SURFACE CONCENTRATION OF TRANSMISSION EIGENFUNCTIONS

YAT TIN CHOW, YOUJUN DENG, HONGYU LIU, AND MAHESH SUNKULA

Abstract. The transmission eigenvalue problem is a type of non-elliptic and non-selfadjoint spectral problem that arises in the wave scattering theory when invisibility/transparency occurs. The transmission eigenfunctions are the interior resonant modes inside the scattering medium. We are concerned with the geometric rigidity of the transmission eigenfunctions and show that they concentrate on the boundary surface of the underlying domain in two senses. This substantiates the recent numerical discovery in [10] on such an intriguing spectral phenomenon of the transmission resonance. Our argument is based on generalized Weyl's law and certain novel ergodic properties of the coupled boundary layer-potential operators which are employed to analyze the generalized transmission eigenfunctions.

Keywords: transmission eigenfunctions; surface concentration; coupled layer-potential operators; quantum ergodicity; wave scattering; invisibility and transparency
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1. Introduction

We first introduce the time-harmonic acoustic wave scattering, which is the physical origin of the transmission eigenvalue problem in our study and moreover shall be used to motivate our mathematical analysis.

Let D be an open connected and bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a $\mathcal{C}^\infty$ smooth boundary ∂D and a connected complement $\mathbb{R}^d \backslash \overline{D}$. In the physical setting, D signifies the support of an inhomogeneous medium scatterer located in an otherwise uniformly homogeneous space. The medium parameter is characterised by the refractive index which is normalised to be 1 in $\mathbb{R}^d \backslash \overline{D}$ and is assumed to be $Q \in \mathbb{R}_+$ and $Q \neq 1$ in D. Set $V = (Q^2 - 1)\chi_D + 0\chi_{\mathbb{R}^d \backslash \overline{D}}$, which is referred to as the scattering potential. Let $\psi_0 \in \mathcal{C}^\infty(\mathbb{R}^d)$ be an impinging wave field which is an entire solution to $(\Delta + \kappa^2)\psi_0 = 0$ in $\mathbb{R}^d$, where $\kappa \in \mathbb{R}_+$ signifies the angular frequency of the wave. The impingement of ψ_0 on the scattering potential (D, V), or equivalently on the scattering medium (D, Q), leads to the following Helmholtz system for the total wave field $\psi \in H^1_{loc}(\mathbb{R}^d)$:

$$\begin{cases} \Delta\psi + \kappa^2(1 + V)\psi = 0 & \text{in } \mathbb{R}^d \\ (\partial_r - \mathrm{i}\kappa)(\psi - \psi_0) = \mathcal{O}(r^{-\frac{d+1}{2}}) & \text{as } r \to \infty, \end{cases} \tag{1.1}$$

where $\mathrm{i} := \sqrt{-1}$ and $r := |x|$ for $x \in \mathbb{R}^d$. The last limit in (1.1) is known as the Sommerfeld radiation condition which holds uniformly in the angular variable $\hat{x} := x/|x| \in \mathbb{S}^{d-1}$ and characterises the outgoing nature of the scattered $\psi^s := \psi - \psi_0$. The well-posedness of the scattering system (1.1) is known (cf. [11]) and in particular it holds that

$$\psi(x) = \psi_0(x) + \frac{e^{\mathrm{i}\kappa r}}{r^{(d-1)/2}}\psi_\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r^{(d+1)/2}}\right) \quad \text{as } r \to +\infty. \tag{1.2}$$

In (1.2), ψ_∞ is referred to as the far-field pattern which encodes the scattering information of the underlying scatterer under the probing of the incident wave ψ_0. An inverse problem of industrial importance is to recover (D, V) by knowledge of ψ_∞. It is clear that the recovery fails if $\psi_\infty \equiv 0$, namely invisibility/transparency occurs. In such a case, one has by the

Rellich theorem [11] that $\psi = \psi_0$ in $\mathbb{R}^d\backslash\overline{D}$. Hence, if setting $u = \psi|_D$ and $v = \psi_0|_D$, it holds that

$$\begin{cases} \Delta u + \kappa^2(1+V)u = 0 & \text{in } D, \\ \Delta v + \kappa^2 v = 0 & \text{in } D, \\ u = v, \quad \partial_\nu u = \partial_\nu v & \text{on } \partial D, \end{cases} \tag{1.3}$$

where and also in what follows $\nu \in \mathbb{S}^{d-1}$ stands for the exterior unit normal to ∂D. That is, if invisibility/transparency occurs, the total and incident wave fields fulfil the spectral system (1.3), which is referred to as the transmission eigenvalue problem in the literature.

Let us consider the spectral study of the transmission eigenvalue problem (1.3). It is clear that $u = v \equiv 0$ are trivial solutions. If there exist nontrivial $u \in L^2(D)$ and $v \in L^2(D)$ such that $u - v \in H_0^2(D)$ and the first two equations in (1.3) are fulfilled, then κ is referred to as a transmission eigenvalue and u, v are the corresponding transmission eigenfunctions. It is emphasised that in this paper, we are mainly concerned with real transmission eigenvalues, namely $\kappa \in \mathbb{R}_+$, though there exist complex transmission eigenvalues. The transmission eigenvalue problem is non-elliptic and non-selfadjoint, and this is partly evidenced by setting $w = u - v$ and verifying that

$$(\Delta + \kappa^2)\big(\Delta + \kappa^2(1+V)\big)w = 0 \quad \text{in } H_0^2(D), \tag{1.4}$$

which is a fourth-order PDE eigenvalue problem and quadratic in $\lambda = \kappa^2$. The following connection of the transmission eigenfunctions with the scattering problem (1.1)–(1.2) shall be a useful observation for our subsequent study.

Theorem 1.1 ([6]). *Suppose that $\kappa \in \mathbb{R}_+$ is a transmission eigenvalue and $u, v \in L^2(D)$ are the associated transmission eigenfunctions to (1.3). Then for any sufficiently small $\varepsilon > 0$, there exists $g_\varepsilon \in L^2(\mathbb{S}^{d-1})$ such that*

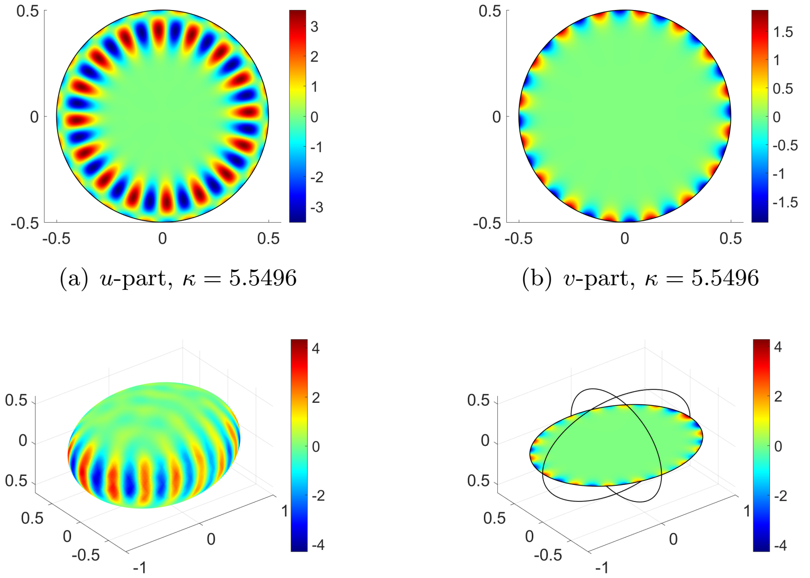
$$\|v_{g_\varepsilon} - v\|_{L^2(D)} < \varepsilon, \quad v_{g_\varepsilon}(x) := \int_{\mathbb{S}^{d-1}} \exp(\mathrm{i}\kappa x \cdot \theta) g_\varepsilon(\theta)\, ds(\theta). \tag{1.5}$$

Moreover, if taking $\psi_0 = v_{g_\varepsilon}$ in (1.1), one has $\|\psi_\infty\|_{L^2(\mathbb{S}^{d-1})} \le C_{V,k}\varepsilon$ and $\|u - \psi\|_{L^2(D)} \le C_{V,k}\varepsilon$, where $C_{V,k}$ is a positive constant depending only on V and k.

In the physical setting, v_{g_ε} is referred to as a Herglotz wave, and Theorem 1.4 states that if u, v are transmission eigenfunctions, they respectively correspond to the total and incident wave fields (restricted in D) from a nearly invisible/transparent scattering scenario.

The transmission eigenvalue problem was first proposed and studied in [9, 23]. The exclusion of the transmission eigenvalues can guarantee the injectivity and dense range of the far-field operator and hence the validity of a certain reconstruction scheme for the inverse scattering problem mentioned earlier. Applications to invisibility cloaking of the transmission eigenfunctions were discussed in [20, 27]. The spectral properties of the transmission eigenvalues have been extensively and intensively studied in the literature, and we refer to [7, 11, 26] for reviews and surveys on the existing developments on this aspect. Recently, several intrinsic geometric properties of the transmission eigenfunctions were discovered and investigated. In [4–6], it is shown the transmission eigenfunctions are generically vanishing around corners or high-curvature places of ∂D. In [10], it is found that "many" transmission eigenfunctions tend to localize around on ∂D in the sense that the L^2-energies of those eigenmodes are concentrated in a neighbourhood of ∂D; see Fig. 1 for two typical numerical illustrations, where the transmission eigenfunctions are plotted associated with different (D, V)'s. It is highly intriguing to have the following observations:

(1) It is clear that the transmission eigenfunctions are interior resonant modes which exhibit highly oscillatory patterns. Interestingly, the high oscillations of these resonant modes are localized on ∂D. The study on the eigenfunction concentration is



(a) u-part, $\kappa = 5.5496$ (b) v-part, $\kappa = 5.5496$

(c) u-part, $\kappa = 3.1023$ (d) v-part, $\kappa = 3.1023$

FIGURE 1. Transmission eigenfunctions to (1.3) associated with $Q = 8$ (or equivalently, $V = 63$) for different D's and κ's. In Figures (a) and (b), the domain D is a central disk of radius 1. In Figure (c), D is an ellipsoid. Figure (d) is the slice plotting of Figure (c) at $x_3 = 0$ for $x = (x_j)_{j=1}^3 \in \mathbb{R}^3$.

a central topic in mathematical physics and spectral theory; see e.g. [36] and the references cited therein. However, the concentration phenomenon presented here is peculiar and different from the existing ones in the literature for the classical eigenvalue problems. Hence, it represents a new spectral phenomenon.

(2) The physical intuition to explain the surface-localizing behaviour can be described as follows. According to Theorem 1.1, the transmission eigenfunctions are (at least approximately) restrictions of the incident and total wave fields when invisibility/transparency occurs. Hence, in order to reach the invisibility/transparency, a 'smart' way for the propagating wave is to 'slide' over the surface of the scattering object, namely (D, V), and return to its original path after passing through the object. This clearly gives rise to the regular pattern depicted in Fig. 1, where the wave fields inside the object clearly propagates along the surface ∂D.

(3) The surface-localization indicates that the transmission eigenfunctions carry the geometric information of the underlying scattering medium and hence is a global geometric rigidity property. This spectral property has been proposed for super-resolution wave imaging, generation of the so-called pseudo plasmon modes with a potential bio-sensing application and the artificial electromagnetic mirage effect [10, 14].

However, the discovery in [10] is mainly based on numerics, though the case within radial geometry is verified rigorously [10, 13] by using the analytic expressions of the transmission eigenfunctions via Bessel and spherical harmonic functions. It is the aim of this paper to derive a theoretical understanding of this peculiar spectral phenomenon. First, we make essential use of the layer-potential operators, especially the so-called Neumann-Poincaré operators, which are used to reformulate the transmission eigenvalue problem (1.3) as a spectral system associated with coupled integral equations. Treating those potential operators as pseudo-differential operators and exploiting their quantitative properties, we introduce a certain generalized transmission eigenvalue problem which approximates the original transmission eigenvalue problem in a certain sense. Second, we show that the (local) Dirichlet energy of the generalised transmission eigenfunctions are localized around ∂D with quantitative characterisations. Then via quantum ergodicity, we can also establish that the generalised transmission eigenfunctions are quantitatively localized around ∂D almost surely. In establishing those quantitative results, generalized Weyl's law and certain novel ergodic properties of the coupled layer potential operators are explored. Finally, we would like to remark briefly that throughout our study, it is assumed that ∂D is $\mathcal{C}^\infty$ smooth and $Q = 1 + V$ is a constant, though the numerics in [10] indicate that the surface-localizing property should hold in more general scenarios with Lipscthiz domains and variable Q; see the numerical illustrations in Fig. 1 as well. However, even in the current setup, the study presents significant technical difficulties and challenges. We believe the theoretical framework developed in this paper can be used to treat similar phenomena for transmission resonances arising in electromagnetic and elastic scattering. Moreover, we shall mainly consider the case that $d > 2$. In fact, the case $d = 2$ can be treated within our framework, but with different calculations and analysis. In order to be focusing in our study, we stick to the case $d \geq 3$, which presents more theoretical challenges.

The rest of the paper is organized as follows. In Section 2, we present the integral reformulation of the transmission eigenvalue problem (1.3) as well as the quantitative properties of the layer-potential operators as pseudo-differential operators. In Section 3, we present the generalized transmission eigenvalue problem and discuss its properties. Section 4 is devoted to the main results on the surface localization as well as the corresponding proofs.

2. Integral formulation and layer-potential operators

2.1. **Preliminaries.** From this section and onward, let us only consider $D \subset \mathbb{R}^d$ with $\partial D \in \mathcal{C}^\infty$ and $d \geq 3$. We discuss the layer potential formulism (cf. [11,16,22,28,34]). For a given $\kappa \in \mathbb{R}_+$, we introduce the single- and double-layer potential operators as follows:

$$\mathcal{S}_{\partial D}^{\kappa}[\phi](x) := \int_{\partial D} G_\kappa(x-y)\phi(y)d\sigma(y), \tag{2.1}$$

$$\mathcal{D}_{\partial D}^{\kappa}[\phi](x) := \int_{\partial D} \partial_{\nu_y} G_\kappa(x-y)\phi(y)d\sigma(y), \tag{2.2}$$

for $x \in \mathbb{R}^d$, where and also in what follows ν_y signifies the exterior unit normal vector at $y \in \partial D$ and G_κ is the outgoing fundamental solution of the PDO $\Delta + \kappa^2$ in $\mathbb{R}^d$ given by

$$G_\kappa(x-y) = \left(C_d \kappa^{d-2} \mathrm{i}\right) (\kappa|x-y|)^{-\frac{d-2}{2}} H^{(1)}_{\frac{d-2}{2}}(\kappa|x-y|). \tag{2.3}$$

Here, C_d is some dimensional constant and $H^{(1)}_{\frac{d-2}{2}}$ is the Hankel function of the first kind and order $(d-2)/2$. It is known that the single-layer potential $\mathcal{S}_{\partial D}^{\kappa}$ satisfies the following jump condition on ∂D:

$$\frac{\partial}{\partial \nu} \left(\mathcal{S}_{\partial D}^{\kappa}[\phi]\right)^{\pm} = (\pm \frac{1}{2} I + \mathcal{K}_{\partial D}^{\kappa}{}^*)[\phi], \tag{2.4}$$

where the superscripts $\pm$ indicate the limits from outside and inside D respectively, and $\mathcal{K}_{\partial D}^{\kappa}{}^* : L^2(\partial D) \to L^2(\partial D)$ is the Neumann-Poincaré operator defined by

$$\mathcal{K}_{\partial D}^{\kappa}{}^*[\phi](x) := \int_{\partial D} \partial_{\nu_x} G_\kappa(x-y)\phi(y)d\sigma(y). \tag{2.5}$$

By an abuse of notations, whenever no confusions arise, we denote the restriction of the layer potential operator onto the boundary with the same notations, i.e. we write $\mathcal{S}^{\kappa}_{\partial D} : L^2(\partial D) \to L^2(\partial D)$ and $\mathcal{D}^{\kappa}_{\partial D} : L^2(\partial D) \to L^2(\partial D)$.

With the above preparation, we consider the transmission eigenvalue problem (1.3) by taking

$$u = \mathcal{S}^{\kappa Q}_{\partial D}[\phi]\,, \quad v = \mathcal{S}^{\kappa}_{\partial D}[\varphi] \text{ on } D, \tag{2.6}$$

where $(\phi, \varphi) \in L^2(\partial D) \times L^2(\partial D)$; and via (2.4), we rewrite (1.3) into the following boundary integral system:

$$\begin{pmatrix} \mathcal{S}^{\kappa Q}_{\partial D} & -\mathcal{S}^{\kappa}_{\partial D} \\ -\frac{1}{2}I + \mathcal{K}^{\kappa Q^*}_{\partial D} & -(-\frac{1}{2}I + \mathcal{K}^{\kappa}_{\partial D}{}^*) \end{pmatrix} \begin{pmatrix} \phi \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.7}$$

or that

$$\varphi = [\mathcal{S}^{\kappa}_{\partial D}]^{-1}\, \mathcal{S}^{\kappa Q}_{\partial D}[\phi]\,, \quad \left((-\frac{1}{2}I + \mathcal{K}^{\kappa Q^*}_{\partial D}) - (-\frac{1}{2}I + \mathcal{K}^{\kappa}_{\partial D}{}^*)\, [\mathcal{S}^{\kappa}_{\partial D}]^{-1}\, \mathcal{S}^{\kappa Q}_{\partial D} \right)[\phi] = 0. \tag{2.8}$$

Instead of (1.3), we shall consider (2.6)–(2.8) in what follows.

We shall treat the layer potential operators in (2.6)–(2.8) as pseudo-differential operators in our subsequent analysis. For clarity and self-containedness, we briefly discuss the pseudo-differential operators and refer to [18, 19] for more relevant details. Throughout the rest of the paper, we let $h := \kappa^{-1}$ and $\Phi\mathrm{SO}^m_h$ denote the pseudo-differential operator with the action $\mathrm{Op}_{a,h} := \mathcal{F}_h^{-1} \circ \mathfrak{M}_a \circ \mathcal{F}_h$, where $\mathcal{F}_h$ is the semi-classical Fourier transform, $\mathfrak{M}_a$ is the action with multiplication by a and a belongs to the symbol class of order m. For clarity, we have

$$\mathcal{F}_h f(x) := (2\pi h)^{-d/2} \int_{\mathbb{R}^d} e^{-\mathrm{i}\frac{x\cdot y}{h}} f(y)\, dy, \tag{2.9}$$

and

$$\mathrm{Op}_{a,h} f(x) := (2\pi h)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\mathrm{i}\frac{(x-y)\cdot\xi}{h}}\, a\left(\frac{x+y}{2}, \xi\right) f(y)\, dy d\xi. \tag{2.10}$$

It is noted that $\mathrm{Op}_{a,h}$ is uniquely defined modulus $h\Phi\mathrm{SO}^{m-1}_h$ if $a \in \tilde{\mathcal{S}}^m(T^*(\partial D)))$, where $T^*(\partial D)$ signifies the cotangent space of ∂D. Here, we briefly mention the following notations and definitions that are used in our study:

$$\begin{aligned}
&\bigcup_i U_i = \partial D\,, \quad F_i : \pi_i^{-1}(U_i) \to U_i \times \mathbb{R}^{d-1}\,, \quad \sum_i \psi_i^2 = 1\,, \quad \mathrm{spt}(\psi_i) \subset U_i\,; \\
&\widetilde{S}^m(T^*(\partial D)) := \Big\{ a : T^*(\partial D) \backslash \partial D \times \{0\} \to \mathbb{C}\,; \\
&\qquad a = \sum_i \psi_i F_i^* \left((F_i^{-1})^*(\psi_i) a_i \right), a_i \in \widetilde{S}^m(U_i \times \mathbb{R}^{d-1}\backslash\{0\}) \Big\}; \\
&\widetilde{S}^m(U_i \times \mathbb{R}^{d-1}\backslash\{0\}) := \Big\{ a : U_i \times (\mathbb{R}^{d-1}\backslash\{0\}) \to \mathbb{C}\,; \\
&\qquad a \in \mathcal{C}^\infty(U_i \times (\mathbb{R}^{d-1}\backslash\{0\}))\,, \ |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le C_{\alpha,\beta}(|\xi|)^{m-|\alpha|} \Big\}.
\end{aligned}$$

Given an operator $\mathscr{A}_h \in \Phi\mathrm{SO}^m_h$, we also denote the symbol (under a given coordinate system):

$$p_{\mathscr{A}_h}(x,\xi) := a(x,\xi) \text{ if } \mathscr{A}_h = \mathrm{Op}_{a,h}. \tag{2.11}$$

Notice that the principal symbol $p_{\mathscr{A}_h}(x,\xi) (\mathrm{mod}\, h\tilde{\mathcal{S}}^{m-1}(T^*(\partial D)))$ is independent of the choice of the coordinate system. In (2.11), we define the operator via the Weyl quantisation. It is remarked that one can also use the left or right quantisation, but this will not affect our subsequent analysis since the principal symbol of the operator is independent of the choice of the quantisation (cf. [3]). For notational sake, we also recall the smooth symbol class of order m, $S^m(T^*(\partial D))$, as

$$\begin{aligned} S^m(T^*(\partial D)) &:= \left\{ a : T^*(\partial D) \to \mathbb{C}\,;\, a = \sum_i \psi_i F_i^* \left((F_i^{-1})^*(\psi_i) a_i \right),\ a_i \in S^m(U_i \times \mathbb{R}^{d-1}) \right\}, \\ S^m(U_i \times \mathbb{R}^{d-1}) &:= \Big\{ a : U_i \times \mathbb{R}^{d-1} \to \mathbb{C}\,; a \in \mathcal{C}^\infty(U_i \times \mathbb{R}^{d-1}) \\ &\qquad |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|^2)^{\frac{m-|\alpha|}{2}} \Big\}. \end{aligned}$$

2.2. **Layer potential operators as ΦSO's.** In this subsection, we compute the principal symbols of of $\mathcal{S}_{\partial D}^{kQ}$ and $\mathcal{K}_{\partial D}^{kQ\,*}$, and also derive some quantitative properties of these operators.

First, we briefly introduce the geometric description of $D \subset \mathbb{R}^d$, and we also refer to [] for a similar treatment. Let $\mathbb{X}(x) : x \in U \subset \mathbb{R}^{d-1} \mapsto \partial D \subset \mathbb{R}^d$ be a regular parametrization of the surface ∂D. We often write the vector $\mathbb{X}_j := \frac{\partial \mathbb{X}}{\partial x_j}$, $j = 1, 2, \ldots, d-1$. One has that the the normal vector is given as $\nu := \times_{j=1}^{d-1} \mathbb{X}_j / | \times_{j=1}^{d-1} \mathbb{X}_j|$. Let $\bar{\nabla}$ denote the standard covariant derivative on the ambient space $\mathbb{R}^d$, and Π be the second fundamental form defined on the tangent space $T(\partial D)$. Next, we introduce the following matrix $A_{ij}(x), x \in \partial D$, defined as

$$A(x) := (A_{ij}(x)) = \langle \mathbf{II}_x(\mathbb{X}_i, \mathbb{X}_j), \nu_x \rangle\,.$$

Let $g = (g_{ij})$ be the induced metric tensor and $(g^{ij}) = g^{-1}$. We write $H(x), x \in \partial D$ as the mean curvature satisfying

$$\mathrm{tr}_{g(x)}(A(x)) := \sum_{i,j=1}^{d-1} g^{ij}(x) A_{ij}(x) := (d-1)H(x)\,.$$

We write ∇ as the Levi-Civita connection on ∂D and $\bar{\nabla}$ that on the ambient Euclidean space. Let Γ_{ij}^k and $\bar{\Gamma}_{ij}^k$ be the Christoffel symbols such that

$$\nabla_{\mathbb{X}_i}\mathbb{X}_j := \sum_{k=1}^{d-1} \Gamma_{ij}^k \mathbb{X}_k, \quad \bar{\nabla}_{e_i} e_j := \sum_{k=1}^{d} \bar{\Gamma}_{ij}^k e_k \quad \text{for} \quad i,j = 1,...,d-1,$$

for the basis $\{e_i\}_{i=1,\ldots,d-1} = \{\mathbb{X}_i\}_{i=1,\ldots,d-1}$ and $e_d = \nu$. It is noted that for a fixed $x \in \partial D$, if one takes the geodesic normal coordinate in a neighborhood of x to give $z \in \mathrm{Dom}(\exp_x) \subset T_x(\partial D) \cong \mathbb{R}^{d-1} \mapsto \mathbb{X}(z) := \exp_x(z) \in \partial D$, then at the point x, we have that $g_{ij}(x) = \boldsymbol{\delta}_{ij}$ and $\Gamma_{ij}^l(x) = 0$ for $i,j,l = 1,...,d-1$. Here and also in what follows, $\boldsymbol{\delta}$ signifies the Kronecker delta. Moreover, if we choose on the ambient space the semi-geodesic normal coordinate in a neighborhood of x, which is given in a neighbourhood of $x \in \partial D$ as (α, s), $\tilde{\mathbb{X}}(\alpha, s) = \exp_x(\alpha) + s\,\nu(\exp_x(\alpha))$, we have $\bar{\Gamma}_{ij}^l(x) = \Gamma_{ij}^l(x) = 0$ for $l \neq d$ and $\bar{\Gamma}_{ij}^d(x) = A_{ij}(x)$. Now, for $y = \exp_x(\delta\,\omega)$ with $\delta \in \mathbb{R}_+$ and $\omega \in \mathbb{S}^{d-1}$, we have (cf. [8, 21, 25]):

$$\begin{aligned} y =& x + \delta\omega - \frac{1}{2}\delta^2 \langle A(x)\omega, \omega\rangle\, \nu(x) + \mathcal{O}(\delta^3)\,, \\ \nu(y) =& \nu(x) + \delta A(x)\omega + \frac{1}{2}\delta^2 \left[\partial_\omega A(x)\omega - |A(x)\omega|^2 \nu(x) \right] + \mathcal{O}(\delta^3)\,, \\ \sqrt{\det(g(y))} =& 1 + \frac{1}{6}\delta^2\, \mathrm{Ric}_x(\omega,\omega) + \mathcal{O}(\delta^3) \\ =& 1 + \frac{1}{6}\delta^2 \left[(d-1)H(x)\langle A(x)\omega, \omega\rangle - |A(x)\omega|^2 \right] + \mathcal{O}(\delta^3)\,, \end{aligned} \tag{2.12}$$

where Ric signifies the Ricci and curvature.

Next, for the fundamental solution G_κ introduced in (2.3), by using the analytic properties of the Hankel function (cf. [24]), one can have by direct calculations that

$$G_\kappa(x-y)=\begin{cases}\kappa^{d-2}\Big[C_d\kappa^{2-d}|x-y|^{2-d}+\tilde{C}_d\kappa^{4-d}|x-y|^{4-d}+\tilde{\tilde{C}}_d\kappa^{6-d}|x-y|^{6-d}\\ \qquad +\mathcal{O}\left(\kappa^{8-d}|x-y|^{8-d}\right)\Big] \text{ when } d>2, d\neq 4,6\,;\\ \kappa^{2}\Big[C_4\kappa^{-2}|x-y|^{-2}+\tilde{C}_4\log(\kappa|x-y|)+\tilde{\tilde{C}}_4\kappa^{2}|x-y|^{2}\\ \qquad +\mathcal{O}\left(\kappa^{4}|x-y|^{4}\right)\Big] \text{ when } d=4\,;\\ \kappa^{4}\Big[C_6\kappa^{-4}|x-y|^{-4}+\tilde{C}_6\kappa^{-2}|x-y|^{-2}+\tilde{\tilde{C}}_6\log(k|x-y|)\\ \qquad +\mathcal{O}\left(\kappa^{2}|x-y|^{2}\right)\Big] \text{ when } d=6\,;\end{cases} \tag{2.13}$$

and when $d>2$:

$$\begin{aligned}\partial_{\nu_x}G_\kappa(x-y)=\kappa^{d-2}\Big[&C_d(2-d)\,\kappa^{2-d}\langle\nu_x,x-y\rangle|x-y|^{-d}\\ &+\tilde{C}_d(4-d)\kappa^{4-d}\langle\nu_x,x-y\rangle|x-y|^{2-d}+\mathcal{O}\left(\kappa^{6-d}|x-y|^{5-d}\right)\Big],\end{aligned} \tag{2.14}$$

where and also in what follows $C_d,\tilde{C}_d,\tilde{\tilde{C}}_d$ signify some dimensional constants.

Using (2.13), (2.14) and (2.12), together with direct though tedious calculations, one has that when $d\neq 4,6$:

$$\begin{aligned}&G_{\kappa Q}(x-y)\,d\sigma(y)\\ =\;&(\kappa Q)^{d-2}\Big[C_dQ^{2-d}(\kappa\delta)^{2-d}+\tilde{C}_dQ^{4-d}(\kappa\delta)^{4-d}+\tilde{\tilde{C}}_d(\kappa\delta)^{6-d}+\mathcal{O}((\kappa\delta)^{8-d})\\ &+C_d\kappa^{-2}Q^{2-d}(\kappa\delta)^{4-d}\Big(\frac{2-d}{8}\langle A(x)\omega,\omega\rangle^2+\frac{1}{6}\,(d-1)H(x)\langle A(x)\omega,\omega\rangle\\ &-\frac{1}{6}|A(x)\omega|^2\Big)+\mathcal{O}(\kappa^{-2}(\kappa\delta)^{5-d})\Big]\,dy;\end{aligned} \tag{2.15}$$

and when $d=4$:

$$\begin{aligned}&G_{\kappa Q}(x-y)\,d\sigma(y)\\ =\;&(\kappa Q)^{2}\Big[C_4Q^{-2}(\kappa\delta)^{-2}+\tilde{C}_4Q\log(\kappa Q\delta)+\tilde{\tilde{C}}_4(\kappa\delta)^{2}+\mathcal{O}((\kappa\delta)^{2})\\ &+C_4\kappa^{-2}Q^{-2}\left(-\frac{1}{4}\langle A(x)\omega,\omega\rangle^2+\frac{1}{2}\,H(x)\langle A(x)\omega,\omega\rangle-\frac{1}{6}|A(x)\omega|^2\right)+\mathcal{O}(\kappa^{-2}(k\delta))\Big]\,dy;\end{aligned} \tag{2.16}$$

and when $d=6$:

$$\begin{aligned}&G_{\kappa Q}(x-y)\,d\sigma(y)\\ =\;&(\kappa Q)^{4}\Big[C_6Q^{-2}(\kappa\delta)^{-4}+\tilde{C}_6(\kappa\delta)^{-2}+\tilde{\tilde{C}}_6Q\log(\kappa Q\delta)+\mathcal{O}((\kappa\delta)^{4})\\ &+C_6\kappa^{-2}Q^{-4}(\kappa\delta)^{2}\left(-\frac{1}{2}\langle A(x)\omega,\omega\rangle^2+H(x)\langle A(x)\omega,\omega\rangle-\frac{1}{6}|A(x)\omega|^2\right)+\mathcal{O}(\kappa^{-2}(\kappa\delta)^{-1})\Big]\,dy;\end{aligned} \tag{2.17}$$

as well as that when $d>2$:

$$\partial_{\nu_x}G_{\kappa Q}(x-y)\,d\sigma(y)$$

$$
\begin{aligned}
= \quad & \frac{(\kappa Q)^{d-2}}{2}\Big[C_d(2-d)Q^{2-d}(\kappa\delta)^{2-d}\langle A(x)\omega,\omega\rangle + \tilde{C}_d(4-d)Q^{4-d}(\kappa\delta)^{4-d}\langle A(x)\omega,\omega\rangle \\
& +\mathcal{O}((\kappa\delta)^{6-d}) + C_d(2-d)\kappa^{-2}Q^{2-d}(\kappa\delta)^{4-d}\Big(\frac{2-d}{8}\langle A(x)\omega,\omega\rangle^3 \\
& +\frac{1}{6}(d-1)H(x)\langle A(x)\omega,\omega\rangle^2 - \frac{1}{6}|A(x)\omega|^2\langle A(x)\omega,\omega\rangle\Big) + \mathcal{O}(\kappa^{-2}(\kappa\delta)^{5-d})\Big]\, dy.
\end{aligned}
\tag{2.18}
$$

Next, by using the Fourier transform (2.9) with respect to $\mathfrak{q} := \kappa\delta\omega = h^{-1}\delta\omega$, together with the use of the results derived in (2.15)–(2.16), one can obtain by direct though tedious calculations the following principal symbols around $x = y$ in the geodesic normal coordinate (here, we also refer to [1, 2, 29, 30] for related results in the literature), when $d \neq 4, 6$:

$$
\begin{aligned}
& p_{\kappa\mathcal{S}_{\partial D}^{\kappa Q}}(x,\xi) \\
= \quad & Q^{d-2}\Big[C_d Q^{2-d}\mathcal{F}_{\mathfrak{q}}\left[|\mathfrak{q}|^{2-d}\right](\xi) + \tilde{C}_d Q^{4-d}\mathcal{F}_{\mathfrak{q}}\left[|\mathfrak{q}|^{4-d}\right](\xi) \\
& +\tilde{\tilde{C}}_d Q^{6-d}\mathcal{F}_{\mathfrak{q}}\left[|\mathfrak{q}|^{6-d}\right](\xi) + \mathcal{O}(|\xi|^{-7}) + C_d h^2 Q^{2-d}\mathcal{F}_{\mathfrak{q}}\Big[\frac{2-d}{8}|\mathfrak{q}|^{-d}\langle A(x)\mathfrak{q},\mathfrak{q}\rangle^2 \\
& +\frac{1}{6}(d-1)H(x)|\mathfrak{q}|^{2-d}\langle A(x)\mathfrak{q},\mathfrak{q}\rangle - \frac{1}{6}|\mathfrak{q}|^{2-d}|A(x)\mathfrak{q}|^2\Big](\xi) + \mathcal{O}(h^2|\xi|^{-4})\Big];
\end{aligned}
\tag{2.19}
$$

and when $d = 4$:

$$
\begin{aligned}
& p_{\kappa\mathcal{S}_{\partial D}^{\kappa Q}}(x,\xi) \\
= \quad & Q^{d-2}\Big[C_d Q^{2-d}\mathcal{F}_{\mathfrak{q}}\left[|\mathfrak{q}|^{2-d}\right](\xi) + \tilde{C}_d Q^{4-d}\mathcal{F}_{\mathfrak{q}}\left[\log(|\mathfrak{q}|)\right](\xi) \\
& +\tilde{\tilde{C}}_d Q^{6-d}\mathcal{F}_{\mathfrak{q}}\left[|\mathfrak{q}|^{6-d}\right](\xi) + \mathcal{O}(|\xi|^{-7}) + C_d h^2 Q^{2-d}\mathcal{F}_{\mathfrak{q}}\Big[\frac{2-d}{8}|\mathfrak{q}|^{-d}\langle A(x)\mathfrak{q},\mathfrak{q}\rangle^2 \\
& +\frac{1}{6}(d-1)H(x)|\mathfrak{q}|^{2-d}\langle A(x)\mathfrak{q},\mathfrak{q}\rangle - \frac{1}{6}|\mathfrak{q}|^{2-d}|A(x)\mathfrak{q}|^2\Big](\xi) + \mathcal{O}(h^2|\xi|^{-4});\Big]
\end{aligned}
\tag{2.20}
$$

and when $d = 6$:

$$
\begin{aligned}
& p_{\kappa\mathcal{S}_{\partial D}^{\kappa Q}}(x,\xi) \\
= \quad & Q^{d-2}\Big[C_d Q^{2-d}\mathcal{F}_{\mathfrak{q}}\left[|\mathfrak{q}|^{2-d}\right](\xi) + \tilde{C}_d Q^{4-d}\mathcal{F}_{\mathfrak{q}}\left[|\mathfrak{q}|^{4-d}\right](\xi) \\
& +\tilde{\tilde{C}}_d Q^{4-d}\mathcal{F}_{\mathfrak{q}}\left[\log(|\mathfrak{q}|)\right](\xi) + \mathcal{O}(|\xi|^{-7} + C_d h^2 Q^{2-d}\mathcal{F}_{\mathfrak{q}}[\frac{2-d}{8}|\mathfrak{q}|^{-d}\langle A(x)\mathfrak{q},\mathfrak{q}\rangle^2 \\
& +\frac{1}{6}(d-1)H(x)|\mathfrak{q}|^{2-d}\langle A(x)\mathfrak{q},\mathfrak{q}\rangle - \frac{1}{6}|\mathfrak{q}|^{2-d}|A(x)\mathfrak{q}|^2](\xi) + \mathcal{O}(h^2|\xi|^{-4})\Big];
\end{aligned}
\tag{2.21}
$$

and when $d > 2$:

$$
\begin{aligned}
& p_{\kappa\mathcal{K}_{\partial D}^{\kappa Q *}}(x,\xi) \\
= \quad & \frac{Q^{d-2}}{2}\Big[C_d(2-d)Q^{2-d}\mathcal{F}_{\mathfrak{q}}\left[|\mathfrak{q}|^{-d}\langle A(x)\mathfrak{q},\mathfrak{q}\rangle\right](\xi) + \tilde{C}_d(4-d)Q^{4-d}\mathcal{F}_{\mathfrak{q}}\left[|\mathfrak{q}|^{2-d}\langle A(x)\mathfrak{q},\mathfrak{q}\rangle\right](\xi) \\
& +\mathcal{O}(|\xi|^5) + C_d(2-d)h^2 Q^{2-d}\mathcal{F}_{\mathfrak{q}}\Big[\frac{2-d}{8}|\mathfrak{q}|^{-2-d}\langle A(x)\mathfrak{q},\mathfrak{q}\rangle^3 \\
& +\frac{1}{6}(d-1)H(x)|\mathfrak{q}|^{-d}\langle A(x)\mathfrak{q},\mathfrak{q}\rangle^2 - \frac{1}{6}|\mathfrak{q}|^{-d}|A(x)\mathfrak{q}|^2\langle A(x)\mathfrak{q},\mathfrak{q}\rangle\Big](\xi) + \mathcal{O}(h^2|\xi|^{-4})\Big].
\end{aligned}
\tag{2.22}
$$

Finally, by letting $\mathfrak{q} = (\mathfrak{q}_j)_{j=1}^d$ and $\xi = (\xi_j)_{j=1}^d$, one can deduce the following relations by direct calculations:

$$
\begin{aligned}
&\mathcal{F}_{\mathfrak{q}}\left[|\mathfrak{q}|^{2-d}\right](\xi) = K_{1,d}|\xi|^{-1}\,, \quad \mathcal{F}_{\mathfrak{q}}\left[|\mathfrak{q}|^{4-d}\right](\xi) = K_{2,d}|\xi|^{-3} \text{ for } d \neq 4\,;\\
&\mathcal{F}_{\mathfrak{q}}\left[|\mathfrak{q}|^{6-d}\right](\xi) = K_{3,d}|\xi|^{-5} \text{ for } d \neq 6\,, \quad \mathcal{F}_{\mathfrak{q}}\left[\log(|\mathfrak{q}|)\right](\xi) = \begin{cases} K_{4,d}|\xi|^{-3} \text{ for } d = 4 \\ K_{4,d}|\xi|^{-5} \text{ for } d = 6 \end{cases}\,;\\
&\mathcal{F}_{\mathfrak{q}}\left[\mathfrak{q}_i\mathfrak{q}_j|\mathfrak{q}|^{-d}\right](\xi) = -K_{5,d}\partial_i\partial_j|\xi|\,, \quad \mathcal{F}_{\mathfrak{q}}\left[\mathfrak{q}_i\mathfrak{q}_j|\mathfrak{q}|^{2-d}\right](\xi) = -K_{1,d}\partial_i\partial_j|\xi|^{-1}\,; \qquad (2.23)\\
&\mathcal{F}_{\mathfrak{q}}\left[\mathfrak{q}_i\mathfrak{q}_j\mathfrak{q}_k\mathfrak{q}_l|\mathfrak{q}|^{-d}\right](\xi) = K_{5,d}\partial_i\partial_j\partial_k\partial_l|\xi|\,;\\
&\mathcal{F}_{\mathfrak{q}}\left[\mathfrak{q}_i\mathfrak{q}_j\mathfrak{q}_k\mathfrak{q}_l\mathfrak{q}_m\mathfrak{q}_n|\mathfrak{q}|^{-2-d}\right](\xi) = -K_{6,d}\partial_i\partial_j\partial_k\partial_l\partial_m\partial_n|\xi|^3\,;
\end{aligned}
$$

where $K_{j,d}$, $j = 1, 2, \ldots, 6$ are dimensional constants. By combining (2.19)–(2.23), we can further obtain:

$$
\begin{aligned}
& p_{\kappa\mathcal{S}_{\partial D}^{\kappa Q}}(x,\xi)\\
= \;& C_d|\xi|^{-1} + \tilde{C}_dQ^2|\xi|^{-3} + \tilde{\tilde{C}}_dQ^4|\xi|^{-5} + \mathcal{O}(|\xi|^{-7})\\
& +h^2\Bigg(\frac{2-d}{8}K_{1,d}\sum_{i,j,k,l=1}^{d-1} A_{ij}(x)A_{kl}(x)\partial_i\partial_j\partial_k\partial_l|\xi| \qquad (2.24)\\
& -\frac{1}{6}K_{2,d}\,(d-1)H(x)\sum_{i,j=1}^{d-1}A_{ij}(x)\partial_i\partial_j|\xi|^{-1} - K_{3,d}\frac{1}{6}\sum_{i,j,k}^{d-1}A_{ik}(x)A_{kj}(x)\partial_i\partial_j|\xi|^{-1}\Bigg)\\
& +\mathcal{O}(h^2|\xi|^{-4}),
\end{aligned}
$$

and

$$
\begin{aligned}
& p_{\kappa\mathcal{K}_{\partial D}^{\kappa Q*}}(x,\xi)\\
= \;& \frac{1}{2}\Bigg[-\widehat{C}_d(2-d)\sum_{i,j=1}^{d-1}A_{ij}(x)\partial_i\partial_j|\xi| - \widehat{\widehat{C}}_d(4-d)Q^2\sum_{i,j=1}^{d-1}A_{ij}(x)\partial_i\partial_j|\xi|^{-1} + \mathcal{O}(|\xi|^{-5})\\
& +(2-d)h^2\Bigg[-\frac{2-d}{8}K_{4,d}\sum_{i,j,k,l,m,n=1}^{d-1}A_{ij}(x)A_{kl}(x)A_{mn}(x)\partial_i\partial_j\partial_k\partial_l\partial_m\partial_n|\xi|^3 \qquad (2.25)\\
& +\frac{1}{6}K_{5,d}\,(d-1)H(x)\sum_{i,j,k,l=1}^{d-1}A_{ij}(x)A_{kl}(x)\partial_i\partial_j\partial_k\partial_l|\xi|\\
& -\frac{1}{6}K_{6,d}\sum_{i,j,k,l,m=1}^{d-1}A_{ij}(x)A_{km}(x)A_{ml}(x)\partial_i\partial_j\partial_k\partial_l|\xi|\Bigg] + \mathcal{O}(h^2|\xi|^{-4})\Bigg],
\end{aligned}
$$

where $\widehat{C}_d, \widehat{\widehat{C}}_d$ are another set of dimensional constants.

Finally, we would like to point out that by tracing back the computations in (2.24), one can see that the derivation of the principal symbol of $k\mathcal{S}_{\partial D}^{kQ}$, namely $p_{k\mathcal{S}_{\partial D}^{kQ}}(x,\xi)$ is contained in:

$$
C_d|\xi|^{-1} + \tilde{C}_dQ^2|\xi|^{-3} + \tilde{\tilde{C}}_dQ^4|\xi|^{-5} + O(|\xi|^{-7}) = Q^{-1}[\mathfrak{F}G_1\big((0,z')\big)](Q^{-1}\xi)\,,
$$

where $G_1(\cdot)$ is $G_\kappa(\cdot)$ with $\kappa = 1$; $(0, z') \in \mathbb{R}^d$ with $z' \in \mathbb{R}^{d-1}$; and $\mathfrak{F}$ is standard Fourier transform, namely $\mathcal{F}_h$ in (2.9) with $h = 1$, but associated with $z' \in \mathbb{R}^{d-1}$.

In summarizing the derivation and discussion so far in this subsection, we present the following lemma.

Lemma 2.1. *The operators $\mathcal{S}_{\partial D}^{\kappa Q}$ and $\mathcal{K}_{\partial D}^{\kappa Q^*}$ can be decomposed as follows:*

$$\mathcal{S}_{\partial D}^{\kappa Q} = h\left(\mathcal{S}_{-1}^{\kappa Q} + h^2 \mathcal{S}_{-3}^{\kappa Q}\right) \quad \text{and} \quad \mathcal{K}_{\partial D}^{\kappa Q^*} = \frac{h}{2}\left(\mathcal{K}_{-1}^{\kappa Q} + h^2 \mathcal{K}_{-3}^{\kappa Q}\right), \tag{2.26}$$

where

$$\mathcal{S}_{-1}^{\kappa Q} \in \Phi\mathrm{SO}_h^{-1}, \, \mathcal{S}_{-3}^{\kappa Q} \in \Phi\mathrm{SO}_h^{-3} \quad \text{and} \quad \mathcal{K}_{-1}^{\kappa Q} \in \Phi\mathrm{SO}_h^{-1}, \, \mathcal{K}_{-3}^{\kappa Q} \in \Phi\mathrm{SO}_h^{-3},$$

and in the geodesic normal coordinate, we have

$$\begin{aligned} & p_{\mathcal{S}_{-1}^{\kappa Q}}(x,\xi) \\ = \; & Q^{-1}[\mathfrak{F}G_1\big((0,z')\big)](Q^{-1}\xi) = C_d|\xi|^{-1} + \tilde{C}_d Q^2|\xi|^{-3} + \tilde{\tilde{C}}_d Q^4 |\xi|^{-5} + \mathcal{O}(|\xi|^{-7}), \end{aligned} \tag{2.27}$$

$$\begin{aligned} & p_{\mathcal{S}_{-3}^{\kappa Q}}(x,\xi) \\ = \; & \frac{2-d}{8}K_{1,d}\sum_{i,j,k,l=1}^{d-1} A_{ij}(x)A_{kl}(x)\partial_i\partial_j\partial_k\partial_l|\xi| - \frac{1}{6}K_{2,d}\,(d-1)H(x)\times \\ & \sum_{i,j=1}^{d-1} A_{ij}(x)\partial_i\partial_j|\xi|^{-1} - K_{3,d}\frac{1}{6}\sum_{i,j,k}^{d-1} A_{ik}(x)A_{kj}(x)\partial_i\partial_j|\xi|^{-1} + \mathcal{O}(|\xi|^{-4}), \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} & p_{\mathcal{K}_{-1}^{\kappa Q}}(x,\xi) \\ = \; & -\widehat{C}_d(2-d)\sum_{i,j=1}^{d-1} A_{ij}(x)\partial_i\partial_j|\xi| - \widehat{\widehat{C}}_d(4-d)Q^2\sum_{i,j=1}^{d-1} A_{ij}(x)\partial_i\partial_j|\xi|^{-1} + \mathcal{O}(|\xi|^{-5}), \end{aligned} \tag{2.29}$$

$$\begin{aligned} & p_{\mathcal{K}_{-3}^{\kappa Q}}(x,\xi) \\ = \; & (2-d)\bigg(-\frac{2-d}{8}K_{4,d}\sum_{i,j,k,l,m,n=1}^{d-1} A_{ij}(x)A_{kl}(x)A_{mn}(x)\partial_i\partial_j\partial_k\partial_l\partial_m\partial_n|\xi|^3 \\ & +\frac{1}{6}K_{5,d}\,(d-1)H(x)\sum_{i,j,k,l=1}^{d-1} A_{ij}(x)A_{kl}(x)\partial_i\partial_j\partial_k\partial_l|\xi| \\ & -\frac{1}{6}K_{6,d}\sum_{i,j,k,l,m=1}^{d-1} A_{ij}(x)A_{km}(x)A_{ml}(x)\partial_i\partial_j\partial_k\partial_l|\xi|\bigg) + \mathcal{O}(|\xi|^{-4}), \end{aligned} \tag{2.30}$$

where $C_d, \tilde{C}_d, \tilde{\tilde{C}}_d, \widehat{C}_d, \widehat{\widehat{C}}_d$ and $K_{j,d}, j = 1, ..., 6$ are dimensional constants.

It is also noted that $Q^{d-2}\mathcal{S}_{-1}^{\kappa Q}$ and $\frac{Q^{d-2}}{2}\mathcal{K}_{-1}^{\kappa Q}$ are the principal parts of the operators $\kappa\mathcal{S}_{\partial D}^{\kappa Q}$ and $\kappa\mathcal{K}_{\partial D}^{\kappa Q^*}$, respectively.

2.3. **Layer potential operators associated with a surface in D.** For the subsequent use, we shall also need to consider layer potential operators defined on surfaces in D, which correspond to the boundaries of certain interior domains. To that end, for a given $R \in \mathbb{R}_+$, let us consider a surface $\Gamma_R \subset D$ which has a global C^∞ diffeomorphism $F_R : \partial D \to \Gamma_R$ and satisfies $\mathrm{dist}(\Gamma_R, \partial D) \geq R$. Consider a smooth vector field X in $D \subset \mathbb{R}^d$, which may coincide with either a tangent vector field or a normal vector field ν when restricted on

Γ_R. Let us also write $X(f) = df(X) = \sum_{i=1}^d X_i(x)\partial_i f(x)$. Then we consider the following operator $\widetilde{\mathcal{S}_{\partial D}^{\kappa}} : L^2(\partial D) \to L^2(\partial D)$ and $\widetilde{X \circ \mathcal{S}_{\partial D}^{\kappa}} : L^2(\partial D) \to L^2(\partial D)$:

$$\widetilde{\mathcal{S}_{\partial D}^{\kappa}}[\phi](x) := \left(\mathcal{S}_{\partial D}^{\kappa}[\phi]\right)(F_R(x)), \quad \left(\widetilde{X \circ \mathcal{S}_{\partial D}^{\kappa}}\right)[\phi](x) := \left[X \circ \left(\mathcal{S}_{\partial D}^{\kappa}[\phi]\right)\right](F_R(x)) \tag{2.31}$$

for $x \in \partial D$ and $\nu_{F_R(x)}$ is a normal of Γ_R at the point $F_R(x)$. For a simple illustration, if D is strictly convex and contains the origin $\mathbf{0} \in \mathbb{R}^d$ with $\operatorname{dist}(\partial D, \mathbf{0}) > 0$, then for any $0 < R < \operatorname{dist}(\partial D, \mathbf{0})$ we can choose $\Gamma_R = \left(1 - \frac{R}{\operatorname{dist}(\partial D, \mathbf{0})}\right)\partial D$ and $F_R(x) = \left(1 - \frac{R}{\operatorname{dist}(\partial D, \mathbf{0})}\right)x$.

Lemma 2.2. *Given $R \in \mathbb{R}_+$ and a surface $\Gamma_R \subset D$ with $\operatorname{dist}(\Gamma_R, \partial D) \geq R$ and a smooth vector field X in D, the operators $\widetilde{\mathcal{S}_{\partial D}^{\kappa Q}}$ and $\widetilde{X \circ \mathcal{S}_{\partial D}^{\kappa}}$ can be decomposed as follows:*

$$\widetilde{\mathcal{S}_{\partial D}^{\kappa Q}} := Q^{\frac{d-1}{2}} \sum_{\alpha \geq 0} h^{\frac{d-1}{2}+|\alpha|} \widetilde{\mathcal{S}^{\kappa Q}}_{\alpha, -d-1-|\alpha|}, \tag{2.32}$$

$$\widetilde{X \circ \mathcal{S}_{\partial D}^{\kappa Q}} := Q^{\frac{d-1}{2}} \sum_{\alpha \geq 0} h^{\frac{d-1}{2}+|\alpha|} (\widetilde{X \circ \mathcal{S}^{\kappa Q}})_{\alpha, -d-1-|\alpha|}, \tag{2.33}$$

where

$$\widetilde{\mathcal{S}^{\kappa Q}}_{\alpha, -d-1-|\alpha|}, \ (\widetilde{X \circ \mathcal{S}^{\kappa Q}})_{\alpha, -d-1-|\alpha|} \in \Phi \mathrm{SO}_h^{-d-1-|\alpha|}, \tag{2.34}$$

and

$$\begin{aligned} \left|p_{\widetilde{\mathcal{S}^{\kappa Q}}_{\alpha,-d-1-|\alpha|}}(x,\xi)\right| &\leq C_{d,\alpha} R^{-\frac{d-3}{2}-|\alpha|} |\xi|^{-d-1-|\alpha|}, \\ \left|p_{(\widetilde{X \circ \mathcal{S}^{\kappa Q}})_{\alpha,-d-1-|\alpha|}}(x,\xi)\right| &\leq C_{d,\alpha,X} R^{-\frac{d-1}{2}-|\alpha|} |\xi|^{-d-1-|\alpha|}, \end{aligned} \tag{2.35}$$

for some constants $C_{d,\alpha}$ and $C_{d,\alpha,X}$.

Proof. First, noting that $G(x-y)$ is real analytic outside the region $|x-y| \geq R$, together with the fact that

$$H^{(1)}_{\frac{d-2}{2}}(x) \sim \sqrt{\frac{2}{\pi x}} e^{\mathrm{i}\left(x - \frac{2d-5}{4}\pi\right)} \quad \text{as } |x| \to \infty,$$

we have that

$$\begin{aligned} G_\kappa(F_R(x) - y) &= \kappa^{d-2} \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_z^\alpha \left[G_1(\kappa(F_R(x) - z))\right]|_{z=x} (y-x)^\alpha \\ &= \kappa^{d-2} \sum_{\alpha \geq 0} \frac{1}{\alpha!} \left[(\partial^\alpha G_1)(\kappa(F_R(x) - x))\right] \kappa^\alpha (y-x)^\alpha \end{aligned} \tag{2.36}$$

where for $\kappa \gg 1$, we have

$$\|(\partial^\alpha G_1)(\kappa(F_R(x) - x))\|_{L^\infty} \leq C_d \kappa^{-\frac{d-3}{2}-|\alpha|} R^{-\frac{d-3}{2}-|\alpha|}. \tag{2.37}$$

Similarly, we have

$$\begin{aligned} &(X_{F_R(x)} \circ G_\kappa)(F_R(x) - y) \\ = \ & -\kappa^{d-2} \sum_{\alpha \geq 0} \sum_{i=1}^d \frac{1}{\alpha!} \partial_z^\alpha \left[X_i(F_R(x)) \partial_{z_i} \left[G_1(\kappa(F_R(x) - z))\right]\right]|_{z=x} (y-x)^\alpha \\ = \ & -\kappa^{d-1} \sum_{\alpha \geq 0} \sum_{i=1}^d \frac{1}{\alpha!} X_i(F_R(x)) \left[(\partial_i \partial^\alpha G_1)(\kappa(F_R(x) - x))\right] \kappa^\alpha (y-x)^\alpha, \end{aligned} \tag{2.38}$$

where for $\kappa \gg 1$, we have

$$\|(\partial_i \partial^\alpha G_1)(\kappa(F_R(x) - x))\|_{L^\infty} \leq C_d \kappa^{-\frac{d-1}{2}-|\alpha|} R^{-\frac{d-1}{2}-|\alpha|}. \tag{2.39}$$

Using (2.36)–(2.39), one can show that

$$\widetilde{\mathcal{S}^{\kappa Q}_{\partial D}} := Q^{\frac{d-1}{2}} \sum_{\alpha\geq 0} h^{\frac{d-1}{2}+|\alpha|} \widetilde{\mathcal{S}^{\kappa Q}}_{\alpha,-d-1-|\alpha|}\,, \quad \widetilde{X\circ \mathcal{S}^{\kappa Q}_{\partial D}} := Q^{\frac{d-1}{2}} \sum_{\alpha\geq 0} h^{\frac{d-1}{2}+|\alpha|} (\widetilde{X\circ \mathcal{S}^{\kappa Q}})_{\alpha,-d-1-|\alpha|},$$

where $\widetilde{\mathcal{S}^{\kappa Q}_{\alpha,-d-1-|\alpha|}}, (\widetilde{X\circ\mathcal{S}^{\kappa Q}})_{\alpha,-d-1-|\alpha|} \in \Phi\mathrm{SO}_h^{-d-1-|\alpha|}$ via checking with Beal's criterion that for linear functions, $\ell_1, ..., \ell_N$,

$$\begin{aligned}
& \|\mathrm{ad}_{\ell_1(x,h\partial)}\circ\cdots\circ \mathrm{ad}_{\ell_N(x,h\partial)} \widetilde{\mathcal{S}^{\kappa Q}}_{\alpha,-d-1-|\alpha|}\|_{L^2\to L^2} \\
& +\|\mathrm{ad}_{\ell_1(x,h\partial)}\circ\cdots\circ \mathrm{ad}_{\ell_N(x,h\partial)} (\widetilde{X\circ\mathcal{S}^{kQ}})_{\alpha,-d-1-|\alpha|}\|_{L^2\to L^2} \\
\leq\;& C_1\kappa^{\frac{d-3}{2}+|\alpha|} \sum_{|\beta|=N} \|\kappa^{-N}\partial_x^\beta[(\partial^\alpha G_1)(\kappa(F_R(x)-x))]\|_{L^\infty} \\
& +C_2\kappa^{\frac{d-1}{2}+|\alpha|} \sum_{|\beta|=N}\sum_{i=1}^d \|\kappa^{-N}\partial_x^\beta[X_i(F_R(x))(\partial_i\partial^\alpha G_1)(\kappa(F_R(x)-x))]\|_{L^\infty} \\
\leq\;& Ch^{2N}\left(R^{-\frac{d-3}{2}-|\alpha|} \sum_{|\beta|=N} \|\partial_x^\beta F_R(x)\|^N R^{-\frac{d-1}{2}-|\alpha|} + \sum_{|\beta|=N+4} \|\partial_x^\beta F_R(x)\|^{2N}\|\partial_x^\beta X(x)\|^N \right) \\
=\;& O(h^{2N}) \leq O(h^N)\,.
\end{aligned}$$

The proof is complete. □

3. Generalized/approximate transmission eigenvalue problem

We first recall the integral formulation of the transmission eigenvalue problem (2.6)–(2.8). Using the results derived in the previous section, we can show that

Lemma 3.1. *There exists a self-adjoint operator $\mathcal{A}^{k,Q}_{\partial D}$ such that we have a non-trivial solution to (2.7) if and only if there exists a solution to the following (eigenvalue) problem for $\mathcal{A}^{k,Q}_{\partial D}$:*

$$0 = \left(\mathcal{A}^{\kappa,Q}_{\partial D} - 1\right)[\phi], \quad \phi\in L^2(\partial D), \tag{3.1}$$

where

$$\mathcal{A}^{\kappa,Q}_{\partial D} = \mathcal{A}^{\kappa}_{-2} + h\mathcal{A}^{\kappa}_{-3}, \tag{3.2}$$

and for $l=2,3$, $\mathcal{A}^{\kappa}_{-l}\in\Phi\mathrm{SO}_h^{-l}$ and in the geodesic normal coordinate, we have

$$\begin{aligned}
p_{\mathcal{A}^{\kappa}_{-2}}(x,\xi) &= 1 - \left| \frac{C_d}{\tilde{C}_d}\frac{(|\xi|^2+1)}{(Q^2-1)}\frac{p_{\mathcal{S}^{\kappa Q}_{-1}}(x,\xi)}{p_{\mathcal{S}^{\kappa}_{-1}}(x,\xi)} \right|^2 \\
&= 1 - \left| \frac{C_d}{\tilde{C}_d}\frac{(|\xi|^2+1)}{(Q^2-1)}\frac{[\mathfrak{F}G_1\big((0,z')\big)](Q^{-1}\xi)}{Q^{-1}[\mathfrak{F}G_1\big((0,z')\big)](\xi)} \right|^2 \\
&= 2\,\Re\left(\frac{C_d\tilde{\tilde{C}}_d(Q^2-1)^{-1}+\tilde{C}_d^2}{C_d\tilde{C}_d}\right)|\xi|^{-2} + \mathcal{O}(|\xi|^{-4})\,.
\end{aligned} \tag{3.3}$$

Here, $\mathcal{A}^{\kappa}_{-2}$ is the principal part of $\mathcal{A}^{\kappa,Q}_{\partial D}$ and $\mathcal{A}^{\kappa,Q}_{\partial D}\leq 1$.

Proof. Using Lemma 2.1, we can calculate in the geodesic normal coordinate that

$$p_{[\mathcal{S}^{\kappa}_{\partial D}]^{-1}\mathcal{S}^{\kappa Q}_{\partial D}}(x,\xi)$$

$$
\begin{aligned}
&= \left(1-\frac{\tilde{C}_d}{C_d}|\xi|^{-2}-\frac{\tilde{\tilde{C}}_d C_d-\tilde{C}_d^2}{C_d^2}|\xi|^{-4}+\mathcal{O}(|\xi|^{-6})+\mathcal{O}(h^2|\xi|^{-2})\right)\times \\
&\quad \left(1+\frac{\tilde{C}_d}{C_d}Q^2|\xi|^{-2}+\frac{\tilde{\tilde{C}}_d}{C_d}Q^4|\xi|^{-4}+O(|\xi|^{-6})+\mathcal{O}(h^2|\xi|^{-2})\right) \qquad (3.4)\\
&= 1+\frac{\tilde{C}_d}{C_d}(Q^2-1)|\xi|^{-2}-\frac{\tilde{\tilde{C}}_d C_d+\tilde{C}_d^2(Q^2-1)}{C_d^2}|\xi|^{-4}+\mathcal{O}(|\xi|^{-6})+\mathcal{O}(h^2|\xi|^{-2}),
\end{aligned}
$$

which further yields that

$$
\begin{aligned}
&p_{(-\frac{1}{2}I+\mathcal{K}_{\partial D}^{\kappa Q\,*})-(-\frac{1}{2}I+\mathcal{K}_{\partial D}^{\kappa\,*})[\mathcal{S}_{\partial D}^{\kappa}]^{-1}\mathcal{S}_{\partial D}^{\kappa Q}}(x,\xi)\\
= &-\frac{1}{2}+\frac{h}{2}\Bigg(-\widehat{C}_d(d-2)\sum_{i,j=1}^{d-1}A_{ij}(x)\partial_i\partial_j|\xi|-\widehat{\widehat{C}}_d(d-4)Q^2\sum_{i,j=1}^{d-1}A_{ij}(x)\partial_i\partial_j|\xi|^{-1}\\
&+\mathcal{O}(|\xi|^{-5})+\mathcal{O}(h^2|\xi|^{-3})\Bigg)+\Bigg(\frac{1}{2}+\frac{h}{2}\Bigg(\widehat{C}_d(d-2)\sum_{i,j=1}^{d-1}A_{ij}(x)\partial_i\partial_j|\xi|\\
&+\widehat{\widehat{C}}_d(d-4)\sum_{i,j=1}^{d-1}A_{ij}(x)\partial_i\partial_j|\xi|^{-1}+O(|\xi|^{-5})+\mathcal{O}(h^2|\xi|^{-3})\Bigg)\Bigg)\\
&\times\left(1+\frac{\tilde{C}_d}{C_d}(Q^2-1)|\xi|^{-2}-\frac{\tilde{\tilde{C}}_d C_d+\tilde{C}_d^2(Q^2-1)}{C_d^2}|\xi|^{-4}+O(|\xi|^{-6})+\mathcal{O}(h^2|\xi|^{-2})\right)\\
= &\frac{\tilde{C}_d}{2C_d}(Q^2-1)|\xi|^{-2}-\frac{C_d\tilde{\tilde{C}}_d+\tilde{C}_d^2(Q^2-1)}{C_d^2}|\xi|^{-4}+\mathcal{O}(|\xi|^{-6})\\
&+\frac{h}{2}(Q^2-1)\Bigg(\left(\widehat{\widehat{C}}_d(d-4)+\widehat{C}_d(d-2)\right)(d-1)H(x)|\xi|^{-3} \qquad (3.5)\\
&+\left(3\widehat{\widehat{C}}_d(d-4)-\widehat{C}_d(d-2)\right)\langle A(x)\xi,\xi\rangle|\xi|^{-5}+O(|\xi|^{-5})\Bigg)+\mathcal{O}(h^3|\xi|^{-2}),
\end{aligned}
$$

and hence

$$
\begin{aligned}
&p_{\frac{2C_d}{\tilde{C}_d}(Q^2-1)^{-1}(-\Delta_{\partial D}+1)\left((-\frac{1}{2}I+\mathcal{K}_{\partial D}^{\kappa Q\,*})-(-\frac{1}{2}I+\mathcal{K}_{\partial D}^{\kappa\,*})[\mathcal{S}_{\partial D}^{\kappa}]^{-1}\mathcal{S}_{\partial D}^{\kappa Q}\right)}(x,\xi)\\
= &\;1-\left(2\,\frac{C_d\tilde{\tilde{C}}_d(Q^2-1)^{-1}+\tilde{C}_d^2}{C_d\tilde{C}_d}-1\right)|\xi|^{-2}+\mathcal{O}(|\xi|^{-4})\\
&+h\,\frac{C_d}{\tilde{C}_d}\Bigg(\left(\widehat{\widehat{C}}_d(d-4)+\widehat{C}_d(d-2)\right)(d-1)H(x)|\xi|^{-1} \qquad (3.6)\\
&+\left(3\widehat{\widehat{C}}_d(d-4)-\widehat{C}_d(d-2)\right)\langle A(x)\xi,\xi\rangle|\xi|^{-3}+\mathcal{O}(|\xi|^{-3})\Bigg)+\mathcal{O}(h^3).
\end{aligned}
$$

By tracing the computations in (3.5)–(3.6), we can obtain a more precise expression of the principal symbol as follows:

$$
1-\left(2\,\frac{C_d\tilde{\tilde{C}}_d(Q^2-1)^{-1}+\tilde{C}_d^2}{C_d\tilde{C}_d}-1\right)|\xi|^{-2}+O(|\xi|^{-4})=\frac{C_d}{\tilde{C}_d}\frac{(|\xi|^2+1)}{(Q^2-1)}\frac{[\mathfrak{F}G_1(0,z)](Q^{-1}\xi)}{Q^{-1}[\mathfrak{F}G_1(0,z)](\xi)}\,. \qquad (3.7)
$$

Hence, noting the fact that $(-\Delta_{\partial D}+1)f=0$ if and only if $f=0$, we have a non-trivial solution to (2.7) if and only if there exists a solution to

$$0=\left(\mathcal{B}^{\kappa,Q}_{\partial D}-1\right)[\phi],\quad \mathcal{B}^{\kappa,Q}_{\partial D}:=\mathcal{B}^{\kappa}_{-2}+h\mathcal{B}^{\kappa}_{-3}, \tag{3.8}$$

with

$$\mathcal{B}^{\kappa}_{-2}\in \Phi\mathrm{SO}_h^{-2}\quad\text{and}\quad \mathcal{B}^{\kappa}_{-3}\in \Phi\mathrm{SO}_h^{-3}, \tag{3.9}$$

where the principal symbol of $\mathcal{B}^{k}_{-1}$ is stated in (3.3), which in turn holds if and only if there exists a solution to

$$0=\left(\mathcal{B}^{\kappa,Q}_{\partial D}-1\right)^*\left(\mathcal{B}^{\kappa,Q}_{\partial D}-1\right)[\phi]. \tag{3.10}$$

Writing

$$\mathcal{A}^{\kappa,Q}_{\partial D}:=\mathcal{B}^{\kappa,Q}_{\partial D}+\left(\mathcal{B}^{\kappa,Q}_{\partial D}\right)^*-\left(\mathcal{B}^{\kappa,Q}_{\partial D}\right)^*\mathcal{B}^{\kappa,Q}_{\partial D}, \tag{3.11}$$

and noticing that

$$p_{\mathcal{A}^{\kappa,Q}_{\partial D}}(x,\xi)=2\,\Re\left(p_{\mathcal{B}^{\kappa,Q}_{\partial D}}(x,\xi)\right)-\overline{p_{\mathcal{B}^{\kappa,Q}_{\partial D}}(x,\xi)}\,p_{\mathcal{B}^{\kappa,Q}_{\partial D}}(x,\xi), \tag{3.12}$$

one can readily complete the proof.

□

Remark 3.2. *It is observed from its principal symbol that $\mathcal{A}^{k,Q}_{\partial D}$ is compact and self-adjoint in $L^2(\partial D,d\sigma)$. We write a sequence of eigen-pairs $\{(\lambda_j(h),\phi_j(h))\}_{j=1}^{\infty}$ where, fixing $h=\kappa^{-1}>0$, the set of eigenfunctions form an orthonormal frame and the set of eigenvalues converges to zero as j goes to infinity. It is easily seen that κ is a transmission eigenvalue if and only if there exists j such that $\lambda_j(\kappa^{-1})=1$. It can also be directly inferred from the Sobolev embedding that $\phi_j(h)\in\mathcal{C}^\infty(\partial D)$.*

Next, we introduce the definition of ε-almost transmission eigen-pairs, which for terminological convenience shall be referred to as the generalized transmission eigen-pairs in what follows. Let $\varepsilon\ll 1$ and $0\le\delta\le\varepsilon$. Consider the following transmission eigenvalue problem:

$$\left(\mathcal{A}^{\kappa,Q}_{\partial D}-1+\delta\right)[\phi_\delta]=0,\quad \|\phi_\delta\|_{L^2(\partial D)}=1. \tag{3.13}$$

Set

$$\varphi_\delta=\left(\mathcal{S}^{\kappa}_{\partial D}\right)^{-1}\mathcal{S}^{\kappa Q}_{\partial D}[\phi_\delta]\ \text{ on }\ \partial D. \tag{3.14}$$

One can directly verify that the pair $(\phi_\delta,\varphi_\delta)\in L^2(\partial D)\times L^2(\partial D)$ satisfies

$$\begin{cases}\mathcal{S}^{\kappa Q}_{\partial D}[\phi_\delta]-\mathcal{S}^{\kappa}_{\partial D}[\varphi_\delta]=0,\\ \|(-\frac{1}{2}I+\mathcal{K}^{\kappa Q}_{\partial D}{}^*)[\phi_\delta]-(-\frac{1}{2}I+\mathcal{K}^{\kappa}_{\partial D}{}^*)[\varphi_\delta]\|^2_{H^2}\le\varepsilon\,\frac{|\tilde{C}_d|^2}{4|C_d|^2}(Q^2-1)^2.\end{cases} \tag{3.15}$$

Hence, if we let $u_\delta,v_\delta\in H^1(D)$ be defined as follows:

$$u_\delta=\mathcal{S}^{\kappa Q}_{\partial D}[\phi_\delta],\quad v_\delta=\mathcal{S}^{\kappa}_{\partial D}[\varphi_\delta]\quad \text{in } D. \tag{3.16}$$

Then they approximately satisfy (1.3) in a sense that

$$\begin{cases}\Delta u_\delta+\kappa^2Q^2u_\delta=0 & \text{in } D,\\ \Delta v_\delta+\kappa^2 v_\delta=0 & \text{in } D,\\ u_\delta=v_\delta,\quad \|\partial_\nu u_\delta-\partial_\nu v_\delta\|^2_{H^2(\partial D)}\le\varepsilon\,\frac{|\tilde{C}_d|^2}{4|C_d|^2}(Q^2-1)^2 & \text{on } \partial D.\end{cases} \tag{3.17}$$

Definition 3.3. *We call $\kappa > 0$ a generalised transmission eigenvalue if for such κ, there exists (δ, ϕ_δ) with $0 \le \delta \le \varepsilon$ and $\varepsilon \ll 1$ such that (3.13) holds. The pair (u_δ, v_δ) defined in (3.16) are called the ε-almost transmission eigenfunctions, in a sense that they satisfy (3.17). For simplicity, they are also referred to as the generalised transmission eigenfunctions. We also denote the generalised transmission eigenspace as follows:*

$$\mathbb{E}(\kappa,\varepsilon) := \bigcup_{0\le\delta\le\varepsilon} \operatorname{Span}\Big\{ \Big(\mathcal{S}^{kQ}_{\partial D}[\phi_\delta], \mathcal{S}^{k}_{\partial D}[\varphi_\delta]\Big) : \phi_\delta \text{ and } \varphi_\delta \text{ are given in (3.13) and (3.14) respectively}\Big\}. \tag{3.18}$$

We also refer to the following quantity as the multiplicity of a generalised transmission eigenvalue κ:

$$\mathfrak{m}(\kappa,\varepsilon) := \dim \bigcup_{0\le\delta\le\varepsilon} \left\{\phi \in L^2(\partial D) \,:\, \left(\mathcal{A}^{\kappa,Q}_{\partial D} - 1 + \delta\right)[\phi] = 0\right\}. \tag{3.19}$$

Remark 3.4. *We notice that following this definition, a candidate ϕ_δ that satisfies (3.13) for some $0 \le \delta \le \varepsilon$ will sit in the subspace*

$$\mathbb{V}_\varepsilon := \overline{\bigcup \left\{ \mathbf{V} \subset L^2(\partial D)\,,\, \left\| \left(\mathcal{B}^{\kappa,Q}_{\partial D} - 1\right)|_{\mathbf{V}} \right\|_{\mathcal{L}(L^2(\partial D),L^2(\partial D))} \le \varepsilon \right\}},$$

since it is immediate to check that (3.13) implies

$$\left\| \left(\mathcal{B}^{\kappa,Q}_{\partial D} - 1\right)[\phi_\delta] \right\|^2_{L^2(\partial D)} = \delta \|\phi_\delta\|^2_{L^2} \le \varepsilon\,.$$

With the above interpretation, we realize that the concept of the generalised transmission eigenvalue in Definition 3.3 is intimately related to the concept of pseudospectrum of the operator $\mathcal{B}^{\kappa,Q}_{\partial D}$.

Remark 3.5. *By (3.2) and (3.3), it can be seen that the operator $\mathcal{A}^{\kappa,Q}_{\partial D}$ is compact and furthermore by (3.11) it is symmetric. Hence, the eigenvalues of the operator $\mathcal{A}^{\kappa,Q}_{\partial D}$ are real and have a cluster point 0, as an approximation to a finite rank operator. Hence, the union of all the eigenspaces for any fixed threshold away from 0 is finite dimensional. In particular, $\mathbb{E}(\kappa,\varepsilon)$ in (3.18) is finite dimensional and $\mathfrak{m}(\kappa,\varepsilon)$ is a finite number.*

Remark 3.6. *It is easily seen that an exact transmission eigenvalue is a generalised transmission eigenvalue. Hence, the generalised eigenspace $\mathbb{E}(\kappa,\varepsilon)$ contains the exact transmission eigenfunctions. Indeed, $\mathbb{E}(\kappa,0)$ is an exact transmission eigenspace. For the numerical finding in [10] as well as the examples presented in Fig. 1, considering the numerical errors, they are actually are certain generalised transmission eigenfunctions.*

4. Surface concentration of generalised transmission eigenfunctions

In this section, we present the main results on the surface concentration of the generalised transmission eigenfunctions. Throughout the rest of the paper, we assume that D is strictly convex. By Remark 3.2, for any fixed $h = \kappa^{-1} > 0$, we let

$$\{(\lambda_j(h), \phi_j(h))\}_{j=1}^{\infty} \quad \text{with} \quad \lim_{j\to\infty} \lambda_j(h) = 0 \text{ and } \|\phi_j(h)\|_{L^2(\partial D)} = 1, \tag{4.1}$$

be the sequence of eigen-pairs of $\mathcal{A}^{k,Q}_{\partial D}$. It is noted that $\kappa \in \mathbb{R}_+$ is a transmission eigenvalue if and only if $\lambda_{j_0}(h) = 1$ for a certain $j_0 \in \mathbb{N}$, or equivalently $\mathfrak{m}(\kappa, 0) > 0$, whereas $1 - \varepsilon \le$

$\lambda_{j_0}(h) \le 1$ corresponds to a generalised transmission eigenvalue according to Definition 3.3. In what follows, we set $\varphi_j(h) := \left[\mathcal{S}^k_{\partial D}\right]^{-1} \mathcal{S}^{kQ}_{\partial D}\left[\phi_j(h)\right]$ and

$$u_j(h) := \mathcal{S}^{\kappa Q}_{\partial D}\left[\phi_j(h)\right], \quad \partial_\nu u_j(h) := (-\frac{1}{2}I + \mathcal{K}^{\kappa Q^*}_{\partial D})\left[\phi_j(h)\right] \quad \text{in } D\,; \tag{4.2}$$

$$v_j(h) := \mathcal{S}^{\kappa}_{\partial D}\left[\varphi_j(h)\right], \quad \partial_\nu v_j(h) := (-\frac{1}{2}I + \mathcal{K}^{\kappa Q^*}_{\partial D})\left[\varphi_j(h)\right] \quad \text{in } D\,. \tag{4.3}$$

4.1. **Surface concentration in an averaging sense.** The first main result of this section is stated as follows.

Theorem 4.1. *Define the averaging functionals $\mathscr{I}^\ell_\zeta(\kappa, x, \varepsilon)$, $\zeta = u, v$ and $\ell = 0, 1$ as follows:*

$$\mathscr{I}^0_\zeta(\kappa, x, \varepsilon) \;:=\; \frac{\sum\limits_{1-\varepsilon\le\lambda_j(\kappa^{-1})\le 1} \left|\left[\zeta_j(\kappa^{-1})\right](x)\right|^2}{\#\left\{1-\varepsilon \le \lambda_j(\kappa^{-1}) \le 1\right\}}, \quad \zeta = u \text{ or } v; \tag{4.4}$$

$$\mathscr{I}^1_\zeta(\kappa, x, \varepsilon) \;:=\; \frac{\sum\limits_{1-\varepsilon\le\lambda_j(\kappa^{-1})\le 1} \left|\nabla\left[\zeta_j(\kappa^{-1})\right](x)\right|^2}{\#\left\{1-\varepsilon \le \lambda_j(\kappa^{-1}) \le 1\right\}}, \quad \zeta = u \text{ or } v;, \tag{4.5}$$

where # counts the number of elements of a given set. Let $\Gamma_R \subset D$ be a closed surface such that $\operatorname{dist}(\Gamma_R, \partial D) := R \ge 0$. Then for any bump function $\gamma(x) \in \mathcal{C}^\infty(\Gamma_R)$, we have as $\kappa \to +\infty$:

$$\int_{\Gamma_R} \gamma(x)\mathscr{I}^0_\zeta(\kappa, x, \varepsilon)\, d\sigma(x) \;\sim\; \kappa^{-2} \qquad \textit{if } \operatorname{supp}(\gamma) \subset \partial D, \quad \zeta = u, v; \tag{4.6}$$

$$\int_{\Gamma_R} \gamma(x)\mathscr{I}^0_u(\kappa, x, \varepsilon)\, d\sigma(x) \;=\; \mathcal{O}(Q^{d-1}R^{3-d}\kappa^{1-d}) \qquad \textit{if } \operatorname{supp}(\gamma) \cap \partial D = \emptyset; \tag{4.7}$$

$$\int_{\Gamma_R} \gamma(x)\mathscr{I}^0_v(\kappa, x, \varepsilon)\, d\sigma(x) \;=\; \mathcal{O}(R^{3-d}\kappa^{1-d}) \qquad \textit{if } \operatorname{supp}(\gamma) \cap \partial D = \emptyset\,; \tag{4.8}$$

and

$$\int_{\Gamma_R} \gamma(x)\mathscr{I}^1_\zeta(\kappa, x, \varepsilon)\, d\sigma(x) \;\sim\; 1 \qquad \textit{if } \operatorname{supp}(\gamma) \subset \partial D, \quad \zeta = u, v; \tag{4.9}$$

$$\int_{\Gamma_R} \gamma(x)\mathscr{I}^1_u(\kappa, x, \varepsilon)\, d\sigma(x) \;=\; \mathcal{O}(Q^{d-1}R^{1-d}\kappa^{1-d}) \qquad \textit{if } \operatorname{supp}(\gamma) \cap \partial D = \emptyset; \tag{4.10}$$

$$\int_{\Gamma_R} \gamma(x)\mathscr{I}^1_v(\kappa, x, \varepsilon)\, d\sigma(x) \;=\; \mathcal{O}(R^{1-d}\kappa^{1-d}) \qquad \textit{if } \operatorname{supp}(\gamma) \cap \partial D = \emptyset\,, \tag{4.11}$$

where the asymptotic constants in the RHS terms of the above relations depend on $\|\gamma\|_{C(\Gamma)}$.

Remark 4.1. *By Theorem 4.1, the surface concentration of the generalised transmission eigenfunctions can be observed as follows. By (4.6) and (4.9), we readily see that the generalised transmission eigenfunctions u and v are highly oscillatory around ∂D (in an averaging sense as described by the averaging functionals). Indeed, by multiplying a normalisation factor κ^2, namely considering $\kappa \cdot (u, v)$, we see that the gradient fields blow up as $\kappa \to +\infty$. Considering $\kappa \cdot (u, v)$, and by (4.6)–(4.8) and (4.9)–(4.11), we readily see that in particular when $d \ge 4$, $\kappa \cdot u(x)$ and $\kappa \cdot v(x)$ decay rapidly when x leaves away from ∂D (again in the averaging sense) for κ sufficiently large. Even in the case with $d = 3$, we can also see that $\kappa \cdot u$ decays when leaving away from ∂D if $0 < Q < 1$; and $\kappa \cdot v$ decays locally around ∂D when $R < 1$, and hence inside D by the unique continuation. According to our discussion in Section 3, the generalised transmission eigenfunctions contain the exact transmission eigenfunctions as subsequences, and hence it is unobjectionable to claim that* *concentration property in Theorem 4.1 holds also for the exact transmission eigenfunctions (in an averaging sense).*

We proceed to give the proof of Theorem 4.1. First, we discuss the generalised Weyl law, which shall be needed in our proof. To that end, we introduce the following Hamiltonian:

$$\mathcal{H}(x,\xi) := p_{\mathcal{A}^{\kappa}_{-2}}(x,\xi) : T^*(\partial D) \to \mathbb{R}. \tag{4.12}$$

Proposition 4.2. *[12, 18, 19, 31, 33, 35, 36] Assuming that D is strictly convex and fixing $\varepsilon > 0$, for any $a \in S^m(T^*(\partial D))$, we have as $h \to +0$,*

$$\begin{aligned}&(2\pi h)^{(d-1)} \sum_{1-\varepsilon \le \lambda_j(h) \le 1} \left\langle \mathrm{Op}_{a,h}\, \phi_j(h), \phi_j(h) \right\rangle_{L^2(\partial D)} \\ &= \int_{\{1-\varepsilon\le\mathcal{H}\le1\}} a\, d\sigma \otimes d\sigma^{-1} + o_\varepsilon(1),\end{aligned} \tag{4.13}$$

where $\phi_j(h)$'s are given in (4.1) and the little-o depends on ε.

Corollary 4.3. *Assuming that D is strictly convex, we have*

$$\mathfrak{m}(\kappa,\varepsilon) = \#\{1-\varepsilon \le \lambda_j(h) \le 1\} = (2\pi h)^{1-d} \left(\int_{\{1-\varepsilon\le\mathcal{H}\le1\}} d\sigma \otimes d\sigma^{-1} + o_\varepsilon(1) \right). \tag{4.14}$$

where $\mathfrak{m}(\kappa,\varepsilon)$ is defined as in (3.19). Hence, for any given $\varepsilon > 0$, there exits a κ_ε such that any $\kappa > \kappa_\varepsilon$ is a generalised transmission eigenfunction with multiplicity $\mathfrak{m}(\kappa,\varepsilon) \sim k^{d-1}$ according to Definition 3.3.

Proof. The first equality in (4.14) comes from the immediate observation that, counting multiplicity,

$$\#\{1-\varepsilon \le \lambda_j(h) \le 1\} = \dim \bigcup_{1-\varepsilon\le\lambda_j(h)\le1} \left\{ \phi \in L^2(\partial D) \,:\, \left(\mathcal{A}^{\kappa,Q}_{\partial D} - \lambda_j(h)\right)[\phi] = 0 \right\} = \mathfrak{m}(\kappa,\varepsilon)\,.$$

The second equality in (4.14) is the classical Weyl's law, which can be easily obtained from Proposition 4.2 by taking $a = 1$ in (4.13). The last conclusion comes from the observation that, given $\varepsilon > 0$, we check that $(2\pi h)^{1-d} \gg 1$ for $h = \kappa^{-1} > 0$ sufficiently small and hence $\mathfrak{m}(\kappa,\varepsilon) \gg 1$. Therefore by definition, we can always find κ_ε such that for all $\kappa > \kappa_\varepsilon$, we have $\mathfrak{m}(\kappa,\varepsilon) > 1$ and therefore κ is a generalised transmission eigenfunction with multiplicity $\mathfrak{m}(\kappa,\varepsilon) \sim k^{d-1}$. □

Proof of Theorem 4.1. By choosing $a(x,\xi)$ as a smooth non-negative bump function $\gamma \in \mathcal{C}^\infty$ either on ∂D or Γ_R multiplied with appropriate choices of symbols, we can obtain the averaged results for $\zeta_j(h)$ and $\nabla\zeta_j(h)$ ($\zeta = u$ or v) with pointwise concentration. Our argument is inspired by that developed in [2] in a different context. In what follows, we denote a global C^∞ diffeomorphism $F_R : \partial D \to \Gamma_R$ (cf. Section 2.3). We also write $q = F_R^{-1}(p)$ and set $\{\chi_{q,\delta}\}_{\delta>0}$ to be a family of smooth nonnegative bump functions compactly supported in $B_\delta(q)$ on Γ. We also write $\sigma_{\{\mathcal{H}=1\}}$ as the surface Liouville measure on $\{\mathcal{H}=1\}$.

First, noting the fact that

$$u_j(h) = \mathcal{S}^{\kappa Q}_{\partial D}[\phi_j(h)] = h\left(\mathcal{S}^{\kappa Q}_{-1} + h^2 \mathcal{S}^{\kappa Q}_{-3}\right)[\phi_j(h)]\,,$$

we can make a choice of symbol $a(x,\xi) = \gamma(x)|p_{\mathcal{S}^{\kappa Q}_{-1}}(x,\xi)|^2$ in Theorem 4.13, together with Corollary 4.14 to obtain as $h \to +0$:

$$\frac{\sum_{1-\varepsilon\le\lambda_j(h)\le1} \int_{\partial D} \gamma(x)|[u_j(h)](x)|^2 d\sigma(x)}{\#\{1-\varepsilon \le \lambda_j(h) \le 1\}}$$

$$
\begin{aligned}
&= h^2 \frac{\sum\limits_{1-\varepsilon\le\lambda_j(h)\le 1} \left\langle \mathrm{Op}_{\gamma(x)|p_{\mathcal{S}_{-1}^{\kappa Q}}(x,\xi)|^2,h}\phi_j(h),\phi_j(h)\right\rangle_{L^2(\partial D)}}{\#\{1-\varepsilon\le\lambda_j(h)\le 1\}} + \mathcal{O}(h^3) \\
&= h^2 \left(\frac{\int\limits_{\{1-\varepsilon\le\mathcal{H}\le 1\}} \gamma(x)|p_{\mathcal{S}_{-1}^{\kappa Q}}(x,\xi)|^2 d\sigma\otimes d\sigma^{-1}}{\int\limits_{\{1-\epsilon\le\mathcal{H}\le 1\}} d\sigma\otimes d\sigma^{-1}} + o_{\varepsilon,\varphi}(1)\right) \\
&= h^2 \left(\frac{\int\limits_{\{\mathcal{H}=1\}} \gamma(x)|p_{\mathcal{S}_{-1}^{\kappa Q}}(x,\xi)|^2 d\sigma_{\{\mathcal{H}=1\}}}{\int\limits_{\{\mathcal{H}=1\}} d\sigma_{\{\mathcal{H}=1\}}} + \mathcal{O}_\gamma(\varepsilon) + o_{\varepsilon,\varphi}(1)\right), \qquad (4.15)
\end{aligned}
$$

which readily gives (4.6) with $\zeta = u$.

In a similar manner, from

$$
\partial_\nu u_j(h) = (-\frac{1}{2}I + \mathcal{K}_{\partial D}^{\kappa Q*})[\phi_j(h)] = \left(-\frac{1}{2}I + \frac{h}{2}\mathcal{K}_{-1}^{\kappa Q} + \frac{h^3}{2}\mathcal{K}_{-3}^{\kappa Q}\right)[\phi_j(h)],
$$

we can make a choice of symbol $a(x,\xi) = \frac{1}{4}\gamma(x)$ in Theorem 4.13, together with Corollary 4.14, to obtain as $h \to +0$:

$$
\begin{aligned}
&\frac{\sum\limits_{1-\varepsilon\le\lambda_j(h)\le 1} \int_{\partial D}\gamma(x)|[\partial_\nu u_j(h)](x)|^2 d\sigma(x)}{\#\{1-\varepsilon\le\lambda_j(h)\le 1\}} \\
&= \frac{1}{4}\frac{\int\limits_{\{\mathcal{H}=1\}} \gamma(x) d\sigma_{\{\mathcal{H}=1\}}}{\int\limits_{\{\mathcal{H}=1\}} d\sigma_{\{\mathcal{H}=1\}}} + \mathcal{O}_\gamma(\varepsilon) + o_{\varepsilon,\gamma}(1). \qquad (4.16)
\end{aligned}
$$

Furthermore, noting that

$$
\partial_{x_i} u_j(h) = \frac{\mathrm{Op}_{\xi_i,h}}{h}\circ \mathcal{S}_{\partial D}^{\kappa Q}[\phi_j(h)],
$$

we can make a choice of symbol $a(x,\xi) = \gamma(x)|\xi|^2|p_{\mathcal{S}_{-1}^{kQ}}(x,\xi)|^2$ in Theorem 4.13, together with Corollary 4.14 to obtain as $h \to +0$:

$$
\begin{aligned}
&\frac{\sum\limits_{1-\varepsilon\le\lambda_j(h)\le 1} \int_{\partial D}\gamma(x)|[\partial_x u_j(h)](x)|^2 d\sigma(x)}{\#\{1-\varepsilon\le\lambda_j(h)\le 1\}} \\
&= \frac{\int\limits_{\{\mathcal{H}=1\}} \gamma(x)|\xi|^2|p_{\mathcal{S}_{-1}^{\kappa Q}}(x,\xi)|^2 d\sigma_{\{\mathcal{H}=1\}}}{\int\limits_{\{\mathcal{H}=1\}} d\sigma_{\{\mathcal{H}=1\}}} + \mathcal{O}_\gamma(\varepsilon) + o_{\varepsilon,\varphi}(1). \qquad (4.17)
\end{aligned}
$$

By combining (4.16) and (4.17), we readily have (4.9) with $\zeta = u$.

Next, recalling the diffeomorphism $F_R : \partial D \to \Gamma_R$, we notice that for $x \in \partial D$, it holds that

$$
u_j(h)(F_R(x)) = \widetilde{\mathcal{S}_{\partial D}^{\kappa Q}}[\phi_j(h)](x), \quad [X\circ u_j(h)](F_R(x)) = \widetilde{X\circ\mathcal{S}_{\partial D}^{\kappa Q}}[\phi_j(h)](x), \qquad (4.18)
$$

$$
v_j(h)(F_R(x)) = \widetilde{\mathcal{S}_{\partial D}^{\kappa}}[\varphi_j(h)](x), \quad [X\circ v_j(h)](F_R(x)) = \widetilde{X\circ\mathcal{S}_{\partial D}^{\kappa}}[\varphi_j(h)](x). \qquad (4.19)
$$

Considering $u_j(h)(F_R(x))$, we can make use of the explicit expressions in Lemma 2.2, as well as the respective choice of symbols $a(x,\xi) = \det(DF_R)^{-1}(F_R(x))\,|p_{\widetilde{\mathcal{S}^{\kappa Q}}_{0,-d-1}}(x,\xi)|^2$ and

$a(x,\xi) = \det(DF_R)^{-1}(F_R(x))\,|p_{(\widetilde{X\circ\mathcal{S}^{\kappa Q}})_{0,-d-1}}|^2$ in Theorem 4.13, together with a change of variable and Corollary 4.14 to obtain as $h \to +0$:

$$\frac{\sum\limits_{1-\varepsilon\le\lambda_j(h)\le 1} \int_{\Gamma_R} \gamma(x)|u_j(h)(x)|^2 d\sigma(x)}{\#\{1-\varepsilon \le \lambda_j(h) \le 1\}} = \mathcal{O}_{\varepsilon,\varphi,\Gamma}(Q^{d-1}R^{3-d}h^{d-1}), \tag{4.20}$$

$$\frac{\sum\limits_{1-\varepsilon\le\lambda_j(h)\le 1} \int_{\Gamma_R} \gamma(x)|[X\circ u_j(h)](x)|^2 d\sigma(x)}{\#\{1-\varepsilon \le \lambda_j(h) \le 1\}} = \mathcal{O}_{\varepsilon,\varphi,X,\Gamma}(Q^{d-1}R^{1-d}h^{d-1})\,. \tag{4.21}$$

Now, one can directly verify that (4.20) gives (4.7), and taking $X = e_j$ in (4.21) and summing them all up gives (4.10).

Finally, with a quick observation that

$$p_{[\mathcal{S}^{\kappa}_{\partial D}]^{-1}\mathcal{S}^{\kappa Q}_{\partial D}}(x,\xi) = \frac{p_{\mathcal{S}^{\kappa Q}_{-1}}(x,\xi)}{p_{\mathcal{S}^{k}_{-1}}(x,\xi)} + \mathcal{O}(h^2|\xi|^2)\,,$$

we obtain the $\zeta = v$ counterparts by multiplying the symbol a in all of the previous five choices to obtain as $h \to +0$:

$$\frac{\sum\limits_{1-\varepsilon\le\lambda_j(h)\le 1} \int_{\partial D} \gamma(x)|[v_j(h)](x)|^2 d\sigma(x)}{\#\{1-\varepsilon \le \lambda_j(h) \le 1\}}$$
$$= h^2 \left(\frac{\int\limits_{\{\mathcal{H}=1\}} \gamma(x)|p_{\mathcal{S}^{\kappa Q}_{-1}}(x,\xi)|^2 d\sigma_{\{H=1\}}}{\int\limits_{\{\mathcal{H}=1\}} d\sigma_{\{\mathcal{H}=1\}}} + \mathcal{O}_\gamma(\varepsilon) + o_{\varepsilon,\gamma}(1) \right),$$

$$\frac{\sum\limits_{1-\varepsilon\le\lambda_j(h)\le 1} \int_{\partial D} \gamma(x)|[\partial_\nu v_j(h)](x)|^2 d\sigma(x)}{\#\{1-\varepsilon \le \lambda_j(h) \le 1\}} = \frac{\int\limits_{\{\mathcal{H}=1\}} \gamma(x)\frac{|p_{\mathcal{S}^{\kappa Q}_{-1}}(x,\xi)|^2}{|p_{\mathcal{S}^{\kappa}_{-1}}(x,\xi)|^2} d\sigma_{\{\mathcal{H}=1\}}}{\int\limits_{\{H=1\}} d\sigma_{\{\mathcal{H}=1\}}} + \mathcal{O}_\gamma(\varepsilon) + o_{\varepsilon,\gamma}(1),$$

$$\frac{\sum\limits_{1-\varepsilon\le\lambda_j(h)\le 1} \int_{\partial D} \gamma(x)|[\partial_x v_j(h)](x)|^2 d\sigma(x)}{\#\{1-\varepsilon \le \lambda_j(h) \le 1\}} = \frac{\int\limits_{\{\mathcal{H}=1\}} \gamma(x)|\xi|^2|p_{\mathcal{S}^{\kappa Q}_{-1}}(x,\xi)|^2 d\sigma_{\{\mathcal{H}=1\}}}{\int\limits_{\{\mathcal{H}=1\}} d\sigma_{\{\mathcal{H}=1\}}} + \mathcal{O}_\gamma(\varepsilon) + o_{\varepsilon,\gamma}(1)\,,$$

and

$$\frac{\sum\limits_{1-\varepsilon\le\lambda_j(h)\le 1} \int_{\Gamma_R} \gamma(x)|v_j(h)(x)|^2 d\sigma(x)}{\#\{1-\varepsilon \le \lambda_j(h) \le 1\}} = \mathcal{O}_{\varepsilon,\gamma,\Gamma}(R^{3-d}h^{d-1}),$$

$$\frac{\sum\limits_{1-\varepsilon\le\lambda_j(h)\le 1} \int_{\Gamma_R} \gamma(x)|[X\circ v_j(h)](x)|^2 d\sigma(x)}{\#\{1-\varepsilon \le \lambda_j(h) \le 1\}} = \mathcal{O}_{\varepsilon,\gamma,X,\Gamma}(R^{1-d}h^{d-1}).$$

The proof is complete. □

4.2. **Quantum ergodicity and surface concentration almost surely.** In this subsection, we move onto obtaining another characterisation of the surface concentration of $\phi_j(h)$ and hence $u_j(h)$, $v_j(h)$.

Before we continue, let us recall the following solution under a Hamiltonian flow:

$$\begin{cases} \frac{\partial}{\partial t} a_{x,\xi}(t) &= \{\mathcal{H}, a_{x,\xi}(t)\}, \\ a_{x,\xi}(0) &= a_0(x,\xi) \in S^m(T^*(\partial D)), \end{cases} \tag{4.22}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket given by $\{f,g\} := X_f\, g = -\omega(X_f, X_g)$ with X_f being the symplectic gradient vector field given by $\iota_{X_f}\,\omega = df$. Hence, we have $\frac{\partial}{\partial t} a_{x,\xi} = X_{\mathcal{H}} a_{x,\xi}$, and $a_{x,\xi}(t) = a_0(\gamma(t), p(t))$ where

$$\begin{cases} \frac{\partial}{\partial t}(\gamma(t), p(t)) & = X_{\mathcal{H}}(\gamma(t), p(t)), \\ (\gamma(0), p(0)) & = (x,\xi) \in T^*(\partial D). \end{cases}$$

Next, we recall the Heisenberg's picture and the lift to the operator level via Egorov's theorem.

Proposition 4.4. *[15, 18, 19] The following operator evolution equation*

$$\begin{cases} \frac{\partial}{\partial t} A_h(t) = \frac{\mathrm{i}}{h}\left[\mathrm{Op}_{\mathcal{H},h}, A_h(t)\right], \\ A_h(0) = \mathrm{Op}_{a_0,h}, \end{cases} \tag{4.23}$$

defines a unique Fourier integral operator (up to $h^\infty\, \Phi\mathrm{SO}_h^{-\infty}$) for $t < C\log(h)$:

$$A_h(t) = e^{-\frac{\mathrm{i}t}{h}\mathrm{Op}_{\mathcal{H},h}}\, A_h(0)\, e^{\frac{\mathrm{i}t}{h}\mathrm{Op}_{\mathcal{H},h}} + O(h\, \Phi\mathrm{SO}_h^{m-1}).$$

Moreover,

$$A_h(t) = \mathrm{Op}_{a_{x,\xi}(t),h} + \mathcal{O}(h\, \Phi\mathrm{SO}_h^{m-1}),$$

or that $p_{A_h(t)}(x,\xi) = a_{(x,\xi)}(t) + \mathcal{O}(|\xi|^{-1})$.

With the notion of $X_{\mathcal{H}}$ at hand, we now consider for $0 \le \delta \le \varepsilon$, the set $M_{X_{\mathcal{H}}}(\{\mathcal{H} = 1-\delta\})$ as the set of invariant measures on $\{\mathcal{H} = 1-\delta\}$ and also $M_{X_{\mathcal{H}},\mathrm{erg}}(\{\mathcal{H} = 1-\delta\})$ as the set of ergodic measures with respect to the Hamiltonian flow generated by $X_{\mathcal{H}}$ on $\{\mathcal{H} = 1-\delta\}$. The Choquet's theorem can be applied to obtain the classical ergodic decomposition theorem:

$$\sigma_{\{\mathcal{H}=1-\delta\}} = \sigma_{\{\mathcal{H}=1-\delta\}}(\{\mathcal{H} = 1-\delta\}) \int_{M_{X_{\mathcal{H}},\mathrm{erg}}(\{\mathcal{H}=1-\delta\})} \mu \, d\nu_{1-\delta}(\mu),$$

where $\sigma_{\{\mathcal{H}=1-\delta\}}$ is the Liouville measure on the surface $\{\mathcal{H} = 1-\delta\}$ and hence the disintegration theorem gives

$$\begin{aligned} &\int_{1-\varepsilon\le\mathcal{H}\le1} f(x,\xi)\sigma\otimes\sigma^{-1}(x,\xi) \\ &= \int_0^\varepsilon \sigma_{\{\mathcal{H}=1-\delta\}}(\{\mathcal{H}=1-\delta\}) \int_{M_{X_{\mathcal{H}},\mathrm{erg}}(\{\mathcal{H}=1-\delta\})} \int_{\mathrm{supp}(\mu)} \frac{f(x,\xi)}{|(\partial_x\mathcal{H}, \partial_\xi\mathcal{H})(x,\xi)|} d\mu(x,\xi)\, d\nu_{1-\delta}(\mu)\, d\delta. \end{aligned}$$

The above conclusion generalizes a related result that was established in our earlier work [2]. The difference lies in that since $\mathcal{H}$ is no longer homogenous, the flow $X_{\mathcal{H}}$ behaves differently on each level surface $\{\mathcal{H} = 1-\delta\}$, and hence $M_{X_{\mathcal{H}},\mathrm{erg}}(\{\mathcal{H} = 1-\delta\})$ is different for each $0 < \delta < \varepsilon$. With the above decomposition, similar to [2], we can obtain the following lemma.

Lemma 4.5. *For any $r \le s$ and all $a_0 \in \mathcal{S}^m(T^*(\partial D))$, we have*

$$\frac{1}{T}\int_0^T a_{x,\xi}(t)dt \to_{a.e.\, d\sigma\otimes d\sigma^{-1} \text{ and } L^2(\{1-\varepsilon\le H\le1\}, d\sigma\otimes d\sigma^{-1})} \bar{a}(x,\xi) \text{ as } T\to\infty,$$

for some $\bar{a} \in L^2(\{1-\varepsilon \le H \le 1\}, d\sigma\otimes d\sigma^{-1})$, and a.e. $d\nu_{1-\delta}\, d\delta$, we have

$$\bar{a}(x,\xi) = \int_{\{H=1-\delta\}} a_0\, d\mu \quad a.e.\ d\mu.$$

Now, we can utilize the Egorov's lift in Proposition 4.4 and our definition of $\bar{a}$ in Lemma 4.5, and follow a similar argument as in [2] to obtain the following quantum ergodicity theorem with some generalization compared to the classical results in and [12, 15, 17, 31–33, 35, 36].

Theorem 4.6. *Fixing $\varepsilon > 0$, we have the following (variance-like) estimate as $h \to +0$,*

$$\frac{1}{\#\{1-\varepsilon \leq \lambda_i(h) \leq 1\}} \sum_{1-\varepsilon \leq \lambda_i(h) \leq 1} \left| \left\langle \mathrm{Op}_{a-\bar{a},h}\, \phi_i(h), \phi_i(h) \right\rangle_{L^2} \right|^2 \to 0. \tag{4.24}$$

As an important consequence of Theorem 4.6, by using Chebeychev's trick and a diagonal argument, one can readily derive the following quantum ergodicity result.

Corollary 4.7. *Given r, s, there exists a subsequence $S(h) \subset J(h) := \{j \in \mathbb{N} : 1-\varepsilon \leq \lambda_j(h) \leq 1\}$ of density 1, i.e.*

$$\frac{\sum_{j \in S(h)} 1}{\sum_{j \in J(h)} 1} = 1 + o_\varepsilon(1),$$

such that whenever $h \to +0$ one has:

$$\max_{j \in S(h)} \left| \left\langle \mathrm{Op}_{a-\bar{a},h}\, \phi_j(h), \phi_j(h) \right\rangle_{L^2} \right| = o_\varepsilon(1) \ \ \text{and} \ \ \frac{\sum_{j \in S(h)} 1}{\sum_{j \in J(h)} 1} = 1 + o_\varepsilon(1)\,.$$

By using the quantum ergodicity result, we can arrive at the following lemma.

Lemma 4.8. *Let Γ_R and $\gamma \in C^\infty(\Gamma)$ be given in Theorem 4.1 with $R = \mathrm{dist}(\Gamma_R, \partial D)$. Given $\varepsilon > 0$, there exists $S(h) \subset J(h) := \{j \in \mathbb{N} : 1-\varepsilon \leq \lambda_j(h) \leq 1\}$ with $\frac{\sum_{j \in S(h)} 1}{\sum_{j \in J(h)} 1} \sim 1$ such that, as $h \to +0$, the following results hold simultaneously:*

$$\max_{j \in S(h)} \left| \int_{\partial D} \gamma(x)\, |u_j(h)(x)|^2 \, d\sigma(x) - h^2 \left\langle \mathrm{Op}_{\overline{a_1},h}\, \phi_j(h), \phi_j(h) \right\rangle_{L^2(\partial D)} \right| = o_\varepsilon(h^2), \tag{4.25}$$

$$\max_{j \in S(h)} \left| \int_{\partial D} \gamma(x)\, |\nabla u_j(h)(x)|^2 \, d\sigma(x) - \left\langle \mathrm{Op}_{\overline{a_2},h}\, \phi_j(h), \phi_j(h) \right\rangle_{L^2(\partial D)} \right| = o_\varepsilon(1), \tag{4.26}$$

and

$$\max_{j \in S(h)} \left| \int_{\Gamma_R} \gamma(x)\, |u_j(h)(x)|^2 \, d\sigma(x) - Q^{d-1} h^{d-1} \left\langle \mathrm{Op}_{\overline{a_3},h}\, \phi_j(h), \phi_j(h) \right\rangle_{L^2(\partial D)} \right| = o_\varepsilon(h^{d-1}), \tag{4.27}$$

$$\max_{j \in S(h)} \left| \int_{\Gamma_R} \gamma(x)\, |\nabla u_j(h)(x)|^2 \, d\sigma(x) - Q^{d-1} h^{d-1} \left\langle \mathrm{Op}_{\overline{a_4},h}\, \phi_j(h), \phi_j(h) \right\rangle_{L^2(\partial D)} \right| = o_\varepsilon(h^{d-1}), \tag{4.28}$$

as well as

$$\max_{j \in S(h)} \left| \int_{\partial D} \gamma(x)\, |v_j(h)(x)|^2 \, d\sigma(x) - h^2 \left\langle \mathrm{Op}_{\overline{a_5},h}\, \phi_j(h), \phi_j(h) \right\rangle_{L^2(\partial D)} \right| = o_\varepsilon(1), \tag{4.29}$$

$$\max_{j \in S(h)} \left| \int_{\partial D} \gamma(x)\, |\nabla v_j(h)(x)|^2 \, d\sigma(x) - \left\langle \mathrm{Op}_{\overline{a_6},h}\, \phi_j(h), \phi_j(h) \right\rangle_{L^2(\partial D)} \right| = o_\varepsilon(h^2), \tag{4.30}$$

and

$$\max_{j \in S(h)} \left| \int_{\Gamma_R} \gamma(x)\, |v_j(h)(x)|^2 \, d\sigma(x) - h^{d-1} \left\langle \mathrm{Op}_{\overline{a_7},h}\, \phi_j(h), \phi_j(h) \right\rangle_{L^2(\partial D)} \right| = o_\varepsilon(h^{d-1}), \tag{4.31}$$

$$\max_{j \in S(h)} \left| \int_{\Gamma_R} \gamma(x)\, |\nabla v_j(h)(x)|^2 \, d\sigma(x) - h^{d-1} \left\langle \mathrm{Op}_{\overline{a_8},h}\, \phi_j(h), \phi_j(h) \right\rangle_{L^2} \right| = o_\varepsilon(h^{d-1}), \tag{4.32}$$

with

$$a_1(x,\xi) \ := \ \gamma(x) |p_{\mathcal{S}_{-1}^{\kappa Q}}(x,\xi)|^2,$$

$$
\begin{aligned}
a_2(x,\xi) &:= \gamma(x)+\varphi(x)|\xi|^2|p_{\mathcal{S}^{\kappa Q}_{-1}}(x,\xi)|^2,\\
a_3(x,\xi) &:= \gamma(F_R(x))\mathrm{det}(DF_R)^{-1}(F_R(x))\,|p_{\widetilde{\mathcal{S}^{\kappa}Q}_{0,-d-1}}(x,\xi)|^2=\mathcal{O}(R^{3-d}|\xi|^{-d-2}),\\
a_4(x,\xi) &:= \gamma(F_R(x))\mathrm{det}(DF_R)^{-1}(F_R(x))\left(\sum_{j=1}^{d}|p_{\widetilde{(e_j\circ\mathcal{S}^{\kappa}Q)}_{0,-d-1}}(x,\xi)|^2\right)=\mathcal{O}(R^{1-d}|\xi|^{-d-2}),\\
a_5(x,\xi) &:= \gamma(x)|p_{\mathcal{S}^{\kappa Q}_{-1}}(x,\xi)|^2,\\
a_6(x,\xi) &:= \gamma(x)\frac{|p_{\mathcal{S}^{\kappa Q}_{-1}}(x,\xi)|^2}{|p_{\mathcal{S}^{\kappa}_{-1}}(x,\xi)|^2}+\gamma(x)|\xi|^2|p_{\mathcal{S}^{\kappa Q}_{-1}}(x,\xi)|^2,\\
a_7(x,\xi) &:= \gamma(F_R(x))\mathrm{det}(DF_R)^{-1}(F_R(x))\,\frac{|p_{\mathcal{S}^{\kappa Q}_{-1}}(x,\xi)|^2}{|p_{\mathcal{S}^{\kappa}_{-1}}(x,\xi)|^2}\,|p_{\widetilde{\mathcal{S}^{\kappa}}_{0,-d-1}}(x,\xi)|^2=\mathcal{O}(R^{3-d}|\xi|^{-d-2}),\\
a_8(x,\xi) &:= \gamma(F_R(x))\mathrm{det}(DF_R)^{-1}(F_R(x))\,\frac{|p_{\mathcal{S}^{\kappa Q}_{-1}}(x,\xi)|^2}{|p_{\mathcal{S}^{\kappa}_{-1}}(x,\xi)|^2}\left(\sum_{j=1}^{d}|p_{\widetilde{(e_j\circ\mathcal{S}^{\kappa})}_{0,-d-1}}(x,\xi)|^2\right)\\
&= \mathcal{O}(R^{1-d}|\xi|^{-d-2}).
\end{aligned}
$$

Proof. We obtain the result by choosing $a(x,\xi)$ as a smooth non-negative bump function $\gamma\in\mathcal{C}^\infty$ either on ∂D or Γ_R with $\mathrm{dist}(\Gamma_R,\partial D)=R>0$ multiplied with appropriate symbols, and then applying Theorem 4.7. For instance, we obtain the descriptions of u_j as follows: (4.2) is obtained by choosing the symbol $|p_{\mathcal{S}^{kQ}_{-1}}(x,\xi)|^2$. (4.26) comes from choosing the symbols $1+|\xi|^2|p_{\mathcal{S}^{kQ}_{-1}}(x,\xi)|^2$. (4.27) is resulted from taking $\det(DF_R)^{-1}(F_R(x))\,|p_{\widetilde{\mathcal{S}^{k}Q}_{0,-d-1-0}}(x,\xi)|^2$ and then apply a change of variable formula. (4.28) comes from taking $\det(DF_R)^{-1}(F_R(x))\,|p_{\widetilde{(X\circ\mathcal{S}^{k}Q)}_{0,-d-1-0}}(x,\xi)|^2$ with X being one of the constant coordinate vectors $\{e_k\}_{k=1}^d$, and then summed over all symbols resulting from $X=e_k$. Their v counterparts (4.29)-(4.32) are obtained similary with a specific choice $Q=1$. The proof is complete. □

By using the above results, we can derive the following theorem which indicates that there are "many" generalised transmission eigenfunctions which are localized around ∂D.

Theorem 4.2. *Given $\varepsilon>0$, for any closed surface $\Gamma_R\subset D$ and any bump function $\gamma(x)\in\mathcal{C}^\infty(\Gamma_R)$ as described in Theorem 4.1, there exists $S(\kappa^{-1})\subset J(\kappa^{-1}):=\{j\in\mathbb{N}:1-\varepsilon\le\lambda_j(\kappa^{-1})\le 1\}$ with $\frac{\sum_{j\in S(h)}1}{\sum_{j\in J(h)}1}\sim 1$ such that we have as $\kappa\to\infty$:*

$$
\begin{aligned}
\int_{\Gamma_R}\gamma(x)\left|u_j(\kappa^{-1})(x)\right|^2 d\sigma(x) &= \mathcal{O}(\kappa^{-2}) && \textit{if } \mathrm{supp}(\gamma)\subset\partial D,\\
\int_{\Gamma_R}\gamma(x)\left|u_j(\kappa^{-1})(x)\right|^2 d\sigma(x) &= \mathcal{O}(Q^{d-1}R^{d-3}\kappa^{1-d}) && \textit{if } \mathrm{supp}(\gamma)\cap\partial D=\emptyset,\\
\int_{\Gamma_R}\gamma(x)\left|v_j(\kappa^{-1})(x)\right|^2 d\sigma(x) &= \mathcal{O}(\kappa^{-2}) && \textit{if } \mathrm{supp}(\gamma)\subset\partial D,\\
\int_{\Gamma_R}\gamma(x)\left|v_j(\kappa^{-1})(x)\right|^2 d\sigma(x) &= \mathcal{O}(R^{d-3}\kappa^{1-d}) && \textit{if } \mathrm{supp}(\gamma)\cap\partial D=\emptyset,
\end{aligned}
$$

and

$$\int_{\Gamma_R} \gamma(x) \left|\nabla u_j(\kappa^{-1})(x)\right|^2 d\sigma(x) = \mathcal{O}(Q^{d-1}R^{d-1}\kappa^{1-d}) \quad \textit{if} \;\; \mathrm{supp}(\gamma)\cap\partial D=\emptyset,$$
$$\int_{\Gamma_R} \gamma(x) \left|\nabla v_j(\kappa^{-1})(x)\right|^2 d\sigma(x) = \mathcal{O}(R^{d-1}\kappa^{1-d}) \quad \textit{if} \;\; \mathrm{supp}(\gamma)\cap\partial D=\emptyset.$$

Moreover, there exists another subsequence $\widetilde{S(\kappa^{-1})} \subset J(\kappa^{-1})$ *such that*

$$\int_{\Gamma_R} \gamma(x) \left|\nabla u_j(\kappa^{-1})(x)\right|^2 d\sigma(x) \geq \mathcal{O}(1) \quad \textit{if} \;\; \mathrm{supp}(\gamma)\subset\partial D,$$
$$\int_{\Gamma_R} \gamma(x) \left|\nabla v_j(\kappa^{-1})(x)\right|^2 d\sigma(x) \geq \mathcal{O}(1) \quad \textit{if} \;\; \mathrm{supp}(\gamma)\subset\partial D.$$

In all of the above relations, the asymptotic constants in the RHS terms depend on $\|\gamma\|_{C(\Gamma)}$.

Proof. From the fact that

$$|\left\langle \mathrm{Op}_{\overline{a},h}\,\phi_i(h),\phi_i(h)\right\rangle_{L^2(\partial D)}| \leq \|\mathrm{Op}_{\overline{a},h}\|_{\mathcal{L}(L^2(\partial D),L^2(\partial D))}\|\phi_i(h)\|^2_{L^2(\partial D)} \leq C_a\,,$$

(bearing in mind that $\|\phi_i(h)\|^2_{L^2}=1$) together with Corollary 4.8, we can arrive at the first 6 conclusions of the theorem. As an example, recalling $\kappa=h^{-1}$, we show directly from that

$$\begin{aligned}
&\max_{j\in S(\kappa^{-1})}\left|\int_{\partial D}\gamma(x)\left||u_j(\kappa^{-1})(x)\right|^2 d\sigma(x)\right| \\
\leq\; &\max_{j\in S(\kappa^{-1})}|\kappa^{-2}\left\langle \mathrm{Op}_{\overline{a_1},h}\,\phi_j(h),\phi_j(h)\right\rangle_{L^2(\partial D)}| \\
&\quad+\max_{j\in S(\kappa^{-1})}\left|\int_{\partial D}\gamma(x)\left|u_j(\kappa^{-1})(x)\right|^2 d\sigma(x)-\kappa^{-2}\left\langle \mathrm{Op}_{\overline{a_1},h}\,\phi_j(h),\phi_j(h)\right\rangle_{L^2(\partial D)}\right| \\
\leq\; & C_{a_1}\,\kappa^{-2}+o_\varepsilon(\kappa^{-2}) \\
=\; & \mathcal{O}(\kappa^{-2}).
\end{aligned}$$

The other five conclusions can be obtained in a similar manner.

The last two conclusions come from applying the pigeonhole principle to the sums (4.8) with $\zeta=u,v$ respectively. In fact, if supposing otherwise that for $\mathrm{spt}(\gamma)\subset\partial D$, we have as $\kappa\to\infty$ that

$$\max_{j\in S(\kappa^{-1})}\left|\int_{\Gamma_R}\gamma(x)\left|\nabla\zeta_j(\kappa^{-1})(x)\right|^2 d\sigma(x)\right|\to 0\,.$$

Then one can directly check that as $h\to\infty$:

$$\left|\frac{\sum\limits_{1-\varepsilon\leq\lambda_j(h)\leq 1}\int_{\partial D}\gamma(x)|[\zeta_j(h)](x)|^2 d\sigma(x)}{\#\{1-\varepsilon\leq\lambda_j(h)\leq 1\}}\right|\to 0\,,$$

which contradicts to (4.8), and therefore the result follows.

The proof is complete. □

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