Analysis of Domain Decomposition Preconditioners on Irregular Regions

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Abstract

We present an algebraic analysis of some domain decomposition preconditioners on irregular regions. We analyze a preconditioner proposed in [3] for the interface system and prove that, for all L-shaped regions and some C-shaped regions, it produces a convergence rate that is independent of gridsize and aspect ratios. We prove that the condition number of the preconditioned capacitance system is bounded by 2.16 for all L-shaped domains. We also give some results for other simple irregular geometries.

1. Introduction

We consider the problem of solving an elliptic partial differential equation on a domain that is broken up into rectangular subregions. By using domain decomposition or substructuring techniques, the problem is reduced to separately solving approximate problems in the subdomains and updating the solution at the interfaces between two or more subregions. For the class of domain decomposition methods considered in this paper, the basic idea consists of the following: the differential operator is discretized on a grid imposed over the domain, which is partitioned into several subregions. Then, by applying block elimination to the discretized equations, a system is derived for the unknowns on the interfaces between subregions. This system is sometimes called the capacitance system. Forming the right hand side for the interface system requires the solution of independent elliptic problems on the subdomains. For certain constant coefficient problems on regular domains, fast direct methods can be applied to the solution of the interface system [3, 6]. Such is not the case, however, for more general operators or irregular domains. For efficiency reasons the system must then be solved by iterative methods, such as the preconditioned conjugate gradient method. Once the solution is known on the interfaces, one more elliptic problem must be solved on each subdomain with the computed values as boundary conditions.

\[ Au = f \]  

Figure 1: The domain \( \Omega \) and its partition

In order to illustrate the method, we will apply the process described above to a simple region \( \Omega \), which can be decomposed into two rectangles \( \Omega_1 \) and \( \Omega_2 \), with interface \( \Gamma_3 \), as shown in fig.1. Let

\[ Au = f \]  

(1.1)

2
represent the discretization of the differential operator on Ω. By reordering the variables, the system (1.1) can be written in block form as:

\[
\begin{pmatrix}
    A_{11} & A_{13} \\
    A_{21} & A_{23} \\
    A_{31} & A_{33}
\end{pmatrix}
\begin{pmatrix}
    u_1 \\
    u_2 \\
    u_3
\end{pmatrix} =
\begin{pmatrix}
    f_1 \\
    f_2 \\
    f_3
\end{pmatrix},
\]

(1.2)

where the indexes for \( u \) and \( f \) correspond to gridpoints in \( Ω_1, Ω_2 \) and \( Γ_3 \), respectively. Based on the following block decomposition of the matrix in (1.2),

\[
A = \begin{pmatrix}
    A_{11} & A_{13} \\
    A_{21} & A_{23} \\
    A_{31} & A_{33}
\end{pmatrix}
= \begin{pmatrix}
    I & A_{11}^{-1} A_{13} \\
    I & A_{22}^{-1} A_{23}
\end{pmatrix}
\]

(1.3)

where \( C \) is the Schur complement of \( A_{33} \) in \( A \), i.e.

\[
C = A_{33} - A_{31} A_{11}^{-1} A_{13} - A_{23} A_{22}^{-1} A_{23},
\]

(1.4)

the system (1.2) can be solved as follows:

**Step 1:** Solve

\[
A_{11} z_1 = f_1
\]

(1.5)

\[
A_{22} z_2 = f_2
\]

(1.6)

**Step 2:** Form

\[
g = f_3 - A_{13} z_1 - A_{23} z_2
\]

(1.7)

and solve

\[
C u_3 = g
\]

(1.8)

**Step 3:** Solve

\[
A_{11} u_1 = f_1 - A_{13} u_3
\]

(1.9)

and

\[
A_{22} u_2 = f_2 - A_{23} u_3
\]

(1.10)

Steps 1 and 3 correspond to solving independent problems on the subdomains. The matrix \( C \) given by (1.4), sometimes called the capacitance matrix, is dense and expensive to compute. It is possible, however, to compute the action of \( C \) on a vector \( v \) at the cost of solving problems on the subdomains with boundary conditions on \( Γ \) given by \( v \). Therefore, the interface system (1.8) is often solved by preconditioned conjugate gradients (PCG). Since each iteration involves solving problems on the subdomains, it is essential to keep the number of iterations low. For this reason, much effort has been devoted recently to the construction of good preconditioners for the capacitance matrix [7, 1, 8, 3, 6]. Many of the preconditioners proposed are spectrally equivalent to the exact boundary operator. They therefore yield convergence rates that are bounded independently of the gridsize. The method is particularly suited to problems for which the subproblems can be solved efficiently, for example, when the operator has separable coefficients. When the subdomain problems cannot be solved efficiently but they can be approximated by separable operators, it is possible to derive block preconditioners for the original system based on preconditioners for the interface system [9, 2, 4].

In [3], the case of a constant coefficient operator on a rectangular domain divided into two strips is analyzed. For this simple case, it is shown that, for many of the preconditioners proposed in the
literature, while the condition number of the preconditioned system can be bounded independently of the gridsize \( h \) for a fixed domain, it can grow as a function of the aspect ratio of the subdomains. Roughly speaking, the aspect ratio of a rectangle is the ratio between its width and its height. For a precise definition, see theorem 2.3 (note: for one of the preconditioners proposed in [1], the bound grows when only one of the subdomains becomes narrow). A fast direct solver for \( C \) based on Fourier analysis can be derived from the exact eigenvalue decomposition of the capacitance matrix. This operator takes aspect ratios into account and solves exactly the interface problem for the case of constant coefficients on a rectangle divided into two strips. It is therefore proposed in [3] to apply it as a preconditioner for interface systems on irregular regions or for variable coefficient operators. We will call this preconditioner \( M_C \). For the case of a five point finite differences discretization of the Poisson equation on a regular grid of size \( h = \frac{1}{n+1} \), \( M_C \) is formally given by the following decomposition:

\[
M_C = W_n \text{diag}(\lambda_j)W_n^T ,
\]

where \( W_n \) is the matrix of sine modes of dimension \( n \). Its elements are given by:

\[
w_{ij}(n) = \sqrt{\frac{2}{n+1}} (\sin ij \pi h)^T ,
\]

for \( i, j = 1, \ldots, n \), and the eigenvalues \( \lambda_j, j = 1, \ldots, n \), are given by

\[
\lambda_j(n, m_1, m_2) = -\left( \frac{1 + \gamma_j^{m_1+1}}{1 - \gamma_j^{m_1+1}} + \frac{1 + \gamma_j^{m_2+1}}{1 - \gamma_j^{m_2+1}} \right) \sqrt{\sigma_j + \frac{\sigma_j^2}{4}}
\]

where \( m_1 \) and \( m_2 \) are the number of grid points in the \( y \)-direction in \( \Omega_1 \) and \( \Omega_2 \) respectively.

\[
\sigma_j = 4 \sin^2 \left( \frac{j \pi}{(n+1) \frac{\pi}{2}} \right)
\]

and

\[
\gamma_j = \left( 1 + \frac{\sigma_j}{2} - \sqrt{\sigma_j + \frac{\sigma_j^2}{4}} \right)^2
\]

The preconditioners proposed in [7] and [8] have the same eigenvectors as (1.11), but the eigenvalues are those of the square root of the one-dimensional discrete Laplacian, namely \( \sqrt{\sigma_j} \) in [7] and \( \sqrt{\sigma_j + \frac{\sigma_j^2}{4}} \) in [8]. One of the preconditioners given in [1] has eigenvalues equal to \( \lambda_j \) of (1.13), with \( m_1 = m_2 \).

In this paper, we are interested in analyzing (1.11) as preconditioner on irregular domains and in particular, we want to study the dependency of the convergence rate on the gridsize and the shape of the domain. Many of the preconditioners, when applied to an L-shaped region, have convergence rates that are bounded independently of the gridsize. The bound, however, depends on the relative aspect ratios of the subdomains. All of the preconditioners, except for \( M_C \), are known to deteriorate when one of the subdomains becomes narrow. In section 2, we show that on any L-shaped region, the preconditioned capacitance matrix for \( M_C \) has a condition number that is bounded by 2.16, independently of gridsize and aspect ratios. Given an L-shaped region, there are two ways of separating it in two rectangular subregions. We prove, also in section 2, an interesting property of the preconditioner \( M_C \), namely that the convergence rate is not seriously affected by the way we choose to subdivide the domain. In section 3, we discuss the extension of
some of the results in section 2 to other shapes. In the proofs of sections 2 and 3, we often use a common operator, which describes the interaction between two perpendicular interior interfaces. This operator is analyzed in detail in the appendix.

2. L-shaped regions

In this section, we want to describe the interface operator and its preconditioners, for the simplest irregular shape that can be decomposed in rectangular subregions, namely an L-shaped domain. Consider the Poisson equation on the region $\Omega$ of fig. 2.

![Figure 2: L-shaped domain](image)

It is clear that either interface, $\Gamma_4$ or $\Gamma_5$, will divide the domain into two rectangles. We might ask ourselves the question: is a particular decomposition better than the other? In this section we will show that, for the particular preconditioner we analyze, the difference between the rates of convergence for the two decompositions, if any, is always very small (see theorem 2.1). We also give a bound for the condition number that is independent of the mesh size and the subdomain aspect ratios.

Let the linear system

$$Au = f$$

represent a standard second order five point discretization of differential equation on a regular grid imposed on the domain $\Omega$, where the gridpoints on the subdomains have been reordered first and then those on the interfaces $\Gamma_4$ and $\Gamma_5$. Consider, for example, the domain $\Omega$ as the union of two rectangles divided by the interface $\Gamma_4$. By the process described by equations (1.5) to (1.10), an interface system of the form

$$C_4u_4 = g$$

can be derived for the variables on $\Gamma_4$. The capacitance matrix $C_4$ is the Schur complement of $A_{44}$ in $A$. The matrix $A$ can be also decomposed as follows:

$$A = \begin{pmatrix} A_\Omega & \frac{1}{A_\Omega} P \\ P^T & C_{45} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

where

$$A_\Omega = \begin{pmatrix} A_{11} & A_{22} \\ A_{22} & A_{33} \end{pmatrix}, \quad P = \begin{pmatrix} A_{14} \\ A_{14} \\ A_{24} & A_{25} \end{pmatrix}$$

$$5$$
and $C_{45}$ is the Schur complement of $A_{44}$ and $A_{55}$ in $A$, i.e.,

$$C_{45} \equiv \begin{pmatrix} A_{44} & \mathbb{I} \\ A_{55} & -P^T A_{11}^{-1} P \end{pmatrix} = \begin{pmatrix} M_4 & -A_{24}^T A_{22}^{-1} A_{25} \\ -A_{24}^T A_{22}^{-1} A_{24} & M_5 \end{pmatrix},$$

(2.4)

with

$$M_4 = A_{44} - A_{14}^T A_{11}^{-1} A_{14} - A_{24}^T A_{22}^{-1} A_{24}$$

(2.5)

and

$$M_5 = A_{55} - A_{25}^T A_{22}^{-1} A_{25} - A_{35}^T A_{33}^{-1} A_{35}.$$  

(2.6)

The matrix $M_4$ would be the capacitance matrix for $\Gamma_4$ if the domain $\Omega_3$ were absent. Similarly, $M_5$ would be the capacitance matrix for $\Gamma_5$ if the domain $\Omega_1$ were absent. In fact, they are nothing but the pre-conditioner $M_C$ described in the previous section. Both $M_4$ and $M_5$ have eigenvalue decompositions of the form (1.11), with eigenvalues given by $\lambda_j(n,m_1,m_2)$ for $j = 1, \ldots, n$ and $\lambda_i(m_3,n,m_3)$ for $i = 1, \ldots, m_2$, respectively.

The matrix $C_4$ of (2.2) is, as we mentioned earlier, the Schur complement of $A_{44}$ in $A$, but it can also be written as the Schur complement of $M_4$ in $C_{45}$. Therefore, we can derive the following expression for $C_4$ in terms of $M_4$:

**Lemma 2.1.** The interface matrix for $\Gamma_4$ in $\Omega$ can be written as

$$C_4 = M_4 + B^T B,$$

(2.7)

where

$$B = (-M_5)^{-1/2} A_{25}^T A_{22}^{-1} A_{24}.$$  

(2.8)

The preconditioner proposed in [3] for $C_4$ would correspond to using $M_C = M_4$. Since $M_4$ is negative definite, we can choose $\sqrt{-M_4}$ as a symmetric pre-conditioner for $C_4$. Let us define the pre-conditioned matrix:

$$\hat{C}_4 = (-M_4)^{-1/2} C_4 (-M_4)^{-1/2},$$

(2.9)

then, by (2.7), we have

$$\hat{C}_4 = -I_n + V^T V,$$

(2.10)

where $V \in \mathbb{R}^{m_2 \times n}$ is

$$V = B (-M_4)^{-1/2}.$$  

(2.11)

Similarly, by deriving expressions for $C_5$ analogous to (2.7) to (2.10) and using $M_C = M_5$ as a pre-conditioner for $C_5$, we can prove that

$$\hat{C}_5 = -I_{m_2} + V V^T.$$  

(2.12)

We can make some immediate observations. First, if $n = m_2$, $\hat{C}_4 = \hat{C}_5$ and therefore, both ways of decomposing the domain would be equivalent. When, for example, $n > m_2$, $V^T V$ is rank deficient, and therefore $\hat{C}_4$ has at least $n - m_2$ eigenvalues that are equal to one. On the other hand, $\beta \neq 0$ is an eigenvalue of $V^T V$ if and only if $\beta$ is an eigenvalue of $VV^T$. Therefore, all eigenvalues of $\hat{C}_4$ are also eigenvalues of $\hat{C}_5$ and vice versa, except from, possibly, the eigenvalue 1. We summarize this in the following:


Table 1: Eigenvalues of preconditioned capacitance system for an L-shaped region

<table>
<thead>
<tr>
<th>( n = 31, m2 = 7 )</th>
<th>( n = 63, m2 = 15 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sv of ( \tilde{V} )</td>
<td>( \sigma(\tilde{C}_4) )</td>
</tr>
<tr>
<td>0.18204</td>
<td>0.96686</td>
</tr>
<tr>
<td>0.03868</td>
<td>0.99850</td>
</tr>
<tr>
<td>0.00514</td>
<td>0.99997</td>
</tr>
<tr>
<td>0.00045</td>
<td>0.99999</td>
</tr>
<tr>
<td>0.00002</td>
<td>1.00000</td>
</tr>
<tr>
<td>0.00000</td>
<td>1.00000</td>
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<td>1.00000</td>
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<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

**Theorem 2.1.** If \( n = m2 \), then \( \tilde{C}_4 = \tilde{C}_5 \). Otherwise, all eigenvalues of \( \tilde{C}_4 \) that are different from one are also eigenvalues of \( \tilde{C}_5 \) and vice versa. Moreover,

\[
K(\tilde{C}_4) \leq \frac{1}{1 - \|VT\|^2_2}.
\]

and

\[
K(\tilde{C}_5) \leq \frac{1}{1 - \|VT\|^2_2}.
\]

**Proof.** The first part of the theorem was proved earlier. Since \( VT \) and \( VTV^T \) are symmetric and non-negative definite and \( \tilde{C}_4 \) and \( \tilde{C}_5 \) are negative definite [5], \( \|VV^T\|^2_2 = \|VT\|^2_2 < 1 \), therefore, the eigenvalues \( \beta_i \) of \( VTV^T \) and \( VV^T \) are in [0,1). Suppose, for example, that \( n > m2 \). Then, all eigenvalues of \( \tilde{C}_4 \) are between 1 - \( \beta_{\text{max}} \) and 1 and all eigenvalues of \( \tilde{C}_5 \) are between 1 - \( \beta_{\text{max}} \) and 1 - \( \beta_{\text{min}} \). Then, we have

\[
K(\tilde{C}_4) = \frac{1}{1 - \beta_{\text{max}}} = \frac{1}{1 - \|VT\|^2_2}.
\]

and

\[
K(\tilde{C}_5) = \frac{1 - \beta_{\text{min}}}{1 - \beta_{\text{max}}} \leq \frac{1}{1 - \|VT\|^2_2}.
\]

From the results of theorem 2.1 it can be shown [5] that, given some equivalence conditions for the initial guess, the difference between the number of iterations when PCG is applied to \( C_4 \) with preconditioner \( M_4 \) and the number of iterations when PCG is applied to \( C_5 \) with preconditioner \( M_5 \) is at most one. In practice, however, both cases should be essentially equivalent when some of the eigenvalues of \( VTV \) are very small. Numerical computations show that the eigenvalues \( \beta_i \) of \( VTV \) and \( VV^T \) decrease very quickly with \( i \). Therefore, in finite precision, only a few eigenvalues of \( \tilde{C}_4 \) and \( \tilde{C}_5 \) are different from one, which leads to rapid convergence of the PCG method when applied to either matrix. Moreover, it also follows that \( K(\tilde{C}_4) \approx K(\tilde{C}_5) \). For example, for the L-shaped region with corners: (0,0), (3,0), (3,0.25), (1,0.25), (1,1.25) and (0,1.25), for \( n = 31 \) and 63, table 1 shows the singular values of \( V \) and the eigenvalues of \( \tilde{C}_5 \), computed in single precision.

We conclude that either way of decomposing an L-shaped region into two rectangles produces almost the same convergence rate, when preconditioner \( M_5 \) is used. Moreover, we will be able to
give an analytical bound on the condition number of the preconditioned capacitance matrix. This bound is derived from a bound on the norm of the operator $V^T V$. But first, the following theorem will give us a useful expression for the elements of a unitary transformation of $V$, where $W_n$ and $W_m$ are defined by (1.12).

**Theorem 2.2.** Let

$$\tilde{V} = W_m V W_n$$

(2.13)

Then, $\|V\|_2 = \|\tilde{V}\|_2$ and the elements of the matrix $\tilde{V}$ are given by

$$v_{ij} = \frac{2}{\sqrt{(n+1)(m+1)}} \frac{\sin \frac{ir}{m_2+1}}{s_j^{(4)}} \frac{\sin \frac{ir}{n_1+1}}{s_i^{(5)}} \left( \sigma_j^{(n)} + \sigma_i^{(m_2)} \right)$$

(2.14)

for $i = 1, \ldots, m_2$ and $j = 1, \ldots, n$, where

$$s_j^{(4)} = \sqrt{|\lambda_j(n_1, n_2)|}, \quad s_i^{(5)} = \sqrt{|\lambda_i(m_2, n, n_3)|}$$

and $\sigma_j^{(n)}$ and $\sigma_i^{(m_2)}$ are given by (1.14).

**Proof.** The operator $A_{25}^{-1} A_{22} A_{24}$ in (2.8) corresponds to $Q_{14}$ of theorem 4.2 (see appendix). Then, by replacing (4.6) and (1.11) in (2.8) and (2.11), we have (2.14).

As theorem 2.1 suggests, in order to find a bound for the condition number of the preconditioned capacitance system, we need to bound the norm of $V$, or $\|\tilde{V}\|$. Since we have an expression for the elements of $\tilde{V}$, we can bound $\|\tilde{V}\|_1$ and $\|\tilde{V}\|_\infty$ and then use the property:

$$\|\tilde{V}\|_2 \leq \sqrt{\|\tilde{V}\|_1 \|\tilde{V}\|_\infty}$$

The results are summarized in the next theorem. A proof can be found in [5]:

**Theorem 2.3.** Define the aspect ratio for domain $\Omega_2$ in fig. 2 as $\alpha = \frac{n_1+1}{m_2+1}$. Then,

a) $\|\tilde{V}\|_1 \leq \sqrt{\alpha} \cdot 0.733$ and $\|\tilde{V}\|_\infty \leq \sqrt{\frac{1}{\alpha}} \cdot 0.733$.

b) $\|V^T V\|_2 \leq \|\tilde{V}\|_2 = \|\tilde{V}\|_2 \leq \|\tilde{V}\|_1 \|\tilde{V}\|_\infty \leq 0.54$.

c) For all gridsizes and all L-shaped regions,

$$K(\tilde{C}_4) \leq 2.16 \quad \text{and} \quad K(\tilde{C}_5) \leq 2.16$$

(2.15)

In our experiments, condition numbers larger than 1.2 have not been observed. The bounds (2.15), however, are fairly tight for the expression $\sqrt{\|\tilde{V}\|_1 \|\tilde{V}\|_\infty}$, as was shown by numerical experiments with large values of $n$ and $m_2$.

We would also like to discuss briefly how the parameter $n_3$ (or, respectively, $m_1$) affects the performance of preconditioner $M_4$ ($M_5$). Clearly, as $n_3$ tends to zero, the domain $\Omega$ approaches the shape of a perfect rectangle. The preconditioner $M_4$ should reflect this by becoming the exact boundary operator. In other words, $K(C_4)$ should approach one. We can verify that this is the case as follows: $v_{ij}$ in (2.14) depends on $n_3$ only through $\lambda_i(m_2, n, n_3)$ (defined in (1.13)), which tends to infinity when the aspect ratio $\frac{n_3+1}{m_2+1}$ tends to zero, and therefore $v_{ij}$ tends to zero. However, we
can see that this dependency is very weak, because \( \lambda_j(m_2, n, n_3) \) tends rapidly to an asymptotic value independent of \( n_3 \) when such aspect ratio grows. Only the fact that

\[
\lambda_j(m_2, n, n_3) \geq 2\sqrt{\sigma_j}
\]

(2.16)
is used in the proof of theorem 2.3, which is true for all values of \( n_3 \). The discussion above implies that the performance of \( M_4 \) as a preconditioner for \( C_4 \) is fairly independent on how irregular the region is.

Incidentally, since only (2.16) was used in the proof of theorem 2.3, the bounds (a) and (b) for \( \|V^T V\| \) hold for other preconditioners as well, as long as (2.16) holds [7, 1, 8]. The bound given in (c), however, does not hold for other preconditioners for which the preconditioned system cannot be written in the form (2.9) or (2.12). In fact, the preconditioned system is always of the form \( X + V^T V \), where the norm of \( X \) may grow when the aspect ratio \( \alpha \) of domain \( \Omega_2 \) tends to zero (see [3] for an example on a T-shaped region).

3. Other irregular regions

Some of the expressions and results of the previous section are more general than they appear and they can be used as basic components for more complicated regions that are unions of rectangles. For example, a C-shaped region can be subdivided as indicated in fig. 3.

![Figure 3: C-shaped domain](image)

Similar to L-shaped domains, the region of fig. 3 can be separated in three rectangles by either \( \Gamma_6 \) and \( \Gamma_7 \), or \( \Gamma_8 \) and \( \Gamma_9 \). A system

\[
C_{67} \begin{pmatrix} u_6 \\ u_7 \end{pmatrix} = g_{67}
\]

can be derived by block elimination for the interfaces \( \Gamma_6 \) and \( \Gamma_7 \). This system can be preconditioned with a multistrip operator \( M_{67} \) described in [6]. \( M_{67} \) solves, exactly, the problem on a rectangle divided into three strips. Similarly, a system

\[
C_{89} \begin{pmatrix} u_8 \\ u_9 \end{pmatrix} = g_{89}
\]

can be derived for the interfaces \( \Gamma_8 \) and \( \Gamma_9 \), which can be preconditioned by a block diagonal preconditioner, with diagonal blocks \( M_8 \) and \( M_9 \), of the form (1.11).
By arguments similar to the previous section's, the preconditioned interface system for $\Gamma_6$ and $\Gamma_7$ can be written in the form

$$\hat{C}_{67} \equiv (-M_{67})^{-1/2}C_{67}(-M_{67})^{-1/2} = -I + V^TV$$

and similarly,

$$\hat{C}_{89} \equiv \begin{pmatrix} -M_8 & 0 \\ 0 & -M_9 \end{pmatrix}^{-1/2}C_{89}\begin{pmatrix} -M_8 & 0 \\ 0 & -M_9 \end{pmatrix}^{-1/2} = -I + VTV^T$$

for certain matrix $V \in \mathbb{R}^{(m_1+m_3)\times 2}$. A unitary transformation $\tilde{V}$ of $V$ can be written as a block two by two matrix whose block elements have expressions similar to the matrix $\tilde{V}$ for L-shaped regions [5]. By theorem 2.1, we have that both ways of dividing the domain are almost equivalent, and $K(\hat{C}_{67})$ and $K(\hat{C}_{89})$ are bounded by

$$\frac{1}{1 - \|V^TV\|_2} \quad (3.1)$$

When $m_1 = m_3 = n$, $K(\hat{C}_{67}) = K(\hat{C}_{89})$. For the case when $m_1 = m_3 \leq m_2$, the results of theorem 2.3 can be applied in the following theorem. We refer the interested reader to [5] for the proof.

**Theorem 3.1.** Let $\rho_1^L(\alpha)$ and $\rho_\infty^L(\alpha)$ be bounds for $\|\tilde{V}\|_1$ and $\|\tilde{V}\|_\infty$ for an L-shaped domain like fig. 2, where $\alpha$ is the aspect ratio for the domain $\Omega_2$ in the picture. Given a C-shaped region like fig. 3, if $m_1 = m_3 \leq m_2$, then

a) $\|\tilde{V}\|_1 \leq \frac{1}{\sqrt{0.866}}\rho_1^L\left(\frac{n+1}{m_1+1}\right)$ and $\|\tilde{V}\|_\infty \leq \frac{1}{\sqrt{0.866}}\rho_\infty^L\left(\frac{n+1}{m_1+1}\right)$.

b) $\|VTV\|_2 \leq \|\tilde{V}\|_2^2 \leq \|\tilde{V}\|_1\|\tilde{V}\|_\infty \leq 0.62$.

c) $K(\hat{C}_{67}) \leq 2.63$ and $K(\hat{C}_{89}) \leq 2.63$ for all gridsizes and all C-shaped regions such that $m_1 = m_3 \leq m_2$.

4. Appendix

The interaction between interior edges

In this appendix we will analyze the operator that represents the interaction between two interfaces of a given subdomain. Let the region $R$ of fig. 4, with edges $\Gamma_k$ for $k = 1$ to 4, be a subdomain of $\Omega$. Let $n_1$ be the number of gridpoints in $\Gamma_1$ and $n_2$, the number of gridpoints in $\Gamma_2$. The corner points are not included in the edges $\Gamma_1$. They may or may not be interior to $\Omega$. 10
Let $A$ be the discrete Laplacian operator defined on the domain $\Omega$. If the interface $\Gamma_k$, for $k \leq 4$, is interior to $\Omega$, then we can define $P_k$, the submatrix of $A$ that represents the coupling between gridpoints of $R$ and gridpoints on $\Gamma_k$. We are interested in describing the operator $Q_{kl}$ which takes a vector $v$ defined on the gridpoints of $\Gamma_1$, solves a Poisson problem on $R$ with the boundary values given by $v$ on $\Gamma_1$ and zero elsewhere and produces the restriction of the solution at the set of gridpoints in $R$ adjacent to $\Gamma_k$. Such operator can be written as follows:

$$Q_{kl} = P_k^T A_R^{-1} P_l$$

(4.1)

where $A_R$ represents the discrete Laplacian operator on $R$. When $\Gamma_k$ and $\Gamma_l$ are parallel, the operator $Q_{kl}$ is diagonalizable by Fourier modes. We illustrate this case by describing $Q_{13}$. The case $Q_{24}$ is completely analogous. The proof of the following theorem can be found in [6].

**Theorem 4.1.** Let $W_{n_1}$ be the matrix of sine modes of dimension $n_1$, (1.12). Let $Q_{13}$ be the operator that represents the coupling between interfaces $\Gamma_1$ and $\Gamma_3$, defined as in (4.1). Then,

$$Q_{13} = W_{n_1} D_{13} W_{n_1}$$

where the matrix $D_{13}$ is diagonal, with diagonal entries given by

$$d_{jj} = \sqrt{\gamma_j^{n_2}} \left( \frac{1 - \gamma_j}{1 - \gamma_j^{n_2+1}} \right)^2$$

(4.2)

with

$$\gamma_j = \left( 1 + \frac{\sigma_j^{(1)}}{2} \sqrt{\frac{\sigma_j^{(1)}}{4} + \left( \frac{\sigma_j^{(1)}}{4} \right)^2} \right)^2$$

(4.3)

and

$$\sigma_j^{(1)} = 4 \sin^2 \frac{j\pi}{2(n_1 + 1)}$$

(4.4)

The operators $Q_{12}$ and $Q_{14}$, on the other hand, are not diagonalizable by Fourier modes. Moreover, they are, in general not square, but $n_1$ by $n_2$ rectangular matrices. It is possible, however, to describe the elements of the matrices

$$\hat{Q}_{12} = W_{n_1} Q_{12} W_{n_2} \quad \text{and} \quad \hat{Q}_{14} = W_{n_1} Q_{14} W_{n_2}$$

where the elements of $W_{n_1}$, $l = 1, 2$, are given by (1.12), as follows:

**Theorem 4.2.** Let the operator $Q_{14}$ that represents the coupling between interfaces $\Gamma_1$ and $\Gamma_4$ be defined as in (4.1). Then, the elements of the matrix $\hat{Q}_{14}$ are given by

$$q_{ij}^{14} = \frac{2}{\sqrt{(n_1 + 1)(n_2 + 1)}} \frac{\sin \frac{ij}{n_1+1}}{\sigma_i^{(1)} + \sigma_j^{(2)}}$$

(4.5)

for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$, where $\sigma_j^{(l)} = 4 \sin^2 \frac{i\pi}{2(n_l+1)}$, for $l = 1, 2$. Similarly, the elements of the matrix $\hat{Q}_{12}$ are given by

$$q_{ij}^{12} = \frac{2}{\sqrt{(n_1 + 1)(n_2 + 1)}} \frac{\sin \frac{ni\pi}{n_1+1}}{\sigma_i^{(1)} + \sigma_j^{(2)}}$$

(4.6)
Proof. In order to simplify the notation, we will use direct (or tensor) products to define the various operators. The eigenvalue decomposition of the matrix \( A_R \) is well known and it is given by

\[
A_R = (W_2 \otimes W_1) A (W_2 \otimes W_1)
\]  

(4.7)

where \( A \) is the \( n_1 n_2 \times n_1 n_2 \) diagonal matrix whose diagonal elements are

\[
\lambda_j = -\sigma_i^{(1)} - \sigma_j^{(2)}
\]

with \( J = (j-1)n_1 + i \), for \( i = 1, \ldots, n_1 \) and \( j = 1, \ldots, n_2 \). The matrices \( P_1 \) and \( P_2 \) can be written as:

\[
P_1 = e_1^{(2)} \otimes I_l
\]

(4.8)

\[
P_2 = I_2 \otimes e_1^{(1)}
\]

(4.9)

where \( I_l \), for \( l = 1, 2 \), is the identity matrix of dimension \( n_l \) and \( e_1^{(1)} \) is the first column of \( I_l \). By replacing equations (4.7) to (4.9) in (4.1) and then applying the following two properties of tensor products:

i) \( (X \otimes Y)^T = X^T \otimes Y^T \) and

ii) \( (X_1 \otimes Y_1) (X_2 \otimes Y_2) = (X_1 X_2) \otimes (Y_1 Y_2) \),

we have:

\[
Q_{14} = \left( (e_1^{(2)T} W_2) \otimes W_1 \right) A_0^{-1} \left( W_2 \otimes (W_1 e_1^{(1)}) \right)
\]

(4.10)

and therefore,

\[
\dot{Q}_{14} = \left( (e_1^{(2)T} W_2) \otimes I_l \right) A_0^{-1} \left( I_2 \otimes (W_1 e_1^{(1)}) \right)
\]

(4.11)

Then we can see that the \( j \)-th column of (4.11) is given by

\[
\sqrt{\frac{2}{n_2 + 1}} \sin \frac{j \pi}{n_2 + 1} \left( \sigma_j^{(2)} I_1 + \text{diag}(\sigma_i^{(1)}) \right)^{-1} W_1 e_i^{(1)}
\]

from which (4.5) follows.

Similarly, (4.6) can be derived by using

\[
P_2 = I_2 \otimes e_{n_1}^{(1)}
\]

(4.12)

instead of (4.9).

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References


