual Symposium on Combinatorial Search*, 2017.
- [4] D. Mellinger, A. Kushleyev, and V. Kumar, “Mixed-integer quadratic program trajectory generation for heterogeneous quadrotor teams,” in *2012 IEEE International Conference on Robotics and Automation*. IEEE, 2012, pp. 477–483.
- [5] F. Augugliaro, A. P. Schoellig, and R. D’Andrea, “Generation of collision-free trajectories for a quadcopter fleet: A sequential convex programming approach,” in *2012 IEEE/RSJ International Conference on Intelligent Robots and Systems*. IEEE, 2012, pp. 1917–1922.
- [6] M. Chen, J. F. Fisac, S. Sastry, and C. J. Tomlin, “Safe sequential path planning of multi-vehicle systems via double-obstacle hamilton-jacobi-isaacs variational inequality,” in *2015 European Control Conference (ECC)*. IEEE, 2015, pp. 3304–3309.
- [7] M. DeBord, W. Hönig, and N. Ayanian, “Trajectory planning for heterogeneous robot teams,” in *2018 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2018, pp. 7924–7931.
- [8] D. R. Robinson, R. T. Mar, K. Estabridis, and G. Hewer, “An efficient algorithm for optimal trajectory generation for heterogeneous multi-agent systems in non-convex environments,” *IEEE Robotics and Automation Letters*, vol. 3, no. 2, pp. 1215–1222, 2018.
- [9] D. P. Bertsekas, *Linear Network Optimization: Algorithms and Codes*. MIT Press, 1991.
- [10] J. R. Marden and A. Wierman, “Distributed welfare games with applications to sensor coverage,” in *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*. IEEE, 2008, pp. 1708–1713.
- [11] G. Arslan, J. R. Marden, and J. S. Shamma, “Autonomous vehicle-target assignment: A game-theoretical formulation,” *Journal of Dynamic Systems, Measurement, and Control*, vol. 129, no. 5, pp. 584–596, 2007.
- [12] J. R. Marden and M. Effros, “The price of selfishness in network coding,” *IEEE Transactions on Information Theory*, vol. 58, no. 4, pp. 2349–2361, 2012.
- [13] T. Roughgarden, *Selfish Routing and the Price of Anarchy*. MIT Press Cambridge, 2005, vol. 174.
- [14] J. R. Marden and T. Roughgarden, “Generalized efficiency bounds in distributed resource allocation,” *IEEE Transactions on Automatic Control*, vol. 59, no. 3, pp. 571–584, 2014.
- [15] M. R. Kirchner, “Towards general performance bounds of the distributed sensor coverage problem,” *Electrical, Computer & Energy Engineering Graduate Theses & Dissertations*, no. 71, 2013.
- [16] W. Hönig, S. Kiesel, A. Tinka, J. W. Durham, and N. Ayanian, “Conflict-based search with optimal task assignment,” in *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems*. International Foundation for Autonomous Agents and Multiagent Systems, 2018, pp. 757–765.
- [17] D. Morgan, G. P. Subramanian, S.-J. Chung, and F. Y. Hadaegh, “Swarm assignment and trajectory optimization using variable-swarm, distributed auction assignment and sequential convex programming,” *The International Journal of Robotics Research*, vol. 35, no. 10, pp. 1261–1285, 2016.
- [18] O. Gross, “The bottleneck assignment problem,” The Rand Corporation, Santa Monica, CA, Technical Report RM-1102, 1959.
- [19] I. Mitchell, A. M. Bayen, and C. J. Tomlin, “A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games,” *IEEE Transactions on Automatic Control*, vol. 50, no. 7, pp. 947–957, 2005.
- [20] S. Osher and R. Fedkiw, *Level Set Methods and Dynamic Implicit Surfaces*. Springer Science & Business Media, 2006, vol. 153.
- [21] I. M. Mitchell, “The flexible, extensible and efficient toolbox of level set methods,” *Journal of Scientific Computing*, vol. 35, no. 2, pp. 300–329, 2008.
- [22] R. E. Bellman, *Adaptive Control Processes: A Guided Tour*. Princeton University Press, 2015.
- [23] —, *Dynamic Programming*. Princeton University Press, 1957, vol. 1, no. 2.
- [24] J. Darbon and S. Osher, “Algorithms for overcoming the curse of dimensionality for certain Hamilton-Jacobi equations arising in control theory and elsewhere,” *Research in the Mathematical Sciences*, vol. 3, no. 1, p. 19, 2016.
- [25] E. Hopf, “Generalized solutions of non-linear equations of first order,” *Journal of Mathematics and Mechanics*, vol. 14, pp. 951–973, 1965.
- [26] M. R. Kirchner, G. Hewer, J. Darbon, and S. Osher, “A primal-dual method for optimal control and trajectory generation in high-dimensional systems,” in *IEEE Conference on Control Technology and Applications*, 2018, pp. 1575–1582.
- [27] M. R. Kirchner, R. Mar, G. Hewer, J. Darbon, S. Osher, and Y. Chow, “Time-optimal collaborative guidance using the generalized Hopf formula,” *IEEE Control Systems Letters*, vol. 2, no. 2, pp. 201–206, 2018.
- [28] A. R. Bryson and Y.-C. Ho, *Applied Optimal Control: Optimization, Estimation and Control*. CRC Press, 1975.
- [29] L. C. Evans, *Partial Differential Equations*. Providence, R.I.: American Mathematical Society, 2010.
- [30] R. Burkhard, M. Dell’Amico, and S. Martello, *Assignment Problems: Revised Reprint*. SIAM, 2012.
- [31] A. I. Subbotin, *Generalized Solutions of First Order PDEs: The Dynamical Optimization Perspective*. Birkhäuser, 1995.
- [32] J.-B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of Convex Analysis*. Springer Science & Business Media, 2012.
- [33] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed. McGraw-Hill New York, 1976.
- [34] A. B. Kurzhanski and P. Varaiya, *Dynamics and Control of Trajectory Tubes: Theory and Computation*. Springer, 2014, vol. 85.
- [35] S. Osher and C.-W. Shu, “High-order essentially nonoscillatory schemes for hamilton-jacobi equations,” *SIAM Journal on numerical analysis*, vol. 28, no. 4, pp. 907–922, 1991.
- [36] M. Fashoro, O. Hajek, and K. Loparo, “Controllability properties of constrained linear systems,” *Journal of Optimization Theory and Applications*, vol. 73, no. 2, pp. 329–346, 1992.
- [37] L. S. Pontryagin, *Mathematical Theory of Optimal Processes*. Routledge, 2018.
- [38] I. M. Mitchell and J. A. Templeton, “A toolbox of Hamilton-Jacobi solvers for analysis of nondeterministic continuous and hybrid systems,” in *HSCC*, vol. 5. Springer, 2005, pp. 480–494.
- [39] J. Nocedal and S. Wright, *Numerical Optimization*. Springer Science & Business Media, 2006.
- [40] C. Ramirez, R. Sanchez, V. Kreinovich, and M. Argaez, “ $\sqrt{x^2 + \mu}$ is the most computationally efficient smooth approximation to $|x|$: a proof,” *Journal of Uncertain Systems*, no. 3, pp. 205–210, 2014.
- [41] M. R. Kirchner, “A level set approach to online sensing and trajectory optimization with time delays,” in *IFAC-PapersOnline*, vol. 52, no. 8, 2019, pp. 301–306.