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COMPUTATIONAL AND APPLIED MATHEMATICS

Singular Perturbation Analysis of Integral Equations

Charles G. Lange
Donald R. Smith

December 1986

CAM Report 86-02

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

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by *Charles G. Lange*[*)] and *Donald R. Smith*[†)]

Abstract

A study is given of certain linear singularly perturbed Volterra or Fredholm vector integral equations, including also certain integrodifferential equations. Several examples are discussed that illustrate the rich diversity of (in some cases surprising) behavior that is possible for such equations. A special class of equations is studied further using a version of the additive multivariable technique, and the results are justified by appropriate error estimates.

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*) Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90024

†) Department of Mathematics, University of California at San Diego, La Jolla, CA 92093

1. *Introduction*

We consider the vector integral equation

$$\epsilon w(x) + h(x, \epsilon) = \int_0^1 K(x, s)w(s)ds, \quad 0 \leq x \leq 1, \tag{1.1}$$

for small positive values of ϵ ($\epsilon \to 0^+$), for a solution function $w = w(x) = w(x, \epsilon)$ that is an m-dimensional real or complex vector-valued function, and where the data functions h and K are real or complex vector-valued and matrix-valued functions with appropriate compatible orders. In operator form the equation becomes $Kw = h + \epsilon w$, and the reduced equation is given by putting $\epsilon = 0$ in (1.1),

$$Kw = h_0 \qquad (h_0(x) := h(x, 0)). \tag{1.2}$$

We assume that the given forcing function h is smooth and has an asymptotic expansion in nonnegative powers of ϵ as

$$h(x, \epsilon) \sim \sum_{j=0}^{\infty} h_j(x)\epsilon^j \quad \text{for} \quad 0 \leq x \leq 1, \tag{1.3}$$

as $\epsilon \to 0^+$, for suitable smooth functions $h_j = h_j(x)$. There is no difficulty in permitting the kernel K to depend similarly on ϵ, but for simplicity we take K to be independent of ϵ. Also for simplicity, here and in the next few sections we consider mainly the scalar case ($m = 1$), with K, h and w suitable real- or complex-valued functions; the vector case is considered in later sections of this paper. By considering the vector case for (1.1), we are able directly to include also various *integrodifferential* equations.

Perhaps the simplest case of (1.1) corresponds to a scalar constant kernel, with $K(x, s) \equiv A$ for some fixed number A, in which case an easy calculation shows that (1.1) has the unique solution

$$w(x,\epsilon) = \frac{-1}{\epsilon(A-\epsilon)}\left[(A-\epsilon)h(x,\epsilon) + A\int_0^1 h(s,\epsilon)ds\right]. \tag{1.4}$$

This solution is generally unbounded as $\epsilon \to 0$, uniformly for $0 \leq x \leq 1$, and *there are no boundary layers*. This behavior is indicative of typical behavior for smooth kernels: in *Appendix A.1*, for various classes of smooth kernels, we show more generally that (1.1) has a unique solution, and the solution does not exhibit boundary-layer behavior, as $\epsilon \to 0$.

Our interest here is in *piecewise* smooth kernels $K(x, s)$ that exhibit discontinuities so as to result in solutions of boundary-layer type. A simple example is given by the piecewise constant kernel

$$K(x, s) := \begin{cases} A, & s < x \\ B, & s > x, \end{cases} \tag{1.5}$$

for given numbers A and B, $A \neq B$. In this case the reduced equation (1.2) has a solution, given as $w(x) = (A-B)^{-1}h_0'(x)$, *if and only if* $h_0 = h_0(x)$ satisfies the condition

$$Bh_0(1) = Ah_0(0). \tag{1.6}$$

However, for all small $\epsilon > 0$, the full equation (1.1) has a unique solution whether or not (1.6) holds. For example, suppose $h(x) = e^{\lambda x}$ for a fixed constant λ, so that the integral equation becomes

$$\epsilon w(x) + e^{\lambda x} = A\int_0^x w(s)ds + B\int_x^1 w(s)ds. \tag{1.7}$$

Upon differentiation of (1.7) we find the equation $\epsilon w' - (A-B)w = -\lambda e^{\lambda x}$, with general solution

$$w(x,\epsilon) = ce^{\frac{1}{\epsilon}(A-B)x} + \frac{\lambda}{A-B-\epsilon\lambda}e^{\lambda x}, \tag{1.8}$$

where c is a constant of integration that is determined by inserting (1.8) back into (1.7), yielding

$$c = \frac{(A-B)(A - Be^{\lambda})}{\epsilon(B-A+\epsilon\lambda)\left[A - Be^{(A-B)/\epsilon}\right]}. \tag{1.9}$$

The resulting solution undergoes rapid oscillations in the case *Re* A = *Re* B, while either boundary-layer behavior or rapid exponential growth can occur in each of the cases *Re* A < *Re* B and *Re* B < *Re* A. Our interest here is in problems exhibiting boundary-layer phenomena. We intend to treat the case of rapid oscillations in a subsequent work.

For example, (1.7) is a Volterra equation for an initial-value problem in the case $B = 0$, with solution

$$w(x, \epsilon) = \frac{1}{\epsilon}\,\frac{Ae^{\frac{A}{\epsilon}x}}{-A+\epsilon\lambda} + \frac{\lambda e^{\lambda x}}{A-\epsilon\lambda}. \tag{1.10}$$

The solution undergoes rapid exponential growth for $x > 0$ if $Re\ A > 0$, while it has a boundary layer at $x = 0$ if $Re\ A < 0$. Analogous results hold when $A = 0$, in which case (1.7) is a backward Volterra equation for a terminal-value problem, and the solution undergoes rapid exponential growth for $x < 1$ if $Re\ B > 0$, and so forth.

Similarly we find from (1.8)-(1.9) the result:

$Re\ A < Re\ B$ implies

$$w(x,\epsilon) \sim \frac{1}{\epsilon}\,\frac{Be^{\lambda} - A}{A}\, e^{\frac{A-B}{\epsilon}x} + \frac{\lambda}{A-B}\, e^{\lambda x}, \qquad \text{if} \quad A \neq 0. \tag{1.11}$$

The first term on the right side of (1.11) represents a boundary-layer correction at $x = 0$, while the last term represents a slowly varying outer solution, with

$$\lim_{\substack{\epsilon \to 0^+ \\ \text{Fixed } x\in(0,1]}} w(x,\ \epsilon) = \frac{\lambda}{A-B}\, e^{\lambda x}. \tag{1.12}$$

This limiting outer solution given by the right side of (1.12) coincides with the solution of the reduced equation (1.2) when this latter equation has a solution (i.e. when (1.6) holds), and in this case the solution of (1.1) is uniformly $O(1)$ as $\epsilon \to 0^+$ because (1.6) implies $Be^{\lambda} = A$ and so the $O(1/\epsilon)$ term vanishes in the boundary-layer term. On the other hand if $Be^{\lambda} \neq A$, then (1.6) fails to hold and the reduced equation (1.2) has no solution. The amplitude of the boundary-layer correction is $O(1/\epsilon)$ in this case ($Be^{\lambda} \neq A$), and the boundary-layer correction acts like a "delta function" when inserted into the integral of (1.1), so that the sum of the terms on the right side of (1.11) provides a useful asymptotic solution to the reduced equation for $x > 0$. An analogous result holds in the case $Re\, B < Re\, A$, $B \neq 0$, with the layer region occurring at $x = 1$ instead of at $x = 0$.

The example (1.7) shows that boundary-layer behavior is possible with discontinuous kernels, and in fact boundary layers are possible for other types kernels possessing both stronger and weaker discontinuities than indicated by (1.5), such as the kernel

$$K(x, s) = \frac{1}{\sqrt{|x-s|}},$$

and also kernels for which $K(x, s)$ is continuous but $\partial^k K(x,s)/\partial x^k$ is discontinuous for some $k > 0$. However, in this paper we restrict consideration mainly to certain of the simplest problems of boundary-layer type for (1.1). Except in Section 3, where less regularity is required, we generally assume that $K(x, s)$ is smooth on $[0, 1]\times[0, 1]$ except for a jump discontinuity along the diagonal $x = s$, with smooth jump (matrix) $J = J(x)$ given as

$$J(x) := K(x, x^-) - K(x, x^+) \quad \text{for} \quad 0 < x < 1, \tag{1.13}$$

where $K(x, x^-) := \lim\limits_{\substack{s\to x\\ s<x}} K(x, s)$ and $K(x, x^+) := \lim\limits_{\substack{s\to x\\ s>x}} K(x, s)$ for $0 < x < 1$, and with $J(0)$ and $J(1)$ given by the respective limits of $J(x)$ as $x \to 0^+$ and $x \to 1^-$. Moreover, except in Section 2, we generally impose additional conditions so as to restrict attention to a class of problems with the simplest type boundary-layer structure. Considerations of various other related problems with solutions of boundary-layer and interior-layer type, problems with kernels having discontinuites along more general curves other than $x = s$, various nonlinear problems, etc., are deferred elsewhere. For the present class of problems considered here, we are able to obtain rigorous results in a clear, precise manner. We shall see that the class of problems is of considerable interest because it includes several important special cases such as the singularly perturbed vector Volterra equation (with $K(x,s) \equiv 0$ for $s > x$, with jump $J(x) = K(x, x^-)$), various integrodifferential equations, and many important singularly perturbed boundary-value problems for differential equations that can be reformulated as integral equations of the type (1.1) with a discontinuous kernel.

For example, the (scalar) Dirichlet problem

$$\epsilon y'' + a(x)y' + b(x)y = f(x) \quad \text{for} \quad 0 \leq x \leq 1,$$

$$y(0, \epsilon) = \alpha_0, \quad y(1, \epsilon) = \alpha_1, \tag{1.14}$$

(smooth real data a, b and f, and real α_0, α_1 and ϵ),

is known to have a unique solution (as $\epsilon \to 0^+$) if $a(x)$ is everywhere nonzero, with resulting solution that has a single boundary layer occurring either at the left endpoint or the right endpoint in the respective cases $a(x) > 0$ or $a(x) < 0$ (cf. Smith [1985; Section 8.3]). A routine calculation shows that the problem (1.14) can be replaced by an equivalent integral equation of the type (1.1) with $m = 1$, $w = y$, and

$$h(x,\epsilon) := -\epsilon[\alpha_0 + (\alpha_1-\alpha_0)x] - \int_0^x (x-1)sf(s)ds - \int_x^1 (s-1)xf(s)ds,$$

$$K(x, s) := \begin{cases} (x-1)[a(s) + s(a'(s) - b(s))] & \text{if} \quad s < x, \\ x[a(s) + (s-1)(a'(s) - b(s))] & \text{if} \quad s > x. \end{cases} \tag{1.15}$$

Here the kernel K has jump (1.13) given as

$$J(x) = -a(x), \tag{1.16}$$

and this problem (1.1), (1.15) is of the type considered here if the function a is nonzero.

Examples such as (1.5) and (1.15) show that (1.1) can have solutions of boundary-layer type if the kernel K has a jump discontinuity across $x = s$ and if suitable stability conditions hold. Moreover, in the case (1.5) the amplitude in the boundary layer is $O(1/\epsilon)$ unless (1.6) holds so that the reduced equation has a solution, in which case the solution of (1.1) (including the amplitude in the boundary layer) is uniformly $O(1)$. If the appropriate stability conditions do not hold, then solutions of (1.1) need not be of the boundary-layer type considered here, but rather, a variety of other behaviors can occur as discussed briefly for (1.7) and as illustrated more fully in Section 2. However, subject to appropriate assumptions, we shall find that (1.1) has a unique solution $w = w(x, \epsilon)$ satisfying

$$w \sim \textit{outer solution} + \textit{boundary-layer correction}, \quad \text{as} \quad \epsilon \to 0^+, \qquad (1.17)$$

where the outer solution is regular and of order unity, while the boundary-layer correction term is generally a linear combination of "delta functions" at the endpoints $x = 0$ and $x = 1$, of order $O(1/\epsilon)$. In particular, subject to the present assumptions, the situation for (1.1) is analogous to that for certain _singular_ singularly perturbed problems for a differential equation, where the reduced problem is not uniquely solvable; cf. Smith [1985, p. 227] and Schmeiser and Weiss [1986] for references on such problems for ordinary and partial differential equations. On the other hand, in the special case (1.15) (with $a(x)$ nonzero) and in various other cases arising similarly from commonly studied singularly perturbed boundary value problems for differential equations (such as (4.2)-(4.3) below), the resulting reduced equation (1.2) is always uniquely solvable, and the amplitudes of the resulting boundary layers are $O(1)$. Hence such commonly studied singularly perturbed boundary value problems for differential equations correspond to a rather special case within the framework of the theory discussed here for the integral equation.

As indicated in (1.17), the solution of (1.1) possesses an *additive decomposition* into the sum of a regular part plus a boundary-layer correction, subject to our assumptions. (Other types of behavior can occur when our assumptions are relaxed, as indicated by the examples of sections 2 and 3.) In the present study of (1.1) we use, for the most part, a direct additive multivariable (boundary layer) technique of a sort that is commonly employed in certain studies of singularly perturbed differential equations primarily since the work of O'Malley [1968, 1969, 1970, 1971a, 1971b, 1974]; see also Latta [1951], Carrier [1953, 1974], Cochran [1962], Erdélyi [1968], Hoppensteadt [1971] and Smith [1971, 1975, 1985] among many possible references. Instead of the direct additive multivariable method employed here for (1.1), we could equally well use the method of matched asymptotic expansions, as illustrated for various differential equations in Carrier [1953, 1974], Eckhaus [1973, 1979], Nayfeh [1973], Bender and Orszag [1978], Kevorkian and Cole [1981], and Lange [1983] among many possible references, and as illustrated in Lange and Miura [1982, 1985a, 1985b] for differential-difference equations. The formal use of the additive multivariable method has in some cases led to spurious solutions for differential equations; cf. Carrier and Pearson [1968], Rosenblat and Szeto [1980], and Lange [1983]. And indeed we find that the method of matched asymptotic expansions is more effective than the additive multivariable method for certain integral equation problems for (1.1) when our present assumptions are relaxed, as in the case of related problems with solutions of interior-layer type, or problems for which the kernel $K(x, s)$ has less regularity than assumed here. We shall defer to a subsequent work a general discussion on the use of matched asymptotic expansions for various singularly perturbed integral equations. The additive multivariable method is a convenient approach when it applies, and for this reason we use the method here.

It is a well-known consequence of a result of Banach [1922] that, for all *large* ϵ, (1.1) is uniquely solvable for any forcing function h, subject to mild regularity conditions on h and K. On the other hand, for *small* ϵ as considered here, the corresponding results in the literature are rather restricted. Olmstead and Handelsman [1972] consider certain scalar linear and nonlinear Volterra equations related to (1.1), with a singular Volterra kernel of convolution type. Erdélyi [1974] considers again the linear equation of Olmstead and Handelsman [1972], using a different method. Sirovich and Knight [1982] consider a class of linear Fredholm equations with kernels that are primarily of convolution type but with a slowly varying nonconvolution part, such as $K(x, s) = V(x-s, \epsilon x)$. Hoppensteadt [1983] considers certain Volterra equations with a small parameter ϵ appearing in the kernels in such a way that the kernels behave as delta functions as $\epsilon \to 0$. The present paper seems to be the first to apply standard singular perturbation methods to Fredholm (including Volterra) integral equations of the type (1.1).

In an important work that has not received adequate attention in the applied literature, Èskin [1973, 1981] uses the Wiener/Hopf method in a study of certain mixed boundary value problems for multidimensional elliptic pseudodifferential equations, including elliptic singular integral equations which in the one-variable case include equations that are related to (1.1). Our methods and results differ from those of Èskin. For example, Èskin expresses his asymptotic representations for solutions in a certain *multiplicative* form rather than the additive multivariable form employed here. Moreover, while our study is restricted to the one-dimensional problem, we are able to obtain more explicit results.

The present paper is organized as follows. In Section 2 we give the exact solution for a model scalar equation depending on several auxiliary parameters, and we show that a diversity of different types behavior can occur for different choices of these parameters. In certain cases this model equation satisfies the stability conditions imposed later in this paper, resulting in solutions of boundary-layer type satisfying (1.17). But in other cases the model equation violates one or more of these conditions, resulting then in an interesting variety of different types behavior. In Section 3 we describe our adaptation of the additive boundary-layer technique to integral equations such as those considered here, in the context of a less restrictive scalar example that satisfies weaker regularity conditions than those we later employ. Section 4 contains examples of vector equations, while Section 5 contains a description of a general class of such vector equations subject to certain conditions and assumptions. The additive multivariable technique is used in Section 6 to construct a formal asymptotic expansion for solutions of (1.1) subject to the conditions of Section 5. The resulting asymptotic expansion is used in Section 7 to prove existence and uniqueness of solutions of boundary-layer type for (1.1), and we prove that the given asymptotic expansion is uniformly valid for $0 \leq x \leq 1$ as $\epsilon \to 0^+$, again subject to suitable conditions. The results are applied to systems of integrodifferential equations in Section 8.

2. *Exact Solution of a Model Equation*

An indication of the rich diversity of forms that is possible for solutions of singularly perturbed integral equations can be obtained by examination of relatively simple model equations. The equation which we focus on here is particularly illuminating and has the advantage that it can be transformed into an easily solvable ordinary differential equation. The insight gained from a study of this and other model equations provides a motivation for the formal perturbation approach utilized in the remainder of the paper.

Consider the linear scalar integral equation

$$\epsilon w(x) + h(x) = A\int_0^x s w(s)ds + B\int_x^1 w(s)ds, \qquad 0 \leq x \leq 1, \tag{2.1}$$

where A, B are specified real constants, $h(x)$ is a given smooth function, and $0 < \epsilon << 1$. This equation is of the form (1.1) with kernel $K(x, s)$ given as

$$K(x, s) := \begin{cases} As & \text{for} \quad s < x \\ B & \text{for} \quad s > x, \end{cases} \tag{2.2}$$

so that the jump $J(x)$ of (1.13) is

$$J(x) = Ax - B \quad \text{for} \quad 0 \le x \le 1. \tag{2.3}$$

The associated reduced problem, obtained by putting $\epsilon = 0$ in (2.1), is

$$h(x) = A\int_0^x su(s)ds + B\int_x^1 u(s)ds, \quad 0 \le x \le 1. \tag{2.4}$$

Differentiation of (2.4) with respect to x yields the potential solution

$$u(x) = \frac{h'(x)}{Ax-B} = J^{-1}(x)h'(x), \quad 0 \le x \le 1. \tag{2.5}$$

By direct substitution of (2.5) into (2.4) we conclude that the reduced problem has a solution (given uniquely by (2.5)) if and only if $h(x)$ satisfies the condition

$$h(0) = B\int_0^1 \frac{h'(s)}{As-B}ds. \tag{2.6}$$

In contrast to the reduced problem, (2.1) has a unique solution for all sufficiently small positive (or negative) ϵ. Indeed, differentiation of (2.1) with respect to x transforms the integral equation into the following differential equation

$$\epsilon w'(x) - J(x)w(x) = -h'(x), \quad 0 \le x \le 1, \tag{2.7}$$

which has the general solution

$$w(x) = e^{\frac{1}{\epsilon}\left[\frac{Ax^2}{2} - Bx\right]}\left[c - \frac{1}{\epsilon}\int_0^x h'(s)e^{-\frac{1}{\epsilon}\left[\frac{As^2}{2} - Bs\right]}ds\right], \tag{2.8}$$

where c is an arbitrary constant. Upon substitution of (2.8) into (2.1), we find that c is given uniquely by

$$c = \frac{h(0) + \frac{B}{\epsilon}\int_0^1 e^{\frac{A}{2\epsilon}\left[s - \frac{B}{A}\right]^2}\left[\int_0^s h'(t)e^{-\frac{A}{2\epsilon}\left[t - \frac{B}{A}\right]^2}dt\right]ds}{-\epsilon + B\int_0^1 e^{\frac{1}{\epsilon}\left[\frac{As^2}{2} - Bs\right]}ds}, \tag{2.9}$$

where we assume here that $A \neq 0$; the simpler case $A = 0$ can be easily handled similarly.

Our task is to interpret the asymptotic behavior of the solution (2.8)-(2.9) for $\epsilon \to 0^+$. Consistent with our assumption that $\epsilon > 0$ and A and B are real, we may assume without loss of generality that $B = -1$, 0, or +1. (If $B \neq 0$, simply divide (2.1) by $|B|$ and redefine ϵ, A and h.) Careful examination of (2.8)-(2.9) reveals distinctly different behavior in the five cases (in increasing degree of complexity):

(i) $B = 0,\ A \neq 0$, **(ii)** $B = +1,\ A < 1\ (A \neq 0)$, **(iii)** $B = -1,\ A > -1\ (A \neq 0)$, (2.10)

(iv) $B = -1,\ A < -1$, and **(v)** $B = +1,\ A > 1$.

We shall discuss each of these cases in turn. For sake of brevity we omit consideration of the interesting transition cases such as $B = +1$, $A = +1$.

Case (i): B = 0 (Volterra equation)

For this special case (2.1) is a Volterra integral equation of the second kind with a unique solution for all $\epsilon > 0$ given by

$$w(x,\epsilon) = -\frac{1}{\epsilon} e^{Ax^2/(2\epsilon)}\left[h(0) + \int_0^x h'(s)e^{-As^2/(2\epsilon)}ds\right], \quad x \geq 0. \tag{2.11}$$

According to (2.6) with $B = 0$, the associated reduced problem has a solution if and only if $h(0) = 0$. The effect of the $\epsilon w(x)$ term in (2.1) depends crucially on the sign of the parameter A; see Figure 1 for typical solution forms.

--- Figure 1 Here ---

If $A < 0$, then it follows from (2.11) that there is an initial layer of width $O(\sqrt{\epsilon})$ at $x = 0$, outside of which the solution is a slowly-varying function of x. (We reserve the term boundary layer for Fredholm equations.) Simple descriptions of the solution which are valid in the initial layer and in the outer region can be obtained by carrying out the appropriate limit processes in (2.11). The inner expansion, corresponding to the limit process $\epsilon \to 0^+$ with $\tilde{x} = x/\sqrt{\epsilon}$ fixed, takes the form

$$w(x,\epsilon) = \frac{1}{\epsilon}\,\eta(\tilde{x},\epsilon) \sim \frac{1}{\epsilon}\sum_{j=0}^{\infty}\eta_j(\tilde{x})\,\epsilon^{j/2}, \quad \tilde{x} = x/\sqrt{\epsilon}, \tag{2.12}$$

where

$$\eta_0(\tilde{x}) = -h(0)e^{A\tilde{x}^2/2}, \quad \eta_1(\tilde{x}) = -h'(0)\int_0^{\tilde{x}} e^{A(\tilde{x}^2-\tilde{s}^2)/2}d\tilde{s},$$
$$\eta_2(\tilde{x}) = \frac{h''(0)}{A}\left[1 - e^{A\tilde{x}^2/2}\right], \ \ldots. \tag{2.13}$$

There are several properties of this expansion which merit discussion. First, we observe that the leading-order term in (2.12) satisfies (2.1) with $h(x)$ replaced by $h(0)$, i.e.,

$$\epsilon\left[\frac{1}{\epsilon}\,\eta_0(\tilde{x})\right] + h(0) = A\int_0^x s\left[\frac{1}{\epsilon}\eta_0(s/\sqrt{\epsilon})\right]ds, \tag{2.14}$$

or, strictly in terms of $\tilde{x}$,

$$\eta_0(\tilde{x}) + h(0) = A\int_0^{\tilde{x}} \tilde{s}\eta_0(\tilde{s})d\tilde{s}. \tag{2.15}$$

In other words, the $\epsilon w(x)$ term enters directly into the dominant balance in the inner equation limit of (2.1). (This latter result holds even if $h(0) = 0$.) Second, in general, the magnitude of the solution is large in the initial layer ($O(1/\epsilon)$ if $h(0) \neq 0$, or $O(1/\sqrt{\epsilon})$ if $h(0) = 0$ but $h'(0) \neq 0$). As a consequence of this large magnitude, the contribution to the integral in (2.1) from the (narrow) initial layer is $O(1)$ as $\epsilon \to 0^+$ with $x > 0$ fixed. More precisely, the leading-order term in (2.12) satisfies the integrated relation

$$A\int_0^{\delta(\epsilon)} s\left[\frac{1}{\epsilon}\eta_0(s/\sqrt{\epsilon})\right]ds \sim h(0) \quad \text{as} \quad \epsilon \to 0^+, \tag{2.16}$$

provided $\delta(\epsilon) \gg \sqrt{\epsilon}$. There is an important connection between this second property and the explicit form of the solution in the outer region. Finally, we note that the higher-order terms in (2.12), starting with $j = 2$, do not decay for large $\tilde{x}$.

The standard outer expansion for $w(x,\epsilon)$ when $A < 0$ is obtained from (2.11) by carrying out the limit process $\epsilon \to 0^+$ with $x > 0$ fixed. It is given by

$$w(x,\epsilon) = y(x,\epsilon) \sim \sum_{j=0}^{\infty} y_j(x)\epsilon^j, \tag{2.17}$$

with

$$y_0(x) = \frac{h'(x)}{Ax}, \quad y_1(x) = \frac{1}{A^2 x}\left[\frac{h'(x)}{x}\right]', \quad \ldots. \tag{2.18}$$

We observe that the leading-order term in (2.17) satisfies the integral equation

$$h(x) - h(0) = A\int_0^x s y_0(s)ds, \tag{2.19}$$

which, unless $h(0) = 0$, is *not* the reduced equation associated with (2.1). Clearly, the $h(0)$ term on the LHS of (2.19) is just what must be added to the reduced equation to guarantee a (unique) solution. In fact, it is not difficult to see that this term arises from the aforementioned contribution to the integral in (2.1) from the initial-layer region (cf. (2.16)). Thus, while the $\epsilon w(x)$ term does not enter directly into the dominant balance in the outer equation limit of (2.1), it does make a fundamental contribution indirectly through the integral term. These concepts are discussed further in the next section.

An interesting feature of the outer expansion (2.17) is the singular behavior of each $y_j(x)$ as $x \to 0^+$. In particular, the integrals $\int_0^x s y_j(s)ds$ do not exist for $j \geq 1$. This singular behavior is a consequence of the fact that the jump $J(x) \equiv K(x, x-)$ vanishes at $x = 0$. Later we show, subject to suitable conditions, that the outer solution is regular for $x \geq 0$ for scalar Volterra equations if $K(x, x-)$ is regular and negative for $x \geq 0$.

Following standard procedures it is easy to obtain an approximation to $w(x, \epsilon)$ in (2.11) which is uniformly valid to order unity. In fact, by adding three terms of the inner expansion (2.12) to one term of the outer expansion (2.17) and subtracting the common part, one can form the following "composite" approximation (Kevorkian and Kole [1981]):

$$w(x, \epsilon) = -\frac{h(0)}{\epsilon} e^{A\tilde{x}^2/2} - \frac{h'(0)}{\sqrt{\epsilon}} \int_0^{\tilde{x}} e^{(A/2)(\tilde{x}^2-\tilde{s}^2)} d\tilde{s}$$
$$- \frac{h''(0)}{A} e^{A\tilde{x}^2/2} + \frac{h'(x) - h(0)}{Ax} + O(\sqrt{\epsilon}) \tag{2.20}$$

as $\epsilon \to 0+$. The asymptotic error estimate $O(\sqrt{\epsilon})$ in (2.20) can be shown to hold uniformly on any compact interval $0 \leq x \leq x_0$.

The situation when $A > 0$ in *Case (i)* $(B = 0)$ is easier to interpret for (2.1). It is clear from (2.11) that the solution has the form of a rapidly-growing exponential for <u>*all*</u> fixed $x > 0$, which is approximated to leading order (assuming $h(0) \neq 0$) by

$$w(x,\epsilon) \sim -\frac{h(0)}{\epsilon} e^{Ax^2/(2\epsilon)} \quad \text{as} \quad \epsilon \to 0^+.$$

In other words, there is no outer region if $A > 0$; the $\epsilon w(x)$ term belongs in the dominant balance in (2.1) for all $x \geq 0$. Behavior such as this can be analyzed by an adaptation of the WKB method, but we omit this analysis here.

Case (ii): $B = 1$, $A < 1$ (boundary layer at $x = 0$)

In what follows we exclude the special case $A = 0$ as the results are similar in that case to those for $B = 0$, $A > 0$. With the present restrictions (2.1) is a Fredholm integral equation. A straightforward asymptotic evaluation of (2.8)-(2.9) leads to the following uniform leading-order approximation to $w(x,\epsilon)$ for $0 \leq x \leq 1$,

$$w(x,\epsilon) = \frac{h'(x)}{Ax-1} + O(\epsilon) + \frac{e^{-x/\epsilon}}{A\epsilon^2}\left[h(0) + \int_0^1 \frac{h'(t)}{1-At}dt + O(\epsilon)\right] \qquad (2.21)$$

as $\epsilon \to 0^+$.

The similarities between this result and the preceding result for $B = 0$, $A < 0$ are apparent. In both instances the solution has a large magnitude in a layer region near $x = 0$ and is slowly varying with an $O(1)$ magnitude away from $x = 0$ (cf. Figure 1a and Figure 2a). However, there are essential differences as well. For the present case the width of the layer is narrower ($O(\epsilon)$ rather than $O(\sqrt{\epsilon})$) and the magnitude of the solution in the layer is larger ($O(1/\epsilon^2)$ rather than $O(1/\epsilon)$) than for the Volterra equation. Moreover, in contrast with the earlier case, the function

$$y_0(x) = \frac{h'(x)}{Ax-1} \qquad (A < 1), \qquad (2.22)$$

which describes the solution to leading order away from $x = 0$, is smooth on $0 \leq x \leq 1$; there is no singularity at $x = 0$ (cf. (2.18)). In fact, in the present case each term of the outer expansion is smooth on $0 \leq x \leq 1$. As a consequence, the solution can be expressed asymptotically in terms of an additiove boundary-layer representation.

--- Figure 2 Here ---

In the next section we show how to derive (2.21) by a perturbation method. To make our approach plausible we briefly comment on certain properties of (2.21). The parallels with the case $B = 0$, $A < 0$ are clear. The function $y_0(x)$ is the unique solution of the Fredholm equation of the first kind

$$h(x) - h(0) - \int_0^1 \frac{h'(t)}{1-At}dt = A\int_0^x s y_0(s)ds + \int_x^1 y_0(s)ds. \tag{2.23}$$

We observe that (2.23) is the reduced equation (2.4) with additional terms on the LHS. These additional terms represent the contribution to the first integral on the RHS of (2.1) from the boundary-layer region. More precisely, it follows from (2.21) that

$$\begin{aligned} A\int_0^{\delta(\epsilon)} sw(s,\epsilon)ds \;&\sim\; A\int_0^{\delta(\epsilon)} s\left[\frac{e^{-s/\epsilon}}{A\epsilon^2}\left[h(0) + \int_0^1 \frac{h'(t)}{1-At}dt\right]\right]dt \\ &\sim\; h(0) + \int_0^1 \frac{h'(t)}{1-At}dt \quad \text{as} \quad \epsilon \to 0^+, \end{aligned} \tag{2.24}$$

provided that $\epsilon \ll \delta(\epsilon) \ll 1$ (cf. (2.16)). When the reduced problem (2.4) has no solution, it is crucial that the magnitude of the solution in the boundary layer be large so that the contribution to the integral in (2.1) from this region enters into the dominant balance for the outer function $y_0(x)$.

Case (iii): $B = -1$, $A > -1$ (boundary layer at $x = 1$)

For this case an asymptotic evaluation of (2.8)-(2.9) yields the following uniform leading-order approximation to $w(x,\epsilon)$ for $0 \leq x \leq 1$,

$$w(x,\epsilon) = \frac{h'(x)}{1+Ax} + O(\epsilon) - \frac{1+A}{\epsilon}\, e^{\frac{x-1}{\epsilon}(1+A)}\left[h(0) + \int_0^1 \frac{h'(t)}{1+At}dt + O(\epsilon)\right] \tag{2.25}$$

as $\epsilon \to 0^+$.

The form of (2.25) is that of an additive boundary-layer representation, with the boundary layer at $x = 1$ rather than at $x = 0$ (see Figure 2b). Moreover, the magnitude of the solution in the layer is $O(1/\epsilon)$ rather than $O(1/\epsilon^2)$ as in the preceding case. Still, the contribution to the integral in (2.1) from this region enters into the dominant balance for the outer solution. To leading order the outer solutions for cases **(ii)** and **(iii)** are essentially the same. In the next section we derive (2.25) by a perturbation approach.

Case (iv): $B = -1$, $A < -1$ (interior layer at $x = -\frac{1}{A}$)

The last two cases require somewhat more care in carrying out an asymptotic evaluation of (2.8)-(2.9). The solution does not possess an elementary additive decomposition as in cases *(ii)* and *(iii)*. In the present case the solution is slowly varying on $0 \leq x \leq 1$ except in an interior-layer region (of width $O(\sqrt{\epsilon})$) at $x = -1/A$ wherein it has the form of a large amplitude, $O(1/\sqrt{\epsilon})$, "spike" (see Figure 2c). Leading-order approximations to $w(x,\epsilon)$ in the outer region and in the interior layer are given by

$$w(x,\epsilon) \sim \begin{cases} \dfrac{h'(x)}{1+Ax}, & x \in [0,\ 1] \quad \text{with} \quad |x + \frac{1}{A}| \gg \sqrt{\epsilon}, \\ -\left[\sqrt{\dfrac{|A|}{2\pi\epsilon}} \left[h(0) + \Pr\displaystyle\int_0^1 \frac{h'(t)}{1+At}dt \right] \right. & \\ \left. + \dfrac{h'(-1/A)}{\sqrt{|A|\epsilon}} \displaystyle\int_0^{\sqrt{\frac{|A|}{\epsilon}}(x+\frac{1}{A})} e^{\frac{t^2}{2}} dt \right] e^{\frac{A}{2\epsilon}(x+\frac{1}{A})^2}, & x + \frac{1}{A} = O(\sqrt{\epsilon}), \end{cases} \tag{2.26}$$

as $\epsilon \to 0^+$. The indicated integral in (2.26) is to be interpreted as a principal-part integral. Again one can obtain a uniform leading order composite expansion analogous to (2.20), but we omit this result here.

Observe that the form of the outer solution agrees to leading order with that in Case *(iii)*, although here the outer solution is singular in the layer region. This nonintegrable singular behavior explains why problems with interior layers must be treated by alternative methods. We defer such a study to a subsequent work.

Case (v): $B = 1$, $A > 1$ (exponentially large solution)

For this case there are no layer regions, but rather the solution is exponentially large in $O(1)$ neighborhoods of the endpoints, and slowly varying in between these neighborhoods (see Figure 2d). The juxtaposition of these regions is a surprising property of the integral equation. Leading-order approximations to $w(x,\epsilon)$ in these three regions depend upon whether $1 < A < 2$ or $A > 2$.

If $1 < A < 2$, we find

$$w(x,\epsilon) \sim \begin{cases} \dfrac{h'(1/A)}{\epsilon^{3/2} A(A-1)} \sqrt{\dfrac{2\pi}{A}}\; e^{-\frac{1}{\epsilon}\left[1-\frac{A}{2}\right]} e^{\frac{A}{2\epsilon}\left[x-\frac{1}{A}\right]^2}, & 0 \le x < \dfrac{1}{A} - \sqrt{\dfrac{2-A}{A}}, \\ \dfrac{h'(x)}{Ax-1}, & \dfrac{1}{A} - \sqrt{\dfrac{2-A}{A}} < x < \dfrac{1}{A}, \\ -\, h'(1/A) \sqrt{\dfrac{2\pi}{\epsilon A}}\; e^{\frac{A}{2\epsilon}\left[x-\frac{1}{A}\right]^2}, & \dfrac{1}{A} < x \le 1, \end{cases} \tag{2.27}$$

while if $A > 2$, then

$$w(x,\epsilon) \sim \begin{cases} h'(1/A) \sqrt{\dfrac{2\pi}{\epsilon A}}\; e^{\frac{A}{2\epsilon}\left[x-\frac{1}{A}\right]^2}, & 0 \le x < \dfrac{1}{A}, \\ \dfrac{h'(x)}{Ax-1}, & \dfrac{1}{A} < x < \dfrac{1}{A} + \sqrt{\dfrac{A-2}{A}}, \\ (1-A)\, h'(\tfrac{1}{A}) \sqrt{2\pi\epsilon A}\; e^{-\frac{1}{\epsilon}\left[-1 + \frac{A}{2}\right]} e^{\frac{A}{2\epsilon}\left[x-\frac{1}{A}\right]^2}, & \dfrac{1}{A} + \sqrt{\dfrac{A-2}{A}} < x \le 1. \end{cases} \tag{2.28}$$

Of course, these results do not hold in small transition zones between the specified regions. Observe that the slowly-varying portion of the solution is identical to that in Case *(ii)* where $A < 1$. Also, within a factor of $1/\epsilon$, the magnitude of $w(x,\epsilon)$ is the same at $x = 0$ and $x = 1$.

We conclude this discussion by noting that if h is a constant, then the solution in this case differs dramatically from the description in (2.27)-(2.28). A uniform leading-order approximation to $w(x,\epsilon)$ for $0 \leq x \leq 1$ is given by

$$w(x,\epsilon) \sim \begin{cases} \dfrac{h}{A\epsilon^2}\, e^{-x/\epsilon}, & (h \text{ constant}, \; 1 < A < 2), \\ \\ \dfrac{h(A-1)}{\epsilon}\, e^{\frac{x-1}{\epsilon}(A-1)}, & (h \text{ constant}, \; A > 2), \end{cases} \tag{2.29}$$

as $\epsilon \to 0^+$. The layer structure described by (2.29) is highly unstable to deviations in h from a constant.

3. *Description of the Perturbation Method*

In this section we describe our adaptation of the additive boundary-layer (multivariable) technique to integral equations of the type (1.1). For simplicity we consider only a single equation, reserving the treatment of systems to following sections. In order to demonstrate the versatility of the technique, we place somewhat fewer restrictions on the behavior of the kernel here than in following sections. The presentation indicates clearly how to extend the basic ideas to an even broader class of problems.

The scalar kernel $K(x, s)$ is assumed to be smooth on $[0, 1]\times(0, 1)$ except for a jump discontinuity along the diagonal $x = s$. We allow $K(x, s)$ to have algebraic behavior as $s \to 0^+$ and as $s \to 1^-$, as prescribed by the following specified asymptotic relations,

$$K(x, s) \sim \begin{cases} A(x)s^{\beta_0} & \text{as} \quad s \to 0^+, \quad 0 < s < x \leq 1, \\ ax^{\alpha_0}s^{\beta_0} & \text{as} \quad x \to 0^+, \quad 0 < s < x, \\ B(x)\cdot(1-s)^{\beta_1} & \text{as} \quad s \to 1^-, \quad 0 \leq x < s < 1, \\ b\cdot(1-x)^{\alpha_1}(1-s)^{\beta_1} & \text{as} \quad x \to 1^-, \quad x < s < 1, \end{cases} \tag{3.1}$$

where $A(x)$ and $B(x)$ are smooth functions of x on $(0, 1]$ and $[0, 1)$, respectively, and α_0, α_1, β_0, β_1, a and b are real constants. It follows from (3.1) that $A(x)$ and $B(x)$ satisfy

$$A(x) \sim ax^{\alpha_0} \quad \text{as} \quad x \to 0^+ \quad \text{and} \quad B(x) \sim b(1-x)^{\alpha_1} \quad \text{as} \quad x \to 1^-. \tag{3.2}$$

Our perturbation ansatz is predicated on $w(x,\epsilon)$ having boundary layers at $x = 0$ and/or $x = 1$ outside of which it is a slowly-varying function. Specifically, we seek $w = w(x) = w(x,\epsilon)$ in the form

$$w(x,\epsilon) = w^{\dagger}(x,\epsilon) + \phi(\epsilon)\tilde{w}(\tilde{x},\epsilon) + \psi(\epsilon)\hat{w}(\hat{x},\epsilon), \tag{3.3}$$

with

$$\tilde{x} := \frac{x}{\mu(\epsilon)}, \quad \hat{x} := \frac{1-x}{\nu(\epsilon)}, \tag{3.4}$$

for a suitable outer approximation $w^{\dagger}$ and suitable boundary-layer correction terms $\tilde{w}$ and $\hat{w}$ depending on the respective boundary-layer variables $\tilde{x}$ and $\hat{x}$, for suitable width order functions μ and ν. The unknown order functions ϕ and ψ describe the magnitude of w in the layers ($\tilde{w}$ and $\hat{w}$ are $O(1)$ as $\epsilon \to 0$ with, respectively, $\tilde{x}$ and

$\hat{x}$ fixed). Moreover, the layer terms $\tilde{w}$ and $\hat{w}$ must satisfy the decay (matching) conditions

$$\tilde{w} \to 0 \quad \text{as} \quad \tilde{x} \to \infty \quad \text{and} \quad \hat{w} \to 0 \quad \text{as} \quad \hat{x} \to \infty. \tag{3.5}$$

Determination of the layer-width order functions μ and ν is also part of the problem. We shall be content with ascertaining the leading-order approximation for each of $w^{\dagger}$, $\tilde{w}$ and $\hat{w}$. An algorithm for constructing higher-order terms is presented in Section 6.

As regards the model equation discussed in Section 2, the form of (3.3) is general enough to include Cases *(i)* – *(iii)*, but not *(iv)* and *(v)*. A more sophisticated approach is necessary for the latter cases. It is reassuring that, in attempting to carry out (3.3) for Cases *(iv)* and *(v)*, we are alerted to the failure of the ansatz. No spurious solutions are generated.

Substitution of (3.3) into (1.1) leads to

$$\begin{aligned} &\int_0^1 K(x, s)w^{\dagger}(s)ds + \phi(\epsilon)[I_1(x,\epsilon) + I_2(x,\epsilon)] + \psi(\epsilon)[I_3(x,\epsilon) + I_4(x,\epsilon)] \\ &\qquad = h(x,\epsilon) + \epsilon[w^{\dagger}(x) + \phi(\epsilon)\tilde{w}(\tilde{x}) + \psi(\epsilon)\hat{w}(\hat{x})], \quad 0 \le x \le 1, \end{aligned} \tag{3.6}$$

where the integrals $I_1, \cdots, I_4$ are defined by

$$\begin{aligned} I_1(x,\epsilon) &= \int_0^x K(x,s)\tilde{w}(\tfrac{s}{\mu})ds, \qquad I_2(x,\epsilon) = \int_x^1 K(x,s)\tilde{w}(\tfrac{s}{\mu})ds, \\ I_3(x,\epsilon) &= \int_0^x K(x,s)\hat{w}(\tfrac{1-s}{\nu})ds, \qquad I_4(x,\epsilon) = \int_x^1 K(x,s)\hat{w}(\tfrac{1-s}{\nu})ds. \end{aligned} \tag{3.7}$$

An asymptotic analysis of (3.6) depends on the decay rates of the layer terms $\tilde{w}$ and $\hat{w}$. To keep the analysis relatively simple we assume that

$\tilde{w}$ and $\hat{w}$ decay rapidly enough so that certain improper integrals exist and so that certain asymptotic estimates hold. Of course, the validity of these assumptions can be checked a'posteriori in individual cases. Even in instances when these assumptions fail to hold, it is still often possible to carry out a modified asymptotic analysis along the lines presented here.

Our first step is to determine an equation for the outer function $w^{\dagger}(x)$. This is accomplished by carrying out the limit process $\epsilon \to 0^+$ with x fixed in (3.6), $0 < x < 1$. Each of the integrals in (3.7) must be evaluated separately. We can rewrite I_1 using (3.1) as

$$I_1(x,\epsilon) = \mu^{1+\beta_0} A(x)\left[c_0 - \int_{\tilde{X}}^{\infty} t^{\beta_0}\tilde{w}(t)dt\right] + \mu\int_0^{\tilde{X}}\left[K(x,\mu t) - A(x)(\mu t)^{\beta_0}\right]\tilde{w}(t)dt, \tag{3.8}$$

where

$$c_0 = \int_0^{\infty} t^{\beta_0}\tilde{w}(t)dt. \tag{3.9}$$

We assume that $\tilde{w}$ decays rapidly enough so that the integral in (3.9) exists. Otherwise, $I_1(x, \epsilon)$ must be evaluated differently. Similarly, we can rewrite I_4 as

$$I_4(x,\epsilon) = \nu^{1+\beta_1} B(x)\left[c_1 - \int_{\hat{X}}^{\infty} t^{\beta_1}\hat{w}(t)dt\right] + \nu\int_0^{\hat{X}}\left[K(x,1-\nu t) - B(x)(\nu t)^{\beta_1}\right]\hat{w}(t)dt, \tag{3.10}$$

where

$$c_1 = \int_0^\infty t^{\beta_1} \hat{w}(t)dt. \tag{3.11}$$

Because of the assumed rapid decay of $\tilde{w}$ and $\hat{w}$, I_2 and I_3 do not contribute to the dominant balance in (3.6) in the limit $\epsilon \to 0^+$ with fixed x, $0 < x < 1$ (in this limit $\tilde{x}$ and $\hat{x} \to \infty$). Using (3.1) we easily derive the asymptotic relations (all based on $\epsilon \to 0^+$ with $0 < x < 1$ fixed),

$$I_1(x,\epsilon) \sim \mu^{1+\beta_0} c_0 A(x), \qquad I_4(x,\epsilon) \sim \nu^{1+\beta_1} c_1 B(x),$$
$$I_2 << I_1, \quad I_3 << I_4. \tag{3.12}$$

Thus, the dominant balance in (3.6) for the outer limit is

$$\int_0^1 K(x, s)w^\dagger(s)ds - h_0(x) \sim -\mu^{1+\beta_0}\phi c_0 A(x) - \nu^{1+\beta_1}\psi c_1 B(x) \tag{3.13}$$

for $0 < x < 1$.

If A and B are linearly independent, then neither term on the RHS of (3.13) can exceed $O(1)$ in magnitude. On the other hand, in order for (3.13) to represent a solvable equation for $w^\dagger$, at least one of these two terms must be $O(1)$ but not $o(1)$. (We are assuming here that the reduced problem (1.2) has no solution — if it does, then our analysis would be modified accordingly, as indicated in Section 7.) Let us assume that

$$\mu^{1+\beta_0}\phi = 1, \quad \nu^{1+\beta_1}\psi = 1, \tag{3.14}$$

and replace the asymptotic relation (3.13) by the equation

$$\int_0^1 K(x,s)w^{\dagger}(s)ds \;-\; h_0(x) = -\,c_0A(x) - c_1B(x), \quad 0 < x < 1. \tag{3.15}$$

We contend that (3.15) represents the appropriate governing equation for the leading-order outer approximation $w^{\dagger}(x) = w^{\dagger}(x,0)$. The constants c_0 and c_1 are connected to the boundary-layer correction terms by (3.9) and (3.11), respectively. The reader can compare the development leading to (3.15) with our corresponding remarks on the outer solution for the model equation in Section 2.

We shall return to (3.15) after consideration of the inner or boundary-layer correction terms. With (3.15), the original equation (3.6) reduces to

$$\begin{aligned}\phi(\epsilon)[I_1(x,\epsilon) - c_0\mu^{1+\beta_0}A(x) + I_2(x,\epsilon)] + \psi(\epsilon)[I_3(x,\epsilon) - c_1\nu^{1+\beta_1}B(x)\\ + I_4(x,\epsilon)] \;=\; h(x,\epsilon) - h_0(x) + \epsilon[w^{\dagger}(x) + \phi\tilde{w}(\tilde{x}) + \psi\hat{w}(\hat{x})],\end{aligned} \tag{3.16}$$

for $0 \le x \le 1$.

First we focus on the layer at $x = 0$. This is accomplished by carrying out the limit process $\epsilon \to 0^+$ with $\tilde{x} = x/\mu(\epsilon)$ fixed, so that in this limit there will hold $x \to 0^+$ and $\hat{x} \to \infty$. From (3.1)-(3.2) and (3.7)-(3.8) the following asymptotic relations hold (all based on $\epsilon \to 0^+$ with $\tilde{x}$ fixed):

$$I_1(x,\epsilon) - c_0\mu^{1+\beta_0}A(x) \;\sim\; -\,\mu^{1+\alpha_0+\beta_0}a\tilde{x}^{\alpha_0}\int_{\tilde{x}}^{\infty} t^{\beta_0}\tilde{w}(t)dt,$$

$$I_2(x,\epsilon) = \mu\int_{\tilde{x}}^{\frac{1}{\mu}} K(\mu x,\mu t)\tilde{w}(t)dt \;\sim\; \mu K(0,0+)\int_{\tilde{x}}^{\infty}\tilde{w}(t)dt,$$

$$I_4(x,\epsilon) - c_1\nu^{1+\beta_1}B(x) \sim \nu\int_0^{\frac{1-\mu\tilde{x}}{\nu}} \left[K(\mu\tilde{x},1-\nu t) - B(\mu\tilde{x})(\nu t)^{\beta_1}\right]\hat{w}(t)dt$$
$$= o(\nu^{1+\beta_1}), \tag{3.17}$$

and $I_3(x, \epsilon) << I_4(x, \epsilon) - c_1\nu^{1+\beta_1}B(x)$, where

$$K(0,0^+) := \underset{\substack{s \to 0 \\ 0<x<s}}{\text{limit}}\ K(x,s). \tag{3.18}$$

For simplicity we assume that the limit in (3.18) is well-defined, although we could allow more general behavior.

With (3.17), the dominant balance in (3.16) for the inner limit at $x = 0$ becomes

$$-\mu^{1+\alpha_0+\beta_0}\phi a\tilde{x}^{\alpha_0}\int_{\tilde{x}}^{\infty} t^{\beta_0}\tilde{w}(t)dt + \mu\phi K(0,0+)\int_{\tilde{x}}^{\infty}\tilde{w}(t)dt$$
$$+ \nu\psi\int_0^{\frac{1-\mu\tilde{x}}{\nu}}\left[K(\mu\tilde{x},1-\nu t) - B(\mu\tilde{x})(\nu t)^{\beta_1}\right]\hat{w}(t)dt \sim \epsilon\phi\tilde{w}(\tilde{x}) \tag{3.19}$$

for $\tilde{x} > 0$. By a similar process we can deduce the dominant balance in (3.16) for the inner limit at $x = 1$; it is given by

$$-\nu^{1+\alpha_1+\beta_1}\psi b\hat{x}^{\alpha_1}\int_{\hat{x}}^{\infty} t^{\beta_1}\hat{w}(t)dt \quad + \nu\psi K(1,1-)\int_{\hat{x}}^{\infty}\hat{w}(t)dt$$

$$+\mu\phi\int_0^{\frac{1-\nu\hat{x}}{\mu}}\left[K(1-\nu\hat{x},\mu t) - A(1-\nu\hat{x})(\mu t)^{\beta_0}\right]\tilde{w}(t)dt \sim \epsilon\psi\hat{w}(\hat{x}) \tag{3.20}$$

for $\hat{x} > 0$, where $K(1,1^-) := \lim_{\substack{s \to 1 \\ s<x<1}} K(x, s)$. We observe that (3.19)-(3.20) represents a coupled system of asymptotic relations for $\tilde{w}$ and $\hat{w}$. Of course, the magnitudes of the individual terms depend on the specific values of the exponents α_0, α_1, β_0 and β_1, and as a consequence, the corresponding system of leading-order equations will often be much simpler.

It is interesting to note that for certain cases, all the terms in (3.19)-(3.20) have the same magnitude. For example, suppose that there hold

$$\beta_0 = -\alpha_0, \quad \beta_1 = -\alpha_1 \quad (\text{with } \alpha_0, \alpha_1 > 0), \tag{3.21}$$

and also, as an extension of (3.1),

$$K(x, s) \sim \begin{cases} A(x)s^{-\alpha_0}\left[1 + \gamma_0 s^{\alpha_1}\right] & \text{as } s \to 0^+, \quad 0 < s < x \leq 1, \\ B(x)(1-s)^{-\alpha_1}\left[1+\gamma_1(1-s)^{\alpha_0}\right] & \text{as } s \to 1^-, \quad 0 \leq x < s < 1, \end{cases} \tag{3.22}$$

for constants γ_0, γ_1. Then it follows from (3.14) and (3.19)-(3.22) that there hold

$$\mu = \nu = \epsilon, \quad \phi = \epsilon^{\alpha_0 - 1}, \quad \psi = \epsilon^{\alpha_1 - 1}, \tag{3.23}$$

and the leading-order equations for the inner functions $\tilde{w}$ and $\hat{w}$ become

$$-a\tilde{x}^{\alpha_0}\int_{\tilde{x}}^{\infty} t^{-\alpha_0}\tilde{w}(t)dt + K(0,0+)\int_{\tilde{x}}^{\infty}\tilde{w}(t)dt + \gamma_1 B(0)\int_0^{\infty} t^{\alpha_0-\alpha_1}\hat{w}(t)dt = \tilde{w}(\tilde{x}) \tag{3.24}$$

for $\tilde{x} > 0$, and

$$-b\hat{x}^{\alpha_1}\int_{\hat{x}}^{\infty} t^{-\alpha_1}\hat{w}(t)dt + K(1,1-)\int_{\hat{x}}^{\infty}\hat{w}(t)dt + \gamma_0 A(1)\int_0^{\infty} t^{\alpha_1-\alpha_0}\tilde{w}(t)dt = \hat{w}(\hat{x}) \tag{3.25}$$

for $\hat{x} > 0$. The system (3.24)-(3.25) is indeed coupled, but in a very special way. Differentiating each equation twice with respect to x yields two uncoupled second-order ordinary differential equations for $\tilde{w}$ and $\hat{w}$, subject to certain coupled boundary conditions. Further consideration of this particular example would lead us too far astray from the central purpose of this section.

It is worthwhile to investigate how our proposed asymptotic development applies to the model equation of Section 2, where the kernel $K(x,s)$ is given as

$$K(x, s) = \begin{cases} As & \text{for } s < x \\ B & \text{for } s > x, \end{cases} \tag{3.26}$$

with jump

$$J(x) = Ax - B \quad \text{for} \quad 0 \leq x \leq 1. \tag{3.27}$$

Comparing these results with (3.1), we have in this case

$$\alpha_0 = 0, \quad \beta_0 = 1, \quad \alpha_1 = \beta_1 = 0. \tag{3.28}$$

With these choices the dominant balances in (3.19)–(3.20) for the boundary-layer correction terms reduce to

$$-\mu^2 A\int_{\tilde{x}}^{\infty} t\tilde{w}(t)dt + \mu B\int_{\tilde{x}}^{\infty}\tilde{w}(t)dt \sim \epsilon\tilde{w}(\tilde{x}), \quad \tilde{x} \geq 0, \tag{3.29}$$

and

$$\nu J(1)\int_{\hat{x}}^{\infty}\hat{w}(t)dt \sim \epsilon\hat{w}(\hat{x}), \quad \hat{x} \geq 0. \tag{3.30}$$

Moreover, the equation for the leading-order outer approximation given in (3.15) takes the special form

$$A\int_0^x sw^{\dagger}(s)ds + B\int_x^1 w^{\dagger}(s)ds = h(x) - c_0A - c_1B \quad \text{for} \quad 0 < x < 1, \tag{3.31}$$

with

$$c_0 = \int_0^\infty t\tilde{w}(t)dt, \quad c_1 = \int_0^\infty \hat{w}(t)dt. \tag{3.32}$$

First, suppose that $B = 0$, so that our model equation is in fact a Volterra equation (case (i) in Section 2). It follows from (3.14), (3.28) and (3.29) that we should set

$$\mu = \sqrt{\epsilon}, \quad \phi = \frac{1}{\epsilon}, \quad . \tag{3.33}$$

with the leading-order equation for the initial-layer correction term $\tilde{w}$ given by

$$\tilde{w}(x) + A\int_{\tilde{x}}^\infty t\tilde{w}(t)dt = 0. \tag{3.34}$$

(We need not consider (3.30) because a boundary-layer correction at $x = 1$ is not relevant for this special case.) Examination of the exact solution (2.11) confirms the correctness of the scalings in (3.33) for the width and magnitude of the initial-layer correction term.

If $A < 0$, then (3.34) has a nontrivial solution given by

$$\tilde{w}(\tilde{x}) = \tilde{c}e^{A\tilde{x}^2/2}, \tag{3.35}$$

with $\tilde{c}$ an arbitrary constant. If $A > 0$, then (3.34) describes a rapid exponential growth in violation of our original assumption on the character of $\tilde{w}$. If $A < 0$, then the decay condition in (3.5) is satisfied. To determine $\tilde{c}$ in the latter case we turn to (3.31) with $B = 0$. According to (2.4)-(2.6) (with $h(x)$ there replaced by $h(x) - c_0A$), the resulting equation for $w^{\dagger}(x)$ has a solution, given uniquely by

$$w^{\dagger}(x) = \frac{h'(x)}{Ax}, \tag{3.36}$$

if and only if there holds

$$h(0) - c_0 A = 0. \tag{3.37}$$

Assuming $h(0) \neq 0$, the value of $\tilde{c}$ in (3.35) readily follows from (3.32) and (3.37) to be

$$\tilde{c} = -h(0). \tag{3.38}$$

In summary, for $B = 0$ our formal approach yields the potential leading-order approximation

$$w(x, \epsilon) \sim -\frac{h(0)}{\epsilon} e^{Ax^2/(2\epsilon)} + \frac{h'(x)}{Ax}. \tag{3.39}$$

Unfortunately, as pointed out in Section 2, this additive initial-layer representation is not uniformly valid for $x \geq 0$. The singular nature of the outer function for x near 0 provides a clear indication that our perturbation ansatz fails for this special case. The difficulty can be traced to the slow decay of the higher-order boundary-layer terms (see (2.12)-(2.13)). It is easy to obtain a uniform composite approximation in (2.20) by a straightforward modification of our ansatz which takes into account this slower decay.

Next, suppose $B \neq 0$. For small μ, the term with B dominates the term with A in (3.29). It follows from (3.14), (3.28), (3.29) and (3.30) that we should set (cf. (3.33))

$$\mu = \nu = \epsilon, \quad \phi = \frac{1}{\epsilon^2}, \quad \psi = \frac{1}{\epsilon} \tag{3.40}$$

with

$$\tilde{w}(\tilde{x}) + J(0)\int_{\tilde{x}}^{\infty}\tilde{w}(t)dt = 0 \tag{3.41}$$

replacing (3.34) as the leading-order equation for $\tilde{w}$.

Solving (3.30) (with $\nu = \epsilon$) and (3.41) we obtain

$$\tilde{w}(\tilde{x}) = \tilde{c}e^{J(0)\tilde{x}} \quad \text{for} \quad \tilde{x} \geq 0, \tag{3.42}$$

and

$$\hat{w}(\hat{x}) = \hat{c}e^{-J(1)\hat{x}} \quad \text{for} \quad \hat{x} \geq 0, \tag{3.43}$$

where $\tilde{c}$ and $\hat{c}$ are constants of integration. The decay conditions in (3.5) are satisfied if and only if $J(0) < 0$ and $J(1) > 0$. For example, if $J(x) < 0$ on $0 \leq x \leq 1$, then a boundary layer (BL) is possible at $x = 0$ but not at $x = 1$. Cases (ii) - (iv) in Section 2 encompass all the possibilities which we list as follows:

(ii) $J(0) < 0, \quad J(1) < 0$ (BL possible only at $x = 0$),

(iii) $J(0) > 0, \quad J(1) > 0$ (BL possible only at $x = 1$),

(3.44)

(iv) $J(0) > 0, \quad J(1) < 0$ (no BL's are possible),

(v) $J(0) < 0, \quad J(1) > 0$ (BL's are possible at both ends).

We discuss these cases in turn.

For case (ii) we must take $\hat{c} = 0$ in (3.43) which implies that $c_1 = 0$ in (3.31). The resulting equation for $w^{\dagger}(x)$ has a solution, given uniquely by

$$w^{\dagger}(x) = \frac{h'(x)}{J(x)} \quad \text{for} \quad 0 < x < 1, \tag{3.45}$$

if and only if (cf. (2.6))

$$h(0) - c_0 A = B\int_0^1 \frac{h'(s)}{J(s)} ds. \tag{3.46}$$

From (3.32), (3.37) and (3.46) we find

$$\tilde{c} = \frac{(J(0))^2}{A}\left[h(0) - B\int_0^1 \frac{h'(s)}{J(s)} ds\right]. \tag{3.47}$$

Putting these results together yields the following leading-order approximation to $w(x, \epsilon)$ for $0 \leq x \leq 1$,

$$w(x,\epsilon) \sim \frac{h'(x)}{J(x)} + e^{-Bx/\epsilon}\frac{B^2}{A\epsilon}\left[h(0) - B\int_0^1 \frac{h'(s)}{J(s)}ds\right], \tag{3.48}$$

which agrees with the exact result in (2.21). The perturbation method described in this section works for Case (ii).

It is straightforward to verify that our perturbation method works for Case (iii) as well. Case (iv) is another matter. The fact that boundary layers are impossible does not preclude the possibility of interior layers. Suppose that we modified (3.3) by adding an interior-layer term of the form

$$\chi(\epsilon)\bar{w}(\bar{x}), \tag{3.49}$$

with

$$\bar{x} := \frac{x-\xi}{\kappa(\epsilon)}, \qquad \xi \text{ fixed in } [0, 1], \tag{3.50}$$

for suitable functions χ and κ. An analysis similar to that leading to (3.19) yields the following differential equation for $\bar{w}$,

$$\bar{w}'(\bar{x}) - J(\xi)\bar{w}(\bar{x}) = 0. \tag{3.51}$$

The solution of (3.51) grows exponentially either for $\bar{x} \to -\infty$ or for $\bar{x} \to +\infty$. Thus, an interior layer is impossible unless $J(\xi) = 0$. But this is just what happens in Case (iv), resulting there in a large amplitude "spike" at $x = \xi \equiv \frac{B}{A}$.

A further indication of the significance of the zeros of the jump function $J(x)$ is given by (3.45). The outer function $w^{\dagger}(x)$ becomes unbounded where $J(x)$ vanishes. This singular behavior is a clear signal of the failure of our ansatz. A more refined treatment of problems with interior layers is under study.

Case (v) seems somewhat more ambiguous since boundary layers appear to be possible at both $x = 0$ and at $x = 1$. The signal that something is amiss again follows from consideration of (3.45) for $w^{\dagger}(x)$. From (3.44) it is clear that the jump J must change sign at a zero in $(0, 1)$. At such a point $w^{\dagger}$ is singular (if h' is nonzero there), and the ansatz fails. As in Case (iv), a more refined analysis is called for. A lesson to be learned from the exact solution in Case (v) is that a solution of an integral equation such as (1.1) need not exhibit boundary-layer behavior even though a formal analysis suggests that boundary layers are possible.

We conclude this section by relating the present results with our treatment of vector integral equations in the remainder of this work. Regarding the scalings in the multivariable expansion (3.3)-(3.4) it follows immediately from (3.14), (3.19) and (3.20) that, in general,

$$\mu = \nu = \epsilon, \quad \phi = \psi = \frac{1}{\epsilon} \tag{3.52}$$

for kernels $K(x, s)$ which are smooth on $[0, 1]\times[0, 1]$ (except for a jump discontinuity along the diagonal $x = s$) and which satisfy the additional condition that

$$K(x, 0) \neq 0, \quad K(x, 1) \neq 0 \qquad \text{for} \quad 0 \leq x \leq 1. \tag{3.53}$$

This result forms the basis for our choice of scalings in Section 6.

Also, our study of the model equation (2.1) reveals that the additive boundary-layer representation is appropriate provided the jump $J(x)$ does not vanish on $[0, 1]$ (cf. cases (ii) and (iii) in (3.44)). If $J(x)$ does vanish on [0, 1], then other phenomena such as interior "spikes" or regions of rapid exponential growth are possible. This result appears to be quite general. For example, it is not difficult to establish that the solution $w^{\dagger}$ of (3.15) can become singular at the zeros of $J(x)$. A generalization of the condition that $J(x) \neq 0$ on [0, 1] plays an important role in our analysis of vector integral eqsuations in Sections 5 and 6.

4. *Examples of Vector Systems*

Vector equations of the type (1.1) with a discontinuous kernel appear in several important areas, including the study of coupled systems of singularly perturbed Volterra integral equations. In addition, many problems involving differential equations and integrodifferential equations can be conveniently reformulated as a vector integral equation of the type (1.1). For example, results analogous to those discussed earlier for the scalar Dirichlet problem (1.14) when the coefficient $a(x)$ there is everywhere nonzero (cf. (1.15), (1.16)) hold also for a vector 2^{nd} order system of the type (1.14) for suitable matrix functions $a(x)$ and $b(x)$, for a given vector function $f(x)$, and for given vectors α_0, α_1; see Smith [1986a] for references. In this latter case the resulting equation (1.1) is a vector Fredholm equation.

Another example occurs for the scalar Dirichlet problem (1.14) in the case $a(x) \equiv 0$, in which case the problem can be written as

$$\epsilon^2 y'' + b(x)y = f(x), \qquad y(0,\epsilon) = \alpha_0, \quad y(1,\epsilon) = \alpha_1, \tag{4.1}$$

where ϵ has been replaced here by ϵ^2. In this case a direct integration shows that (4.1) can be replaced by an equivalent integral equation of the type (1.1) with $m = 2$, $w = \begin{bmatrix} y \\ \epsilon y' \end{bmatrix}$, and with vector function

$$h(x,\epsilon) := -\begin{bmatrix} \epsilon\alpha_0 \\ \epsilon^2(\alpha_1-\alpha_0) + \int_0^x sf(s)ds + \int_x^1 (s-1)f(s)ds \end{bmatrix}, \tag{4.2}$$

and matrix kernel function

$$K(x, s) := \begin{cases} \begin{bmatrix} 0 & 1 \\ -sb(s) & 0 \end{bmatrix} & \text{for } s < x, \\ \begin{bmatrix} 0 & 0 \\ (1-s)b(s) & 0 \end{bmatrix} & \text{for } s > x. \end{cases} \tag{4.3}$$

The jump (1.13) is given as

$$J(x) = \begin{bmatrix} 0 & 1 \\ -b(x) & 0 \end{bmatrix}. \tag{4.4}$$

Again, an analogous result holds for a vector 2^{nd} order problem of the type (4.1).

As another example, consider the first order system of differential equations

$$\epsilon\frac{dw}{dx} = A(x)w + f(x) \qquad \text{for} \quad 0 \le x \le 1, \tag{4.5}$$

subject to the coupled boundary condition (cf. pp. 3–4 of Smith [1985])

$$Lw(0, \epsilon) + Rw(1, \epsilon) = \gamma, \tag{4.6}$$

for an m-vector solution $w = w(x, \epsilon)$, for given smooth data functions $\underset{m\times m}{A} = A(x)$ and $\underset{m\times 1}{f} = f(x)$, and for given m×m boundary matrices L and

R, and given vector γ. The case $L = I_m$, $R = 0$ corresponds to the initial value problem, while the case $L = 0$, $R = I_m$ corresponds to the terminal value problem, and other cases yield various other boundary value problems. A direct integration of (4.5) shows that the problem (4.5)-(4.6) can be formulated equivalently as the system of integral equations (1.1) with

$$h(x, \epsilon) := -\epsilon M^{-1}\gamma - \int_0^x M^{-1}Lf(s)ds + \int_x^1 M^{-1}Rf(s)ds, \tag{4.7}$$

and

$$K(x, s) := \begin{cases} M^{-1}LA(s), & s < x \\ -M^{-1}RA(s), & s > x, \end{cases} \tag{4.8}$$

provided the matrix $M := L + R$ *is invertible*. Even if this matrix is singular, the problem can often be repackaged in various equivalent forms so that the present procedure can still be used to obtain an equivalent integral equation of the form (1.1). The jump matrix J of (1.13) is given here as

$$J(x) = A(x). \tag{4.9}$$

Finally, we show that an integrodifferential equation of the form

$$\epsilon^2 u'(x) + g(x, \epsilon) = \int_0^1 E(x, s)u(s)ds, \quad 0 \leq x \leq 1, \tag{4.10}$$

along with a suitable auxiliary condition (such as an initial condition, terminal condition, or some other boundary condition), can be reformulated as a vector integral equation. If (4.10) is a *scalar* integrodifferential equation, then the resulting integral equation (1.1) will be a two-dimensional *vector* equation. More generally if (4.10) is a first-order vector differential-integral equation for an m-dimensional vector solution, then the corresponding equation (1.1) will be a vector integral equation for a 2m-dimensional solution: the dimension of the problem doubles. We shall consider the vector case here because it is just as easy as the scalar case.

Hence we consider (4.10) for an m-dimensional solution function $u = u(x) = u(x,\epsilon)$, where g and E are vector and matrix data functions with appropriate compatible orders, and where the given kernel function $E = E(x, s)$ has a jump discontinuity $j = j(x)$ given as (cf. (1.13))

$$j(x) = E(x,x^-) - E(x, x^+). \tag{4.11}$$

Along with the (4.10) we also impose on u a boundary condition of the type (4.6),

$$Lu(0, \epsilon) + Ru(1, \epsilon) = \gamma, \tag{4.12}$$

where γ is a given vector, and the given data quantities L and R are $m \times m$ matrices. As with (4.5)-(4.6), so also here, the case $L = I_m$, $R = 0$ in (4.12) corresponds to the initial value problem for (4.10), while the case $L = 0$, $R = I_m$ corresponds to the terminal value problem, and other cases yield various other problems with spatially coupled boundary conditions.

We set

$$v(x) := \epsilon u'(x), \tag{4.13}$$

so that $u(x) = u(0) + (1/\epsilon)\int_0^x v$. This last result with (4.12) implies $(L + R)u(0) = \gamma - \frac{1}{\epsilon}\int_0^1 Rv(s)ds$, and then we find

$$\epsilon u(x) - \epsilon(L+R)^{-1}\gamma = \int_0^x (L+R)^{-1}Lv(s)ds - \int_x^1 (L+R)^{-1}Rv(s)ds, \tag{4.14}$$

provided the matrix $L + R$ is invertible. Also, from (4.10) and (4.13) we have

$$\epsilon v(x) + g(x, \epsilon) = \int_0^1 E(x, s)u(s)ds. \tag{4.15}$$

The equations (4.14) and (4.15) can be written in the form (1.1) with the earlier dimension m of (1.1) replaced now with $2m$, and with

$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \quad h(x,\epsilon) = \begin{bmatrix} -\epsilon(L+R)^{-1}\gamma \\ g(x,\epsilon) \end{bmatrix}, \tag{4.16}$$

and

$$K = \underset{2m\times 2m}{K(x, s)} = \begin{cases} \begin{bmatrix} 0 & (L+R)^{-1}L \\ E(x,s) & 0 \end{bmatrix} & \text{for } s < x, \\ \begin{bmatrix} 0 & -(L+R)^{-1}R \\ E(x,s) & 0 \end{bmatrix} & \text{for } s > x. \end{cases} \tag{4.17}$$

The jump matrix J of (1.13) becomes here

$$J = J(x) = \begin{bmatrix} 0 & I_m \\ j(x) & 0 \end{bmatrix}, \tag{4.18}$$

where *j(x)* is the jump in E(x, s) as in (4.11).

Similarly, we can reformulate higher-order differential-integral equations as vector integral equations with increased dimension. For exaample, a *second-order* differential-integral equation can be formulated as a vector integral equation with *triple* the dimension.

5. *A Class of Vector Equations with Solutions of Boundary-Layer Type*

We introduce a special class of vector integral equations (1.1) for which we are able to obtain rigorous results in a clear, precise manner. The ideas and techniques used here can be more broadly applied, but for simplicity we restrict consideration to the stated class of problems. The special class of vector equations (1.1) considered here is characterized by the requirement that the kernel $K = K^{m\times m}(x, s)$ is smooth on $[0, 1]\times[0, 1]$ except for a jump discontinuity along $x = s$, with smooth jump matrix $J = J(x)$ given by (1.13), subject to the basic assumption that this jump matrix is *uniformly invertible* (nonsingular). In fact, we make the stronger assumption that ***all eigenvalues*** $\lambda = \lambda(x)$ ***of*** J ***have nonzero real parts,*** with

$$|Re\ \lambda(x)| \geq \kappa_1 > 0 \tag{5.1}$$

uniformly for $0 \leq x \leq 1$, for some fixed $\kappa_1 > 0$. Hence each eigenvalue of J satisfies precisely one of the two inequalities $Re\ \lambda \leq -\kappa_1$ or $Re\ \lambda \geq +\kappa_1$, and the smoothness of J guarantees that no eigenvalue $\lambda(x)$ can switch from one of these inequalities to the other as x ranges over [0, 1]. The assumption (5.1) corresponds to the assumption of elliptic type in Èskin [1981]. We also generally assume that the matrix function $\bar{K}$ defined by (5.5) below is uniformly invertible, and the auxiliary equation (5.6) is assumed to be uniquely solvable, although these last assumptions are not always essential, as indicated later in the discussion following

(6.23). These assumptions lead to boundary-layer behavior of a particular type for the solution of (1.1). Eigenvalues of J with negative real part lead to boundary layers at the left endpoint $x = 0$, while eigenvalues with positive real part lead to layers at the right endpoint $x = 1$ (cf. example (5.12)-(5.13) below).

The smoothness of J and the assumption (5.1) imply that there is a smooth, invertible $m \times m$ matrix function $T(x)$ that will transform $J(x)$ into block diagonal form as (see Chang and Coppel [1969, p. 279]) $T(x)J(x)T^{-1}(x) = \mathrm{diag}[J_{neg}(x), J_{pos}(x)]$, where the matrix functions $J_{neg}(x)$ and $J_{pos}(x)$ take values respectively in $\mathbb{C}^{n \times n}$ and $\mathbb{C}^{p \times p}$ with $n + p = m$, and where the eigenvalues of $J_{neg}(x)$ and $J_{pos}(x)$ have negative and positive real parts respectively. The transformation

$$w^*(x) = T(x)w(x), \quad h^*(x) = T(x)h(x), \quad K^*(x,s) = T(x)K(x,s)T^{-1}(s) \tag{5.2}$$

can be applied to (1.1), yielding the transformed equation $\epsilon w^* + h^* = K^*w^*$, with corresponding jump matrix $J^*(x) := K^*(x, x^-) - K^*(x, x^+)$ which is in block diagonal form, $J^*(x) = \mathrm{diag}[J_{neg}(x), J_{pos}(x)]$. We assume that such a transformation has already been performed and then we drop the asterisks on w^*, h^* and K^* to lighten the notation. Hence, subject to our present assumptions, there is no loss in taking the jump matrix to be in the form

$$\overset{m\times m}{J(x)} = \mathrm{diag}\left[\overset{n\times n}{J_{neg}(x)}, \overset{p\times p}{J_{pos}(x)}\right], \quad \text{for} \quad 0 \le x \le 1, \quad m = n+p, \tag{5.3}$$

where all eigenvalues of J_{neg} and J_{pos} have negative and positive real parts respectively, as described earlier. The cases $m = n$ (with $p = 0$) and $m = p$ (with $n = 0$) are included.

We could dispense with taking J in the block diagonal form (5.3), as indicated by the example in Appendix A.2, and indeed in practice this may be a convenient approach in some cases. However, for a theoretical study it is convenient to use (5.3), and we shall do so here.

In view of (5.3) it is convenient to introduce the projection $\overset{m\times m}{\mathcal{P}}$ given as

$$\mathcal{P} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad I_m - \mathcal{P} = \begin{bmatrix} 0 & 0 \\ 0 & I_p \end{bmatrix}, \tag{5.4}$$

and we shall also need the matrix function $\bar{K}(x)$ defined as

$$\bar{K}(x) := K(x, 0)\mathcal{P} + K(x, 1)(I - \mathcal{P}) \quad \text{for} \quad 0 < x < 1. \tag{5.5}$$

We assume that $\bar{K}(x)$ is invertible, and the following auxiliary equation

$$J(x)w(x) + \phi(x) = - \int_0^1 \left\{ \bar{K}(x)\frac{d}{dx}[\bar{K}^{-1}(x)K(x, s)] \right\} w(s)\, ds \tag{5.6}$$

is assumed to be uniquely solvable for a solution w, for an arbitrary given (regular) m-vector function $\phi = \phi(x)$, where $J(x)$ is the jump matrix given by (1.13). This required assumption, that (5.6) must be solvable, will hold automatically if the invertible jump matrix J is large enough. For example, it is a consequence of a result of Banach [1922] that (5.6) is uniquely solvable if the matrix $J(x)$ is invertible and there holds

$$\sup_{\substack{0<s<1 \\ s\neq x}} \left| \bar{K}(x)\frac{d}{dx}[\bar{K}^{-1}(x)K(x, s)] \right| \leq \gamma\, |J(x)| \quad \text{for} \quad 0 \leq x \leq 1, \tag{5.7}$$

for some fixed $0 \leq \gamma < 1$, where $|\cdot|$ denotes any convenient matrix norm. In particular, (5.7) always holds in the piecewise constant case in which the kernel function K is componentwise piecewise constant for $x \neq s$.

If the original equation (1.1) is a Volterra equation, with $K(x, s) \equiv 0$ for $s > x$, and if $K(x, x^-) = J(x)$ satisfies the eigenvalue condition (5.1), then the matter of the invertibility of $\bar{K}$ is generally crucial for the resulting behavior of solutions. If $\bar{K}$ is invertible, then solutions will exhibit initial-layer behavior, while if $\bar{K}$ is *not* invertible, then solutions may exhibit rapid exponential growth, as illustrated earlier for the scalar example in Section 1 (cf. (1.10)).

Because of the decomposition (5.3), it is sometimes convenient in the general case to write (1.1) explicitly as a coupled system in terms of block components as

$$\epsilon u(x) + f(x,\epsilon) = \int_0^1 [A(x,s)u(s) + B(x,s)v(s)]ds,$$
$$\epsilon v(x) + g(x,\epsilon) = \int_0^1 [C(x,s)u(s) + D(x,s)v(s)]ds, \tag{5.8}$$

for suitable block components $\overset{n\times 1}{u}$, $\overset{p\times 1}{v}$ of $\overset{m\times 1}{w}$; components $\overset{n\times 1}{f}$, $\overset{p\times 1}{g}$ of $\overset{m\times 1}{h}$; and components $\overset{n\times n}{A}$, $\overset{n\times p}{B}$, $\overset{p\times n}{C}$, $\overset{p\times p}{D}$ of $\overset{m\times m}{K}$:

$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \quad h = \begin{bmatrix} f \\ g \end{bmatrix}, \quad \text{and} \quad K = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{5.9}$$

These components are characterized in terms of the projection $\mathcal{P}$ as $u =$

$\begin{bmatrix} u \\ 0 \end{bmatrix} = \mathcal{P}w, \quad v = \begin{bmatrix} v \\ 0 \end{bmatrix} = (I-\mathcal{P})w, \quad f = \begin{bmatrix} f \\ 0 \end{bmatrix} = \mathcal{P}h, \quad g = \begin{bmatrix} g \\ 0 \end{bmatrix} = (I-\mathcal{P})h, \quad A = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ $= \mathcal{P}K\mathcal{P}$, $B = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} = \mathcal{P}K(I-\mathcal{P})$, and so forth. For notational brevity here and elsewhere we sometimes identify a projected vector with its (generally lower dimensional) value restricted to the range of the projection, as $\mathcal{P}w = u$, $(I-\mathcal{P})w = v$, and so forth, where the intended meaning will always be clear from the context.

The jumps in A, B, C and D are given by (1.13), (5.3) and (5.9) as

$$A(x,x^-) - A(x,x^+) = J_{neg}(x), \qquad B(x,x^-) - B(x,x^+) = 0,$$
$$C(x,x^-) - C(x,x^+) = 0, \qquad D(x,x^-) - D(x,x^+) = J_{pos}(x), \tag{5.10}$$

so that, in particular, the off-diagonal blocks B and C of K are continuous. In fact we assume that $B(x, s)$ and $C(x, s)$ are as smooth as required on $[0, 1]\times[0, 1]$.

We assume that the forcing function $h = h(x, \epsilon) = \begin{bmatrix} f \\ g \end{bmatrix}$ in (1.1) (or (5.8)) is smooth and has an asymptotic representation of the type

$$h(x, \epsilon) = \begin{bmatrix} f(x,\epsilon) \\ g(x,\epsilon) \end{bmatrix} \sim \sum_{j=0}^{\infty} h_j(x)\epsilon^j \sim \sum_{j=0}^{\infty} \begin{bmatrix} f_j(x) \\ g_j(x) \end{bmatrix} \epsilon^j \quad \text{with}$$
$$h'(x, \epsilon) \sim \sum_{j=0}^{\infty} h_j'(x)\epsilon^j, \quad \text{for} \quad 0 \le x \le 1, \tag{5.11}$$

as $\epsilon \to 0^+$, for suitable smooth functions $h_j = h_j(x)$, $f_j(x)$, and $g_j(x)$. There is no difficulty in permitting the kernel K to depend similarly on ϵ, but for simplicity we take K to be independent of ϵ.

A simple example is given by the following piecewise constant kernel $(m = 2)$

$$K(x, s) := \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} & \text{for } s < x, \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} & \text{for } s > x, \end{cases} \tag{5.12}$$

with jump $J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, so that here J is already in the block form (5.3), with $n = p = 1$. The eigenvalues of J satisfy (5.1), with one eigenvalue having negative real part while the other has positive real part. The projection $\mathcal{P}$ of (5.4) is given here as $\mathcal{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and the matrix $\bar{K}$ of (5.5) is $\bar{K} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. All of the assumptions described in the first paragraph of this section are seen to hold. In this case the integral equation (1.1) can be easily solved exactly (by considering the related differential equation obtained by differentiating the integral equation), and we find boundary layers of amplitude $O(1/\epsilon)$ at both endpoints $x = 0$ and $x = 1$. Specifically, if we denote the two (scalar) components of w as u and v, with $w = \begin{bmatrix} u(x, \epsilon) \\ v(x, \epsilon) \end{bmatrix}$, then in this case $u(x, \epsilon)$ has a boundary layer at $x = 0$ while $v(x, \epsilon)$ has a layer at $x = 1$, with

$$\begin{aligned} u(x, \epsilon) &= \frac{1}{\epsilon} e^{-x/\epsilon}\left[g_0(0) + f_0(1) - f_0(0)\right] + O(1) \\ v(x, \epsilon) &= \frac{1}{\epsilon} e^{-(1-x)/\epsilon}\left[g_0(0) - g_0(1) + f_0(1)\right] + O(1) \end{aligned} \tag{5.13}$$

as $\epsilon \to 0^+$, uniformly for $0 \le x \le 1$, where f_0 and g_0 denote the components of $h_0(x) = \begin{bmatrix} f_0(x) \\ g_0(x) \end{bmatrix}$, and where outside the layer regions at $x = 0$ and $x = 1$, the $O(1)$ terms involve slowly varying functions (cf. (1.11)). Details are omitted.

Similarly, the earlier piecewise constant scalar case (1.5) can be easily extended to the general vector case, with A and B arbitrary constant m×m matrices. The integral equation (1.1) can be easily solved in this case (up to quadrature), and one generally finds boundary layers at both endpoints, subject to appropriate conditions as considered here. For example, there holds $J = A-B$, so the condition (5.1) requires that the matrix $A-B$ should be nonsingular. The condition (1.6) on the solvability of the reduced equation becomes

$$B(A - B)^{-1}h_0(1) = A(A - B)^{-1}h_0(0). \tag{5.14}$$

In particular, if $A-B$ is stable (all eigenvalues having negative real part) and if A is nonsingular, then $\bar{K}$ is invertible and the vector solution function $w(x,\epsilon)$ has a boundary layer only at $x = 0$, and there holds

$$w(x,\epsilon) = \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(A-B)x}(A-B)A^{-1}\left[B(A-B)^{-1}h(1,\epsilon) - A(A-B)^{-1}h(0,\epsilon)\right] + O(1) \tag{5.15}$$

as $\epsilon \to 0^+$, uniformly for $0 \leq x \leq 1$, where $e^{\frac{1}{\epsilon}(A-B)x}$ is the matrix exponential. Further details are omitted.

It is interesting to reconsider briefly several of the earlier examples of Section 4. For example, the scalar Dirichlet problem (4.1) leads to a vector integral equation (1.1) with jump matrix $J(x)$ given by (4.4). One sees directly that this $J(x)$ satisfies the condition (5.1) if the real coefficient $b(x)$ of the differential equation in (4.1) is *negative*,

$$b(x) < 0 \quad \text{for} \quad 0 \leq x \leq 1. \tag{5.16}$$

It is well known that this condition (5.16) guarantees the unique solvability of (4.1) as $\epsilon \to 0$, and the resulting solution generally exhibits boundary-layer behavior at both endpoints (cf. Smith [1985; Section 8.2]). As an aside we mention that the matrix

$$T(x) = \begin{bmatrix} 1 & -1/\sqrt{-b(x)} \\ 1 & +1/\sqrt{-b(x)} \end{bmatrix} \tag{5.17}$$

can be used in this case in (5.2) so as to diagonalize $J(x)$. The resulting (transformed) matrix $\bar{K}^*(x)$ corresponding to (5.5) is a singular matrix in this case, but a minor modification of the construction of Section 6 can be used successfully for the integral equation with data (4.2)-(4.3).

Turning to the boundary-value problem (4.5)-(4.6) for the first-order system of differential equations $\epsilon\frac{dw}{dx} = A(x)w + f(x)$, we have a vector integral equation of the type (1.1) with jump matrix given by (4.9) as $J(x) = A(x)$. In this case the condition (5.1) is satisfied if the given matrix function $A(x)$ is nonsingular with all eigenvalues λ satisfying (5.1). Similarly the condition on the invertibility of $\bar{K}$ and the condition on the solvability of the auxiliarly equation (5.6) become conditions here on the given matrix function $A(x)$.

In the case of the integrodifferential equation (4.10) subject to the boundary condition (4.12), we obtain a vector integral equation (1.1) with resulting jump matrix $J = \overset{2m\times 2m}{J(x)}$ given by (4.18) in terms of the jump $j = \overset{m\times m}{j(x)}$ associated with the kernel $E(x, s)$ of the given integrodifferential equation. The matrix $J(x)$ in (4.18) is invertible *if and only if* $j(x)$ in (4.11) is invertible. We can give direct sufficient conditions (in terms of the kernel E of the given integrodifferential) that will guarantee the validity of all of the conditions for the corresponding integral equation, stated earlier in terms of $J(x)$ and $\bar{K}(x)$.

For this purpose, for simplicity we assume that the $m\times m$ matrix $j(x)$ has a complete set of m linearly independent eigenvectors $z = z(x)$,

$$z_k = z_k(x) \quad \text{for} \quad k = 1,2,\cdots,m, \tag{5.18}$$

which can be taken to be smooth functions of x, with corresponding eigenvalues $\lambda = \lambda_k(x)$. No assumption is made concerning the multiplicities of the eigenvalues λ_k, where

$$j(x)z_k = \lambda_k z_k \qquad \text{(no sum on } k) \tag{5.19}$$

for $k = 1,2,\cdots,m$. We now transform the jump matrix $J(x)$ of (4.18) to block diagonal form, using (5.2) with transformation matrix $T = \overset{2m\times 2m}{T(x)}$ given as (cf. Smith [1985, p. 230; 1986b])

$$T = \frac{1}{2}\begin{bmatrix} Z^{-1} & -\Omega^{-1}Z^{-1} \\ Z^{-1} & \Omega^{-1}Z^{-1} \end{bmatrix}, \quad \text{with} \quad T^{-1} = \begin{bmatrix} Z & Z \\ -Z\Omega & Z\Omega \end{bmatrix}, \tag{5.20}$$

where $Z = \overset{m\times m}{Z(x)}$ is constructed with its columns given by the eigenvectors z_k of $j(x)$, while $\Omega = \Omega(x)$ is a diagonal matrix with diagonal elements given by square roots $\sqrt{\lambda_k}$ of corresponding eigenvalues, so that

$$j(x)Z(x) = Z(x)(\Omega(x))^2. \tag{5.21}$$

A direct calculation with (4.18) and (5.20)-(5.21) now gives

$$T(x)J(x)T^{-1}(x) = \begin{bmatrix} -\Omega(x) & 0 \\ 0 & +\Omega(x) \end{bmatrix}, \tag{5.22}$$

where the matrix on the right side here is in an appropriate block diagonal form of the type (5.3) if the eigenvalues of $j(x)$ satisfy a suitable condition. Further details are given later in Section 8.

Finally, it is interesting to reconsider briefly the scalar model equation of Section 2. In ***Case (i)*** of that model equation the jump (2.3) satisfies $J(x) < 0$ except at $x = 0$ where $J(0) = 0$, so that the condition (5.1) fails to hold, resulting (in the case $A < 0$) in a layer of width $O(\sqrt{\epsilon})$, rather than $O(\epsilon)$ as would be indicated by (6.1)-(6.2) below. In ***Case (ii)*** there holds $J(x) \leq -\kappa_1 < 0$ for $x \in [0, 1]$, with $\kappa_1 = \min\{1, 1-A\}$, so that (5.1) is satisfied. However, in this case the quantity $\bar{K}$ of (5.5) vanishes, $\bar{K}(x) \equiv 0$, and the resulting solution (2.21) has an $O(1/\epsilon^2)$ magnitude in the layer, instead of $O(1/\epsilon)$ as would be indicated by (6.1)-(6.2). In ***Case (iii)*** there holds $J(x) \geq \kappa_1 > 0$ for $x \in [0, 1]$, with $\kappa_1 = \min\{1, 1+A\}$, so that (5.1) is satisfied. In addition there holds $\bar{K}(x) \equiv -1$, and the equation (5.6) is uniquely solvable (with solution $w(x) = -\phi(x)/[1+Ax]$). In this case all of the assumptions outlined earlier in this section are seen to hold, and indeed the solution (2.25) is of the type (6.1)-(6.2). In ***Case (iv)*** there holds $J(x) = 1 + Ax$ with $A < -1$, so that $J(x_0) = 0$ for $x_0 = -(1/A) \in [0, 1]$, with $J(x) > 0$ for $x \in [0, x_0)$ and $J(x) < 0$ for $x \in (x_0, 1]$. The assumption (5.1) fails to hold, and indeed the behavior (6.1)-(6.2) also fails to hold because there is an interior layer of width $O(\sqrt{\epsilon})$ at x_0, as shown by (2.26). Finally, in ***Case (v)*** there holds $J(x) = -1 + Ax$ with $A > 1$, so that $J(x_0) = 0$ for $x_0 = 1/A$, with $J(x) < 0$ for $x \in [0, x_0)$ and $J(x) > 0$ for $x \in (x_0, 1]$. Hence (5.1) fails to hold, and rather than (6.1)-(6.2), one finds here the remarkable behavior indicated by (2.27) and (2.28).

6. Formal Construction for Vector Equations

Subject to the general assumptions introduced in Section 5, we seek a representation for the solution $w(x) = w(x, \epsilon)$ of (1.1) or (5.8) in the form (see (5.9))

$$w(x, \epsilon) = \begin{bmatrix} u(x,\epsilon) \\ v(x,\epsilon) \end{bmatrix} \sim \begin{bmatrix} y(x,\epsilon) \\ z(x,\epsilon) \end{bmatrix} + \frac{1}{\epsilon}\begin{bmatrix} \eta(\tilde{x},\epsilon) \\ \zeta(\hat{x},\epsilon) \end{bmatrix} \tag{6.1}$$

as $\epsilon \to 0^+$, with boundary-layer variables

$$\tilde{x} := x/\epsilon \quad \text{and} \quad \hat{x} := (1-x)/\epsilon, \tag{6.2}$$

and for suitable outer solution functions $y(x, \epsilon)$, $z(x, \epsilon)$ and suitable boundary-layer corrections η, ζ, given asymptotically as

$$\begin{bmatrix} y(x, \epsilon) \\ z(x, \epsilon) \end{bmatrix} \sim \sum_{j=0}^{\infty} \begin{bmatrix} y_j(x) \\ z_j(x) \end{bmatrix} \epsilon^j \quad \text{and} \quad \begin{bmatrix} \eta(\tilde{x},\epsilon) \\ \zeta(\hat{x},\epsilon) \end{bmatrix} \sim \sum_{j=0}^{\infty} \begin{bmatrix} \eta_j(\tilde{x}) \\ \zeta_j(\hat{x}) \end{bmatrix} \epsilon^j, \tag{6.3}$$

for suitable functions y_j, z_j, η_j, and ζ_j that will be determined below in this section. Our anticipated scalings in (6.1)-(6.2) are based on the considerations discussed in Section 3. It is important to note that if the general assumptions in Section 5 fail to hold, then it still may be possible to represent the solution in an additive multivariable form similar to (6.1)-(6.2), but with different solution magnitudes in the layers and different layer widths; see Section 3 and the final paragraph of Section 5.

In order now to determine the functions y_j, z_j, η_j and ζ_j, we insert (5.11), (6.1) and (6.3) into (1.1) and find

$$\sum_{j=0}^{\infty} \left\{ \begin{bmatrix} y_{j-1}(x) \\ z_{j-1}(x) \end{bmatrix} + \begin{bmatrix} f_j(x) \\ g_j(x) \end{bmatrix} - \int_0^1 K(x,s) \begin{bmatrix} y_j(s) \\ z_j(s) \end{bmatrix} ds \right\} \epsilon^j$$

$$= \sum_{j=0}^{\infty} \left\{ - \begin{bmatrix} \eta_j(\tilde{x}) \\ \zeta_j(\hat{x}) \end{bmatrix} + \frac{1}{\epsilon} \int_0^1 K(x,s) \begin{bmatrix} \eta_j(\tilde{s}) \\ \zeta_j(\hat{s}) \end{bmatrix} ds \right\} \epsilon^j, \tag{6.4}$$

where $y_{-1} \equiv 0$, $z_{-1} \equiv 0$, and in the integral on the right side there holds (following (6.2)) $\tilde{s} := s/\epsilon$ and $\hat{s} := (1-s)/\epsilon$.

We now examine the integral on the right side of (6.4), given as (see (5.9))

$$\frac{1}{\epsilon}\int_0^1 K(x,s)\begin{bmatrix}\eta_j(\tilde{s})\\ \zeta_j(\hat{s})\end{bmatrix}ds = \frac{1}{\epsilon}\int_0^1\begin{bmatrix}A(x,s)\,\eta_j(\tilde{s}) + B(x,s)\,\zeta_j(\hat{s})\\ C(x,s)\,\eta_j(\tilde{s}) + D(x,s)\,\zeta_j(\hat{s})\end{bmatrix}ds, \tag{6.5}$$

where we shall examine the various terms separately on the right side here. Consider first the term involving B,

$$\frac{1}{\epsilon}\int_0^1 B(x,s)\,\zeta_j(\hat{s})\;ds \underset{(s=1-\epsilon\hat{s})}{=} \int_0^{1/\epsilon} B(x,1-\epsilon\hat{s})\,\zeta_j(\hat{s})\;d\hat{s}. \tag{6.6}$$

We Taylor expand $B(x, 1-\epsilon\hat{s})$ with respect to ϵ about $\epsilon = 0$, and find by a routine calculation from (6.6),

$$\frac{1}{\epsilon}\int_0^1 B(x,s)\,\zeta_j(\hat{s})\;ds \sim \sum_{k=0}^{\infty}\frac{\partial_s^k B(x,1)}{k!}\left[\int_0^{\infty}\zeta_j(\hat{s})(-\hat{s})^k d\hat{s}\right]\epsilon^k, \tag{6.7}$$

where $\partial_s^k B(x,1) \equiv \partial^k B(x,s)/\partial s^k\big|_{s=1}$, and where here and below we replace various integrals $\int_0^{1/\epsilon}\cdot\,d\hat{s}$ by corresponding (asymptotically equivalent) integrals $\int_0^{\infty}\cdot\,d\hat{s}$ because of the expected exponential decay of ζ_j, as justified later by the error estimate (6.25). Similarly, using $s = \epsilon\tilde{s}$, we find

$$\frac{1}{\epsilon}\int_0^1 C(x,s)\eta_j(\tilde{s})\,ds \sim \sum_{k=0}^{\infty} \frac{\partial_s^k C(x,0)}{k!}\left[\int_0^{\infty}\eta_j(\tilde{s})\tilde{s}^k d\tilde{s}\right]\epsilon^k, \tag{6.8}$$

where $\partial_s^k C(x,0) \equiv \partial^k C(x,s)/\partial s^k\big|_{s=0}$.

The terms in (6.5) involving A and D must be handled slightly differently because of the jump discontinuities exhibited by these functions. For the term involving A we write

$$\begin{aligned}\frac{1}{\epsilon}\int_0^1 A(x,s)\eta_j(\tilde{s})\,ds &\sim \sum_{k=0}^{\infty} \frac{\partial_s^k A(x,0)}{k!}\left[\int_0^{1/\epsilon}\eta_j(\tilde{s})\tilde{s}^k d\tilde{s}\right]\epsilon^k \\ &+ \int_0^{1/\epsilon}\tilde{A}(\epsilon,\tilde{x},\tilde{s})\eta_j(\tilde{s})d\tilde{s},\end{aligned} \tag{6.9}$$

where

$$\tilde{A}(\epsilon,\ \tilde{x},\ \tilde{s}) := A(\epsilon\tilde{x},\ \epsilon\tilde{s}) - \sum_{k=0}^{\infty} \frac{\partial_s^k A(\epsilon\tilde{x},0)}{k!}\,\tilde{s}^k\epsilon^k, \tag{6.10}$$

and where (6.9) is simply an identity obtained by adding and subtracting the first summation on the right side there, and we have also used $x = \epsilon\tilde{x}$ and $s = \epsilon\tilde{s}$ in the last term on the right side of (6.9). Because of the jump in A, we write the last term on the right side of (6.9) as

$$\int_0^{1/\epsilon} \tilde{A}(\epsilon,\tilde{x},\tilde{s})\eta_j(\tilde{s})d\tilde{s} = \int_0^{\tilde{x}} \tilde{A}(\epsilon,\tilde{x},\tilde{s})\eta_j(\tilde{s})d\tilde{s} + \int_{\tilde{x}}^{1/\epsilon} \tilde{A}(\epsilon,\tilde{x},\tilde{s})\eta_j(\tilde{s})d\tilde{s}, \tag{6.11}$$

and we Taylor expand $\tilde{A}(\epsilon,\tilde{x},\tilde{s})$ in ϵ about $\epsilon = 0$, separately for each of the two integrals on the right side of (6.11). From (6.10) we find $\frac{\partial^k}{\partial\epsilon^k} \tilde{A}(\epsilon,\tilde{x},\tilde{s})\big|_{\epsilon=0+} = 0$ for $0 < \tilde{s} < \tilde{x}$, while $\frac{\partial^k}{\partial\epsilon^k} \tilde{A}(\epsilon,\tilde{x},\tilde{s})\big|_{\epsilon=0+} = (\tilde{x}\partial_x + \tilde{s}\partial_s)^k\left[A(x,s)\big|_{(0,0+)} - A(x,s)\big|_{(0,0-)}\right]$ for $\tilde{s} > \tilde{x}$. Hence we find from (6.9), (6.10) and (6.11),

$$\frac{1}{\epsilon}\int_0^1 A(x,s)\eta_j(\tilde{s})\, ds \sim \sum_{k=0}^{\infty} \frac{\partial_s^k A(x,0)}{k!}\left[\int_0^{\infty}\eta_j(\sigma)\sigma^k d\sigma\right]\epsilon^k$$
$$+ \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\int_{\tilde{x}}^{\infty}\left[(\tilde{x}\partial_x + \sigma\partial_s)^k\left[A(0,0+)-A(0,0-)\right]\right]\eta_j(\sigma)d\sigma, \tag{6.12}$$

where $\partial_x = \partial/\partial x$, $\partial_s = \partial/\partial s$, $(\tilde{x}\partial_x+\sigma\partial_s)^k A(0,0+) := \left[(\tilde{x}\partial_x+\sigma\partial_s)^k A(x,s)\right]_{(x,s)=(0,0+)}$, and so forth. Similarly we have

$$\frac{1}{\epsilon}\int_0^1 D(x,s)\zeta_j(\hat{s})\, ds \sim \sum_{k=0}^{\infty} \frac{\partial_s^k D(x,1)}{k!}\left[\int_0^{\infty}\zeta_j(\sigma)(-\sigma)^k d\sigma\right]\epsilon^k$$
$$+ \sum_{k=0}^{\infty} \frac{(-\epsilon)^k}{k!}\int_{\hat{x}}^{\infty}\left[(\hat{x}\partial_x + \sigma\partial_s)^k\left[D(1,1-)-D(1,1+)\right]\right]\zeta_j(\sigma)d\sigma, \tag{6.13}$$

where $(\hat{x}\partial_x+\sigma\partial_s)^k D(1,1-) := \left[(\hat{x}\partial_x+\sigma\partial_s)^k D(x,s)\right]_{(x,s)=(1,1-)}$, and so forth.

We now insert (6.7), (6.8), (6.12) and (6.13) into (6.5) and in this way we are lead with (6.4) (after interchanging several orders of repeated summations) to the following conditions for the outer functions (for $j = 0,1,2,\cdots$)

$$\begin{bmatrix} y_{j-1}(x) \\ z_{j-1}(x) \end{bmatrix} + \begin{bmatrix} f_j(x) \\ g_j(x) \end{bmatrix} - \int_0^1 K(x,s) \begin{bmatrix} y_j(s) \\ z_j(s) \end{bmatrix} ds$$

$$= \sum_{k=0}^{j} \frac{1}{(j-k)!} \bar{K}_{j-k}(x) \int_0^\infty \begin{bmatrix} \eta_k(\sigma) \\ (-1)^{j-k}\zeta_k(\sigma) \end{bmatrix} \sigma^{j-k} d\sigma \qquad (6.14)_j$$

for $0 \leq x \leq 1$, where the matrix function $\bar{K}_i(x)$ is defined (for $i = 0,1,2,\cdots$) as

$$\begin{aligned} \bar{K}_i(x) &:= \frac{\partial^i}{\partial s^i}\left[K(x,0)\mathcal{P} + K(x,1)(I_m-\mathcal{P})\right] \\ &\equiv \left[\frac{\partial^i K(x,s)}{\partial s^i}\right]\Big|_{s=0}\mathcal{P} + \left[\frac{\partial^i K(x,s)}{\partial s^i}\right]\Big|_{s=1}(I_m-\mathcal{P}) \qquad (6.15)_i \\ &= \begin{bmatrix} \partial_s^i A(x,0) & \partial_s^i B(x,1) \\ \partial_s^i C(x,0) & \partial_s^i D(x,1) \end{bmatrix}; \end{aligned}$$

while for the boundary-layer correction terms we have similarly the conditions (recall (5.10))

$$\eta_j(\tilde{x}) + J_{neg}(0)\int_{\tilde{x}}^{\infty} \eta_j(\sigma)d\sigma$$

$$= -\sum_{k=0}^{j-1} \frac{1}{(j-k)!} \int_{\tilde{x}}^{\infty}\Big[(\tilde{x}\partial_x + \sigma\partial_s)^{j-k}(A(0,0-)-A(0,0+))\Big]\eta_k(\sigma)d\sigma \qquad (6.16)_j$$

for $\tilde{x} \geq 0$, and

$$\zeta_j(\hat{x}) - J_{pos}(1)\int_{\hat{x}}^{\infty} \zeta_j(\sigma)d\sigma$$

$$= \sum_{k=0}^{j-1} \frac{(-1)^{j-k}}{(j-k)!} \int_{\hat{x}}^{\infty}\Big[(\hat{x}\partial_x + \sigma\partial_s)^{j-k}(D(1,1-)-D(1,1+))\Big]\zeta_k(\sigma)d\sigma \qquad (6.17)_j$$

for $\hat{x} \geq 0$, where the summations are put equal to zero on the right sides of $(6.16)_0$ and $(6.17)_0$ in the case $j = 0$, and where the terms in square brackets in the integrals on the right sides are defined as earlier in the lines immediately following (6.12) and (6.13). We now show that the conditions $(6.14)_j$, $(6.16)_j$, and $(6.17)_j$ can be used recursively to determine the functions $y_j(x)$, $z_j(x)$, $\eta_j(\tilde{x})$ and $\zeta_j(\hat{x})$ for $j = 0,1,\cdots$.

First, $(6.14)_0$ implies (note that $y_{-1} \equiv 0$ and $z_{-1} \equiv 0$, while $(6.15)_0$ and (5.5) yield $\bar{K}_0 = \bar{K}$)

$$\begin{bmatrix} f_0(x) \\ g_0(x) \end{bmatrix} - \int_0^1 K(x,s)\begin{bmatrix} y_0(s) \\ z_0(s) \end{bmatrix} ds = \bar{K}(x)\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} \qquad (6.18)$$

for suitable constant vectors α_0 and β_0 given as

$$\alpha_0 = \int_0^\infty \eta_0 \quad \text{and} \quad \beta_0 = \int_0^\infty \zeta_0. \tag{6.19}$$

We now show that the equation (6.18) always has one and only one solution, for suitable values of α_0 and β_0.

To this end, differentiate (6.18) and find the following equation of second kind,

$$\begin{bmatrix} f_0'(x) \\ g_0'(x) \end{bmatrix} - J(x)\begin{bmatrix} y_0(x) \\ z_0(x) \end{bmatrix} - \int_0^1 K_x(x,s)\begin{bmatrix} y_0(s) \\ z_0(s) \end{bmatrix} ds = \bar{K}_x(x)\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}. \tag{6.20}$$

where the jump $J(x)$ is given by (1.13). The matrix function $\bar{K}(x)$ is assumed to be invertible, and so we can solve (6.18) for (α_0, β_0), and the resulting values can be used to eliminate α_0 and β_0 in (6.20). In this way we find that the leading outer functions $y_0(x)$ and $z_0(x)$ satisfy

$$J(x)\begin{bmatrix} y_0(x) \\ z_0(x) \end{bmatrix} - \bar{K}(x)\frac{d}{dx}\left[\bar{K}^{-1}(x)\begin{bmatrix} f_0(x) \\ g_0(x) \end{bmatrix}\right] = -\int_0^1 \left\{\bar{K}(x)\frac{d}{dx}[\bar{K}^{-1}(x)K(x,s)]\right\}\begin{bmatrix} y_0(s) \\ z_0(s) \end{bmatrix} ds, \tag{6.21}$$

which shows that $(y_0(x), z_0(x)) = w(x)$ satisfies (5.6) with $\phi(x) = -\bar{K}(x)\frac{d}{dx}\left[\bar{K}^{-1}(x)\begin{bmatrix} f_0(x) \\ g_0(x) \end{bmatrix}\right]$. In particular $y_0(x)$ and $z_0(x)$ are uniquely determined by (6.21) because (5.6) is assumed to be uniquely solvable. Moreover, a direct calculation shows that the resulting functions y_0 and z_0 determined by (6.21) satisfy

$$\frac{d}{dx}\left\{\bar{K}^{-1}(x)\left[\begin{Bmatrix} f_0(x) \\ g_0(x) \end{Bmatrix} - \int_0^1 K(x,s)\begin{Bmatrix} y_0(s) \\ z_0(s) \end{Bmatrix} ds\right]\right\} \equiv 0 \quad \text{for} \quad 0 \leq x \leq 1, \qquad (6.22)$$

so that these functions y_0, z_0 automatically satisfy (6.18), for suitable integration constants α_0, β_0 which are determined now by simply evaluating (6.18) at any fixed x, yielding

$$\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} = \bar{K}^{-1}(x_0)\left[\begin{Bmatrix} f_0(x_0) \\ g_0(x_0) \end{Bmatrix} - \int_0^1 K(x_0,\ s)\begin{Bmatrix} y_0(s) \\ z_0(s) \end{Bmatrix} ds\right] \qquad (6.23)$$

for any $x_0 \in [0, 1]$. This construction is based on the assumptions that $\bar{K}$ is invertible and (5.6) is uniquely solvable. We use these assumptions to obtain suitable outer functions y_0 and z_0 from equation (6.18). These assumptions are sufficient (along with (5.1)), _but not necessary,_ for the validity of our construction. Indeed, in some cases (6.18) is solvable for certain data with singular $\bar{K}$, as occurs in the important example (4.2)-(4.3). In such cases a minor modification of our construction carries *through for such a singular $\bar{K}$. [Note that the key lemma given in Appendix* A.4 makes no reference to (5.6) and does not require $\bar{K}$ to be invertible.]

Turning now to the determination of the boundary-layer correction terms η_0 and ζ_0, we find directly from $(6.16)_0$, $(6.17)_0$ and (6.19) the results,

$$\eta_0(\tilde{x}) = -e^{J_{neg}(0)\tilde{x}} J_{neg}(0)\alpha_0 \quad \text{and} \quad \zeta_0(\hat{x}) = e^{-J_{pos}(1)\hat{x}} J_{pos}(1)\beta_0, \qquad (6.24)$$

for suitable constants of integration α_0 and β_0. We select α_0 and β_0 in accordance with (6.23), and then one sees directly that the resulting functions $y_0(x)$, $z_0(x)$, $\eta_0(\tilde{x})$, and $\zeta_0(\hat{x})$ constructed here do indeed satisfy the required conditions $(6.14)_0$, $(6.16)_0$, and $(6.17)_0$. Finally, it follows directly from (6.24), along with the negativity of the real parts of the eigenvalues of $J_{neg}(0)$ and $-J_{pos}(1)$, that the boundary-layer correction functions η_0 and ζ_0 decay exponentially toward zero with increasing positive values of their arguments $\tilde{x}$ and $\hat{x}$.

The procedure can be continued recursively so as to provide y_j, z_j, η_j, and ζ_j for as many $j = 0,1,\cdots$ as permitted by the smoothness of the data. Details are omitted. The boundary-layer correction terms η_j and ζ_j will satisfy estimates of the form

$$|\eta_j(\sigma)| \leq C_j e^{-\tilde{\kappa}\sigma} \quad \text{and} \quad |\zeta_j(\sigma)| \leq C_j e^{-\hat{\kappa}\sigma} \quad \text{for all} \quad \sigma \geq 0, \qquad (6.25)_j$$

for $j = 0,1,\cdots$, for suitable constants C_j, for any fixed positive constant $\tilde{\kappa}$ satisfying $Re\,\tilde{\lambda} < -\tilde{\kappa} < 0$ for all eigenvalues $\tilde{\lambda}$ of $J_{neg}(x)$ at $x = 0$, and for any fixed positive constant $\hat{\kappa}$ satisfying $Re\,\hat{\lambda} > \hat{\kappa} > 0$ for all eigenvalues $\hat{\lambda}$ of $J_{pos}(x)$ at $x = 1$.

7. Existence, Uniqueness, and Error Estimates

Subject to the conditions of Section 5, we assume without loss here as in Section 5 that the jump matrix J is in a diagonal block form as in (5.3). We rewrite (1.1) in terms of the function W given as

$$W \equiv \begin{bmatrix} U \\ V \end{bmatrix} = W(x, \epsilon) := w(x, \epsilon) - w_N(x, \epsilon), \qquad (7.1)$$

where we suppress the obvious dependence of W on N, and where w_N

$= w_N(x, \epsilon)$ is the proposed approximate solution defined as

$$w_N(x, \epsilon) := \sum_{j=0}^{N} \left[\begin{bmatrix} y_j(x) \\ z_j(x) \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} \eta_j(x/\epsilon) \\ \zeta_j((1-x)/\epsilon) \end{bmatrix} \right] \epsilon^j \qquad (7.2)_N$$

in terms of the functions y_j, z_j, η_j and ζ_j constructed in Section 6. A direct calculation based on the results of Section 6 shows that w_N satisfies (1.1) approximately, in the sense that the resulting residual ρ_N is small, where $\rho_N = \rho_N(x, \epsilon)$ is defined as

$$\rho_N(x,\epsilon) := \epsilon w_N(x,\epsilon) + h(x,\epsilon) - \int_0^1 K(x,s) w_N(s,\epsilon) ds \quad \text{for} \quad 0 \le x \le 1. \qquad (7.3)$$

In fact a direct calculation as indicated in Appendix A.3 shows that ρ_N satisfies the estimates

$$\begin{aligned} &\rho_N(x,\epsilon) = O(\epsilon^{N+1}) \text{ and } \rho_N'(x,\epsilon) = O(\epsilon^N), \text{ uniformly for } 0 \le x \le 1, \\ &\text{and also } \int_0^1 |\rho_N'(x,\epsilon)| \, dx = O(\epsilon^{N+1}), \quad \text{as } \epsilon \to 0+. \end{aligned} \qquad (7.4)$$

It follows from (1.1), (7.1) and (7.3) that the function W satisfies the equation

$$\epsilon W(x,\epsilon) + \rho_N(x,\epsilon) = \int_0^1 K(x,s) W(s,\epsilon) ds. \qquad (7.5)$$

We assume that the jump J of (1.13) is large enough so that the condition (A.4.7) of Appendix A.4 holds. If the integer N satisfies $N \ge 1$, it then follows directly from (7.4) and (7.5) along with the lemma of Appendix A.4 (with w and h replaced respectively by W and ρ_N in that lemma) that (7.5) has one and only one solution W for all small $\epsilon > 0$, and this solution satisfies $W(x, \epsilon) = O(\epsilon^N)$ as $\epsilon \to 0^+$, uniformly for $0 \leq x \leq 1$. Hence, in terms of w in (7.1), we find that (1.1) has precisely one solution, and this solution satisfies

$$w(x, \epsilon) = w_N(x, \epsilon) + O(\epsilon^N) \quad \text{as} \quad \epsilon \to 0^+, \qquad (7.6)_N$$

uniformly for $0 \leq x \leq 1$. In this way we are lead to the following theorem.

Theorem 7.1. Let the forcing function h be of class $C^{N+2}([0, 1])$ and possess asymptotic expansions of the type (5.11), at least through order $N+2$, for some $N \geq 0$. Let the kernel $K(x, s)$ be of class $C^{N+2}([0, 1]\times[0, 1])$ except for a jump discontinuity along $x = s$ as in (1.13) and (5.3), and let the off-diagonal blocks B and C of K (see (5.9)) be smooth (of class C^{N+2}) without any jumps. Assume that $J(x)$ satisfies (5.1) and (5.3), assume that the jump J is large enough so that (A.4.7) holds, assume that $\bar{K}(x)$ of (5.5) is invertible, and assume that the auxiliary equation (5.6) is uniquely solvable (for every forcing function ϕ). Then there is a fixed number $\epsilon_0 > 0$ such that (1.1) has precisely one solution $w = w(x, \epsilon)$ for $0 < \epsilon \leq \epsilon_0$, and this solution satisfies

$$w(x,\epsilon) = \frac{1}{\epsilon}\begin{bmatrix} \eta_0(x/\epsilon) \\ \zeta_0((1-x)/\epsilon) \end{bmatrix} + \sum_{j=0}^{N}\begin{bmatrix} y_j(x) + \eta_{j+1}(x/\epsilon) \\ z_j(x) + \zeta_{j+1}((1-x)/\epsilon) \end{bmatrix}\epsilon^j + O(\epsilon^{N+1}) \qquad (7.7)$$

uniformly for $(x,\epsilon)\in[0, 1]\times(0, \epsilon_0]$, where y_j, z_j, η_j, ζ_j are the functions constructed in Section 6, and where the boundary-layer correction terms η_j, ζ_j satisfy the estimates (6.25). The $\frac{1}{\epsilon}$ term in (7.7) vanishes, with $\eta_0 \equiv 0$ and $\zeta_0 \equiv 0$, if and only if the reduced forcing function $h_0(x)$ is in the range of the integral operator of (1.1), with

$$h_0(x) = \int_0^1 K(x, s)\bar{w}(s)ds, \quad 0 \le x \le 1, \tag{7.8}$$

for some fixed function $\bar{w}$. If (7.8) holds, then the function $\bar{w}$ is unique, and the leading outer solution determined from (6.21) is given directly in terms of $\bar{w}$ as

$$\begin{bmatrix} y_0(x) \\ z_0(x) \end{bmatrix} = \bar{w}(x), \tag{7.9}$$

and in this latter case the leading outer solution (7.9) satisfies the reduced equation (1.2).

Proof: The earlier discussion demonstrates the existence of precisely one solution w satisfying $(7.6)_N$, for any $N \ge 1$. The stated assumptions permit the construction of w_{N+1} for any $N \ge 0$, and then the stated result (7.7) follows directly from $(7.6)_{N+1}$ and $(7.2)_{N+1}$.

There remain to be proved only the assertions regarding the $1/\epsilon$ term in (7.7). First, suppose there holds $\eta_0 \equiv 0$, $\zeta_0 \equiv 0$. Then (6.24) implies $\alpha_0 = 0$, $\beta_0 = 0$, and (6.23) shows that (7.8)-(7.9) holds at some fixed $x = x_0$. But (6.22) then implies that (7.8)-(7.9) holds for all x.

Conversely, suppose that (7.8) holds for some $\bar{w}$. Then (6.21) and (7.8) imply

$$J(x)\begin{bmatrix} y_0(x) \\ z_0(x) \end{bmatrix} - \bar{K}(x)\frac{d}{dx}\left[\bar{K}^{-1}(x)\int_0^1 K(x, s)\bar{w}(s)ds\right]$$
$$= -\int_0^1\left\{\bar{K}(x)\frac{d}{dx}[\bar{K}^{-1}(x)K(x, s)]\right\}\begin{bmatrix} y_0(s) \\ z_0(s) \end{bmatrix}ds, \tag{7.10}$$

where a direct calculation gives

$$\bar{K}(x)\frac{d}{dx}\left[\bar{K}^{-1}(x)\int_0^1 K(x, s)\bar{w}(s)ds\right]$$
$$= J(x)\bar{w}(x) + \bar{K}(x)\int_0^1 \frac{d}{dx}[\bar{K}^{-1}(x)K(x, s)]\bar{w}(s)ds. \tag{7.11}$$

From (7.10) and (7.11) it follows that the function $w := \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} - \bar{w}$ satisfies (5.6) with zero forcing term, $\phi = 0$, and then the unique solvability of (5.6) implies the result (7.9). But then (7.8), (7.9) and (6.23) imply $\alpha_0 = 0$ and $\beta_0 = 0$, and the construction in Section 6 then yields $\eta_0 \equiv 0$, $\zeta_0 \equiv 0$. ■

If the jump matrix J satisfies the eigenvalue condition (5.1) but if J is _not_ in a block diagonal form such as (5.3), then ***Theorem 7.1*** remains true with the obvious modifications if the transformed kernel K^* of (5.2) satisfies the appropriate conditions of the theorem, with $\bar{K}^*$ invertible. In this case, for small ϵ $(\epsilon \to 0^+)$, the equation (1.1) has a unique solution $w = w(x, \epsilon)$, and this solution is such that $T(x)w(x, \epsilon)$ can be represented by the right side of (7.7). Hence there holds

$$w(x, \epsilon) = \frac{1}{\epsilon}T^{-1}(x)\begin{bmatrix} \eta_0(x/\epsilon) \\ \zeta_0((1-x)/\epsilon) \end{bmatrix}$$
$$+ \sum_{j=0}^{N} T^{-1}(x)\begin{bmatrix} y_j(x) + \eta_{j+1}(x/\epsilon) \\ z_j(x) + \zeta_{j+1}((1-x)/\epsilon) \end{bmatrix}\epsilon^j + O(\epsilon^{N+1}), \tag{7.12}$$

where $T = T(x)$ is the matrix function that transforms $J(x)$ into block diagonal form, appearing in (5.2). In this case each component of the solution function generally has boundary layers at both endpoints because of the coupling due to T^{-1} in (7.12), as illustrated by the example in Appendix A.2.

In summary, solutions of boundary-layer type have been shown to exist for a special class of vector integral equations of the form (1.1) for all small enough $\epsilon > 0$. The data are assumed to be at least of class C^2 except for a jump discontinuity in the kernel which is assumed to possess an invertible jump discontinuity (1.13) that is sufficiently large (e.g. the condition (A.4.7) of Appendix A.4 suffices, and (5.1) is assumed to hold), and if the auxiliary equation (5.6) is uniquely solvable, with $\bar{K}$ of (5.5) invertible. If the jump matrix J is in the block diagonal form (5.3), then the resulting solution satisfies

$$w(x,\epsilon) \equiv \begin{bmatrix} u(x,\epsilon) \\ v(x,\epsilon) \end{bmatrix} = \begin{bmatrix} y_0(x) \\ z_0(x) \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} \eta_0\left[\frac{x}{\epsilon}\right] \\ \zeta_0\left[\frac{1-x}{\epsilon}\right] \end{bmatrix} + \begin{bmatrix} \eta_1\left[\frac{x}{\epsilon}\right] \\ \zeta_1\left[\frac{1-x}{\epsilon}\right] \end{bmatrix} + O(\epsilon) \quad (7.13)$$

as $\epsilon \to 0+$, uniformly for $0 \leq x \leq 1$, so that the solution w can be decomposed into components u and v that have separate boundary layers respectively only at $x = 0$ and at $x = 1$. The boundary layer correction terms decay exponentially away from the endpoints, in accordance with (6.25), so that the solution satisfies

$$\lim_{\substack{\epsilon \to 0+ \\ [\text{Fixed } x \in (0,\, 1)]}} w(x,\, \epsilon) = \begin{bmatrix} y_0(x) \\ z_0(x) \end{bmatrix}, \quad (7.14)$$

where $(y_0(x),\, z_0(x))$ is the leading term in the outer solution.

If the reduced forcing function h_0 is not in the range of the operator K, then existence fails for the reduced equation (1.2) obtained by putting $\epsilon = 0$ in (1.1) or in (5.8). In this case of course the limiting outer solution (7.14) does not satisfy the reduced system (1.2), but rather it is determined as the solution of the auxiliary equation (6.21). Also in this case, the $\frac{1}{\epsilon}$ boundary layer terms η_0 and ζ_0 are nontrivial near the appropriate endpoints (with $\alpha_0 \neq 0$ and $\beta_0 \neq 0$ in (6.24)), and the u-component of w behaves like a delta function near $x = 0$ while the v-component behaves like a delta function near $x = 1$.

If h_0 is in the range of K, then (1.2) has precisely one solution (as a consequence of our assumption on the unique solvability of (5.6)), and this reduced solution coincides with the limiting outer solution (7.14). Moreover in this case the $\frac{1}{\epsilon}$ boundary-layer terms vanish, $\eta_0 \equiv 0$ and $\zeta_0 \equiv 0$, so that (7.13) becomes $u(x, \epsilon) = y_0(x) + \eta_1(x/\epsilon) + O(\epsilon)$, $v(x, \epsilon) = z_0(x) + \zeta_1((1-x)/\epsilon) + O(\epsilon)$ as $\epsilon \to 0+$, uniformly for $0 \leq x \leq 1$. In particular the boundary values become

$$
\begin{aligned}
w(0, \epsilon) &= \begin{bmatrix} y_0(0) + \eta_1(0) \\ z_0(0) \end{bmatrix} + O(\epsilon), \\
w(1, \epsilon) &= \begin{bmatrix} y_0(1) \\ z_0(1) + \zeta_1(0) \end{bmatrix} + O(\epsilon),
\end{aligned}
\tag{7.15}
$$

where η_1 and ζ_1 are found to be given by the same formulas as in (6.24) but with α_0 and β_0 replaced by suitable integration constants $\alpha_1 = \int_0^\infty \eta_1$ and $\beta_1 = \int_0^\infty \zeta_1$. The actual values of these constants are given by a formula of the type (6.23) but with f_0 and g_0 replaced there by $f_1 + y_0$ and $g_1 + z_0$ respectively, and with y_0 and z_0 replaced there by y_1 and z_1 respectively. The required functions y_1 and z_1 are found to be determined by (5.6) with

$$\phi(x) = -\bar{K}(x)\frac{d}{dx}\left[\bar{K}^{-1}(x)\begin{bmatrix} f_1(x) + y_0(x) \\ g_1(x) + z_0(x) \end{bmatrix}\right].$$

If J is not in the block diagonal form (5.3) but if the eigenvalue condition (5.1) still holds and if the transformed kernel K^* of (5.2) yields an invertible $\bar{K}^*$ (as in (5.5), with K replaced there by K^*), then (1.1) still has a unique solution $w(x, \epsilon)$ for all small enough $\epsilon \neq 0$. The solution is of boundary-layer type, with all components of w generally having boundary layers at both endpoints as indicated by (7.12). The situation is illustrated by the example of Appendix A.2.

8. *Integrodifferential Equations*

We saw in Section 4 that the following first-order vector integrodifferential equation (see (4.10))

$$\epsilon^2 u'(x) + g(x, \epsilon) = \int_0^1 E(x, s)u(s)ds, \qquad 0 \leq x \leq 1, \tag{8.1}$$

with discontinuous kernel $E = \overset{m\times m}{E(x, s)}$ and subject to the boundary condition (see (4.12))

$$Lu(0, \epsilon) + Ru(1, \epsilon) = \gamma, \tag{8.2}$$

can be reduced to the related vector integral equation (1.1) with data functions h and K given by (4.16)-(4.17), provided that the matrix $L + R$ is invertible.

The resulting jump matrix J is given by (4.18), and is *not* in a suitable block diagonal form (5.3). However, it follows from (5.22) that $J(x)$ can be block-diagonalized by the matrix $T(x)$ of (5.20), and the resulting block-diagonal matrix (5.22) can be taken to satisfy the conditions of (5.3) if the eigenvalues $\lambda_k(x)$ of $j(x)$ are *excluded from a region in the complex plane* that contains a neighborhood of the origin and a neighborhood of the negative real axis. In this case we need only take the positive square root of λ for each element $\sqrt{\lambda}$ =

$|\lambda|^{1/2}\exp\{\frac{1}{2}i(\arg\lambda)\}$ of the diagonal matrix Ω in (5.20), so that every such element $\sqrt{\lambda}$ of Ω satisfies $Re\sqrt{\lambda} > 0$. Hence, in the terminology of (5.3), the block-diagonal matrix of (5.22) satisfies $n = p = m$, and the projection $\mathcal{P}$ of (5.4) is

$$\mathcal{P} = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}. \tag{8.3}$$

The matrix $\bar{K}^*$ becomes (see (5.5))

$$\bar{K}^*(x) = K^*(x, 0)\mathcal{P} + K^*(x, 1)(I_{2m} - \mathcal{P}) \tag{8.4}$$

where the kernel K^* is obtained from (5.2) as $K^*(x, s) = T(x)K(x,s)T^{-1}(s)$. Hence (8.4) yields

$$\bar{K}^*(x) = T(x)\left[K(x, 0)T^{-1}(0)\mathcal{P} + K(x, 1)T^{-1}(1)(I_{2m} - \mathcal{P})\right] \tag{8.5}$$

with $K(x, s)$ given by (4.17), and it follows that

$$\begin{gathered} \bar{K}^*(x) \ \underline{\textit{is invertible if and only if}} \\ K(x, 0)T^{-1}(0)\mathcal{P} + K(x, 1)T^{-1}(1)(I_{2m} - \mathcal{P}) \ \underline{\textit{is invertible.}} \end{gathered} \tag{8.6}$$

For example if we consider the initial-value problem for (8.1) with $L = I_m$ and $R = 0$ in (8.2), then (4.17) implies

$$K(x, 0)T^{-1}(0)\mathcal{P} + K(x, 1)T^{-1}(1)(I_{2m} - \mathcal{P}) = \begin{bmatrix} -Z(0)\Omega(0) & 0 \\ E(x,0)Z(0) & E(x,1)Z(1) \end{bmatrix}, \tag{8.7}$$

which with (8.6) shows that $\bar{K}^*(x)$ is invertible for the initial-value problem *if and only if* $E(x, 1)$ is invertible. In particular, we conclude that $\bar{K}^*(x)$ is a singular matrix if E is a Volterra kernel with $E(x, s) \equiv 0$ for $s > x$. Based on the results of Section 5 - 7, if E is a Volterra kernel we would then generally not expect to obtain a solution of boundary-layer type for the initial-value problem for the integrodifferential equation (8.1). On the other hand if we consider the *terminal-value* problem with $L = 0$ and $R = I_m$, then (4.17) implies

$$K(x, 0)T^{-1}(0)\mathcal{P} + K(x, 1)T^{-1}(1)(I_{2m} - \mathcal{P}) = \begin{bmatrix} 0 & -Z(1)\Omega(1) \\ E(x,0)Z(0) & E(x,1)Z(1) \end{bmatrix}, \tag{8.8}$$

which with (8.6) shows that $\bar{K}^*(x)$ is invertible for the terminal-value problem *if and only if* $E(x, 0)$ is invertible. Hence, if E is a Volterra kernel, we expect to find a solution of boundry-layer type for the terminal-value problem (subject to the further condition that (5.6) is solvable).

These results are sharp, as can be seen by the scalar example

$$\epsilon^2 u'(x) + 1 = \int_0^x u(s)ds \quad \text{for} \quad 0 \leq x \leq 1. \tag{8.9}$$

For the initial-value problem the unique solution is

$$u(x,\epsilon) = \frac{1}{2}\left[\gamma + \frac{1}{\epsilon}\right]e^{-x/\epsilon} + \frac{1}{2}\left[\gamma - \frac{1}{\epsilon}\right]e^{+x/\epsilon} \tag{8.10}$$

which is not of boundary-layer type but rather exhibits rapid exponential growth for $x > 0 \quad (\epsilon \to 0^+)$. On the other hand, for the terminal-value problem the unique solution of (8.9) is

$$\begin{aligned} u(x,\epsilon) &= \frac{e^{-x/\epsilon}}{\epsilon} + \frac{\gamma-(1/\epsilon)e^{-1/\epsilon}}{1+e^{-2/\epsilon}}\left[e^{-(1+x)/\epsilon} + e^{-(1-x)/\epsilon}\right] \\ &\sim \frac{1}{\epsilon}\, e^{-x/\epsilon} + \gamma e^{-(1-x)/\epsilon}, \end{aligned} \tag{8.11}$$

which is of boundary-layer type, with layers at both endpoints.

By way of comparison, the Fredholm integrodifferential equation

$$\epsilon^2 u'(x) + 1 = 2\int_0^x u(s)ds + \int_x^1 u(s)ds \tag{8.12}$$

has a unique solution of boundary-layer type for the initial-value problem, satisfying

$$u(x,\epsilon) \sim \gamma e^{-x/\epsilon} + \left[\frac{1}{\epsilon} - 2\gamma\right] e^{-(1-x)/\epsilon}, \tag{8.13}$$

and (8.12) also has a unique solution of boundary-layer type for the terminal-value problem, satisfying

$$u(x,\epsilon) \sim \frac{1}{2}\left[\frac{1}{\epsilon} - \gamma\right] e^{-x/\epsilon} + \gamma e^{-(1-x)/\epsilon}. \tag{8.14}$$

These explicit results for (8.12) are also in agreement with the asymptotic results of Section 5 – 7 because the 2X2 matrix $\bar{K}^*(x)$ is seen to be invertible for (8.12) both for the initial-value problem and for the terminal-value problem.

Appendix A.1 Certain Results for Continuous Kernels

We consider several classes of problems for (1.1) with continuous kernels. Consider first the scalar equation (1.1) with a *degenerate* kernel,

$$K(x, s) = \sum_{j=1}^{m} \alpha_j(x)\beta_j(s) \quad \text{for} \quad x, s \in [0, 1], \tag{A.1.1}$$

for given functions α_j, β_j $(j = 1,2,\cdots,m)$ that are continuous on $[0, 1]$, for a fixed positive integer m. In this case (1.1) implies

$$w(x) = \frac{1}{\epsilon}\left[-h(x,\epsilon) + \sum_{j=1}^{m} c_j\alpha_j(x)\right] \quad \text{for} \quad x \in [0, 1], \tag{A.1.2}$$

with constants c_j given as

$$c_j = \langle \beta_j, w \rangle \quad \text{for} \quad j = 1,2,\cdots,m, \tag{A.1.3}$$

where $\langle \cdot,\cdot \rangle$ is the L_2 inner product, $\langle f, g \rangle = \int_0^1 f(s)g(s)ds$. Using (A.1.2) in the right side of (A.1.3), we find the matrix equation

$$(A - \epsilon I)c = b \quad \text{for} \quad c = (c_1,c_2,\cdots,c_m)^T, \tag{A.1.4}$$

where the vector $b = (b_i)$ is given by $b_i := \langle \beta_i, h \rangle$ for $i = 1,2,\cdots,m$, and the matrix $A = (a_{ij})$ is given by

$$a_{ij} := \langle \beta_i, \alpha_j \rangle \quad \text{for} \quad i,j = 1,2,\cdots,m. \tag{A.1.5}$$

It follows from *Lemma A.1.1* (the latter lemma being proved below, at the end of this Appendix A.1) that (A.1.4) has a unique solution c for all small nonzero ϵ, and this solution satisfies $c = c(\epsilon) = O(\epsilon^{-m})$ as $\epsilon \to 0$. We conclude with (A.1.2) that ***the integral equation (1.1), (A.1.1) has a unique solution for all small nonzero*** ϵ, ***and this solution satisfies***

$$w(x) = w(x, \epsilon) = O\left[\epsilon^{-(m+1)}\right] \quad \text{as} \quad \epsilon \to 0, \tag{A.1.6}$$

uniformly for all $x \in [0, 1]$, provided that h is bounded. Hence the solution of (1.1) does not exhibit any boundary layer behavior for a continuous degenerate kernel (A.1.1); rather the solution generally becomes unbounded uniformly for $0 \leq x \leq 1$.

The proof of *Lemma A.1.1* given below actually provides more detailed information than (A.1.6) on the behavior of the solution of (1.1) in the case of a degenerate kernel. The result (A.1.6) represents the worst case among several possibilities, where this worst case occurs precisely when the principal invariants of the matrix $A = (a_{ij})$ of (A.1.5) all vanish, as in the following example with degenerate kernel ($m = 1$)

$$K(x,s) = 1 - 2x, \quad \text{with} \quad \alpha_1(x) = 1 - 2x \quad \text{and} \quad \beta_1(s) \equiv 1. \tag{A.1.7}$$

In this case the solution of (1.1) is given as

$$w(x,\epsilon) = -\frac{c_1}{\epsilon^2}\,\alpha_1(x) - \frac{1}{\epsilon}\,h(x,\epsilon), \quad \text{with} \quad c_1 = \langle \beta_1, h\rangle = \int_0^1 h. \tag{A.1.8}$$

Here there generally holds $w(x,\epsilon) = O(\epsilon^{-2}) = O(\epsilon^{-(m+1)})$ unless h satisfies $\int_0^1 h_0 = 0$ (h_0 orthogonal to β_1), and again there are no boundary layers.

More generally, for a continuous _non_degenerate kernel there may be a sequence of eigenvalues $\epsilon_j \longrightarrow 0^+$, with $Kw_j = \epsilon_j w_j$ for $j = 1,2,\cdots$ (cf. pp. 116–17 of Tricomi [1957]). In such a case one expects that special care must be taken in the study of (1.1) for small ϵ near the eigenvalues; see Lange [1986] for a related situation involving a Sturm/Liouville problem. It follows from our ***Lemma A.4.1*** that this type behavior cannot occur for appropriate _piecewise_ continuous kernels as considered here.

As another example of a continuous kernel, consider the ***constant*** $m\times m$ kernel, $K(x, s) \equiv A$ for some fixed matrix $A\in\mathbb{C}^{m\times m}$. In this case (1.1) implies $\epsilon w(x) = -h(x,\epsilon) + Ac$ with $c = \int_0^1 w$, and one sees directly that this vector c will satisfy the previous (A.1.4) with right hand side $b = \int_0^1 h$. Hence we find with ***Lemma A.1.1*** the result $Ac = O(\epsilon^{-m+1})$. It follows that *the vector integral equation (1.1) with constant $m\times m$ kernel has a unique solution for all small ϵ, and this solution satisfies*

$$w(x) = w(x, \epsilon) = O(\epsilon^{-m}) \quad \text{as} \quad \epsilon \to 0, \tag{A.1.9}$$

uniformly for $x\in[0, 1]$. Again, (A.1.9) represents the worst possible case, and this case occurs when all principal invariants of A vanish, as in the example ($m = 3$ with $A^3 = 0$)

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad \text{with}$$

$$w(x,\epsilon) = -\frac{1}{\epsilon}h(x,\epsilon) - \frac{1}{\epsilon^3}(A^2 + \epsilon A)\int_0^1 h. \tag{A.1.10}$$

In this example there generally holds $w(x,\epsilon) = O(\epsilon^{-3}) = O(\epsilon^{-m})$, unless the integral $\int_0^1 h_0$ is in the null space of A^2.

We close this appendix with a proof of the lemma used earlier.

Lemma A.1.1. *The matrix* $A - \epsilon I$ *is invertible for any given fixed* $A \in \mathbb{C}^{m\times m}$, *for all complex* ϵ *in a deleted neighborhood of* $\epsilon = 0$, $0 < |\epsilon| \leq \epsilon_0$ *for some positive* $\epsilon_0 = \epsilon_0(A)$. *The solution* c *of (A.1.4) satisfies*

$$A^j c = O(\epsilon^{-m+j}) \quad \text{for} \quad j = 0,1,\cdots,m, \quad \text{as} \quad \epsilon \to 0, \tag{A.1.11}$$

for any fixed $b \in \mathbb{C}^{m\times 1}$ *in (A.1.4).*

Proof. The lemma is trivially true if A is nonsingular, but we give a direct proof that handles simultaneously all cases, A nonsingular and A singular. For this purpose we use the Hamilton/Cayley theorem which implies the result

$$A^m = \sum_{j=1}^{m} \mu_j A^{m-j} \tag{A.1.12}$$

for any $m\times m$ A, where the coefficients μ_j are given as $\mu_j := (-1)^{j+1} I_j(A)$ and where $I_j(A)$ is the j^{th} principal invariant of A, given by the coefficient of λ^{m-j} in the characteristic polynomial $\det(A+\lambda I)$. We also use

$$A^{j+1}c = A^j(\epsilon c + b) \quad \text{for any solution } c \text{ of (A.1.4)}, \qquad (A.1.13)_j$$

for $j = 0,1,\cdots$, obtained by multiplying (A.1.4) by A^j.

From (A.1.12) and (A.1.13) one finds directly the result

$$p_m(A, \epsilon)c = -\left[\sum_{j=0}^{m-1} p_j(A, \epsilon)A^{m-1-j}\right]b \qquad (A.1.14)$$

for any solution c of (A.1.4), where the scalar coefficients p_j are defined recursively as

$$\begin{aligned} &p_0(A, \epsilon) := -1, \quad p_1(A, \epsilon) := \mu_1(A) - \epsilon, \quad \text{and} \\ &p_j(A, \epsilon) := \mu_j(A) + \epsilon p_{j-1}(A, \epsilon) \quad \text{for } j = 2,3,\cdots,m. \end{aligned} \qquad (A.1.15)$$

For the derivation of (A.1.14), first apply both sides of (A.1.12) to the vector c, and eliminate the term $A^m c$ between the resulting equation and $(A.1.13)_{m-1}$ to find $p_1 A^{m-1}c = A^{m-1}b - \sum_{j=2}^{m} \mu_j A^{m-j}c$. Next eliminate the term $A^{m-1}c$ between this last result and $(A.1.13)_{m-2}$ to give $p_2 A^{m-2}c = (A^{m-1} - p_1 A^{m-2})b - \sum_{j=3}^{m} \mu_j A^{m-j}c$, and so forth, continuing in this fashion for a total of m steps until (A.1.14) is obtained for $p_m c$; details are omitted. We note however that the quantities $p_j(A, \epsilon)$ of (A.1.15) are all nonzero, for all small enough nonzero ϵ.

One sees directly that the resulting unique solution $c = c(\epsilon)$ of (A.1.14) gives also the unique solution of (A.1.4). The estimates of (A.1.11) follow as special cases of the more general results provided directly by (A.1.14)-(A.1.15). For example, if A is nonsingular, with $\det A \neq 0$, then also $\mu_m(A) \neq 0$, and in this case we have $\frac{1}{p_m(\epsilon)} = O(1)$. Hence the solution c from (A.1.14) satisfies $c(\epsilon) = O(1)$ as $\epsilon \to 0$, with $A^j c = O(1)$, so that (A.1.11) is trivially true. At the other extreme is the case in which all principal invariants vanish, $\mu_j = 0$ for all j. In this case there holds $p_m = -\epsilon^m$, and also $c(\epsilon) = O(1/\epsilon^m)$. The estimates (A.1.11) follow then directly from $(A.1.13)_j$. Further details are omitted. ■

Appendix A.2 An Example with Jump Matrix not in Block Diagonal Form

Consider a 2-dimensional ($m = 2$) system (1.1) with piecewise constant kernel K given as

$$K(x, s) = \begin{cases} \begin{bmatrix} 3 & -1 \\ -1 & -1 \end{bmatrix} & \text{for } s < x, \\ \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} & \text{for } s > x. \end{cases} \tag{A.2.1}$$

The jump matrix J of (1.13) becomes

$$J = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}, \tag{A.2.2}$$

so that J is not in the block diagonal form (5.3).

We give here a construction of an asymptotic expansion for the solution w of (1.1), using a suitable multivariable additive representation applied directly to (1.1) without the preliminary transformation (5.2). We seek $w = w(x) = w(x, \epsilon)$ in the form

$$w(x, \epsilon) = w^{\dagger}(x, \epsilon) + \frac{1}{\epsilon}\left[\tilde{w}(\tilde{x}, \epsilon) + \hat{w}(\hat{x}, \epsilon)\right] \quad \text{with} \quad \tilde{x} := \frac{x}{\epsilon}, \quad \hat{x} := \frac{1-x}{\epsilon}, \tag{A.2.3}$$

for a suitable outer approximation $w^{\dagger}$ and suitable boundary layer correction terms $\tilde{w}$ and $\hat{w}$ depending on the respective boundary layer variables $\tilde{x}$ and $\hat{x}$.

We assume that these functions $w^{\dagger}$, $\tilde{w}$ and $\hat{w}$ have asymptotic expansions (at least up to some finite order, determined by the regularity of the data) analogous to those of (6.3), and we insert these expansions into (1.1) and proceed as in Section 6. Omitting the details, we find for the leading terms $w_0^{\dagger}(x) = w^{\dagger}(x, 0)$, $\tilde{w}_0(\tilde{x}) = \tilde{w}(\tilde{x}, 0)$ and $\hat{w}_0(\hat{x}) = \hat{w}(\hat{x}, 0)$ the conditions

$$h_0(x) - \int_0^1 K(x, s)w_0^{\dagger}(s)ds = K(x,0)\int_0^{\infty}\tilde{w}_0(\sigma)d\sigma + K(x,1)\int_0^{\infty}\hat{w}_0(\sigma)d\sigma, \tag{A.2.4}$$

$$\tilde{w}_0(\tilde{x}) + J(0)\int_{\tilde{x}}^{\infty}\tilde{w}_0(\sigma)d\sigma = 0, \tag{A.2.5}$$

and

$$\hat{w}_0(\hat{x}) - J(1)\int_{\hat{x}}^{\infty}\hat{w}_0(\sigma)d\sigma = 0. \tag{A.2.6}$$

The equation (A.2.4) corresponds to (6.18)-(6.19), while the equations (A.2.5) and (A.2.6) correspond respectively to $(6.16)_0$ and $(6.17)_0$. In the present case the jump matrix is constant, so that $J(0) = J(1) = J$, as given by (A.2.2). In keeping with the assumed existence of the improper integrals in (A.2.4), (A.2.5), and (A.2.6), we have the matching conditions

$$\operatorname*{Limit}_{\tilde{x} \to \infty} \tilde{w}_0(\tilde{x}) = 0 \quad \text{and} \quad \operatorname*{Limit}_{\hat{x} \to \infty} \hat{w}_0(\hat{x}) = 0. \tag{A.2.7}$$

From (A.2.5) and (A.2.6) we find (cf. (6.24))

$$\tilde{w}_0(\tilde{x}) = e^{J(0)\tilde{x}} \alpha \quad \text{and} \quad \hat{w}_0(\hat{x}) = e^{-J(1)\hat{x}} \beta, \tag{A.2.8}$$

for suitable (vector) integration constants α and β which must, if possible, be chosen so as to be compatible with the matching conditions of (A.2.7). That is, we require α and β to be in the (asymptotically) stable initial manifolds of the respective equation (A.2.5) and (A.2.6). In the present case from (A.2.2) and (A.2.8) we have

$$\tilde{w}_0(\tilde{x}) = \frac{\alpha_1 - \alpha_2}{2} e^{2\tilde{x}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{\alpha_1 + \alpha_2}{2} e^{-2\tilde{x}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{A.2.9}$$

and

$$\hat{w}_0(\hat{x}) = \frac{\beta_1 + \beta_2}{2} e^{2\hat{x}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\beta_1 - \beta_2}{2} e^{-2\hat{x}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{A.2.10}$$

where α_1 and α_2 are the components of $\alpha = (\alpha_1, \alpha_2)$, and similarly β_1 and β_2 are the components of $\beta = (\beta_1, \beta_2)$. The matching conditions of (A.2.7) imply with (A.2.9) and (A.2.10) the results

$$\alpha_1 = \alpha_2 \quad \text{and} \quad \beta_1 = -\beta_2, \tag{A.2.11}$$

so that $\tilde{w}_0$ and $\hat{w}_0$ become

$$\tilde{w}_0(\tilde{x}) = \alpha_1 e^{-2\tilde{x}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{w}_0(\hat{x}) = \beta_1 e^{-2\hat{x}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{A.2.12}$$

where these functions now satisfy (A.2.5) and (A.2.6), for arbitray (scalar) constants α_1 and β_1 which will now be specified with (A.2.4).

We use (A.2.1) and (A.2.12) in (A.2.4) and find

$$h_0(x) - \int_0^1 K(x, s) w_0^{\dagger}(s) ds = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \quad \text{for} \quad 0 < x < 1, \tag{A.2.13}$$

where this last equation is analogous to (6.18). Upon differentiation, (A.2.13) yields $h_0'(x) = J(x) w_0^{\dagger}(x)$, or

$$w_0^{\dagger}(x) = J^{-1} h_0'(x) = -\frac{1}{2} \begin{bmatrix} g_0'(x) \\ f_0'(x) \end{bmatrix}, \tag{A.2.14}$$

where f_0 and g_0 are the components of $h_0 = (f_0, g_0)$. A direct calculation now shows that (A.2.14) does indeed provide the solution of (A.2.13) ***if and only if*** there hold

$$\alpha_1 = \frac{1}{2}\Big[-2g_0(0) + f_0(1) + g_0(1)\Big] \quad \text{and} \quad \beta_1 = \frac{1}{2}\Big[2g_0(1) + f_0(0) - g_0(0)\Big]. \tag{A.2.15}$$

Hence we impose (A.2.15), and then (A.2.12), (A.2.14) and (A.2.15) provide the leading terms for the solution (A.2.3). The procedure can be continued so as to determine further (higher order) terms in the appropriate asymptotic expansions, subject to suitable regularity requirements on $h(x, \epsilon)$, but no further terms are obtained here.

In particular, we find that the solution $w = (u, v)$ satisfies

$$w(x,\epsilon) \equiv \begin{bmatrix} u(x,\epsilon) \\ v(x,\epsilon) \end{bmatrix} = \epsilon^{-1}\alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2x/\epsilon} + \epsilon^{-1}\beta_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2(1-x)/\epsilon}$$
$$- \frac{1}{2}\begin{bmatrix} g_0'(x) \\ f_0'(x) \end{bmatrix} + \tilde{w}_1(x/\epsilon) + \hat{w}_1((1-x)/\epsilon) + O(\epsilon), \tag{A.2.16}$$

where $\tilde{w}_1$ and $\hat{w}_1$ denote the next terms in the expansions for the boundary layer corrections. Hence there holds

$$\underset{\substack{\epsilon \to 0^+ \\ (\text{Fixed } 0 < x < 1)}}{\text{Limit}} \quad w(x, \epsilon) = -\frac{1}{2}\begin{bmatrix} g_0'(x) \\ f_0'(x) \end{bmatrix}, \tag{A.2.17}$$

and a direct calculation shows that the leading outer solution given by the right side of (A.2.17) satisfies the reduced equation (1.2) if and only if there hold

$$-2g_0(0) + f_0(1) + g_0(1) = 0 \quad \text{and} \quad 2g_0(1) + f_0(0) - g_0(0) = 0, \tag{A.2.18}$$

where these conditions coincide with (5.14). Moreover, (A.2.18) is seen with (A.2.15)-(A.2.16) to be the necessary and sufficient condition for the solution w to be bounded, $w = O(1)$.

In the present case the matrix $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ can be used in the transformation (5.2) to reduce J to diagonal form, and then the results of Section 5 can be applied to the transformed problem. One finds that the resulting expansion (7.12) coincides with the corresponding expansion obtained directly here without (5.2). (The transformed problem is closely related here to the earlier problem (1.1), (5.12) which can be explicitly solved.)

Appendix A.3 Estimates on Residuals

We give an indication of a derivation for the estimates (7.4) on the residual $\rho_N(x, \epsilon)$ of (7.3), assuming always sufficient smoothness on the data. To this end, insert $(7.2)_N$ into the right side of (7.3) and find

$$\rho_N(x,\epsilon) = \left[h(x,\epsilon) - \sum_{j=0}^{N} h_j(x)\epsilon^j\right] + \begin{bmatrix} y_N(x) \\ z_N(x) \end{bmatrix} \epsilon^{N+1}$$

$$+ \sum_{j=0}^{N} \left[\begin{bmatrix} y_{j-1}(x) + f_j(x) \\ z_{j-1}(x) + g_j(x) \end{bmatrix} + \int_0^1 K(x,s) \begin{bmatrix} y_j(s) \\ z_j(s) \end{bmatrix} ds\right] \epsilon^j \qquad \text{(A.3.1)}$$

$$+ \sum_{j=0}^{N} \left[\begin{bmatrix} \eta_j(x/\epsilon) \\ \zeta_j((1-x)/\epsilon) \end{bmatrix} - \frac{1}{\epsilon}\int_0^1 K(x,s) \begin{bmatrix} \eta_j(s/\epsilon) \\ \zeta_j((1-s)/\epsilon) \end{bmatrix} ds\right] \epsilon^j$$

with $y_{-1} = 0$, $z_{-1} = 0$. This result and the construction of the functions given in Section 6 gives (cf. $(6.14)_j$, and also (5.9), (5.11))

$$\rho_N(x,\epsilon) = O(\epsilon^{N+1}) + \sum_{j=0}^{N} \left[\sum_{k=0}^{j} \left[\frac{1}{(j-k)!} \bar{K}_{j-k}(x) \int_0^\infty \begin{bmatrix} \eta_k(\sigma) \\ (-1)^{j-k} \zeta_k(\sigma) \end{bmatrix} d\sigma\right]\right. \qquad \text{(A.3.2)}$$

$$\left. + \begin{bmatrix} \eta_j(x/\epsilon) \\ \zeta_j((1-x)/\epsilon) \end{bmatrix} - \int_0^{1/\epsilon} \begin{bmatrix} A(x,\epsilon\sigma)\eta_j(\sigma) + B(x,1-\epsilon\sigma)\zeta_j(\sigma) \\ C(x,\epsilon\sigma)\eta_j(\sigma) + D(x,1-\epsilon\sigma)\zeta_j(\sigma) \end{bmatrix} d\sigma\right] \epsilon^j,$$

where several changes of integration variables have been made, and where the $O(\epsilon^{N+1})$ term is uniform in x $(0 \le x \le 1)$ as $\epsilon \to 0^+$.

The first estimate of (7.4) for ρ_N will follow directly from (A.3.2) and the result

$$\sum_{j=0}^{N}\left[\begin{pmatrix}\eta_j(x/\epsilon)\\ \zeta_j((1-x)/\epsilon)\end{pmatrix} - \int_0^{1/\epsilon}\begin{bmatrix}A(x,\epsilon\sigma)\eta_j(\sigma)+B(x,1-\epsilon\sigma)\zeta_j(\sigma)\\ C(x,\epsilon\sigma)\eta_j(\sigma)+D(x,1-\epsilon\sigma)\zeta_j(\sigma)\end{bmatrix}d\sigma\right.$$

$$\left. + \sum_{k=0}^{j}\left[\frac{1}{(j-k)!}\bar{K}_{j-k}(x)\int_0^{\infty}\begin{pmatrix}\eta_k(\sigma)\\ (-1)^{j-k}\zeta_k(\sigma)\end{pmatrix}d\sigma\right]\right]\epsilon^j = O(\epsilon^{N+1}), \tag{A.3.3}$$

<u>*and so now we need only prove*</u> (A.3.3). A direct calculation can be given to show that (A.3.3) follows from (6.16) and (6.17); we indicate the details only in the case $N = 0$.

In fact we consider only the first block-component of the vector equation (A.3.3) in the case $N = 0$; that is, we indicate a proof of the result (cf. $(6.15)_0$)

$$\eta_0(x/\epsilon) - \int_0^{1/\epsilon}\left[A(x,\epsilon\sigma)\eta_0(\sigma) + B(x,1-\epsilon\sigma)\zeta_0(\sigma)\right]d\sigma$$

$$+ A(x,0)\int_0^{\infty}\eta_0(\sigma)d\sigma + B(x,1)\int_0^{\infty}\zeta_0(\sigma)d\sigma \overset{?}{=} O(\epsilon) \tag{A.3.4}$$

for $0 \le x \le 1$ as $\epsilon \to 0^+$. The proof of the corresponding result for the second block-component will follow along the same lines.

Adding and subtracting terms, we first compute

$$\int_0^{1/\epsilon} B(x,1-\epsilon\sigma)\zeta_0(\sigma)d\sigma = B(x,1)\int_0^{\infty}\zeta_0(\sigma)d\sigma - B(x,1)\int_{1/\epsilon}^{\infty}\zeta_0(\sigma)d\sigma$$

$$+ \int_0^{1/\epsilon}\Big[B(x,1-\epsilon\sigma) - B(x,1)\Big]\zeta_0(\sigma)d\sigma, \tag{A.3.5}$$

from which there follows (see $(6.25)_0$)

$$\int_0^{1/\epsilon} B(x,1-\epsilon\sigma)\zeta_0(\sigma)d\sigma - B(x,1)\int_0^{\infty}\zeta_0(\sigma)d\sigma = O(e^{-\hat{\kappa}/\epsilon})$$

$$+ \int_0^{1/\epsilon}\Big[B(x,1-\epsilon\sigma) - B(x,1)\Big]\zeta_0(\sigma)d\sigma. \tag{A.3.6}$$

The regularity of B and Taylor's theorem yield $B(x,1-\epsilon\sigma) - B(x,1) = O(1)\epsilon\sigma$, and then $(6.25)_0$ and (A.3.6) yield

$$\int_0^{1/\epsilon} B(x,1-\epsilon\sigma)\zeta_0(\sigma)d\sigma - B(x,1)\int_0^{\infty}\zeta_0(\sigma)d\sigma = O(\epsilon), \tag{A.3.7}$$

uniformly for $0 \le x \le 1$ as $\epsilon \to 0^+$. This last result shows that (A.3.4) will follow now from

$$\eta_0(x/\epsilon) - \int_0^{1/\epsilon} A(x,\epsilon\sigma)\eta_0(\sigma)d\sigma + A(x,0)\int_0^{\infty}\eta_0(\sigma)d\sigma \overset{?}{=} O(\epsilon), \tag{A.3.8}$$

and so we turn to a proof of (A.3.8).

An analogous argument as that used in obtaining (A.3.6) shows similarly the result

$$\int_0^{1/\epsilon} A(x,\epsilon\sigma)\eta_0(\sigma)d\sigma - A(x,0)\int_0^{\infty}\eta_0(\sigma)d\sigma = O(e^{-\hat{\lambda}/\epsilon}) + \int_0^{1/\epsilon}\Big[A(x,\epsilon\sigma) - A(x,0)\Big]\eta_0(\sigma)d\sigma. \tag{A.3.9}$$

In estimating the last term on the right side of (A.3.9) we must take into account the jump discontinuity (see (5.10))

$$A(x,x^-) - A(x,x^+) = J_{neg}(x), \tag{A.3.10}$$

and so we write

$$\int_0^{1/\epsilon}\Big[A(x,\epsilon\sigma) - A(x,0)\Big]\eta_0(\sigma)d\sigma = \int_0^{x/\epsilon}\Big[A(x,\epsilon\sigma) - A(x,0)\Big]\eta_0(\sigma)d\sigma \tag{A.3.11}$$

$$+ \int_{x/\epsilon}^{1/\epsilon} \Big[A(x,\epsilon\sigma) - A(x,0)\Big] \eta_0(\sigma) d\sigma.$$

$A = A(x,s)$ is smooth for $0 < s < x$, and so a Taylor theorem argument and $(6.25)_0$ imply

$$\int_0^{x/\epsilon} \Big[A(x,\epsilon\sigma) - A(x,0)\Big] \eta_0(\sigma) d\sigma = O(\epsilon), \qquad \text{(A.3.12)}$$

which with (A.3.11) yields

$$\int_0^{1/\epsilon} \Big[A(x,\epsilon\sigma) - A(x,0)\Big] \eta_0(\sigma) d\sigma = O(\epsilon) + \int_{x/\epsilon}^{1/\epsilon} \Big[A(x,\epsilon\sigma) - A(x,0)\Big] \eta_0(\sigma) d\sigma. \qquad \text{(A.3.13)}$$

The left side of (A.3.8) can be evaluated with (A.3.9), (A.3.13), $(6.16)_0$ and $(6.25)_0$, and we find

$$\eta_0(x/\epsilon) - \int_0^{1/\epsilon} A(x,\epsilon\sigma)\eta_0(\sigma)d\sigma + A(x,0)\int_0^{\infty} \eta_0(\sigma)d\sigma$$

$$= O(\epsilon) - \int_{x/\epsilon}^{1/\epsilon} \Big[J_{neg}(0) + \Big[A(\epsilon\tilde{x},\epsilon\sigma) - A(\epsilon\tilde{x},0)\Big]\Big] \eta_0(\sigma) d\sigma. \qquad \text{(A.3.14)}$$

A Taylor theorem argument can be used with (A.3.10) to show $J_{neg}(0)$ +

$\left[A(\epsilon\tilde{x},\epsilon\sigma) - A(\epsilon\tilde{x},0)\right] = O(1)\epsilon\sigma$ uniformly for $\sigma > \tilde{x}$, and the desired result of (A.3.8) follows then from (A.3.14) and $(6.25)_0$.

Further details are omitted (cf. Smith [1985] for details in various related calculations).

Appendix A.4 Existence, Uniqueness, and Estimates

For small ϵ, we give here a direct proof of existence and uniqueness for solutions of (1.1) and we obtain appropriate a *priori* estimates for the solutions, subject to the eigenvalue condition (5.1) and subject to a requirement that the jump matrix J is large in a certain sense. To this end we require the fundamental solutions $F_{neg} = F_{neg}(x,s)$ and $F_{pos} = F_{pos}(x,s)$ characterized as

$$\begin{aligned} \epsilon\frac{\partial}{\partial x}F_{neg}(x,\ s) &= J_{neg}(x)F_{neg}(x,\ s) \quad \text{for} \quad x \neq s, \quad F_{neg}(x,x) = I_n, \\ \epsilon\frac{\partial}{\partial x}F_{pos}(x,\ s) &= J_{pos}(x)F_{pos}(x,\ s) \quad \text{for} \quad x \neq s, \quad F_{pos}(x,x) = I_p, \end{aligned} \tag{A.4.1}$$

where J_{neg} and J_{pos} are the diagonal blocks of the block diagonal jump matrix $J(x)$ as in (5.3), with eigenvalues satisfying

$$\begin{aligned} &Re\ \lambda(x) \leq -\kappa_1 < 0 \quad \text{for all eigenvalues} \quad \lambda(x) \quad \text{of} \quad J_{neg}(x), \quad \text{and} \\ &Re\ \lambda(x) \geq +\kappa_1 > 0 \quad \text{for all eigenvalues} \quad \lambda(x) \quad \text{of} \quad J_{pos}(x), \end{aligned} \tag{A.4.2}$$

uniformly for $0 \leq x \leq 1$, for some fixed constant $\kappa_1 > 0$ as in (5.1). The fundamental solutions $F_{neg}(x,s)$ and $F_{pos}(x,s)$ satisfy the appropriate adjoint equations with respect to s,

$$\epsilon\frac{\partial}{\partial s}F_{neg}(x, s) = -F_{neg}(x, s)J_{neg}(s),$$
$$\epsilon\frac{\partial}{\partial s}F_{pos}(x, s) = -F_{pos}(x, s)J_{pos}(s), \tag{A.4.3}$$

and the following well known estimates of Flatto and Levinson [1955] hold (cf. Exercise 7.1.5 of Smith [1985]):

$$|F_{neg}(x,s)|, \quad |F_{pos}(s,x)| \quad \leq C_0\ e^{-\kappa_0(x-s)/\epsilon} \quad \text{for} \quad 0 \leq s \leq x \leq 1, \tag{A.4.4}$$

as $\epsilon \to 0^+$, for any fixed constant κ_0 satisfying

$$0 < \kappa_0 < \kappa_1, \tag{A.4.5}$$

and for some fixed positive constant C_0, where κ_0 and C_0 are independent of ϵ as $\epsilon \to 0^+$. In the special case that $J_{neg}(x)$ and $J_{pos}(x)$ are *diagonal* matrix functions, then the fundamental solutions F_{neg} and F_{pos} are also diagonal, given as the matrix exponentials $F_{neg}(x,s) = \exp \frac{1}{\epsilon}\int_s^x J_{neg}$ and a similar formula for F_{pos}. In this case, in (A.4.4) one can take

$$C_0 = 1 \quad \text{(in the diagonal case)} \tag{A.4.6}$$

provided that one uses a matrix norm $|\cdot|$ satisfying $|\mathrm{Diag}[d_j]| = \max|d_j|$ for any diagonal matrix.

The following condition (A.4.7) seems natural for the problem at hand because this condition simply requires that *the jump discontinuiuty should be sufficiently large,* as measured here by the positive constant κ_1 in

(5.1) and (A.4.2). For example the single eigenvalue $\lambda(x)$ of $J(x)$ coincides with $J(x)$ in the scalar case ($m = 1$), so that the condition (A.4.7) requires directly in this case that the real part of the jump must be sufficiently large.

Lemma A.4.1. *Let the data h and K in the system (1.1), or equivalently the data f, g, A, B, C, and D in the system (5.8), be piecewise differentiable, except for jump discontinuities as in (5.10), where J_{neg} and J_{pos} satisfy (A.4.2). Let the constants C_0 and κ_0 in (A.4.4) satisfy*

$$C_0\left[\|\mathcal{P}K(0,\cdot)\|_\infty + \|(I-\mathcal{P})K(1,\cdot)\|_\infty + \|K_x\|_\infty\right] < \kappa_0 \tag{A.4.7}$$

where $\mathcal{P}$ is the projection of (5.4). Then there is a positive number ϵ_0 such that (1.1) has precisely one solution w for $0 < \epsilon \leq \epsilon_0$, and this solution satisfies the integral estimate

$$\|w\|_1 \leq \text{const.}\left[|\mathcal{P}h(0,\epsilon)| + |(I-\mathcal{P})h(1,\epsilon)| + \|h'\|_1\right] \tag{A.4.8}$$

and the pointwise estimate

$$|w(x,\epsilon)| \leq \frac{1}{\epsilon}\,\text{const.}\left[|\mathcal{P}h(0,\epsilon)| + |(I-\mathcal{P})h(1,\epsilon)| + \|h'\|_1\right] \tag{A.4.9}$$

for $0 \leq x \leq 1$, for a fixed constant independent of ϵ as $\epsilon \to 0^+$.

Proof: The system (5.8) can be differentiated with (5.10) to give

$$\begin{aligned} \epsilon u'(x) - J_{neg}(x)u(x) &= -f'(x) + \int_0^1 [A_x(x,s)u(s) + B_x(x,s)v(s)]ds, \\ \epsilon v'(x) - J_{pos}(x)v(x) &= -g'(x) + \int_0^1 [C_x(x,s)u(s) + D_x(x,s)v(s)]ds, \end{aligned} \tag{A.4.10}$$

where we suppress the dependence on ϵ of f, g, u, and v. This system (A.4.10) can be integrated using the fundamental solutions of (A.4.1) and (A.4.3) to give

$$\begin{aligned}\epsilon u(x) = \epsilon F_{neg}(x,0)u(0) - \int_0^x F_{neg}(x,s)f'(s)ds \\ + \int_0^x F_{neg}(x,s)\int_0^1 [A_x(s,t)u(t) + B_x(s,t)v(t)]dt\ ds\end{aligned} \tag{A.4.11}$$

and

$$\begin{aligned}\epsilon v(x) = \epsilon F_{pos}(x,1)v(1) + \int_x^1 F_{pos}(x,s)g'(s)ds \\ - \int_x^1 F_{pos}(x,s)\int_0^1 [C_x(s,t)u(t) + D_x(s,t)v(t)]dt\ ds,\end{aligned} \tag{A.4.12}$$

where $A_x(s,t) = A_x(x,t)\big|_{x=s}$ and so forth. The two equations of (5.8) can be evaluated separately at $x = 0$ and at $x = 1$ and then used to eliminate the terms u(0) and v(1) in (A.4.11)-(A.4.12), and in this way we find the system

$$\begin{aligned}u(x) = \mu(x,\ \epsilon) + \mathcal{R}_\epsilon \begin{bmatrix} u \\ v \end{bmatrix}(x), \\ v(x) = \nu(x,\ \epsilon) + \mathcal{S}_\epsilon \begin{bmatrix} u \\ v \end{bmatrix}(x),\end{aligned} \tag{A.4.13}$$

where again we suppress the dependence on ϵ of u and v, and where the functions μ and ν are given as

$$\mu = \mu(x,\ \epsilon) := \frac{1}{\epsilon}\Big[-F_{neg}(x,0)f(0,\ \epsilon) - \int_0^x F_{neg}(x,s)f'(s,\ \epsilon)ds\Big],$$

$$\nu = \nu(x,\ \epsilon) := \frac{1}{\epsilon}\Big[-F_{pos}(x,1)g(1,\ \epsilon) + \int_x^1 F_{pos}(x,s)g'(s,\ \epsilon)ds\Big], \tag{A.4.14}$$

and where the operators $\mathcal{R}_\epsilon$ and $\mathcal{S}_\epsilon$ are defined by the formulas

$$\mathcal{R}_\epsilon w(x) \equiv \mathcal{R}_\epsilon \begin{bmatrix} u \\ v \end{bmatrix}(x) := \frac{1}{\epsilon}\Big[F_{neg}(x,0)\int_0^1 [A(0,s)u(s) + B(0,s)v(s)]ds$$

$$+ \int_0^x F_{neg}(x,s)\int_0^1 [A_x(s,t)u(t) + B_x(s,t)v(t)]dt\ ds\Big], \tag{A.4.15}$$

and

$$\mathcal{S}_\epsilon w(x) \equiv \mathcal{S}_\epsilon \begin{bmatrix} u \\ v \end{bmatrix}(x) := \frac{1}{\epsilon}\Big[F_{pos}(x,1)\int_0^1 [C(1,s)u(s) + D(1,s)v(s)]ds$$

$$- \int_x^1 F_{pos}(x,s)\int_0^1 [C_x(s,t)u(t) + D_x(s,t)v(t)]dt\ ds, \tag{A.4.16}$$

for any vectors $w = \begin{bmatrix} u \\ v \end{bmatrix}$. We use the Banach space $\mathcal{B}$ of continuous vector functions $w = \begin{bmatrix} u \\ v \end{bmatrix}$ where $u = u(x)$ and $v = v(x)$ are continuous vector functions of respective dimensions n and p on $0 \leq x \leq 1$, with norm

$$\|w\| := \|u\|_1 + \|v\|_1 \tag{A.4.17}$$

where $\|u\|_1 := \int_0^1 |u|$, $\|v\|_1 := \int_0^1 |v|$, with $|u|$ and $|v|$ taken in terms of any convenient vector norms on $\mathbb{C}^n$ and $\mathbb{C}^p$. The system (A.4.13) can be written more briefly as

$$w = \psi + \mathcal{L}w \quad \text{for} \quad w \in \mathcal{B}, \tag{A.4.18}$$

with

$$\psi := \begin{bmatrix} \mu \\ \nu \end{bmatrix} \quad \text{and} \quad \mathcal{L}w \equiv \mathcal{L}\begin{Bmatrix} u \\ v \end{Bmatrix} := \begin{bmatrix} \mathcal{R}_\epsilon w \\ \mathcal{S}_\epsilon w \end{bmatrix} \tag{A.4.19}$$

for any vector w.

Routine calculations using (5.4), (A.4.4), (A.4.14)-(A.4.17) and (A.4.19) yield the estimates

$$\begin{aligned} \|\psi\| &\leq \frac{C_0}{\kappa_0}\Big[\, |\mathcal{P}h(0,\epsilon)| + |(I-\mathcal{P})h(1,\epsilon)| + \|h'\| \Big], \\ \|\mathcal{L}\| &\leq \frac{C_0}{\kappa_0}\Big[\|\mathcal{P}K(0,\cdot)\|_\infty + \|(I-\mathcal{P})K(1,\cdot)\|_\infty + \|K_x\|_\infty\Big], \end{aligned} \tag{A.4.20}$$

where $\|\mathcal{L}\|$ denotes the induced operator norm while $\|\cdot\|_\infty$ denotes the maximum norm. The assumption (A.4.7) and the estimate of (A.4.20) for $\|\mathcal{L}\|$ yield now the condition $\|\mathcal{L}\| \leq \gamma$ for a fixed positive constant $\gamma < 1$, with γ independent of ϵ. It follows directly that (A.4.18) has one and only one solution w, given by the Neumann series $w = \sum_{k=0}^{\infty} \mathcal{L}^k \psi$, and this solution satisfies the bound

$$\|w\| \leq \frac{1}{1-\gamma} \|\psi\|. \tag{A.4.21}$$

Moreover, from (A.4.18) one has the pointwise inequality

$$|w(x, \epsilon)| \leq |\psi(x, \epsilon)| + |\mathcal{L}w(x, \epsilon)|, \tag{A.4.22}$$

and the results (A.4.14)-(A.4.17) and (A.4.19)-(A.4.22) yield directly a bound of the type

$$|w(x,\epsilon)| \leq \frac{1}{\epsilon} \text{ const.}\left[|\mathcal{P}h(0,\epsilon)| + |(I-\mathcal{P})h(1,\epsilon)| + \|h'\| \right] \tag{A.4.23}$$

for $0 \leq x \leq 1$, $0 < \epsilon \leq \epsilon_0$, for some fixed $\epsilon_0 > 0$, where $\|\cdot\|$ is defined as in (A.4.17) and (5.9). These results show that the system (A.4.13) has one and only one solution, and the solution satisfies estimates of the type (A.4.8) and (A.4.9).

This solution of (A.4.13) also satisfies (A.4.10), which implies directly the result

$$\begin{aligned} \epsilon u(x) + f(x,\epsilon) &= \int_0^1 [A(x,s)u(s) + B(x,s)v(s)]ds + \alpha \\ \epsilon v(x) + g(x,\epsilon) &= \int_0^1 [C(x,s)u(s) + D(x,s)v(s)]ds + \beta \end{aligned} \tag{A.4.24}$$

for suitable vector constants of integration α and β. The two equations of (A.4.13) can be evaluated separately at $x = 0$ and at $x = 1$, and we find directly with (A.4.14)-(A.4.16) and (A.4.1) the results $\alpha = 0$ and $\beta = 0$ for the constants in (A.4.24). We conclude that the solution of (A.4.13) also solves (5.8), and the stated results of the lemma follow immediately. ■

Acknowledgments

We appreciate the helpful comments of G. I. Èskin, J. B. Keller, J. Kevorkian, R. M. Miura, and H. Weinitschke. The work of the first author was supported in part by the National Science Foundation under Grant DMS 85-04358 and by the Office of Naval Research under Grant N000 14-86-K-0691. The work of the second author was supported in part by the Schwerpunktprogram of the German National Science Foundation (DFG).

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