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COMPUTATIONAL AND APPLIED MATHEMATICS

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May, 1987

CAM Report 87-05

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Invited talk at the Sixth IMACS International Symposium on Computer Methods for Partial Differential Equations. June 23-26, 1987 at Lehigh University, Bethlehem, PA 18015.

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Abstract

We present an algebraic analysis of some domain decomposition preconditioners on irregular regions. We analyze a preconditioner proposed in [3] for the interface system and prove that, for all L-shaped regions and some C-shaped regions, it produces a convergence rate that is independent of gridsize and aspect ratios. We prove that the condition number of the preconditioned capacitance system is bounded by 2.16 for *all* L-shaped domains. We also give some results for other simple irregular geometries.

1. Introduction

We consider the problem of solving an elliptic partial differential equation on a domain that is broken up into rectangular subregions. By using *domain decomposition* or *substructuring* techniques, the problem is reduced to separately solving approximate problems in the subdomains and updating the solution at the interfaces between two or more subregions. For the class of domain decomposition methods considered in this paper, the basic idea consists of the following: the differential operator is discretized on a grid imposed over the domain, which is partitioned into several subregions. Then, by applying block elimination to the discretized equations, a system is derived for the unknowns on the interfaces between subregions. This system is sometimes called the capacitance system. Forming the right hand side for the interface system requires the solution of independent elliptic problems on the subdomains. For certain constant coefficient problems on regular domains, fast direct methods can be applied to the solution of the interface system [3, 6]. Such is not the case, however, for more general operators or irregular domains. For efficiency reasons the system must then be solved by iterative methods, such as the preconditioned conjugate gradient method. Once the solution is known on the interfaces, one more elliptic problem must be solved on each subdomain with the computed values as boundary conditions.

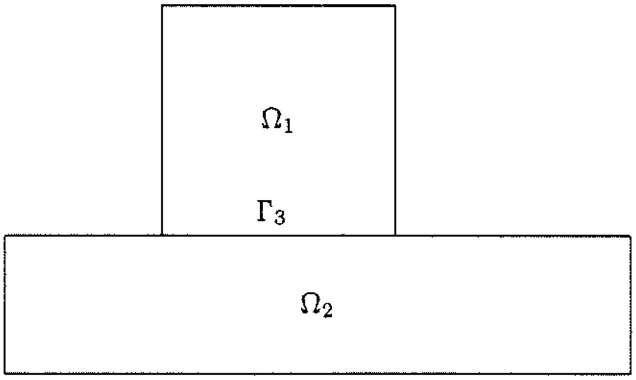

Figure 1: The domain Ω and its partition

In order to illustrate the method, we will apply the process described above to a simple region Ω, which can be decomposed into two rectangles Ω_1 and Ω_2, with interface Γ_3, as shown in fig.1. Let

$$Au = f \tag{1.1}$$

represent the discretization of the differential operator on Ω. By reordering the variables, the system (1.1) can be written in block form as:

$$\begin{pmatrix} A_{11} & & A_{13} \\ & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \quad , \tag{1.2}$$

where the indexes for u and f corespond to gridpoints in Ω_1, Ω_2 and Γ_3, respectively. Based on the following block decomposition of the matrix in (1.2),

$$A = \begin{pmatrix} A_{11} & & \\ & A_{22} & \\ A_{31} & A_{32} & C \end{pmatrix} \begin{pmatrix} I & & A_{11}^{-1}A_{13} \\ & I & A_{22}^{-1}A_{23} \\ & & I \end{pmatrix} \quad , \tag{1.3}$$

where C is the Schur complement of A_{33} in A, i.e.

$$C = A_{33} - A_{13}^T A_{11}^{-1} A_{13} - A_{23}^T A_{22}^{-1} A_{23} \quad , \tag{1.4}$$

the system (1.2) can be solved as follows:

Step 1: Solve

$$A_{11} z_1 = f_1 \tag{1.5}$$

$$A_{22} z_2 = f_2 \tag{1.6}$$

Step 2: Form

$$g = f_3 - A_{13}^T z_1 - A_{23}^T z_2 \tag{1.7}$$

and solve

$$C u_3 = g \tag{1.8}$$

Step 3: Solve

$$A_{11} u_1 = f_1 - A_{13} u_3 \tag{1.9}$$

and

$$A_{22} u_2 = f_2 - A_{23} u_3 \tag{1.10}$$

Steps 1 and 3 correspond to solving independent problems on the subdomains. The matrix C given by (1.4), sometimes called the capacitance matrix, is dense and expensive to compute. It is possible, however, to compute the action of C on a vector v at the cost of solving problems on the subdomains with boundary conditions on Γ given by v. Therefore, the interface system (1.8) is often solved by preconditioned conjugate gradients (PCG). Since each iteration involves solving problems on the subdomains, it is essential to keep the number of iterations low. For this reason, much effort has been devoted recently to the construction of good preconditioners for the capacitance matrix [7, 1, 8, 3, 6]. Many of the preconditioners proposed are spectrally equivalent to the exact boundary operator. They therefore yield convergence rates that are bounded independently of the gridsize. The method is particularly suited to problems for which the subproblems can be solved efficiently, for example, when the operator has separable coefficients. When the subdomain problems cannot be solved efficiently but they can be approximated by separable operators, it is possible to derive block preconditioners for the original system based on preconditioners for the interface system [9, 2, 4].

In [3], the case of a constant coefficient operator on a rectangular domain divided into two strips is analyzed. For this simple case, it is shown that, for many of the preconditioners proposed in the literature, while the condition number of the preconditioned system can be bounded independently of the gridsize h for a fixed domain, it can grow as a function of the aspect ratio of the subdomains. Roughly speaking, the aspect ratio of a rectangle is the ratio between its width and its height. For a precise definition, see theorem 2.3 (note: for one of the preconditioners proposed in [1], the bound grows when only one of the subdomains becomes narrow). A fast direct solver for C based on Fourier analysis can be derived from the exact eigenvalue decomposition of the capacitance matrix. This operator takes aspect ratios into account and solves exactly the interface problem for the case of constant coefficients on a rectangle divided into two strips. It is therefore proposed in [3] to apply it as a preconditioner for interface systems on irregular regions or for variable coefficient operators. We will call this preconditioner M_C. For the case of a five point finite differences discretization of the Poisson equation on a regular grid of size $h = \frac{1}{n+1}$, M_C is formally given by the following decomposition:

$$M_C = W_n \operatorname{diag}(\lambda_j) W_n^T \quad , \tag{1.11}$$

where W_n is the matrix of sine modes of dimension n. Its elements are given by:

$$w_{ij}(n) = \sqrt{\frac{2}{n+1}} (\sin ij\pi h)^T \tag{1.12}$$

for $i, j = 1, \ldots, n$, and the eigenvalues λ_j, $j = 1, \ldots, n$, are given by

$$\lambda_j(n, m_1, m_2) = -\left(\frac{1 + \gamma_j^{m_1+1}}{1 - \gamma_j^{m_1+1}} + \frac{1 + \gamma_j^{m_2+1}}{1 - \gamma_j^{m_2+1}} \right) \sqrt{\sigma_j + \frac{\sigma_j^2}{4}} \tag{1.13}$$

where m_1 and m_2 are the number of grid points in the y-direction in Ω_1 and Ω_2 respectively,

$$\sigma_j = 4 \sin^2 \left(\frac{j}{(n+1)} \frac{\pi}{2} \right) \tag{1.14}$$

and

$$\gamma_j = \left(1 + \frac{\sigma_j}{2} - \sqrt{\sigma_j + \frac{\sigma_j^2}{4}} \right)^2 \quad . \tag{1.15}$$

The preconditioners proposed in [7] and [8] have the same eigenvectors as (1.11), but the eigenvalues are those of the square root of the one-dimensional discrete Laplacian, namely $\sqrt{\sigma_j}$ in [7] and $\sqrt{\sigma_j + \frac{\sigma_j^2}{4}}$ in [8]. One of the preconditioners given in [1] has eigenvalues equal to λ_j of (1.13), with $m_1 = m_2$.

In this paper, we are interested in analyzing (1.11) as preconditioner on irregular domains and in particular, we want to study the dependency of the convergence rate on the gridsize *and* the shape of the domain. Many of the preconditioners, when applied to an L-shaped region, have convergence rates that are bounded independently of the gridsize. The bound, however, depends on the relative aspect ratios of the subdomains. All of the preconditioners, except for M_C, are known to deteriorate when one of the subdomains becomes narrow. In section 2, we show that on any L-shaped region, the preconditioned capacitance matrix for M_C has a condition number that is bounded by 2.16, independently of gridsize *and aspect ratios*. Given an L-shaped region, there are two ways of separating it in two rectangular subregions. We prove, also in section 2, an interesting property of the preconditioner M_C, namely that the convergence rate is not seriously affected by the way we choose to subdivide the domain. In section 3, we discuss the extention of some of the results in section 2 to other shapes. In the proofs of sections 2 and 3, we often use a common operator, which describes the interaction between two perpendicular interior interfaces. This operator is analyzed in detail in the appendix.

2. L-shaped regions

In this section, we want to describe the interface operator and its preconditioners, for the simplest irregular shape that can be decomposed in rectangular subregions, namely an L-shaped domain. Consider the Poisson equation on the region Ω of fig. 2.

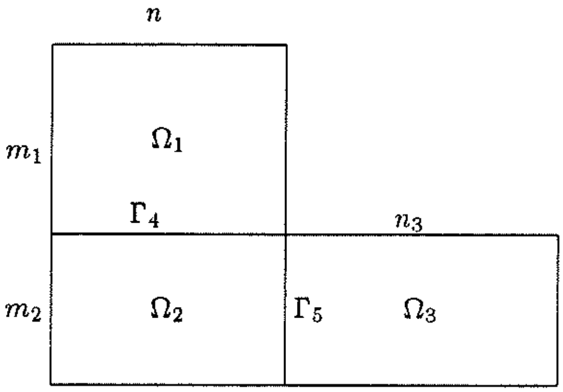

Figure 2: L-shaped domain

It is clear that either interface, Γ_4 or Γ_5, will divide the domain into two rectangles. We might ask ourselves the question: is a particular decomposition better than the other? In this section we will show that, for the particular preconditioner we analyze, the difference between the rates of convergence for the two decompositions, if any, is always very small (see theorem 2.1). We also give a bound for the condition number that is independent of the mesh size and the subdomain aspect ratios.

Let the linear system

$$Au = f \tag{2.1}$$

represent a standard second order five point discretization of differential equation on a regular grid imposed on the domain Ω, where the gridpoints on the subdomains have been reordered first and then those on the interfaces Γ_4 and Γ_5. Consider, for example, the domain Ω as the union of two rectangles divided by the interface Γ_4. By the process described by equations (1.5) to (1.10), an interface system of the form

$$C_4 u_4 = g \tag{2.2}$$

can be derived for the variables on Γ_4. The capacitance matrix C_4 is the Schur complement of A_{44} in A. The matrix A can be also decomposed as follows:

$$A = \begin{pmatrix} A_\Omega & \\ P^T & C_{45} \end{pmatrix} \begin{pmatrix} I & A_\Omega^{-1} P \\ & I \end{pmatrix} \quad , \tag{2.3}$$

where

$$A_\Omega = \begin{pmatrix} A_{11} & & \\ & A_{22} & \\ & & A_{33} \end{pmatrix} \quad , \qquad P = \begin{pmatrix} A_{14} & \\ A_{24} & A_{25} \\ & A_{35} \end{pmatrix}$$

and C_{45} is the Schur complement of A_{44} and A_{55} in A, i.e.,

$$C_{45} \equiv \begin{pmatrix} A_{44} & \\ & A_{55} \end{pmatrix} - P^T A_\Omega^{-1} P = \begin{pmatrix} M_4 & -A_{24}^T A_{22}^{-1} A_{25} \\ -A_{25}^T A_{22}^{-1} A_{24} & M_5 \end{pmatrix} \quad , \tag{2.4}$$

with

$$M_4 = A_{44} - A_{14}^T A_{11}^{-1} A_{14} - A_{24}^T A_{22}^{-1} A_{24} \tag{2.5}$$

and

$$M_5 = A_{55} - A_{25}^T A_{22}^{-1} A_{25} - A_{35}^T A_{33}^{-1} A_{35}. \tag{2.6}$$

The matrix M_4 would be the capacitance matrix for Γ_4 if the domain Ω_3 were absent. Similarly, M_5 would be the capacitance matrix for Γ_5 if the domain Ω_1 were absent. In fact, they are nothing but the preconditioner M_C described in the previous section. Both M_4 and M_5 have eigenvalue decompositions of the form (1.11), with eigenvalues given by $\lambda_j(n, m_1, m_2)$ for $j = 1, \ldots, n$ and $\lambda_i(m_2, n, n_3)$ for $i = 1, \ldots, m_2$, respectively.

The matrix C_4 of (2.2) is, as we mentioned earlier, the Schur complement of A_{44} in A, but it can also be written as the Schur complement of M_4 in C_{45}. Therefore, we can derive the following expression for C_4 in terms of M_4:

Lemma 2.1. *The interface matrix for Γ_4 in Ω can be written as*

$$C_4 = M_4 + B^T B \quad , \tag{2.7}$$

where

$$B = (-M_5)^{-1/2} A_{25}^T A_{22}^{-1} A_{24} \quad . \tag{2.8}$$

∎

The preconditioner proposed in [3] for C_4 would correspond to using $M_C = M_4$. Since M_4 is negative definite, we can choose $\sqrt{-M_4}$ as a symmetric preconditioner for C_4. Let us define the preconditioned matrix:

$$\hat{C}_4 = (-M_4)^{-1/2} C_4 (-M_4)^{-1/2} \quad , \tag{2.9}$$

then, by (2.7), we have

$$\hat{C}_4 = -I_n + V^T V \quad , \tag{2.10}$$

where $V \in R^{m_2 \times n}$ is

$$V = B(-M_4)^{-1/2} \quad . \tag{2.11}$$

Similarly, by deriving expressions for C_5 analogous to (2.7) to (2.10) and using $M_C = M_5$ as a preconditioner for C_5, we can prove that

$$\hat{C}_5 = -I_{m_2} + V V^T \quad . \tag{2.12}$$

We can make some inmediate observations. First, if $n = m_2$, $\hat{C}_4 = \hat{C}_5$ and therefore, both ways of decomposing the domain would be equivalent. When, for example, $n > m_2$, $V^T V$ is rank defficient, and therefore $\hat{C}_4$ has at least $n - m_2$ eigenvalues that are equal to one. On the other hand, $\beta \neq 0$ is an eigenvalue of $V^T V$ if and only if β is an eigenvalue of $V V^T$. Therefore, all eigenvalues of $\hat{C}_4$ are also eigenvalues of $\hat{C}_5$ and viceversa, except from, possibly, the eigenvalue 1. We summarize this in the following:

$n = 31, m2 = 7$		$n = 63, m2 = 15$	
sv of V	$\sigma(\hat{C}_5)$	sv of V	$\sigma(\hat{C}_5)$
0.18204	0.96686	2.165E-01	0.95312
0.03868	0.99850	6.816E-02	0.99535
0.00514	0.99997	1.578E-02	0.99975
0.00045	0.99999	2.971E-03	0.99999
0.00002	1.00000	4.607E-04	0.99999
0.00000	1.00000	5.863E-05	1.00000
0.00000	1.00000	6.082E-06	1.00000
		5.093E-07	1.00000
		3.610E-08	1.00000
		...	...

Table 1: Eigenvalues of preconditioned capacitance system for an L-shaped region

Theorem 2.1. *If $n = m_2$, then $\hat{C}_4 = \hat{C}_5$. Otherwise, all eigenvalues of $\hat{C}_4$ that are different from one are also eigenvalues of $\hat{C}_5$ and viceversa. Moreover,*

$$\mathcal{K}(\hat{C}_4) \leq \frac{1}{1 - \|V^T V\|_2} \quad .$$

and

$$\mathcal{K}(\hat{C}_5) \leq \frac{1}{1 - \|V^T V\|_2} \quad .$$

Proof. The first part of the theorem was proved earlier. Since $V^T V$ and VV^T are symmetric and non-negative definite and $\hat{C}_4$ and $\hat{C}_5$ are negative definite [5], $\|VV^T\|_2 = \|V^T V\|_2 < 1$, therefore, the eigenvalues β_j of $V^T V$ and VV^T are in $[0, 1)$. Suppose, for example, that $n > m_2$. Then, all eigenvalues of $\hat{C}_4$ are between $1 - \beta_{max}$ and 1 and all eigenvalues of $\hat{C}_5$ are between $1 - \beta_{max}$ and $1 - \beta_{min}$. Then, we have

$$\mathcal{K}(\hat{C}_4) = \frac{1}{1 - \beta_{max}} = \frac{1}{1 - \|V^T V\|_2} \quad .$$

and

$$\mathcal{K}(\hat{C}_5) = \frac{1 - \beta_{min}}{1 - \beta_{max}} \leq \frac{1}{1 - \|V^T V\|_2} \quad .$$

∎

From the results of theorem 2.1 it can be shown [5] that, given some equivalence conditions for the initial guess, the difference between the number of iterations when PCG is applied to C_4 with preconditioner M_4 and the number of iterations when PCG is applied to C_5 with preconditioner M_5, is at most one. In practice, however, both cases should be essentially equivalent when some of the eigenvalues of VV^T are very small. Numerical computations show that the eigenvalues β_i of $V^T V$ and VV^T decrease very quickly with i. Therefore, in finite precision, only a few eigenvalues of $\hat{C}_4$ and $\hat{C}_5$ are different from one, which leads to rapid convergence of the PCG method when applied to either matrix. Moreover, it also follows that $\mathcal{K}(\hat{C}_4) \approx \mathcal{K}(\hat{C}_5)$. For example, for the L-shaped region with corners: $(0,0), (3,0), (3,0.25), (1,0.25), (1,1.25)$ and $(0,1.25)$, for $n = 31$ and 63, table 1 shows the singular values of V and the eigenvalues of $\hat{C}_5$, computed in single precision.

We conclude that either way of decomposing an L-shaped region into two rectangles produces almost the same convergence rate, when preconditioner M_C is used. Moreover, we will be able to give an analytical bound on the condition number of the preconditioned capacitance matrix. This bound is derived from a bound on the norm of the operator $V^T V$. But first, the following theorem will give us a useful expression for the elements of a unitary transformation of V, where W_n and W_{m_2} are defined by (1.12).

Theorem 2.2. *Let*

$$\tilde{V} = W_{m_2} V W_n \quad . \tag{2.13}$$

Then, $\|V\|_2 = \|\tilde{V}\|_2$ *and the elements of the matrix* $\tilde{V}$ *are given by*

$$v_{ij} = \frac{2}{\sqrt{(n+1)(m_2+1)}} \frac{\sin \frac{i\pi}{m_2+1} \quad \sin \frac{jn\pi}{n+1}}{s_j^{(4)} s_i^{(5)} \left(\sigma_j^{(n)} + \sigma_i^{(m_2)}\right)} \tag{2.14}$$

for $i = 1, \ldots, m_2$ *and* $j = 1, \ldots, n$, *where*

$$s_j^{(4)} = \sqrt{|\lambda_j(n, m_1, m_2)|} \quad , \qquad s_i^{(5)} = \sqrt{|\lambda_i(m_2, n, n_3)|}$$

and $\sigma_j^{(n)}$ *and* $\sigma_j^{(m_2)}$ *are given by (1.14).*

Proof. The operator $A_{25}^T A_{22}^{-1} A_{24}$ in (2.8) corresponds to Q_{14} of theorem 4.2 (see appendix). Then, by replacing (4.6) and (1.11) in (2.8) and (2.11), we have (2.14).

∎

As theorem 2.1 suggests, in order to find a bound for the condition number of the preconditioned capacitance system, we need to bound the norm of V, or $\|\tilde{V}\|$. Since we have an expression for the elements of $\tilde{V}$, we can bound $\|\tilde{V}\|_1$ and $\|\tilde{V}\|_\infty$ and then use the property:

$$\|\tilde{V}\|_2 \leq \sqrt{\|\tilde{V}\|_1 \|\tilde{V}\|_\infty} \quad .$$

The results are summarized in the next theorem. A proof can be found in [5]:

Theorem 2.3. *Define the aspect ratio for domain* Ω_2 *in fig. 2 as* $\alpha = \frac{n+1}{m_2+1}$. *Then,*

- *a)* $\|\tilde{V}\|_1 \leq \sqrt{\alpha}\, 0.733$ *and* $\|\tilde{V}\|_\infty \leq \sqrt{\frac{1}{\alpha}}\, 0.733$.
- *b)* $\|V^T V\|_2 \leq \|V\|_2^2 = \|\tilde{V}\|_2^2 \leq \|\tilde{V}\|_1 \|\tilde{V}\|_\infty \leq 0.54$.
- *c) For all gridsizes and all L-shaped regions,*

$$\mathcal{K}(\hat{C}_4) \leq 2.16 \quad \text{and} \quad \mathcal{K}(\hat{C}_5) \leq 2.16 \quad . \tag{2.15}$$

In our experiments, condition numbers larger than 1.2 have not been observed. The bounds (2.15), however, are fairly tight for the expression $\sqrt{\|\tilde{V}\|_1 \|\tilde{V}\|_\infty}$, as was shown by numerical experiments with large values of n and m_2.

We would also like to discuss briefly how the parameter n_3 (or, respectively, m_1) affects the performance of preconditioner M_4 (M_5). Clearly, as n_3 tends to zero, the domain Ω approaches the shape of a perfect rectangle. The preconditioner M_4 should reflect this by becoming the exact boundary operator. In other words, $\mathcal{K}(\hat{C}_4)$ should approach one. We can verify that this is the case as follows: v_{ij} in (2.14) depends on n_3 only through $\lambda_i(m_2, n, n_3)$ (defined in (1.13)), which tends to infinity when the aspect ratio $\frac{n_3+1}{m_2+1}$ tends to zero, and therefore v_{ij} tends to zero. However, we can see that this dependency is very weak, because $\lambda_j(m_2, n, n_3)$ tends rapidly to an asymptotic value independent of n_3 when such aspect ratio grows. Only the fact that

$$\lambda_j(m_2, n, n_3) \geq 2\sqrt{\sigma_j} \tag{2.16}$$

is used in the proof of theorem 2.3, which is true for all values of n_3. The discussion above implies that the performance of M_4 as a preconditioner for C_4 is fairly independent on how irregular the region is.

Incidentally, since only (2.16) was used in the proof of theorem 2.3, the bounds (a) and (b) for $\|V^TV\|$ hold for other preconditioners as well, as long as (2.16) holds [7, 1, 8]. The bound given in (c), however, does not hold for other preconditioners for which the preconditioned system cannot be written in the form (2.9) or (2.12). In fact, the preconditioned system is always of the form $X + V^TV$, where the norm of X may grow when the aspect ratio α of domain Ω_2 tends to zero (see [3] for an example on a T-shaped region).

3. Other irregular regions

Some of the expressions and results of the previous section are more general than they appear and they can be used as basic components for more complicated regions that are unions of rectangles. For example, a C-shaped region can be subdivided as indicated in fig. 3.

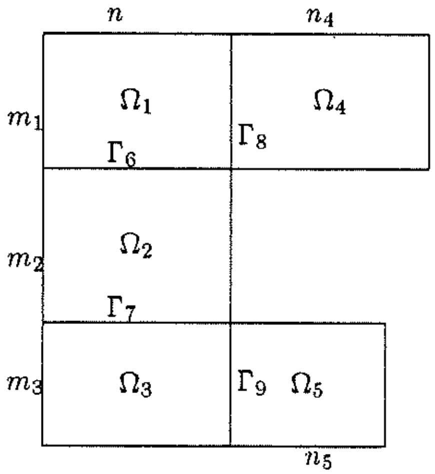

Figure 3: C-shaped domain

Similar to L-shaped domains, the region of fig. 3 can be separated in three rectangles by either Γ_6 and Γ_7, or Γ_8 and Γ_9. A system

$$C_{67}\begin{pmatrix} u_6 \\ u_7 \end{pmatrix} = g_{67}$$

can be derived by block elimination for the interfaces Γ_6 and Γ_7. This system can be preconditioned with a multistrip operator M_{67} described in [6]. M_{67} solves, exactly, the problem on a rectangle divided into three strips. Similarly, a system

$$C_{89}\begin{pmatrix} u_8 \\ u_9 \end{pmatrix} = g_{89}$$

can be derived for the interfaces Γ_8 and Γ_9, which can be preconditioned by a block diagonal preconditioner, with diagonal blocks M_8 and M_9, of the form (1.11).

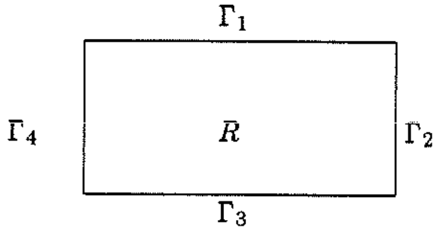

Figure 4: Interaction between interfaces

By arguments similar to the previous section's, the preconditioned interface system for Γ_6 and Γ_7 can be written in the form

$$\hat{C}_{67} \equiv (-M_{67})^{-1/2} C_{67} (-M_{67})^{-1/2} = -I + V^T V$$

and similarly,

$$\begin{aligned} \hat{C}_{89} &\equiv \begin{pmatrix} -M_8 & 0 \\ 0 & -M_9 \end{pmatrix}^{-1/2} C_{89} \begin{pmatrix} -M_8 & 0 \\ 0 & -M_9 \end{pmatrix}^{-1/2} , \\ &= -I + V V^T \end{aligned}$$

for certain matrix $V \in R^{(m_1+m_3)\times 2n}$. A unitary tranformation $\tilde{V}$ of V can be written as a block two by two matrix whose block elements have expressions similar to the matrix $\tilde{V}$ for L-shaped regions [5]. By theorem 2.1, we have that both ways of dividing the domain are almost equivalent, and $\mathcal{K}(\hat{C}_{67})$ and $\mathcal{K}(\hat{C}_{89})$ are bounded by

$$\frac{1}{1 - \|V^T V\|_2} \qquad (3.1) \quad .$$

When $m_1 = m_3 = n$, $\mathcal{K}(\hat{C}_{67}) = \mathcal{K}(\hat{C}_{89})$. For the case when $m_1 = m_3 \leq m_2$, the results of theorem 2.3 can be applied in the following theorem. We refer the interested reader to [5] for the proof.

Theorem 3.1. *Let $\rho_1^L(\alpha)$ and $\rho_\infty^L(\alpha)$ be bounds for $\|\tilde{V}\|_1$ and $\|\tilde{V}\|_\infty$ for an L-shaped domain like fig. 2, where α is the aspect ratio for the domain Ω_2 in the picture. Given a C-shaped region like fig.3, if $m_1 = m_3 \leq m_2$, then*

- *a)* $\|\tilde{V}\|_1 \leq \frac{1}{\sqrt{0.866}} \rho_1^L(\frac{n+1}{m_1+1})$ *and* $\|\tilde{V}\|_\infty \leq \frac{1}{\sqrt{0.866}} \rho_\infty^L(\frac{n+1}{m_1+1})$.
- *b)* $\|V^T V\|_2 \leq \|V\|_2^2 = \|\tilde{V}\|_2^2 \leq \|\tilde{V}\|_1 \|\tilde{V}\|_\infty \leq 0.62$.
- *c)* $\mathcal{K}(\hat{C}_{67}) \leq 2.63$ *and* $\mathcal{K}(\hat{C}_{89}) \leq 2.63$ *for all gridsizes and all C-shaped regions such that* $m_1 = m_3 \leq m_2$.

∎

4. Appendix

The interaction between interior edges

In this appendix we will analyze the operator that represents the interaction between two interfaces of a given subdomain. Let the region R of fig. 4, with edges Γ_k for $k = 1$ to 4, be a subdomain of Ω. Let n_1 be the number of gridpoints in Γ_1 and n_2, the number of gridpoints in Γ_2. The corner points are not included in the edges Γ_i. They may or may not be interior to Ω.

Let A be the discrete Laplacian operator defined on the domain Ω. If the interface Γ_k, for $k \leq 4$, is interior to Ω, then we can define P_k, the submatrix of A that represents the coupling between gridpoints of R and gridpoints on Γ_k. We are interested in describing the operator Q_{kl} which takes a vector v defined on the gridpoints of Γ_l, solves a Poisson problem on R with the boundary values given by v on Γ_l and zero elsewhere and produces the restriction of the solution at the set of gridpoints in R adjacent to Γ_k. Such operator can be written as follows:

$$Q_{kl} = P_k^T A_R^{-1} P_l \quad , \tag{4.1}$$

where A_R represents the discrete Laplacian operator on R. When Γ_k and Γ_l are parallel, the operator Q_{kl} is diagonalizable by Fourier modes. We illustrate this case by describing Q_{13}. The case Q_{24} is completely analogous. The proof of the following theorem can be found in [6].

Theorem 4.1. *Let W_{n_1} be the matrix of sine modes of dimension n_1, (1.12). Let Q_{13} be the operator that represents the coupling between interfaces Γ_1 and Γ_3, defined as in (4.1). Then,*

$$Q_{13} = W_{n_1} D_{13} W_{n_1}$$

where the matrix D_{13} is diagonal, with diagonal entries given by

$$d_{jj} = \sqrt{\gamma_j^{n_2}} \left(\frac{1-\gamma_j}{1-\gamma_j^{n_2+1}} \right) \quad , \tag{4.2}$$

with

$$\gamma_j = \left(1 + \frac{\sigma_j^{(1)}}{2} - \sqrt{\sigma_j^{(1)} + \frac{\left(\sigma_j^{(1)}\right)^2}{4}} \right)^2 \tag{4.3}$$

and

$$\sigma_j^{(1)} = 4 \sin^2 \frac{j\pi}{2(n_1+1)} \tag{4.4}$$

∎

The operators Q_{12} and Q_{14}, on the other hand, are not diagonalizable by Fourier modes. Moreover, they are, in general not square, but n_1 by n_2 rectangular matrices. It is possible, however, to describe the elements of the matrices

$$\hat{Q}_{12} = W_{n_1} Q_{12} W_{n_2} \qquad \text{and} \qquad \hat{Q}_{14} = W_{n_1} Q_{14} W_{n_2} \quad ,$$

where the elements of W_{n_l}, $l = 1, 2$, are given by (1.12), as follows:

Theorem 4.2. *Let the operator Q_{14} that represents the coupling between interfaces Γ_1 and Γ_4 be defined as in (4.1). Then, the elements of the matrix $\hat{Q}_{14}$ are given by*

$$q_{ij}^{14} = \frac{2}{\sqrt{(n_1+1)(n_2+1)}} \, \frac{\sin \frac{i\pi}{n_1+1} \quad \sin \frac{j\pi}{n_2+1}}{\sigma_i^{(1)} + \sigma_j^{(2)}} \tag{4.5}$$

for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$, where $\sigma_j^{(l)} = 4 \sin^2 \frac{j\pi}{2(n_l+1)}$, for $l = 1, 2$. Similarly, the elements of the matrix $\hat{Q}_{12}$ are given by

$$q_{ij}^{12} = \frac{2}{\sqrt{(n_1+1)(n_2+1)}} \, \frac{\sin \frac{n_1 i\pi}{n_1+1} \quad \sin \frac{j\pi}{n_2+1}}{\sigma_i^{(1)} + \sigma_j^{(2)}} \tag{4.6}$$

Proof. In order to simplify the notation, we will use direct (or tensor) products to define the various operators. The eigenvalue decomposition of the matrix A_R is well known and it is given by

$$A_R = (W_2 \otimes W_1) \quad \Lambda \quad (W_2 \otimes W_1) \tag{4.7}$$

where Λ is the $n_1 n_2 \times n_1 n_2$ diagonal matrix whose diagonal elements are

$$\lambda_J = -\sigma_i^{(1)} - \sigma_j^{(2)} \quad ,$$

with $J = (j-1)n_1 + i$, for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. The matrices P_1 and P_2 can be written as:

$$P_1 = e_1^{(2)} \otimes I_1 \tag{4.8}$$

$$P_4 = I_2 \otimes e_1^{(1)} \tag{4.9}$$

where I_l, for $l = 1, 2$, is the identity matrix of dimension n_l and $e_1^{(l)}$ is the first column of I_l. By replacing equations (4.7) to (4.9) in (4.1) and then applying the following two properties of tensor products:

$$\begin{aligned} &i) \quad (X \otimes Y)^T = X^T \otimes Y^T \text{and} \\ &ii) \quad (X_1 \otimes Y_1)(X_2 \otimes Y_2) = (X_1 X_2) \otimes (Y_1 Y_2) \quad , \end{aligned}$$

we have:

$$Q_{14} = \left((e_1^{(2)T} W_2) \otimes W_1 \right) \Lambda_\Omega^{-1} \left(W_2 \otimes (W_1 e_1^{(1)}) \right) \tag{4.10}$$

and therefore,

$$\hat{Q}_{14} = \left((e_1^{(2)T} W_2) \otimes I_1 \right) \Lambda_\Omega^{-1} \left(I_2 \otimes (W_1 e_1^{(1)}) \right) \tag{4.11}$$

Then we can see that the j-th column of (4.11) is given by

$$\sqrt{\frac{2}{n_2+1}} \sin \frac{j\pi}{n_2+1} \left(\sigma_j^{(2)} I_1 + \operatorname{diag}(\sigma_i^{(1)}) \right)^{-1} W_1 \; e_{n_1}^{(1)} \quad ,$$

from which (4.5) follows.

Similarly, (4.6) can be derived by using

$$P_2 = I_2 \otimes e_{n_1}^{(1)} \tag{4.12}$$

instead of (4.9).

∎

Acknowledgments:

The authors were supported in part by the Department of Energy under contract DE-AC02-81ER10996.

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