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Abstract Particle methods approximating hyperbolic partial differential equations with oscillatory solutions are studied. Convergence is proved for approximations for which the continuous solution is not well resolved on the computational grid. Highly oscillatory solutions to the Broadwell and variable coefficient Carleman model are considered. Homogenization results are given and the approximations of more general systems are discussed. Numerical examples are presented.

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1. Introduction

The main difficulty in practical computations of solutions to partial differential equations is often the presence of very different scales in the problem. On a grid that must cover the whole domain of the independent variables, it may be impossible to well resolve highly oscillatory components in the solution. A natural question is if some averaged quantities of the solutions still can be accurately computed.

In [7] it was proved that for a class of hyperbolic differential equations it is possible to get an accurate approximation to a moving space average of the solution even if there are oscillatory components in this solution that are not resolved on the computational grid. For these types of problems standard finite difference or finite element methods are often not useful [7,8]. Errors due to numerical dissipation and dispersion may create O(1) errors even after averaging. Particle methods can have much better convergence properties [7,16]. The errors for particle methods are not of dissipative or dispersive type. They are errors in the approximation of the characteristics and in the approximation of lower order terms. We will see from the analysis and numerical examples in this paper that these errors do not accumulate in the same way as in the standard finite difference schemes when applied to problems with oscillatory solutions.

It is known that high frequency components in the solution of the semilinear Carleman equations [4],

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + u^2 - v^2 = 0, \\[2mm] \dfrac{\partial v}{\partial t} - \dfrac{\partial v}{\partial x} - u^2 + v^2 = 0, \end{cases}$$

may interact through the nonlinear terms and create low frequency components which affect the averaged solutions [19]. In [7] it was proved that this process is correctly described by an appropriate particle method.

In this paper approximation to the Broadwell model [3],

$$(1.2)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + uv - w^2 = 0, \\[2mm] \dfrac{\partial v}{\partial t} - \dfrac{\partial v}{\partial x} + uv - w^2 = 0, \\[2mm] \dfrac{\partial w}{\partial t} \qquad\quad - uv + w^2 = 0, \end{cases}$$

is studied. This set of equations is a more realistic kinetic model than the simple Carleman model. It also describes the important resonance phenomena in which oscillatory components (in u and v) interact and create high frequency oscillations in another component (w) [18]. The generation of high frequencies in a solution due to nonlinear interactions occurs in many problems. We will show that this process may be well represented by a particle method even on a coarse grid.

More precisely, we will show in section 2 that for oscillatory initial values

$$
(1.3) \qquad \begin{cases} u(x,0) = u_0(x, \frac{x}{\varepsilon}), \\ v(x,0) = v_0(x, \frac{x}{\varepsilon}), \\ w(x,0) = w_0(x, \frac{x}{\varepsilon}), \end{cases}
$$

where $u_0(x,y)$, $v_0(x,y)$, $w_0(x,y)$ are 1-periodic in y, there are particle methods which converge essentially independent of ε as the stepsize in the numerical method decreases. The convergence essentially independent of ε is defined as follows.

Definition: Let v^n represent the numerical approximation to u at time t_n ($t_n = n\Delta t$). The approximation v^n converges as $\Delta t \to 0$ essentially independent of ε to u if for any $\delta > 0$, $T > 0$ there exists a set $s(\varepsilon,\Delta t_0) \subset (0,\Delta t_0)$ with measure $(s(\varepsilon,\Delta t_0)) \geq (1-\delta)\Delta t_0$ such that

$$\|u(\cdot,t_n) - v^n\| \leq \delta \,, \quad 0 \leq t_n \leq T$$

is valid for all $\Delta t \in s(\varepsilon,\Delta t_0)$ and where Δt_0 is independent of ε.

The convergence concept of essentially independent of ε is strong enough to mimic the practical case where the high frequency oscillations are not well resolved on the grid. The standard convergence when $\Delta t \to 0$ for a fixed $\varepsilon > 0$ is trivial for these problems. On the other hand our concept is weak enough to make a proof possible. A small set of values of Δt has to be removed in order to avoid resonance between the Δt and ε scales. Compare the almost always convergence for Monte Carlo methods [13].

Section 2 also contains some homogenization results for the Broadwell equations. These homogenization results are used in the convergence proof of particle methods. The convergence results mean that the discrete approximation and the continuous problem have the same homogenized limit as $\varepsilon \to 0$.

The hyperbolic part in (1.2) is solved exactly by the particle method. No averaging is therefore needed in the convergence result. In section 3 we analyze a numerical approximation of the Carleman equations with variable coefficients. The scheme is designed such that particle interaction can be accounted for without introducing interpolation. In our particle scheme, the numerical solution is computed only at those points at which a leftward characteristic line intersects a rightward one. There will be errors in the particle method approximation of the linear part of the system since the characteristic equations for the flow trajectories are solved numerically. As a result, the convergence can only be proved for moving space averages. The convergence proofs for Carleman equations and the Broadwell equations have one feature in common. The local truncation errors in both cases are of order $O(\Delta t)$. In order to show convergence, one needs to take into account cancellation of the local errors at different time levels.

The approximation of oscillatory solutions to more general forms of discrete Boltzmann equations are discussed in section 4. This Section also contains an analysis of particle approximations of scalar multidimensional problems with high frequency solutions. The setting is similar to vortex method approximations of the Euler equations for incompressible flow [1,5,15]. For the analysis of the full vortex method along these lines, see [6,10].

Section 5 contains numerical examples. Quantitative and Qualitative results are given for the convergence properties of the different particle approximations.

2. The Broadwell Model

The Broadwell equation [3] describes a three-dimensional model of rarefied gas in which the particles travel with speed c in either direction along a coordinate axis. If particles traveling in opposite directions collide, they are equally likely to move in each of the three coordinate directions after collision, with velocities of opposite sigh. Other collisions can lead to an exchange of velocities. We denote $N_1^+(x,y,z,t)$ as the number density of particles with velocity (c, 0, 0) at the point (x,y,z) and time t, N_1^- as the density with velocity (-c, 0, 0), and similarly for $N_2\pm$, $N_3^\pm$. Then the resulting equations are

$$\frac{\partial N_1^+}{\partial t} + c\frac{\partial N_1^+}{\partial x} = \frac{\sigma}{3}(N_2^+N_2^- + N_3^+N_3^- - 2N_1^+N_1^-),$$

$$\frac{\partial N_1^-}{\partial t} - c\frac{\partial N_1^-}{\partial x} = \frac{\sigma}{3}(N_2^+N_2^- + N_3^+N_3^- - 2N_1^+N_1^-),$$

ect., where σ is the frequency of collision.

Here we consider the special case of one-dimensional motions in which the N's are independent of y,z , and furthermore $N_2^+ = N_2^- = N_3^+ = N_3^-$. Setting $N_1 = u(x,t)$, $N_2 = v(x,t)$, $N_3 = w(x,t)$ and rescaling the variables so that c=1, σ = 3/2 . We arrive at the Broadwell equation (1.2). In addition we assume that the initial data are given by (1.3), where $u_0(x,y)$, $v_0(x,y)$, $w_0(x,y)$ in (1.3) are 1-periodic functions in the y variable. Moreover we assume that $u_0(x,y)$, $v_0(x,y)$, $w_0(x,y)$ are nonnegative smooth functions in x and y and have compact supports with respect to x.

2.1 The Homogenization Results

The global existence theorem of equation (1.2) was first obtained by Nishida and Mimura [14] in 1974, when the initial data are small in some sense. Subsequently the global existence results without the smallness assumption on the initial data have been obtained by Tartar and Crandall [17], Beale [2] among others.

The properties of the Broadwell model with highly oscillatory initial data are very much different from those of the Carleman model [18]. With the presence of term uv on the right hand side of (1.2), the high frequency components of u and v interact and create oscillation on w. And the w^2 term will then generate low frequency contribution to u and v, thus affecting the average of the solution. This is described by the following homogenization result by Tartar and Papanicolaou [19].

Theorem 2.1 *The solution to* (1.2-3) *converges uniformly to that of the homogenized equation* (2.1-2)

$$u(x,t) - U(x,\frac{x-t}{\varepsilon},t) \to 0 \quad as \quad \varepsilon \to 0,$$

$$v(x,t) - V(x,\frac{x+t}{\varepsilon},t) \to 0 \quad as \quad \varepsilon \to 0,$$

$$w(x,t) - W(x,\frac{x}{\varepsilon},t) \to 0 \quad as \quad \varepsilon \to 0 \quad \text{for } 0 \le t \le T,$$

where U(x,y,t), V(x,y,t) and W(x,y,t) are the solutions of the homogenized equation (2.1-2):

$$(2.1) \qquad \begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{\partial U}{\partial x} + U\displaystyle\int_0^1 V dy - \int_0^1 W^2 = 0, \\ \dfrac{\partial V}{\partial t} - \dfrac{\partial V}{\partial x} + V\displaystyle\int_0^1 U dy - \int_0^1 W^2 = 0, \\ \dfrac{\partial W}{\partial t} + W^2 - \displaystyle\int_0^1 U(x,y-z,t)V(x,y+z,t)dz = 0, \end{cases}$$

with initial values given by

$$(2.2) \qquad \begin{cases} U(x,y,0) = u_0(x,y), \\ V(x,y,0) = v_0(x,y), \\ W(x,y,0) = w_0(x,y). \end{cases}$$

Here we have assumed that a smooth and bounded global solution of (2.1-2) exists up to time T.

Remark 2.1 The homogenization result for the Broadwell equation (1.2) has been obtained by Papanicolaou and Tartar (see e.g. [18]). They can show that the solution in (1.2-3) converges to to that of (2.1-2) strongly in the L^p sense ($p<\infty$). For our purpose, we need convergence in the L^∞-norm and therefore a proof of uniform convergence is included.

Remark 2.2 The local existence result of the homogenized equation (1.1-2) can be obtained by the classical analysis for smooth and bounded initial data. The global existence result for the homogenized equation can then be obtained by combining the known global existence results of (1.2-3) [14,18,2] and the Homogenization Theorem 2.1. Therefore the value of T in Theorem 2.1 is arbitrarily large.

Lemma 2.1 *Let $f(x)$, $g(x,y) \in C^1$. Assume further that $g(x,y)$ is 1-periodic in y and satisfies the relation $\int_0^1 g(x,y)dy = 0$. Then for any constant a and b, we have*

$$\left| \int_a^b f(x) g(x, \frac{x}{\varepsilon}) dx \right| \le C\varepsilon.$$

proof : Express g($x, \frac{x}{\varepsilon}$) as

$$(2.3) \qquad g(x,\frac{x}{\varepsilon}) = \frac{d}{dx}\int_a^x g(x,\frac{s}{\varepsilon})ds - \int_a^x \frac{\partial g}{\partial x}(x,\frac{s}{\varepsilon})ds.$$

Since for any integer n,

$$\int_n^{n+1} \frac{\partial g}{\partial x}(x,y)dy = \frac{\partial}{\partial x}\int_n^{n+1} g(x,y)dy \equiv 0.$$

We conclude that

$$(2.4) \qquad \left| \int_a^b g(y,\frac{s}{\varepsilon})ds \right| \le C_1\varepsilon, \qquad \left| \int_a^x \frac{\partial g}{\partial x}(x,\frac{s}{\varepsilon})ds \right| \le C_2\varepsilon.$$

From equation (2.3) and (2.4), we deduce that

$$\left| \int_a^b f(x)g(x,\frac{x}{\varepsilon})dx \right| =$$

$$\left| f(b)\int_a^b g(b,\frac{s}{\varepsilon})ds - \int_a^b(\int_a^x g(x,\frac{s}{\varepsilon})ds)\frac{df(x)}{dx}dx - \int_a^b f(x)(\int_a^x \frac{\partial g}{\partial x}(x,\frac{s}{\varepsilon})ds)dx \right| \le C\varepsilon.$$

This completes the proof of the Lemma.

proof of Theorem 2.1 :

Subtracting the last equation of (2.1) from that of (1.2) and integrating the resulting equation along their characteristics from 0 to t, we obtain

$$w - W(x,\frac{x}{\varepsilon},t) = -\int_0^t (w(x,s)^2 - W(x,\frac{x}{\varepsilon},s)^2)ds$$

$$(2.5) \qquad +\int_0^t (u(x,s)v(x,s) - U(x,\frac{x-s}{\varepsilon},s)V(x,\frac{x+s}{\varepsilon},s))ds$$

$$+\int_0^t (U(x,\frac{x}{\varepsilon}-\frac{s}{\varepsilon},s)V(x,\frac{x}{\varepsilon}+\frac{s}{\varepsilon},s) - \int_0^1 U(x,\frac{x}{\varepsilon}-z,s)V(x,\frac{x}{\varepsilon}+z,s)dz)ds.$$

For fixed x and t, define

$$f(s) = 1,$$

$$g(s,y) = U(x,\frac{x}{\varepsilon}-y,s)V(x,\frac{x}{\varepsilon}+y,s) - \int_0^1 U(x,\frac{x}{\varepsilon}-z,s)V(x,\frac{x}{\varepsilon}+z,s)dz,$$

then the last term in (2.5) becomes

$$\int_0^t f(s)g(s,\frac{s}{\varepsilon})ds.$$

Obviously f and g satisfy the assumption of Lemma 2.1 . Thus Lemma 2.1 implies that the last term in (2.5) is bounded by $C\varepsilon$ uniformly. The constant C is independent of ε .

Define

$$M = \sup_{0\le t\le T}\{ \sup_{x,y}\{ |u_\varepsilon(x,t)|, |U(x,y,t)|, \dots |W(x,y,t)| \} \}.$$

We deduce from (2.5) that

$$(2.6) \qquad |w_\varepsilon(x,t) - W(x,\frac{x}{\varepsilon},t)| \le 2M\int_0^t G(x,s)ds + C\varepsilon,$$

where G(x,t) is defined by

$$(2.7) \qquad G(x,t) = | u_\varepsilon(x,t) - U(x,\frac{x-t}{\varepsilon},t) | + | v_\varepsilon(x,t) - V(x,\frac{x+t}{\varepsilon},t) | + | w_\varepsilon(x,t) - W(x,\frac{x}{\varepsilon},t) | .$$

Similarly, we obtain

$$(2.8) \qquad | u_\varepsilon(x,t) - U(x,\frac{x-t}{\varepsilon},t) | \le 2M\int_0^t G(x-t+s,s)ds + C\varepsilon,$$

$$(2.9) \qquad | v_\varepsilon(x,t) - V(x,\frac{x+t}{\varepsilon},t) | \le 2M\int_0^t G(x+t-s,s)ds + C\varepsilon.$$

(2.6) + (2.8) + (2.9) yields

$$(2.10) \qquad G(x,t) \le 2M\int_0^t (G(x-t+s,s) + G(x+t-s,s) + G(x,s))ds + 3C\varepsilon.$$

Define

$$E(t) = \sup_x \{ G(x,t) \}$$

It follows immediately from (2.10) that

$$(2.11) \qquad E(t) \le 6M\int_0^t E(s)ds + 3C\varepsilon$$

and the application of the Gronwall inequality to (2.11) then proves the Theorem.

2.2 Convergence of the Particle Scheme

Our goal is to develop a numerical method such that useful results still could be obtained even if the oscillation in a solution are not well resolved on the computational grid. However one can not expect that such a numerical method converges for all grids completely independent of ε.

Let $\Delta x = \Delta t$, $x_i = i\Delta x$, $t_n = n\Delta t$.

Denote u_i^n, v_i^n and w_i^n as the approximations of $u(x_i,t_n)$, $v(x_i,t_n)$ and $w(x_i,t_n)$ respectively. Our particle scheme is given by

$$(2.12) \qquad \begin{cases} u_i^{n+1} = u_{i-1}^n + \Delta t\ ((w_{i-1}^n)^2 - u_{i-1}^n v_{i-1}^n), \\ v_i^{n+1} = v_{i+1}^n + \Delta t\ ((w_{i+1}^n)^2 - u_{i+1}^n v_{i+1}^n), \\ w_i^{n+1} = w_i^n - \Delta t\ ((w_i^n)^2 - u_i^n v_i^n), \end{cases}$$

with initial condition given by

$$(2.13) \qquad \{ \ u_i^0 = u(x_i,0), \quad v_i^0 = v(x_i,0), \quad w_i^0 = w(x_i,0).$$

Theorem 2.2 *Let $m_0 = \sup_x \{ u_0, v_0, w_0 \}$ and let L be the diameter of the support of* u_0, v_0 and w_0. *Define $p = \min \{ 4Tm_0, 3Lm_0 \}$. If $p<1$, then the numerical solution of (2.12) and (2.13) converges strongly to the exact solution of (2.1-2) in L^∞-norm as $\Delta t \to 0$ essentially independent of ε for all $\Delta t \le 0.1(1-p)/m_0$.*

Before we prove Theorem 2.2 , we need to prove a few technical lemmas.

Lemma 2.2 *Define a sequence* $b_{n+1} = b_n + (b_n)^2$ *with* $0.3 \geq b_1 > 0$. *Denote* $m(b_1)$ *as the integer such that* $b_{m(b_1)+1} > 1$ *but* $b_{m(b_1)} \leq 1$. *Then we have*

$$(m(b_1))b_1 \geq 1 - b_1 \geq 0.7 \ .$$

proof: Define a sequence $\{a_n\}$ recursively by

$$a_{n+1} = \frac{\sqrt{1+4a_n} - 1}{2} \ , \ a_1 = 1 \ .$$

First of all, we show that $na_n \geq 1$ by induction.

The case n=1 is trivially true. Assume that $na_n \geq 1$, then

$$a_{n+1} = \frac{\sqrt{1+4a_n} - 1}{2} \geq \frac{\sqrt{1+4/n} - 1}{2} \ .$$

Consider a function f(x) defined by

$$f(x) = 2 - \frac{1+\sqrt{1+4x}}{1+x} \ , \quad \text{for } x \geq 0 \ .$$

It can be shown easily that f(x) is a monotonely increasing function for $x \geq 0$. Thus $f(1/n) \geq f(0) = 0$, for *all* $n \geq 0$. Notice that inequality $f(1/n) \geq 0$ is equivalent to

(2.14)
$$\frac{\sqrt{1+4/n} - 1}{2} \geq \frac{1}{n+1} \ .$$

Therefore we have proven that

(2.15)
$$(n+1)a_{n+1} \geq 1 \ .$$

Thus by induction, $na_n \geq 1$ for all n.

Observe that $a_{n+1} = \dfrac{\sqrt{1+a_n} - 1}{2}$ is the inverse iteration of $b_{n+1} = b_n + (a_n)^2$ for $a_1, b_1 > 0$. Suppose $b_1 \in [a_{n+1}, a_n]$, then it is easy to see that

$$n \leq m(b_1) \leq n+1 \ .$$

Therefore, we obtain

$$(m(b_1))b_1 \geq nb_1 \geq na_{n+1} = (n+1)a_{n+1} - a_{n+1} \geq 1 - b_1 \geq 0.7 \ .$$

Hence the Lemma 2.2 is proved.

Lemma 2.3 *Given the initial data (2.2), if* $0 < \Delta t(m_0) \leq 0.3$, *then the solution of (2.12) remains nonnegative for* $t \in [0, 0.7/m_0]$.

proof: We will prove the Lemma by induction.

The case n=1 is trivial by the assumption on the initial data. Assume that for $k \leq n$ with $\Delta t(n+1) \leq 0.7/m_0$ we have proven

$$u_i^k, v_i^k, w_i^k \geq 0 \qquad \text{for } all \ \ i \ .$$

Denote $b_k = \Delta t \max_i \{ u_i^k, v_i^k, w_i^k \}$. Multiplying (2.12) by Δt and using the induction assumption, we obtain

$$b_{k+1} \le b_k + (b_k)^2, \text{ for } 1 \le k \le n,$$

with $b_1 \equiv \Delta t\, m_0 \le 0.3$. By Lemma 2.3, we have

$$b_k \le 1, \quad \text{for } k \le m(b_1),$$

where $m(b_1) \ge 0.7/b_1 = 0.7/(m_0 \Delta t)$. Thus $b_k \le 1$ for $k \le n+1$. As a result, we have

$$u_i^{n+1} = u_{i-1}^n (1 - \Delta t v_{i-1}^n) + \Delta t (w_{i-1}^n)^2 \ge 0 .$$

Similarly, v_i^{n+1}, $w_i^{n+1} \ge 0$. The Lemma 2.3 is then proved by induction.

Lemma 2.4 *Under the assumption of Theorem 2.2, the numerical solutions of (2.12) are bounded by*

$$0 \le u_i^n,\ v_i^n,\ w_i^n \le 3m_0/(1-p), \qquad \text{for } n\Delta t \le T,$$

where m_0, p are defined as in the Theorem 2.2.

proof: Since $m_0 \Delta t \le 0.3$, by Lemma 2.3 we know that

(2.16) $$u_i^n,\ v_i^n,\ w_i^n \ge 0 \quad \text{for } n\Delta t \le 0.7/m_0 .$$

Define

$$T^* = sup\{\, t \mid 0 \le t \le T;\ u_i^n,\ v_i^n,\ w_i^n \ge 0 \text{ for } all\ n\Delta t \le t \,\}.$$

Inequality (2.16) implies $T \ge T^* \ge 0.7/m_0$.

Claim : $T^* = T$.

Suppose otherwise, i.e. $T^* < T$. Rewrite (2.12) as

(2.17) $$u_i^n = u_{i-n}^0 + \Delta t \sum_{k=0}^{n-1} (w^2 - uv)_{i+k-n}^k ,$$

(2.18) $$v_i^n = v_{i+n}^0 + \Delta t \sum_{k=0}^{n-1} (w^2 - uv)_{i-k+n}^k ,$$

(2.19) $$w_i^n = w_i^0 - \Delta t \sum_{k=0}^{n-1} (w^2 - uv)_i^k .$$

Denote $M(\Delta t)$ as

$$M(\Delta t) = \max \{\, u_i^k,\ v_i^k,\ w_i^k \mid \text{for } all\ i \text{ and } 0 \le k \le T^*/\Delta t \,\}.$$

Since $u_i^k, v_i^k, w_i^k \ge 0$ for $0 \le k\Delta t < T^*$, (2.17) + (2.18) + (2.19) gives

(2.20) $$u_i^n + v_i^n + w_i^n \le u_{i-n}^0 + v_{i+n}^0 + w_i^0 + \Delta t\ M(\Delta t) \sum_{k=0}^{n-1} (w_{i+k-n}^k + w_{i-k+n}^k + u_i^k) .$$

It is a easy matter to show the following identity

(2.21) $$\sum_{k=0}^{n-1} (\, w_{i+k-n}^k + w_{i-k+n}^k + u_i^k + v_i^k \,) = \sum_{k=0}^{n-1} (\, w_{i+k-n}^0 + w_{i-k+n}^0 + u_{i-k}^0 + v_{i+k}^0 \,) .$$

Thus (2.20) and (2.21) imply

$$u_i^n + v_i^n + w_i^n \le 3m_0 + pM(\Delta t) ,$$

where constant p is given as in Theorem 2.2 . Since $p < 1$ by assumption, we obtain

$$0 \le u_i^n,\ v_i^n,\ w_i^n \le M(\Delta t) \le 3m_0/(1-p) , \quad \text{for } n\Delta t < T^* . \tag{2.22}$$

Inequality (2.22) is valid up to $n\Delta t \le T_1 \equiv T^* - \Delta t$. Apply Lemma 2.3 to the scheme at $t = T_1$ with m_0 replaced by $m_0^* \equiv 3m_0/(1-p)$. By the assumption on Δt , $\Delta t m_0^* \le 0.3$. So that we have

$$u_i^n,\ v_i^n,\ w_i^n \ge 0 \quad \text{for } n\Delta t \le T_1 + 0.7/m_0^* .$$

However,

$$T_1 + 0.7/m_0^* = T^* - \Delta t + 0.7/m_0^* \ge T^* + \Delta t > T^* .$$

This contradicts the definition of T^*. Therefore $T^* = T$ and

$$0 \le u_i^n,\ v_i^n,\ w_i^n \le 3m_0/(1-p), \qquad \text{for } 0 \le n\Delta t \le T.$$

This completes the proof of Lemma 2.4.

Lemma 2.5 *Let g(s) and w(x,y) be smooth functions with bounded derivatives. Assume further that w(x,y) is a 1-periodic function in y and satisfies* $\int_0^1 w(x,y)dy = 0$. *Then for* $x_k = k\Delta x,\ 0 \le m < n \le \frac{T}{\Delta x}$,

$$\left| \Delta x \sum_{k=m}^{n} g(x_k) w(x_k, \frac{x_k}{\varepsilon}) \right| \to 0 \quad as \quad \Delta x \to 0 ,$$

essentially independent of ε .

proof: By Abel summation formula,

$$\sum_{k=m}^{n} g(x_k) w(x_k, \frac{x_k}{\varepsilon}) = \sum_{k=m}^{n} w(x_k, \frac{x_k}{\varepsilon}) g(x_n) + \sum_{j=m}^{n-1} \left(\sum_{k=m}^{j} w(x_k, \frac{x_k}{\varepsilon}) \right) (g(x_j) - g(x_{j+1})) .$$

In the proof of Theorem 1 (part a)[6], Engquist has showed that

$$\max_{m \le j \le n} \left| \Delta x \sum_{k=m}^{j} w(x_k, \frac{x_k}{\varepsilon}) \right| \to 0 \quad as \quad \Delta x \to 0 ,$$

essentially independent of ε .

On the other hand, the smoothness of g(x) implies

$$| g(x_j) - g(x_{j+1}) | \le C\Delta x .$$

This completes the proof of Lemma 2.5 .

Lemma 2.6 : *Let $E_n, R_{m,n} \in l^\infty$ with the properties:*

$$E_{n+1} = (P + \Delta t\, A_n)E_n + R_{n,n+1},\ n = 0,1,...,$$

$$E_0 = 0\ ,$$

where the operators P and A_n satisfy

$$\|P\| = 1,\ \|A_n\| \le C\ ,$$

$$R_{n,n+1} + PR_{m,n} = R_{m,n+1},\quad 0 \le m < n\ ,$$

Then $\|E_n\| \le exp(Ct_n) \max\limits_{0 \le m < k \le n} \|R_{m,k}\|.$

proof: For a proof, see [6].

Now we can present the proof of Theorem 2.2.

proof of Theorem 2.2 :

Integration of (1.2) from t_n to t_{n+1} along the characteristics gives

$$\text{(2.23)}\qquad \begin{cases} u(x_i, t_{n+1}) = u(x_{i-1}, t_n) + \int_{t_n}^{t_{n+1}} (w^2 - uv)(x_i - t_{n+1} + s, s)\, ds\ , \\ v(x_i, t_{n+1}) = v(x_{i+1}, t_n) + \int_{t_n}^{t_{n+1}} (w^2 - uv)(x_i + t_{n+1} - s, s)\, ds\ , \\ w(x_i, t_{n+1}) = w(x_i, t_n) - \int_{t_n}^{t_{n+1}} (w^2 - uv)(x_i, s)\, ds\ . \end{cases}$$

Let e , f and g denote the errors in the approximation of u, v and w respectively

$$\begin{cases} e_i^n = u(x_i, t_n) - u_i^n\ , \\ f_i^n = v(x_i, t_n) - v_i^n\ , \\ g_i^n = w(x_i, t_n) - w_i^n\ . \end{cases}$$

Subtracting (2.12) from (2.23), we obtain

$$e_i^{n+1} = e_{i-1}^n + \Delta t(\, w(x_{i-1}, t_n)^2 - (w_{i-1}^n)^2) + I_{+,i}^{n,n+1}(w^2)$$

$$- \Delta t(\, u(x_{i-1}, t_n)v(x_{i-1}, t_n) - u_{i-1}^n v_{i-1}^n\ ) - I_{+,i}^{n,n+1}(uv)\ ,$$

which can further be written as

$$\text{(2.24)}\quad e_i^{n+1} = e_{i-1}^n - \Delta t\, \alpha_{i-1}^n e_{i-1}^n - \Delta t\, \beta_{i-1}^n f_{i-1}^n + \Delta t\, \gamma_{i-1}^n g_{i-1}^n + I_{+,i}^{n,n+1}(w^2) - I_{+,i}^{n,n+1}(uv)\ ,$$

where α_i^n, β_i^n and γ_i^n bounded by the upper bound of the solutions of (1.2) and (2.12), which are independent of ε by Lemma 2.5 and the global existence results for (1.2).

Similarly, we have

(2.25) $f_i^{n+1} = f_{i+1}^n - \Delta t\,\bar{\alpha}_{i+1}^n e_{i+1}^n - \Delta t\,\bar{\beta}_{i+1}^n f_{i+1}^n + \Delta t\,\tilde{\gamma}_{i+1}^n g_{i+1}^n + I_{-,i}^{n,n+1}(w^2) - I_{-,i}^{n,n+1}(uv)$,

(2.26) $$g_i^{n+1} = g_i^n + \Delta t\,\hat{\alpha}_i^n e_{i+1}^n + \Delta t\,\hat{\beta}_i^n f_i^n - \Delta t\,\hat{\gamma}_i^n g_i^n - I_{0,i}^{n,n+1}(w^2) + I_{0,i}^{n,n+1}(uv) ,$$

where

$$e_i^0 = f_i^0 = g_i^0 = 0 \ , \ i = ...-1,0,1....$$

Here $\bar{\alpha}_i^n, ... \hat{\gamma}_i^n$ etc. are bounded independent of ε as before.

$I_{\pm,i}^{m,n}(w)$, $I_{0,i}^{m,n}(w)$ are defined by (m < n)

(2.27) $$\begin{cases} I_{+,i}^{m,n}(w) = \int_{t_m}^{t_n} w(x_i - t_n + s, s)\,ds - \Delta t \sum_{k=m}^{n-1} w(x_i - t_n + t_k, t_k) , \\ I_{-,i}^{m,n}(w) = \int_{t_m}^{t_n} w(x_i + t_n - s, s)\,ds - \Delta t \sum_{k=m}^{n-1} w(x_i + t_n - t_k, t_k) , \end{cases}$$

(2.28) $$I_{0,i}^{m,n}(w) = \int_{t_m}^{t_n} w(x_i, s)\,ds - \Delta t \sum_{k=m}^{n-1} w(x_i, t_k) .$$

Define

$$E_n = (...,g_i^n, f_i^n, e_i^n,...) ,$$

$$PE_n = (...,g_i^n, f_{i+1}^n, e_{i-1}^n,...) ,$$

$$A_n E_n = (...,\hat{\alpha}_i^n e_i^n + \hat{\beta}_i^n f_i^n - \hat{\gamma}_i^n g_i^n, -\bar{\alpha}_{i+1}^n e_{i+1}^n - \bar{\beta}_{i+1}^n f_{i+1}^n + \tilde{\gamma}_{i+1}^n g_{i+1}^n$$
$$, -\alpha_{i-1}^n e_{i-1}^n - \beta_{i-1}^n f_{i-1}^n + \gamma_{i-1}^n g_{i-1}^n ,...) ,$$

$$R_{m,n} = (...,-I_{0,i}^{m,n}(w^2) + I_{0,i}^{m,n}(uv), I_{-,i}^{m,n+1}(w^2) - I_{-,i}^{m,n+1}(uv)$$
$$, I_{+,i}^{m,n}(w^2) - I_{+,i}^{m,n}(uv),...) .$$

Obviously, $\|P\| = 1$ and $\|A_n\| \le C$, since α_i^n, β_i^n etc. are all bounded. Moreover, it can be verified directly from (2.27) and (2.28) that

(2.29) $$\begin{cases} I_{+,i}^{n,n+1}(w) + I_{+,i-1}^{m,n}(w) = I_{+,i}^{m,n+1}(w) , \\ I_{-,i}^{n,n+1}(w) + I_{-,i+1}^{m,n}(w) = I_{-,i}^{m,n+1}(w) , \\ I_{0,i}^{n,n+1}(w) + I_{0,i}^{m,n}(w) = I_{0,i}^{m,n+1}(w) , \end{cases}$$

from which we obtain

$$R_{n,n+1} + PR_{m,n} = R_{m,n+1} .$$

Thus Lemma 2.6 applies and

(2.30) $$\|E_n\| \le exp(CT) \max_{0 \le m < k \le n} \|R_{m,k}\| .$$

It remains to show that $R_{m,k} \rightarrow 0$ *as* $\Delta t \rightarrow 0$ essentially independent of ε.

Let's study each element of $R_{m,n}$ in details.

Case(i)
$$I_{0,i}^{m,n}(w^2) = \int_{t_m}^{t_n} w(x_i, s)^2 ds - \Delta t \sum_{k=m}^{n-1} w(x_k, t_k)^2 .$$

Since $\frac{d}{ds} w(x_i, s)^2 = 2w(uv - w^2)$ is bounded independent of ε, classical error analysis implies that

$$| I_{0,i}^{m,n}(w^2) | \le C \Delta t \rightarrow 0 \text{ as } \Delta t \rightarrow 0 .$$

Case(ii)
$$I_{0,i}^{m,n}(uv) = \int_{t_m}^{t_n} u(x_i, s) v(x_i, s) ds - \Delta t \sum_{k=m}^{n-1} u(x_i, t_k) . v(x_k, t_k)$$

By Theorem 2.1, it suffices to replace $u(x,s)$ by $U(x, \frac{x-s}{\varepsilon}, s)$, $v(x,s)$ by $V(x, \frac{x+s}{\varepsilon}, s)$. Introduce functions f(s) and $g(s, s/\varepsilon)$ as follows

$$f(s) = \int_0^1 U(x_i, x_i/\varepsilon - y, s) V(x_i, x_i/\varepsilon + y, s) dy ,$$

$$g(s, s/\varepsilon) = U(x_i, x_i/\varepsilon - s/\varepsilon, s) V(x_i, x_i/\varepsilon + s/\varepsilon, s) - f(s) .$$

Since U(x,y,t), V(x,y,t) are 1-periodic functions in y variable, g(s,y) is 1-periodic function in y and satisfies

$$\int_0^1 g(s, y) dy = 0 .$$

Since f(s) has bounded derivatives in s variable, we have

$$| I_{0,i}^{m,n}(f) | \le c \Delta t .$$

For $g(s, \frac{s}{\varepsilon})$, if $\Delta t^{3/4} \le \varepsilon$, then

$$| I_{0,i}^{m,n}(g) | \le C \Delta t / \varepsilon \le C \Delta t^{1/4} .$$

If $\varepsilon < \Delta t^{3/4}$, applying Lemma 2.1 and Lemma 2.5 to the first and second term in $I_{0,i}^{m,n}(g)$ respectively, we conclude that

$$I_{0,i}^{m,n}(g) \rightarrow 0 \text{ as } \Delta t \rightarrow 0 \text{ essentially independent of } \varepsilon .$$

Hence we have proved that

$$I_{0,i}^{m,n}(uv) \rightarrow 0 \text{ as } \Delta t \rightarrow 0 \text{ essentially independent of } \varepsilon .$$

Similarly we can show that other elements in $R_{m,n}$ converge to zero as Δt goes to zero essentially independent of ε. This completes the proof of Theorem 2.2.

3. Variable Coefficient Carleman Model

The Carleman equations (1.1) are the simplest model of the discrete Boltzmann equations. They describe a one dimensional rarefied gas whose molecules can only have two distinct velocities which change under collision [4]. The independent variables $u(x,t)$, $v(x,t)$ denote the number densities of molecules at point x and time t with velocity +1 and -1 respectively.

For our understanding of general particle method approximation, it is important to extend the problem to variable coefficients. In the variable coefficient case, the approximation of the principal part of the equations is not exact, which means that we may introduce $O(1)$ errors for oscillatory solutions and local averages have to be considered.

Suppose that in the Carleman model the discrete constant velocities +1 and -1 are replaced by smooth functions $a(x,t)$ and $-b(x,t)$ respectively such that

$$\begin{cases} 0 < a_{\min} \le a(x,t) \le a_{\max} < \infty , \\ 0 < b_{\min} \le b(x,t) \le b_{\max} < \infty . \end{cases}$$

We then obtain the so called Carleman equations with variable velocities:

$$\begin{cases} \dfrac{\partial u}{\partial t} + a(x,t)\dfrac{\partial u}{\partial x} = v^2 - u^2 , \\ \dfrac{\partial v}{\partial t} - b(x,t)\dfrac{\partial v}{\partial x} = u^2 - v^2 . \end{cases} \tag{3.1}$$

Let the initial value be of the form:

$$\begin{cases} u(x,0) = u_0(x,\dfrac{x}{\varepsilon}) , \\ v(x,0) = v_0(x,\dfrac{x}{\varepsilon}) . \end{cases} \tag{3.2}$$

where $u_0(x,y)$, $v_0(x,y) \in C^3$ are of compact support with respect to x and 1*−periodic* in y.

Remark 3.1 Following the proof of theorem 1 by Illner in [12], one can show that equation (3.1) and (3.2) have bounded solutions in C^3 for all time. Moreover, if $0 \le u_0 , v_0 \le M < \infty$, then $0 \le u(x,t), v(x,t) \le M$ for $t > 0$.

Let $\phi(t,s;x)$ and $\psi(t,s;x)$ be the solutions of characteristic equations (3.3) and (3.4) respectively:

$$\begin{cases} \dfrac{d\phi(t,s;x)}{dt} = a(\phi(t,s;x),t) , \\ \phi(s,s;x) = x , \end{cases} \tag{3.3}$$

$$\begin{cases} \dfrac{d\psi(t,s;x)}{dt} = -b(\psi(t,s;x),t) , \\ \psi(s,s;x) = x . \end{cases} \tag{3.4}$$

Here $\phi(t,s;x)$ represents the location of a particle at time t which starts at point x at time

s. Thus ϕ maps a point (x,s) to a point $(\phi(t,s;x),t)$ and $\phi(s,t;x)$ can be regarded as an inverse mapping of $\phi(t,s;x)$.

3.1 Homogenization result

In describing the propagation and nonlinear interaction of the oscillatory solutions, we have the following homogenization result which is analogous to Tartar's result in [20].

Theorem 3.1 *The solution to (3.1) and (3.2) converges uniformly to that of the homogenized equation (3.5) and (3.6)*

$$u_\varepsilon(x,t) - U(x,\frac{\phi(0,t;x)}{\varepsilon},t) \to 0 \quad as \quad \varepsilon \to 0,$$

$$v_\varepsilon(x,t) - V(x,\frac{\psi(0,t;x)}{\varepsilon},t) \to 0 \quad as \quad \varepsilon \to 0, \quad \text{for } 0 \le t \le T,$$

where U(x,y,t) and V(x,y,t) are the solution of the homogenized equation (3.5) and (3.6):

$$\begin{cases} \dfrac{\partial U}{\partial t} + a(x,t)\dfrac{\partial U}{\partial x} + U^2 - \displaystyle\int_0^1 V^2 dy = 0, \\ \dfrac{\partial V}{\partial t} - b(x,t)\dfrac{\partial V}{\partial x} + V^2 - \displaystyle\int_0^1 U^2 dy = 0, \end{cases} \tag{3.5}$$

$$\begin{cases} U(x,y,0) = u_0(x,y), \\ V(x,y,0) = v_0(x,y). \end{cases} \tag{3.6}$$

Remark 3.2 By slightly modifying Illner's proof of theorem 1 in [12], one can show that if $0 \le u_0, v_0 \le M$, then $0 \le U(x,y,t), V(x,y,t) \le M$ for $t>0$.

proof of Theorem 3.1:

Subtracting the first equation of (3.5) from that of (3.1) and integrating the resulting equations along the characteristics from 0 to t, we obtain

$$u(x,t) - U(x,\frac{\phi(0,t;x)}{\varepsilon},t)$$

$$\begin{aligned} &= -\int_0^t \left(u(\phi(s,t;x),s)^2 - U(\phi(s,t;x),\frac{\phi(0,s;\phi(s,t;x))}{\varepsilon},s)^2 \right) ds \\ &+ \int_0^t \left(v(\phi(s,t;x),s)^2 - V(\phi(s,t;x),\frac{\psi(0,s;\phi(s,t;x))}{\varepsilon},s)^2 \right) ds \\ &+ \int_0^t \left(V(\phi(s,t;x),\frac{\psi(0,s;\phi(s,t;x))}{\varepsilon},s)^2 - \int_0^1 V(\phi(s,t;x),y,s)^2 dy \right) ds, \end{aligned} \tag{3.7}$$

where we have used the identity

$$\frac{\partial\phi(0,t;\,x)}{\partial t} + a(x,t)\,\frac{\partial\phi(0,t;\,x)}{\partial x} = 0\,,$$

which follows from differentiating $\phi(0,t;\,\phi(t,0;\,z)) = z$ with respect to t.

For fixed x and t, define

$$f(s) = \psi(0,s;\,\phi(s,t;\,x))\,,$$

$$w(s,z) = V(\phi(s,t;\,x),z,s)^2 - \int_0^1 V(\phi(s,t;\,x),y,s)^2\,dy\,,$$

then the last term in (3.7) becomes

$$\int_o^t w(s,\frac{f(s)}{\varepsilon})\,ds\,.$$

Notice that

$$\frac{df(s)}{ds} = (a(\phi(s,t;x),s) + b(\phi(s,t;x),s))\frac{\partial\psi}{\partial x}(0,s;\phi(s,t;x))\,,$$

$$\frac{\partial\psi}{\partial x}(t,s;x) = exp\,(\int_s^t -b(\psi(h,s;x),h)dh\,) > 0\,.$$

We conclude that f(s) is a strictly increasing function of s. Therefore $f^{-1}(s)$ exists. Applying Lemma 2.1, we obtain

$$(3.8)\qquad \left|\int_o^t w(s,\frac{f(s)}{\varepsilon})\,ds\right| = \left|\int_{f^{-1}(0)}^{f^{-1}(t)} \frac{df^{-1}(s)}{ds} w(f^{-1}(s),\frac{s}{\varepsilon})ds\right| \le C\varepsilon\,.$$

Since u, v, U, V are all bounded by M, we deduce from (3.7) and (3.8) that

$$(3.9)\qquad \left| u(x,t) - U(x,\frac{\phi(0,t;\,x)}{\varepsilon},t)\right| \le 2M\int_0^t G(\phi(s,t;\,x),s)ds + C\varepsilon\,,$$

where G(x,t) is defined by

$$(3.10)\qquad G(x,t) = \left| u(x,t) - U(x,\frac{\phi(0,t;\,x)}{\varepsilon},t)\right| + \left| v(x,t) - V(x,\frac{\psi(0,t;\,x)}{\varepsilon},t)\right|\,.$$

Similarly, we have

$$(3.11)\qquad \left| v(x,t) - V(x,\frac{\psi(0,t;\,x)}{\varepsilon},t)\right| \le 2M\int_0^t G(\psi(s,t;\,x),s)ds + C\varepsilon\,.$$

(3.9) + (3.11) yields

$$(3.12)\qquad G(x,t) \le 2M\int_0^t (G(\phi(s,t;\,x),s) + G(\psi(s,t;\,x),s))ds + 2C\varepsilon\,.$$

Let $E(t) = \sup_x\{\,G(x,t)\,\}$. It follows immediately from (3.12) that

(3.13) $$\begin{cases} E(t) \le 4M\int_0^t E(s)ds + 2C\varepsilon , \\ E(0) = 0 . \end{cases}$$

Application of the Gronwall inequality to (3.13) then proves the Theorem.

3.2 Convergence of Variable Velocity Particle Methods

The particle paths, which are approximation of the characteristics, generate a generalized grid. By 'generalized grid', we mean that we compute the solution only at those points at which a rightward characteristic intersects a leftward one.

We need to find out the coordinates for the corresponding grid points. This can be done by parametrizing t in terms of x as below.

Let $t_+(x_i, x)$ be the unique solution of

(3.14) $$\phi(t_+(x_i, x), 0; x_i) = \psi(t_+(x_i, x), 0; x) , \quad x \ge x_i .$$

Let $t_-(x_i, x)$ be the unique solution of

(3.15) $$\phi(t_-(x_i, x), 0; x) = \psi(t_-(x_i, x), 0; x_i) , \quad x \le x_i .$$

It follows from (3.14) and (3.15) that

(3.16) $$t_+(x_i, x_{i+n}) = t_-(x_{i+n}, x_i) .$$

Since the $\phi(t,0; x)$, $\psi(t,0; x)$ are smooth, $t_\pm(x_i, x)$ are smooth functions of x. Moreover, it can be shown that

$$\frac{d}{dx}t_+(x_i, x) = \frac{\dfrac{\partial\psi(t,0; x)}{\partial x}}{a(\phi(t,0; x_i),t) + b(\psi(t,0; x),t)} , \quad with \quad t = t_+(x_i, x) ,$$

$$\frac{d}{dx}t_-(x_i, x) = -\frac{\dfrac{\partial\phi(t,0; x)}{\partial x}}{a(\phi(t,0; x),t) + b(\psi(t,0; x_i),t)} , \quad with \quad t = t_-(x_i, x) .$$

For simplicity, we introduce the notation

$$\begin{cases} t_i^n = t_+(x_{i-n}, x_{i+n}) , \\ x_i^n = \phi(t_i^n, 0; x_{i-n}) , \qquad x_i = i\Delta x . \end{cases}$$

Let the approximation of $u(x_i^n, t_i^n)$ and $v(x_i^n, t_i^n)$ at the grid (x_i^n, t_i^n) be given by u_i^n and v_i^n respectively. Then our particle algorithm is given as follows :

(3.17) $$\begin{cases} u_i^{n+1} = u_{i-1}^n + (t_i^{n+1} - t_{i-1}^n)((v_{i-1}^n)^2 - (u_{i-1}^n)^2) , \\ v_i^{n+1} = v_{i+1}^n + (t_i^{n+1} - t_{i+1}^n)((u_{i+1}^n)^2 - (v_{i+1}^n)^2) , \end{cases}$$

with initial value given by

(3.18)
$$\begin{cases} u_i^0 = u_0(x_i, \frac{x_i}{\varepsilon}) , \\ v_i^0 = v_0(x_i, \frac{x_i}{\varepsilon}) , \end{cases}$$

where $i = ...-1,0,1...$, $n = 0,1,2....$

In algorithm (3.17), we assume that (x_i^n, t_i^n) are given exactly. There is no time-discretization involved. We refer to algorithm (3.17) as semi-discrete particle methods for equation (3.1).

Theorem 3.2 : *The solution of the particle algorithm (3.17) and (3.18) converges strongly to u, v in L^∞-norm as $\Delta x \to 0$ essentially independent of ε.*

Lemma 3.1 *Let u_i^n, v_i^n be the solution of (3.17) and (4.18) with $0 \le u_0, v_0 \le M$. Denote*

$$\Delta t = \max_{i,n} \{ \mid t_i^{n+1} - t_{i-1}^n \mid , \mid t_i^{n+1} - t_{i+1}^n \mid \}.$$

If $\Delta t \le \frac{1}{2M}$, *then* $0 \le u_i^n, v_i^n \le M$ *for* $n = 1,2,....$

proof: This can be proved easily by induction. We omit the proof.

Now we can present the proof of Theorem 3.2

proof of Theorem 3.2 :

Integration of (3.1) along the characteristics gives

(3.19)
$$u(x_i^{n+1}, t_i^{n+1}) = u(x_{i-1}^n, t_{i-1}^n) + \int_{t_{i-1}^n}^{t_i^{n+1}} (v(\phi(s,0; x_{i-n-1}),s)^2 - u(\phi(s,0; x_{i-n-1}),s)^2)ds ,$$

(3.19)
$$v(x_i^{n+1}, t_i^{n+1}) = v(x_{i+1}^n, t_{i+1}^n) + \int_{t_{i+1}^n}^{t_i^{n+1}} (u(\psi(s,0; x_{i+n+1}),s)^2 - v(\psi(s,0; x_{i+n+1}),s)^2)ds .$$

Let e and f denote the errors in the approximation of u and v respectively

$$\begin{cases} e_i^n = u(x_i^n, t_i^n) - u_i^n , \\ f_i^n = v(x_i^n, t_i^n) - v_i^n . \end{cases}$$

Subtracting the first equation of (3.17) from that of (3.19), we obtain

$$e_i^{n+1} = e_{i-1}^n + (t_i^{n+1} - t_{i-1}^n)(v(x_{i-1}^n, t_{i-1}^n)^2 - (v_{i-1}^n)^2)$$

$$= (t_i^{n+1} - t_{i-1}^n)(u(x_{i-1}^n, t_{i-1}^n)^2 - (u_{i-1}^n)^2) + I_{+,i}^{n,n+1}(v^2) - I_{+,i}^{n,n+1}(u^2) ,$$

which can further be written as

(3.20) $$e_i^{n+1} = e_{i-1}^n - \Delta x\, \alpha_{i-1}^n e_{i-1}^n + \Delta x\, \beta_{i-1}^n f_{i-1}^n + I_{+,i}^{n,n+1}(v^2) - I_{+,i}^{n,n+1}(u^2) ,$$

where $I_{+,i}^{n,n+1}(w)$ is defined by (3.22), α_i^n, β_i^n are bounded by $2M\|\partial_x t_\pm\|_{L^\infty}$.

Similarly, we have

(3.21) $$f_i^{n+1} = f_{i+1}^n + \Delta x\, \tilde{\alpha}_{i+1}^n e_{i+1}^n - \Delta x\, \tilde{\beta}_{i+1}^n f_{i+1}^n + I_{-,i}^{n,n+1}(u^2) - I_{-,i}^{n,n+1}(v^2) .$$

Here $\tilde{\alpha}_i^n$, $\tilde{\beta}_i^n$ are bounded in the same way as α_i^n, β_i^n as above.

$$e_i^0 = 0 , \quad f_i^0 = 0 \quad ; \; i = \ldots -1,0,1 \ldots$$

$I_{+,i}^{m,n}(w)$, $I_{-,i}^{m,n}(w)$ are defined by (m < n)

(3.22) $$I_{+,i}^{m,n}(w) = \int_{t_{i-n+m}^m}^{t_i^n} w(\phi(s,0;x_{i-n}),s)\, ds - \sum_{k=m}^{n-1} (t_{i-n+k+1}^{k+1} - t_{i-n+k}^k) w(x_{i-n+k}^k, t_{i-n+k}^k) ,$$

(3.23) $$I_{-,i}^{m,n}(w) = \int_{t_{i+n-m}^m}^{t_i^n} w(\psi(s,0;x_{i+n}),s)\, ds - \sum_{k=m}^{n-1} (t_{i+n-k-1}^{k+1} - t_{i+n-k}^k) w(x_{i+n-k}^k, t_{i+n-k}^k) .$$

Define

$$E_n = (\ldots, f_0^n, e_0^n, f_1^n, e_1^n, \ldots) ,$$

$$PE_n = (\ldots, f_{i+1}^n, e_{i-1}^n, \ldots) ,$$

$$A_n E_n = (\ldots, \tilde{\alpha}_{i+1}^n e_{i+1}^n - \tilde{\beta}_{i+1}^n f_{i+1}^n , -\alpha_{i-1}^n e_{i-1}^n + \beta_{i-1}^n f_{i-1}^n, \ldots) ,$$

$$R_{m,n} = (\ldots, I_{-,i}^{m,n}(u^2) - I_{-,i}^{m,n}(v^2), -I_{+,i}^{m,n}(u^2) + I_{+,i}^{m,n}(v^2), \ldots) .$$

Obviously, $\|P\| = 1$ and $\|A_n\| \le C$, since α_i^n, β_i^n etc. are all bounded. Moreover, (3.22) and (3.23) imply

$$\begin{cases} I_{+,i}^{m,n}(w) + I_{+,i-1}^{n,n+1}(w) = I_{+,i}^{m,n+1}(w) , \\ I_{-,i}^{m,n}(w) + I_{-,i+1}^{n,n+1}(w) = I_{-,i}^{m,n+1}(w) , \end{cases}$$

from which we obtain

$$R_{n,n+1} + PR_{m,n} = R_{m,n+1} .$$

Thus Lemma 2.6 in section 2 applies and

$$\| E_n \| \leq exp(CT) \max_{0 \leq m < k \leq n} \| R_{m,k} \| . \tag{3.24}$$

It remains to show that $R_{m,k} \to 0$ *as* $\Delta x \to 0$ essentially independent of ε . Observe that

$$| \frac{d}{ds} u^2(\phi(s,0;\, x_{i-n}),s) \mid = 2 \mid uv^2 - u^3 \mid \leq 2M^3 ,$$

$$| \frac{d}{ds} v^2(\psi(s,0;\, x_{i+n}),s) \mid = 2 \mid vu^2 - v^3 \mid \leq 2M^3 .$$

Classical error estimate gives

$$| I^{m,n}_{+,i}(u^2) \mid \leq C\,\Delta x \;\to\; 0 \quad as \quad \Delta x \;\to\; 0 ,$$

$$| I^{m,n}_{-,i}(v^2) \mid \leq C\,\Delta x \;\to\; 0 \quad as \quad \Delta x \;\to\; 0 .$$

However, it is more difficult to show

$$I^{m,n}_{+,i}(v^2) \text{ and } I^{m,n}_{-,i}(u^2) \to 0 \quad as \quad \Delta x \to 0 .$$

There are two cases:

Case I: $\Delta x^{3/4} \leq \varepsilon$, then classical analysis implies

$$| I^{m,n}_{+,i}(v^2) \mid \leq C\,\Delta x \max_{0 \leq s \leq T} \mid \frac{d}{ds} v^2(\phi(s,0;\, x_{i-n}),s) \mid$$

$$\leq \bar{C}\frac{\Delta x}{\varepsilon} = \bar{C}\,\Delta x^{1/4}$$

Case II: $\Delta x^{3/4} > \varepsilon$.

By the Theorem 3.1 , it is enough to replace v(x,t) by $V(x, \frac{\psi(0,t;\, x)}{\varepsilon}, t)$.

Split $V(x,y,t)^2$ into two parts:

$$V(x,y,t)^2 = (\int_0^1 V(x,y,t)^2 dy \;) + \overline{V^2}(x,y,t) ,$$

with $\overline{V^2}(x,y,t) \equiv V(x,y,t)^2 - \int_0^1 V^2 dy$.

It suffices to study $I^{m,n}_{+,i}(\overline{V^2})$. Apply Lemma 2.1 with

$$f(s) \;= \psi(0,s;\, \phi(s,0;\, x_{i-n})) ,$$

$$w(s,y) = \overline{V^2}(\phi(s,0;\, x_{i-n}),y,s) .$$

We get

$$| \int_{t^m_{i-n+m}}^{t^n_i} \overline{V^2}(\phi(s,0;\, x_{i-n}), \frac{\psi(0,s;\, \phi(s,0;\, x_{i-n}))}{\varepsilon}, s) ds \mid$$

$$= \left| \int_{t_{i-n+m}^{m}}^{t_i^n} w\left(s, \frac{f(s)}{\varepsilon}\right) ds \right| \le C\varepsilon \le \Delta x^{3/4} .$$

We are left to show

$$\sum_{k=m}^{n-1} (t_{i-n+k+1}^{k+1} - t_{i-n+k}^{k}) \overline{V^2}\left(x_{i-n+k}^{k} , \frac{\psi(0, t_{i-n+k}^{k} ; x_{i-n+k}^{k})}{\varepsilon} , t_{i-n+k}^{k}\right) \to 0 .$$

Note that

$$\psi(0, t_{i-n+k}^{k}; x_{i-n+k}^{k}) = \psi(0, t_{i-n+k}^{n}; \psi(t_{i-n+k}^{k}, 0; x_{i-n+2k})) = x_{i-n+2k} ,$$

$$t_{i-n+k+1}^{k+1} - t_{i-n+k}^{k} = 2\Delta x \frac{\partial t_+}{\partial x}(x_{i-n}, x_{i-n+2k}) + O(\Delta x^2) .$$

Applying Lemma 2.5 with

$$g(x) = 2\frac{\partial t_+}{\partial x}(x_{i-n}, x) ,$$

$$w(x,y) = \overline{V^2}(\phi(t_+(x_{i-n}, x), 0; x_{i-n}), y, t_+(x_{i-n}, x)) ,$$

we conclude that

$$\left| \sum_{k=m}^{n-1} (t_{i-n+k+1}^{k+1} - t_{i-n+k}^{k}) \overline{V^2}\left(x_{i-n+k}^{k}, \frac{x_{i-n+2k}}{\varepsilon}, t_{i-n+k}^{k}\right) \right|$$

$$= \left| \Delta x \sum_{k=m}^{n-1} g(x_{i-n+2k}) w\left(x_{i-n+2k}, \frac{x_{i-n+2k}}{\varepsilon}\right) \right| \to 0 ,$$

essentially independent of ε as $\Delta x \to 0$.

Hence, $I_{+,i}^{m,n}(v^2) \to 0$ *as* $\Delta x \to 0$ essentially independent of ε.

Similarly, we can show $I_{-,i}^{m,n}(u_i^2) \to 0$ *as* $\Delta x \to 0$ essentially independent of ε .

This completes the proof of Theorem 3.2 .

Remark 3.3 : As we see in the proof of Theorem 3.2 ,the error constants depend on the bound of

$$\frac{\partial t_+}{\partial x}(x_i, x) \text{ and } \frac{\partial t_-}{\partial x}(x_i, x) ,$$

which in terms depend on

$$\frac{\partial \phi(t, 0; x)}{\partial x} \text{ and } \frac{\partial \psi(t, 0; x)}{\partial x} .$$

As a result, we would expect a large error in the approximation if a severe stretching has occurred.

4. Time Discretization Scheme

In the particle algorithm (3.17-18), we assume that t_i^n has been solved exactly. This is not the case in practical computation. Naturally one would ask the following question:

Suppose we use a simple numerical method to approximate the characteristic equation (3.3) and (3.4), do we still have an analogous convergence result as in Theorem 3.2 for the corresponding particle scheme ?

Let $(\bar{x}_i^n, \bar{t}_i^n)$ be the approximation of grid (x_i^n, t_i^n) by Euler forward difference method.

$$(4.1)\qquad \begin{cases} \bar{x}_i^{n+1} = \bar{x}_{i-1}^n + (\bar{t}_i^{n+1} - \bar{t}_{i-1}^n) a(\bar{x}_{i-1}^n, \bar{t}_{i-1}^n) , \\ \bar{x}_i^{n+1} = \bar{x}_{i+1}^n - (\bar{t}_i^{n+1} - \bar{t}_{i+1}^n) b(\bar{x}_{i+1}^n, \bar{t}_{i+1}^n) . \end{cases}$$

Let $\bar{u}_i^n$, $\bar{v}_i^n$ be the particle solution using the approximated grid

$$(4.2)\qquad \begin{cases} \bar{u}_i^{n+1} = \bar{u}_{i-1}^n + (\bar{t}_i^{n+1} - \bar{t}_{i-1}^n)((\bar{v}_{i-1}^n)^2 - (\bar{u}_{i-1}^n)^2) , \\ \bar{v}_i^{n+1} = \bar{v}_{i+1}^n + (\bar{t}_i^{n+1} - \bar{t}_{i+1}^n)((\bar{u}_{i+1}^n)^2 - (\bar{v}_{i+1}^n)^2) . \end{cases}$$

$$(4.3)\qquad \begin{cases} \bar{u}_i^0 = u_0(x_i, \frac{x_i}{\varepsilon}) , \\ \bar{v}_i^0 = v_0(x_i, \frac{x_i}{\varepsilon}) . \end{cases}$$

Theorem 4.1 : *Let $\bar{u}_i^n$, $\bar{v}_i^n$ be the solution of particle algorithm (4.1-3). Then we have*

$$\| \bar{u}_i^n - u(x_i^n, t_i^n) \| \to 0 \quad as \quad \Delta x \to 0 ,$$

$$\| \bar{v}_i^n - v(x_i^n, t_i^n) \| \to 0 \quad as \quad \Delta x \to 0 ,$$

strongly in L^∞–norm essentially independent of ε .

Remark 4.1 : Theorem 4.1 tells us that $\bar{u}_i^n$, $\bar{v}_i^n$ converge to the continuous solution u, v at the exact grid, not the approximated one. As a matter of fact, the error at the approximated grid will be of order O(1). In particular if a(x,t)=b(x,t)=1 , the above particle scheme is identical to the particle scheme used in [7].

proof of Theorem 4.1 : Rewrite (4.1) as

$$(4.4)\qquad \begin{cases} \bar{t}_i^{n+1} = \dfrac{\bar{x}_{i+1}^n - \bar{x}_{i-1}^n + \bar{t}_{i-1}^n a(\bar{x}_{i-1}^n, \bar{t}_{i-1}^n) + \bar{t}_{i+1}^n b(\bar{x}_{i+1}^n, \bar{t}_{i+1}^n)}{a(\bar{x}_{i-1}^n, \bar{t}_{i-1}^n) + b(\bar{x}_{i+1}^n, \bar{t}_{i+1}^n)} , \\ \bar{x}_i^{n+1} = \bar{x}_{i-1}^n + (\bar{t}_i^{n+1} - \bar{t}_{i-1}^n) a(\bar{x}_{i-1}^n, \bar{t}_{i-1}^n) . \end{cases}$$

Integrating (3.3) from t_{i-1}^n to t_i^{n+1} and (3.4) from t_{i+1}^n to t_i^{n+1}, we arrive at

$$(4.5)\qquad \begin{cases} x_i^{n+1} = x_{i-1}^n + (\ t_i^{n+1} - t_{i-1}^n\ )a\,(x_{i-1}^n, t_{i-1}^n) + O\,(\Delta x^2)\ , \\ x_i^{n+1} = x_{i+1}^n + (t_i^{n+1} - t_{i+1}^n)b\,(x_{i+1}^n, t_{i+1}^n) + O\,(\Delta x^2)\ . \end{cases}$$

Since both a(x,t) and b(x,t) are bounded below from zero, (4.5) can be written as

$$(4.6)\qquad \begin{aligned} t_i^{n+1} &= \frac{x_{i+1}^n - x_{i-1}^n + t_{i-1}^n a\,(x_{i-1}^n, t_{i-1}^n) + t_{i+1}^n b\,(x_{i+1}^n, t_{i+1}^n)}{a\,(x_{i-1}^n, t_{i-1}^n) + b\,(x_{i+1}^n, t_{i+1}^n)} + O\,(\Delta x^2)\ , \\ x_{+,i}^{n+1} &= x_{i-1}^n + (t_i^{n+1} - t_{i-1}^n)a\,(x_{i-1}^n, t_{i-1}^n) + O\,(\Delta x^2)\ . \end{aligned}$$

Define the error in the grid as ex and et

$$\begin{cases} ex_i^n = \mid x_i^n - \bar{x}_i^n \mid\ , \\ et_i^n = \mid t_i^n - \bar{t}_i^n \mid . \end{cases}$$

The difference of (4.4) and (4.6) gives

$$(4.7)\qquad \begin{cases} et_i^{n+1} \le K(\ ex_{i+1}^n + ex_{i-1}^n + et_{i+1}^n + et_{i-1}^n\ ) + O\,(\Delta x^2)\ , \\ ex_i^{n+1} \le K(\ ex_{i+1}^n + ex_{i-1}^n + et_{i+1}^n + et_{i-1}^n\ ) + O\,(\Delta x^2)\ , \end{cases}$$

where the constant K depends on the bound of the first order partial derivatives of a(x,t) and b(x,t).

Define $E_n = \max_i(\ et_i^n + ex_i^n\ )$. (4.7) implies

$$(4.8)\qquad \begin{cases} E_{n+1} \le 4K\ E_n + O\,(\Delta x^2)\ , \\ E_0 = 0\ . \end{cases}$$

Thus $E_n \le O\,(\Delta x)$, that is

$$(4.9)\qquad \begin{cases} ex_i^n \le O\,(\Delta x)\ , \quad i = \ldots -1,0,1\ldots, \\ et_i^n \le O\,(\Delta x)\ , \quad n = 0,1,2\ldots \end{cases}$$

On the other hand, in approximating (3.3), we have the following error estimate

$$(4.10)\qquad \begin{cases} x_i^{n+1} = \bar{x}_i^{n+1} + \delta_i\,(t_i^{n+1})\Delta t_i + O\,(\Delta x^2)\ , \\ x_{i-1}^n = \bar{x}_{i-1}^n + \delta_i\,(t_{i-1}^n)\Delta t_i + O\,(\Delta x^2)\ , \end{cases}$$

where the principal error function $\delta_i(t)$ satisfies

$$\begin{cases} \dfrac{d\delta_i(t)}{dt} = \dfrac{\partial a}{\partial x}\delta_i(t) - \dfrac{1}{2}\theta(t)\left(\dfrac{\partial a}{\partial t} + \dfrac{\partial a}{\partial x}a(\phi(t,0;x_{i-n-1}),t)\right), \\ \delta_i(0) = 0 , \end{cases}$$

where $\theta_i(t)$ is defined such that

$$t_{i-n+k}^{k+1} - t_{i-n-1+k}^{k} = \Delta t_i \ \theta(t_{i-n-1+k}^{k}) ,$$

$$\Delta t_i = \max_k (t_{i-n+k}^{k+1} - t_{i-n+k-1}^{k}) = O(\Delta x) .$$

$\delta_i(t)$ is smooth in t , since a(x,t) is assumed to be smooth. Thus we deduce from (4.10)

$$x_i^{n+1} - x_{i-1}^{n} = \bar{x}_i^{n+1} - \bar{x}_{i-1}^{n} + O(\Delta x^2) . \tag{4.11}$$

(4.1), (4.5) and (4.11) then give

$$(t_i^{n+1} - t_{i-1}^{n})a(x_{i-1}^{n}, t_{i-1}^{n}) = (\bar{t}_i^{n+1} - \bar{t}_{i-1}^{n})a(\bar{x}_{i-1}^{n}, \bar{t}_{i-1}^{n}) + O(\Delta x^2) . \tag{4.12}$$

Since a(x,t) is bounded below from zero, (4.9) and (4.12) then give us the following key error estimate in the time grid

$$(t_i^{n+1} - t_{i-1}^{n}) = (\bar{t}_i^{n+1} - \bar{t}_{i-1}^{n}) + O(\Delta x^2) . \tag{4.13}$$

Similarly, we have

$$(t_i^{n+1} - t_{i+1}^{n}) = (\bar{t}_i^{n+1} - \bar{t}_{i+1}^{n}) + O(\Delta x^2) . \tag{4.14}$$

$\bar{u}_i^n$, $\bar{v}_i^n$ are bounded for small Δx . Noting that algorithm (4.2-3) are identical to (3.17-18) except for a local error of order $O(\Delta x^2)$ according to (4.13-14), we conclude that

$$\begin{cases} | \bar{u}_i^n - u_i^n | = O(\Delta x) , & i = ...-1,0,1..., \\ | \bar{v}_i^n - v_i^n | = O(\Delta x) , & n = ...0,1,2... \end{cases} \tag{4.15}$$

Here u_i^n, v_i^n are the particle solution corresponding to the exact x_i^n and t_i^n .

Hence Theorem 4.1 follows from Theorem 3.2.

Remark 4.6 Theorem 4.3 tells us that particle solutions with approximate grid capture the most important features of oscillatory solutions, such as the speed of propagation and the amplitude of the oscillatory solutions. Although we do not know the locations of the exact grid, we can show that after averaging the numerical solutions converge to the homogenized solutions. This is good enough for practical purpose , and is the best we could hope for.

For more details on the averaging numerical solutions, see discussion at the end of section 5.

5. Generalization

Naturally one would like to generalize the previous results to the discrete Boltzmann equation with finite many velocities $u_1, u_2, ..., u_p$ *in* R^n *space*

$$
(5.1) \quad \begin{cases} \dfrac{\partial N_i}{\partial t} + u_i \cdot \nabla N_i = \dfrac{1}{2} \sum\limits_{j,k,l} (A_{k,l}^{i,j} N_k N_l - A_{i,j}^{k,l} N_i N_j) , \\ N_i(x,0) = N_{i,0}(x,\dfrac{x}{\varepsilon}) , \quad i=1,2...p , \end{cases}
$$

where $N_i = N_i(x,t)$ denotes the number density of particles with velocity u_i at the point x and time t. $A_{i,j}^{k,l}$ is a transition probability associated with the collision " u_i, u_j to u_k, u_l ". $A_{i,j}^{k,l} N_i N_j$ is the number of collisions u_i, u_j to u_k, u_l per unit time and per unit volume.

For a derivation of (5.1), see Ref. [11] .

Approximation of (5.1) is investigated in [10]. Basicly, if the velocity field in (5.1) can be normalized to be integers, then the techniques used for the Broadwell model can be generalized to (5.1). However, there are some essential difficulties in approximating (5.1) if some of the velocity components are irrational numbers for p>2. This is not shared by the Carleman model.

Let's take the Broadwell model as an example. This is a three dimensional model with six allowed velocities. For our purpose, we choose $u_i^{\pm}$ (i=1,2,3) as follows

$$
\begin{aligned} u_1^+ &= (1,0,0), \quad & u_1^- &= (-1,0,0) \\ u_2^+ &= (\alpha,1,0), \quad & u_2^- &= (\alpha,-1,0) \\ u_3^+ &= (\alpha,0,1), \quad & u_3^- &= (\alpha,0,-1) \end{aligned}
$$

If the initial values for $N_i^{\pm}$ (i=1,2,3) are functions of x alone, then $N_i^{\pm}(x,y,z,t)$ will be a function of x alone.

Denote $u = N_1^+$, $v = N_1^-$, $w = N_2^+ = N_2^- = N_3^+ = N_3^-$. Then (5.1) is reduced to the equation of the Broadwell type

$$
(5.2) \quad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + uv - w^2 = 0 , \\ \dfrac{\partial v}{\partial t} - \dfrac{\partial v}{\partial x} + uv - w^2 = 0 , \\ \dfrac{\partial w}{\partial t} + \alpha \dfrac{\partial w}{\partial x} - uv + w^2 = 0 . \end{cases}
$$

As before, we assume oscillatory initial values of the form

$$
(5.3) \quad \left\{ u(x,0) = u_0(x,\frac{x}{\varepsilon}) , \ v(x,0) = v_0(x,\frac{x}{\varepsilon}) , \ w(x,0) = w_0(x,\frac{x}{\varepsilon}) . \right.
$$

The weak limits of solution $u_\varepsilon, v_\varepsilon$ and w_ε as $\varepsilon \to 0$ are very sensative to the coefficient α . For α = irrational number, it is showed in [9,10] the solution of (5.2-3) will converge uniformly to the following homogenized equation in a similar fashion as that in Theorem 3.1.

(5.4)
$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{\partial U}{\partial x} + U\int_0^1 V dy - \int_0^1 W^2 = 0 , \\ \dfrac{\partial V}{\partial t} - \dfrac{\partial V}{\partial x} + V\int_0^1 U dy - \int_0^1 W^2 = 0 , \\ \dfrac{\partial W}{\partial t} + \alpha\dfrac{\partial W}{\partial x} - (\int_0^1 U(x,y,t)dy)(\int_0^1 V(x,y,t)dy) + W^2 = 0 , \end{cases}$$

with intial condition given by

(5.5)
$$\begin{cases} U(x,y,0) = u_0(x,y) , \\ V(x,y,0) = v_0(x,y) , \\ W(x,y,0) = w_0(x,y) . \end{cases}$$

Suppose that w_0 is y independent. The oscillation of u and v will create oscillation on w at later time. The homogenized equation (2.3) indicates that w remains oscillatory as $\varepsilon \to 0$ if $\alpha = 0$. However, if α = irrational number, (5.4) implies that W(x,y,t) is y independent. Thus we expect there is some kind of singularity in the high order powers of the solution. Since this is a nonlinear equation, such a singularity would affect the local average of the solution.

If α is a function of x and t, it is difficult to choose the computational grid . This is because the three characteristics may never meet at one point. No local interpolation is allowed since the solution is highly oscillatory. This makes the problem much more complicated if one wants to use the characteristic type of methods. In a multi-dimensional system, we are faced with the similar difficulty.

However, for a scalar problem, there is only one family of characteristics. The problem becomes much simpler.

Consider the Cauchy problem for a semi-linear hyperbolic equation

(5.6)
$$\begin{cases} \dfrac{\partial u}{\partial t} + \sum_{i=1}^{n} a_i(x,t)\dfrac{\partial u}{\partial x_i} = F(u,x,t) \quad , \ t > 0 , \\ u(x,0) = u_0(x,\dfrac{x}{\varepsilon}) , \qquad x \in R^n , \quad u \in R , \end{cases}$$

where $u_0(x,y)$ is 1-periodic in each of component of y.

Equation (5.6) is similar to the vorticity equation in incompressible Euler equations with vorticity $u(x,t)$ and velocity vector $\vec{a}(x,t)$. The particle methods we use here are closely related to vortex methods for incompressible Euler equations. For analysis of the full vortex method along these lines, we refer to [6,10].

We assume that u_0 is a smooth function of compact support. $a_i(x,t)$ and F are smooth functions given in such a way that (5.6) has a unique smooth solution. Furthermore we assume that F(u,x,t) is Liptchitz continuous in u variable.

Let X(t,s; x) be the solution of

$$\begin{cases} \dfrac{\partial X(t,s;x)}{\partial t} = a(X(t,s;x),t) & , \quad a = (a_1,\ldots,a_n) , \\ X(s,s;x) = x \quad , & X = (X_1, \ldots, X_n) . \end{cases}$$

We have the following homogenized equation for (5.6)

$$(5.7) \qquad \begin{cases} \dfrac{\partial U}{\partial t} + \sum_{i=1}^{n} a_i(x,t)\dfrac{\partial U}{\partial x_i} = F(U,x,t) , \quad t > 0 , \\ U(x,y,0) = u_0(x,y) , \qquad x, y \in R^n , \quad U \in R . \end{cases}$$

Theorem 5.1 *Under the assumption on a and F as above, we have*

$$u(x,t) \equiv U(x, \frac{X(0,t;x)}{\varepsilon}, t) .$$

proof: We first substitute $U(x, \frac{X(0,t;x)}{\varepsilon}, t)$ into (5.7), subtract equation (5.6) from (5.7). Then integration of the resulting equation along characteristics from 0 to t will give

$$(5.8) \qquad | u(x,t) - U(x, \frac{X(0,t;x)}{\varepsilon}, t) |$$

$$= | \int_0^t (F(u(X(s,t;x),s), X(s,t;x), s) - F(U(X(s,t;x), \frac{X(0,t;x)}{\varepsilon}, s), X(s,t;x), s))\, ds \; |$$

$$\leq L \int_0^t | u(X(s,t;x),s) - U(X(s,t;x), \frac{X(0,t;x)}{\varepsilon}, s) | \, ds .$$

Denote

$$E_\varepsilon(t) = \max_x | u(x,t) - U(x, \frac{X(0,t;x)}{\varepsilon}, t) | .$$

Since X(0,t;x) = X(0,s; X(s,t;x)), we obtain from (5.8) that

$$(5.9) \qquad \begin{cases} E_\varepsilon(t) \leq L \displaystyle\int_0^t E_\varepsilon(s) ds , \\ E_\varepsilon(0) = 0 . \end{cases}$$

The Gronwall inequality then implies $E_\varepsilon(t) \equiv 0$. This completes the proof of Theorem 5.1

Let X_i^n be a first order numerical approximation of $X(t_n, 0; x_i)$, where $t_n = n\Delta t$, $x_i = i\Delta x$, with $i \in Z^n$. u_i^n denotes the approximation of $u(X(t_n,0;x_i),t_n)$ Then a particle algorithm for (5.6) is given by

$$(5.10) \qquad \begin{cases} u_i^{n+1} = u_i^n + \Delta t F(u_i^n, X_i^n, t_n) , \\ u_i^0 = u_0(x_i, \frac{x_i}{\varepsilon}) . \end{cases}$$

Theorem 5.2 *Under the assumption on a(x,t) and F(u,x,t) as before, we have*

$$\| u_i^n - u(X(t_n, 0 ; x_i), t_n) \| \rightarrow 0 , \qquad \text{as } \Delta t \text{ and } \Delta x \rightarrow 0 ,$$

$$\text{for } \quad i \in Z^n , \qquad n=0,1...,$$

strongly in L^∞*–norm essentially independent of* ε .

proof: This can be proved as in the proof of Theorem 3.2 case I. We omit the proof.

In many cases people are interested in obtaining a correct averaged solutions for oscillatory problems. This depends on what kind of average formula one uses. Here we study one version of these average formula which is used commonly in practice.

The discrete average for a numerical solution u_i^n of (5.6) is defined as

$$(5.11) \qquad \overline{u_j^n} = \Delta x \sum_k \theta_\sigma(x_j - X_k^n)\, u_k^n \frac{\partial X}{\partial x}(t_n, 0; x_k) ,$$

where

$$\theta_\sigma(x) = \frac{1}{\sigma^n}\theta(\frac{x}{\sigma^n}) , \qquad \textit{with} \quad \sigma = \Delta t^\beta, \;\; 0 < \beta < 1/2 .$$

Here $\theta(x)$ is nonnegative C^1 function with support inside the unit ball centered at origin, and it is normalized such that $\int \theta(x) dx = 1$. $\frac{\partial X}{\partial x}$ is the determinant of the Jacobian matrix $\frac{\partial X(t,0;x)}{\partial x}$ which can be calculated by

$$\begin{cases} \frac{d}{dt}(\frac{\partial X}{\partial x}) = (\, div(a))(\frac{\partial X}{\partial x}) , \\ \frac{\partial X}{\partial x}\big|_{t=0} = 1 . \end{cases}$$

As u_i^n converges to the exact solution in L^∞*–norm* , one can show that

$$\overline{u_j^n} \rightarrow \int_{R^n} \theta_\sigma(x_j - X(t_n, 0; y))\, u(X(t_n, 0; y), t_n) \frac{\partial X}{\partial x}(t_n, 0; y) dy$$

$$= \int_{R^n} \theta_\sigma(x_j - z) u(z, t_n)\, dz ,$$

as $\Delta t \rightarrow 0$ essentially independent of ε .

Furthermore, one can show

$$\int_{R^n} \theta_\sigma(x_j - z) u(z, t_n)\, dz \rightarrow \int_{[0,1]^n} U(x_j, y, t_n) dy \quad \textit{as} \quad \Delta t \rightarrow 0 .$$

Thus we have

$$\overline{u_j^n} \to \int_{[0,1]^n} U(x_j, y, t_n)\, dy \quad as \quad \Delta t \to 0 . \tag{5.12}$$

essentially independent of ε . That is, $\overline{u_j^n}$ converges strongly to the weak limit of solution $u_\varepsilon(x,t)$ essentially independent of ε .

6. Numerical Results

The numerical experiments are carried out for the variable coefficient Carleman model (3.1) and the Broadwell model (1.2). Moreover, for the Carleman equation, we test numerically how the oscillation structure would effect the numerical solutions.

For the variable coefficient Carleman model (3.1), we choose the initial data to be

$$u(x,0) = \begin{cases} 0.5sin^4(\pi(x-3)/2)(1 + sin(2\pi(x-3)/\varepsilon)) & , \ |x-4| < 1 , \\ 0 & , \ |x-4| \geq 1 , \end{cases} \tag{6.1}$$

$$v(x,0) = \begin{cases} 0.5sin^4(\pi(x-4)/2)(1 + sin(2\pi(x-4)/\varepsilon)) & , \ |x-5| < 1 , \\ 0 & , \ |x-5| \geq 1 . \end{cases} \tag{6.2}$$

Moreover we choose the variable coefficients to be

$$a(x) = 1 + 0.5sin(xt)$$
$$b(x) = 1 + 0.2cos(3xt)$$

In the numerical experiment, we take $\varepsilon \cong 0.0175$, and compute the solutions using algorithm (4.1-3) with a sequence of gridsizes. We then compare the coarse grid solutions with an 'accurate' solution obtained by using a fine grid. In our calculation, we choose the fine gridsize to be $\Delta x = 0.0052$. All the numerical solutions are obtained after $N(\Delta x)$ time iterations ($N(\Delta x) = 1.28/\Delta x$))

In figure 4, we display a sample grid network that we are using for algorithm (4.1-3). Since the grid generated by (4.1) is no longer uniform, we need to use the average formula (5.11). In our calculation, we choose $\sigma = 0.433$ and

$$\theta(x) = \begin{cases} (1 + cos(\pi x))/(2\pi) & , \ |x| < 1 , \\ 0 & , \ |x| \geq 1 . \end{cases}$$

In table 1, we give a list of errors in the averages of the coarse grid solution u compared to the fine grid solution. We can see that the averages of the numerical solutions converge to that of the 'accurate' solution essentially independent of ε . The rate of convergence is basically of order $O(\Delta x)$.

In figure 1a and 1b, we display the initial pulse for u for Δx=0.031 and Δx=0.0052 respectively. In figure 2a and 2b, solutions for the coarse and fine grid are given at the $1.28/\Delta x$ time level. The smooth part of solution u to the left of the oscillatory pulse is due to the interaction with v through the nonlinear lower order terms. In the smooth region, we can see that the coarse grid solution approximates the fine grid one very well. Figure 3a and 3b display the moving averages of solution u for these two gridsize. We observe the average of the coarse grid solution agrees with that of the fine grid solution in the oscillatory region.

In the numerical experiment for the Broadwell model (1.2), we choose the initial values for u and v as in (6.1) and (6.2) and zero initial value for w. We take $\varepsilon \equiv 0.0097$, and compute the solutions at time t=2.48 using algorithm (2.12-13) with a sequence of gridsizes. In our calculation, we choose the fine gridsize to be $\Delta x = 0.0013$, that is, we have about 10 particles per wavelength. The moving space averages of a solution are computed by the same smoothing function as before.

In table 2 and 3, we give a list of errors in the averages of the coarse grid solutions u and w compared to the fine grid solution. As before we observe that the moving space average of the numerical solution converges essentially independent of ε .

Figure 5a and 5b display the solution w at time t=2.48 for Δx=0.031 and Δx=0.0013 respectively. As we see, the nonlinear interaction of uv term generates a oscillatory pulse on w. In Figure 6a and 6b, we plot the moving space averages of the two above mentioned solution w. We can see that in the region of oscillation, the average of the coarse grid solutions agrees with that of the fine grid solution very well.

In the constant coefficient Carleman model (a(x,t)=b(x,t)=1), Tartar [19] has shown that if

$$(6.3) \qquad \int_0^1 u_A(x,y,0)^m \, dy = \int_0^1 u_B(x,y,0)^m \, dy \quad , \quad \text{for } all \ integer \ m \geq 0,$$

$$(6.4) \qquad \int_0^1 v_A(x,y,0)^m \, dy = \int_0^1 v_B(x,y,0)^m \, dy \quad , \quad \text{for } all \ integer \ m \geq 0,$$

then we have

$$\int_0^1 U_A(x,y,t)^m \, dy = \int_0^1 U_B(x,y,t)^m \, dy \quad , \quad \text{for } all \ integer \ m \geq 0 \ \text{ and } t > 0,$$

$$\int_0^1 V_A(x,y,t)^m \, dy = \int_0^1 V_B(x,y,t)^m \, dy \quad , \quad \text{for } all \ integer \ m \geq 0 \ \text{ and } t > 0,$$

where U and V are the solutions for the corresponding homogenized equations.

From our convergence Theorem 4.1, our particle solutions also have the similar properties. The difference is that integration in y is now replaced by the moving space average of the numerical solution. Of course the equality is true only up to the numerical accuracy.

To test this result for our particle scheme, we consider two types of initial data. Introduce function $\phi_A(x)$ and $\phi_B(x)$ as follows:

$$\phi_A(x) = \begin{cases} 5x & , \ 0 \leq x \leq 0.2 , \\ -5(x-0.2)+1 & , \ 0.2 < x \leq 0.6 , \\ 5(x-0.6)-1 & , \ 0.6 < x \leq 0.8 , \\ 0 & , \ 0.8 < x \leq 1.0 , \end{cases}$$

$$\phi_B(x) = \begin{cases} 5x & , \ 0 \le x \le 0.2 , \\ -5(x-0.2)+1 & , \ 0.2 < x \le 0.4 , \\ 0.0 & , \ 0.4 < x \le 0.6 , \\ -5(x-0.6) & , \ 0.6 < x \le 0.8 , \\ 5(x-0.8)-1 & , \ 0.8 < x \le 1.0 . \end{cases}$$

We extend the $\phi_A(x)$ and $\phi_B(x)$ to be 1-periodic functions in the real line.

Define two types initial oscillatory data as below:

Initial Condition Type A:

$$u_A(x,0) = \begin{cases} 0.5sin^4(\pi(x-3)/2)(1+\phi_A((x-3)/\varepsilon)) & , \ |x-4| < 1 , \\ 0 & , \ |x-4| \ge 1 , \end{cases} \tag{6.5}$$

$$v_A(x,0) = \begin{cases} 0.5sin^4(\pi(x-4)/2)(1+\phi_A((x-4)/\varepsilon)) & , \ |x-5| < 1 , \\ 0 & , \ |x-5| \ge 1 , \end{cases} \tag{6.6}$$

Initial Condition Type B:

$$u_B(x,0) = \begin{cases} 0.5sin^4(\pi(x-3)/2)(1+\phi_B((x-3)/\varepsilon)) & , \ |x-4| < 1 , \\ 0 & , \ |x-4| \ge 1 , \end{cases} \tag{6.7}$$

$$v_B(x,0) = \begin{cases} 0.5sin^4(\pi(x-4)/2)(1+\phi_B((x-4)/\varepsilon)) & , \ |x-5| < 1 , \\ 0 & , \ |x-5| \ge 1 . \end{cases} \tag{6.8}$$

Obviously the initial data type A , type B satisfy the relations (6.3-4). The two types of data are only piecewise smooth. Table 4 indicates the convergence of the moving space average of the numerical solutions essentially independent of ε for the data of type A. In table 5 , we compare the moving space average of the numerical solutions corresponding to the two types of initial data. And we observe that they agree with one another up to the corresponding numerical accuracy.

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FIGURE CAPTIONS

Figure 1a : The initial value for u in the variable coefficient Carleman model, $\Delta x = 0.031$, $\varepsilon \equiv 0.0175$.

Figure 1b : The initial value for u in the variable coefficient Carleman model, $\varepsilon \equiv 0.0175$, where the solid line is for Δx=0.0052, the o's are for Δx=0.031.

Figure 2a : Particle solution u for variable velocity Carleman equations, Δx =0.031, $\varepsilon \equiv 0.0175$.

Figure 2b : Particle solution u for variable velocity Carleman equations, $\varepsilon \equiv 0.0175$, where the solid line is for Δx=0.0052, the o's are for Δx=0.031.

Figure 3a : Averaged particle solution u for variable velocity Carleman equations, Δx =0.031; $\varepsilon \equiv 0.0175$

Figure 3b : Averaged particle solution u for variable velocity Carleman equations, $\varepsilon \equiv 0.0175$, where the solid line is for Δx=0.0052, the o's are for Δx=0.031.

Figure 4 : The particle paths for the Carleman equations, Δx =0.155,

Figure 5a : Particle solution w for the Broadwell equations, Δx =0.031, $\varepsilon \equiv 0.0097$,

Figure 5b : Particle solution w for the Broadwell equations, $\varepsilon \equiv 0.0097$, where the solid line is for Δx=0.0013, the o's are for Δx=0.031.

Figure 6a : Averaged particle solution w for the Broadwell equations, Δx =0.031, $\varepsilon \equiv$0.0097,

Figure 6b : Averaged particle solution w for the Broadwell equations, $\varepsilon \equiv$0.0097 , where the solid line is for Δx=0.0013, the o's are for Δx=0.031.

Table 1 : Errors in the averaged solution of u for the variable coefficient Carleman Model. Here $\varepsilon \equiv 0.0175$. $a(x,t)=1+0.5\sin(xt)$, $b(x,t)=1+0.2\cos(3xt)$. The solutions are obtained after $1.28/\Delta x$ time iterations. The 'accurate' solution is computed with $\Delta x = 0.0052$.

Gridsize	L1-norm	L2-norm	Max-norm
0.062	0.022760	0.021525	0.039000
0.031	0.007753	0.005596	0.008300
0.0155	0.004208	0.003510	0.006800
0.0109	0.001573	0.001121	0.001600

Table 2 : Errors in the averaged solution of u for the Broadwell Model. Here $\varepsilon \equiv 0.0097$, $t=2.48$. The 'accurate' solution is computed with $\Delta x = 0.0013$

Gridsize	L1-norm	L2-norm	Max-norm
0.062	0.035219	0.031217	0.040100
0.031	0.019681	0.016854	0.026000
0.0155	0.007600	0.006725	0.008900
0.0052	0.002005	0.001754	0.002500

Table 3 : Errors in the averaged solution of w for the Broadwell Model. Here $\varepsilon \equiv 0.0097$, $t=2.48$. The 'accurate' solution is computed with $\Delta x = 0.0013$

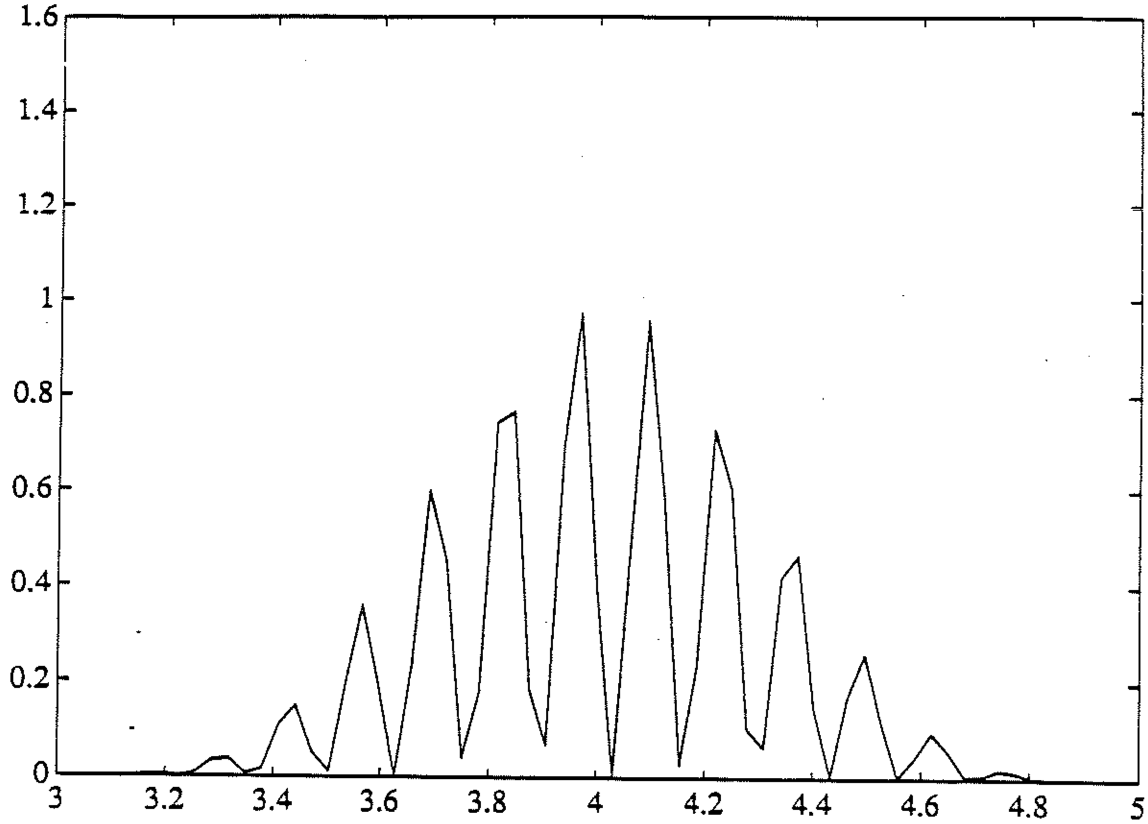
Gridsize	L1-norm	L2-norm	Max-norm
0.062	0.005227	0.005182	0.008100
0.031	0.003625	0.003782	0.006000
0.0155	0.001427	0.001360	0.001900
0.0052	0.000586	0.000637	0.001600

Table 4 : Errors in the averaged solution of u using initial data type A for the Carleman Model. Here $\varepsilon \equiv 0.011$, time t=2.48.
The 'accurate' solution is computed with Δx =0.0025

Gridsize	L1-norm	L2-norm	Max-norm
0.031	0.024112	0.018053	0.021700
0.0155	0.010288	0.007732	0.009700
0.00775	0.004078	0.003156	0.004100

Table 5 : Differences between the averaged solution u of I.C. type A and B for the Carleman Model. Here $\varepsilon \equiv 0.011$, time t = 2.48

Gridsize	L1-norm	L2-norm	Max-norm
0.031	0.002966	0.003353	0.006700
0.0155	0.001472	0.001555	0.003400
0.00775	0.000770	0.000874	0.002200



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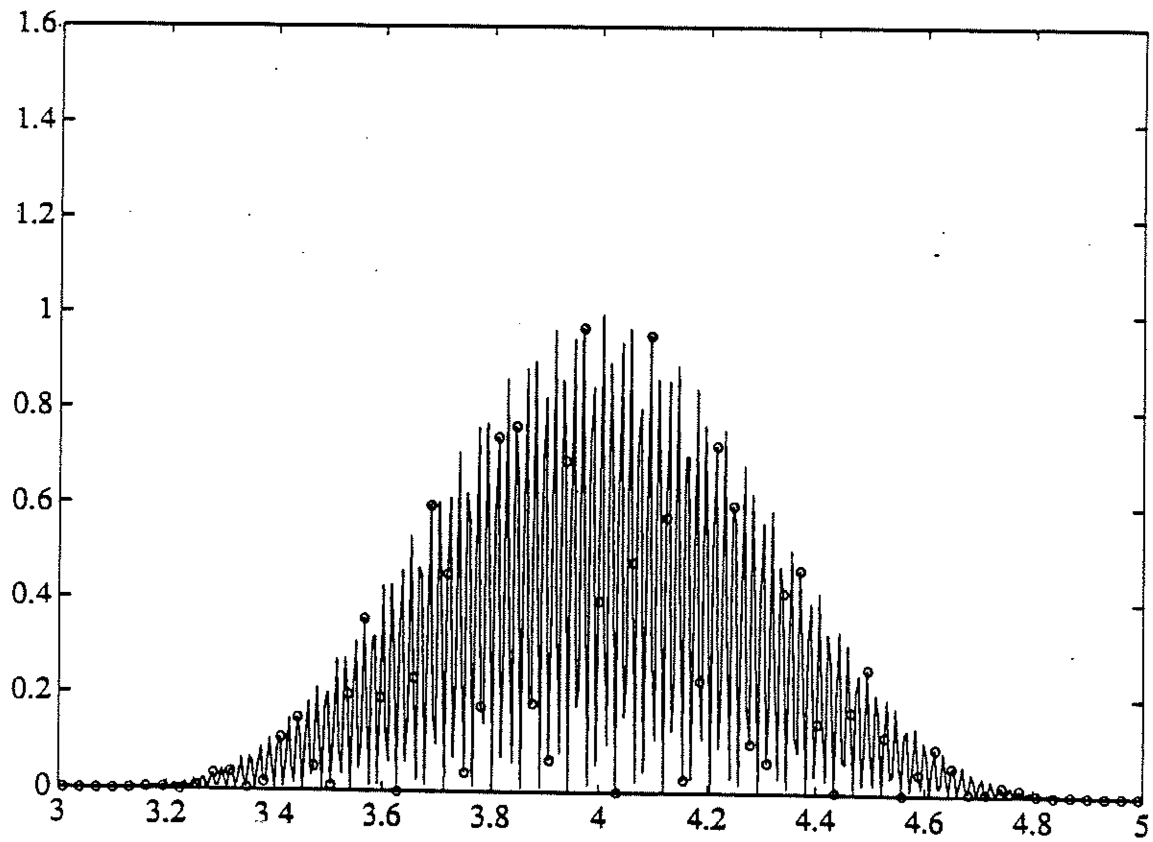

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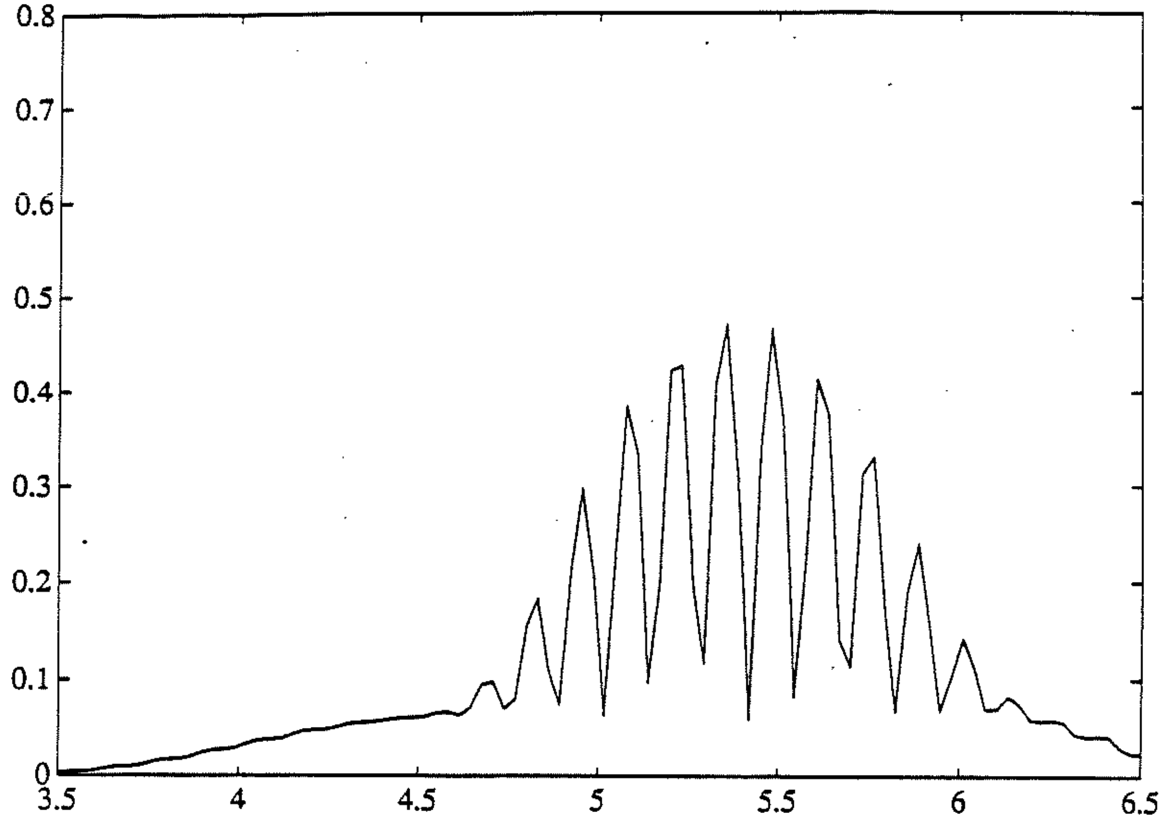

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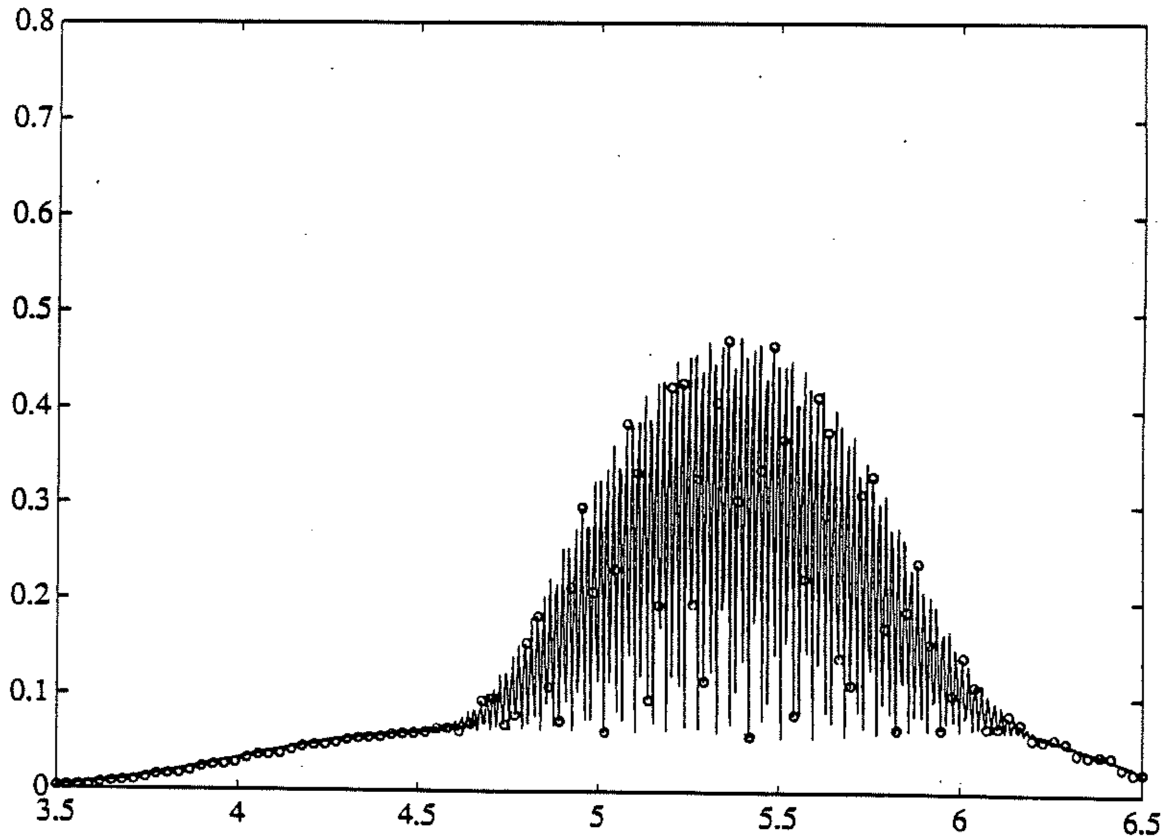

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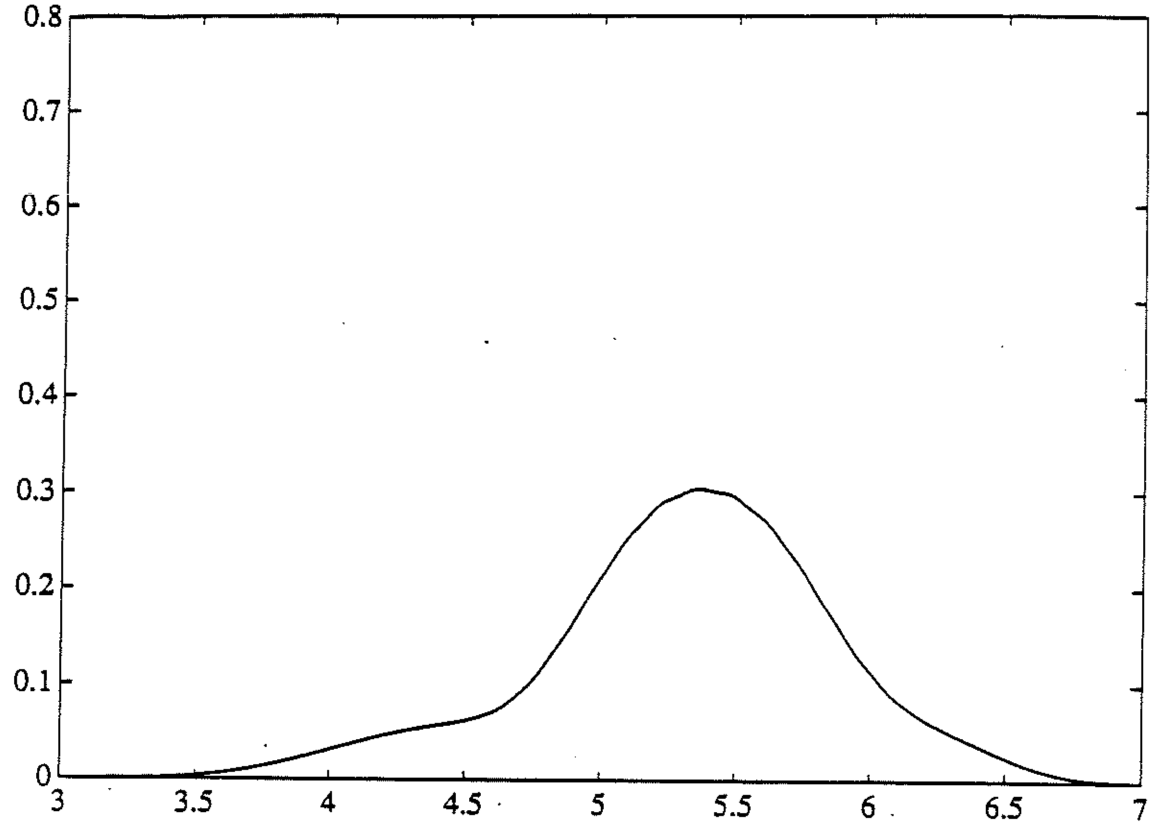
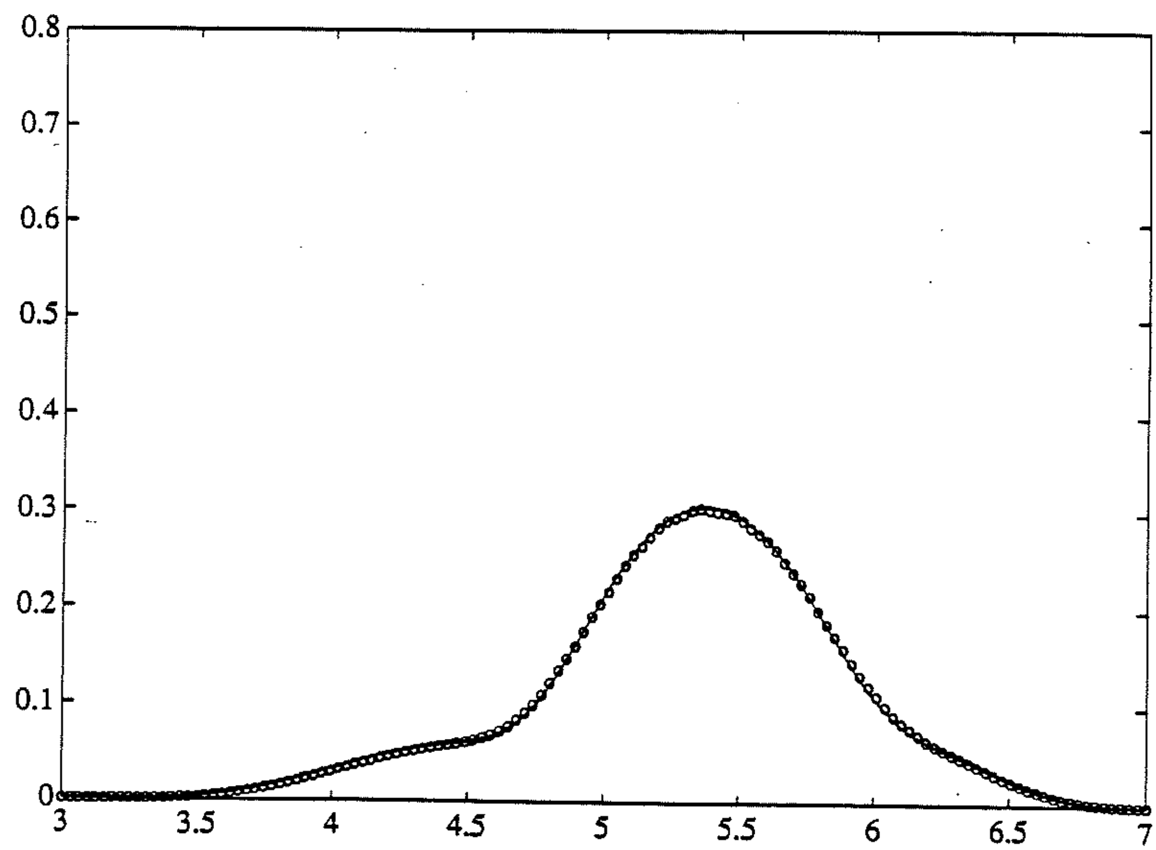


Figure 3a



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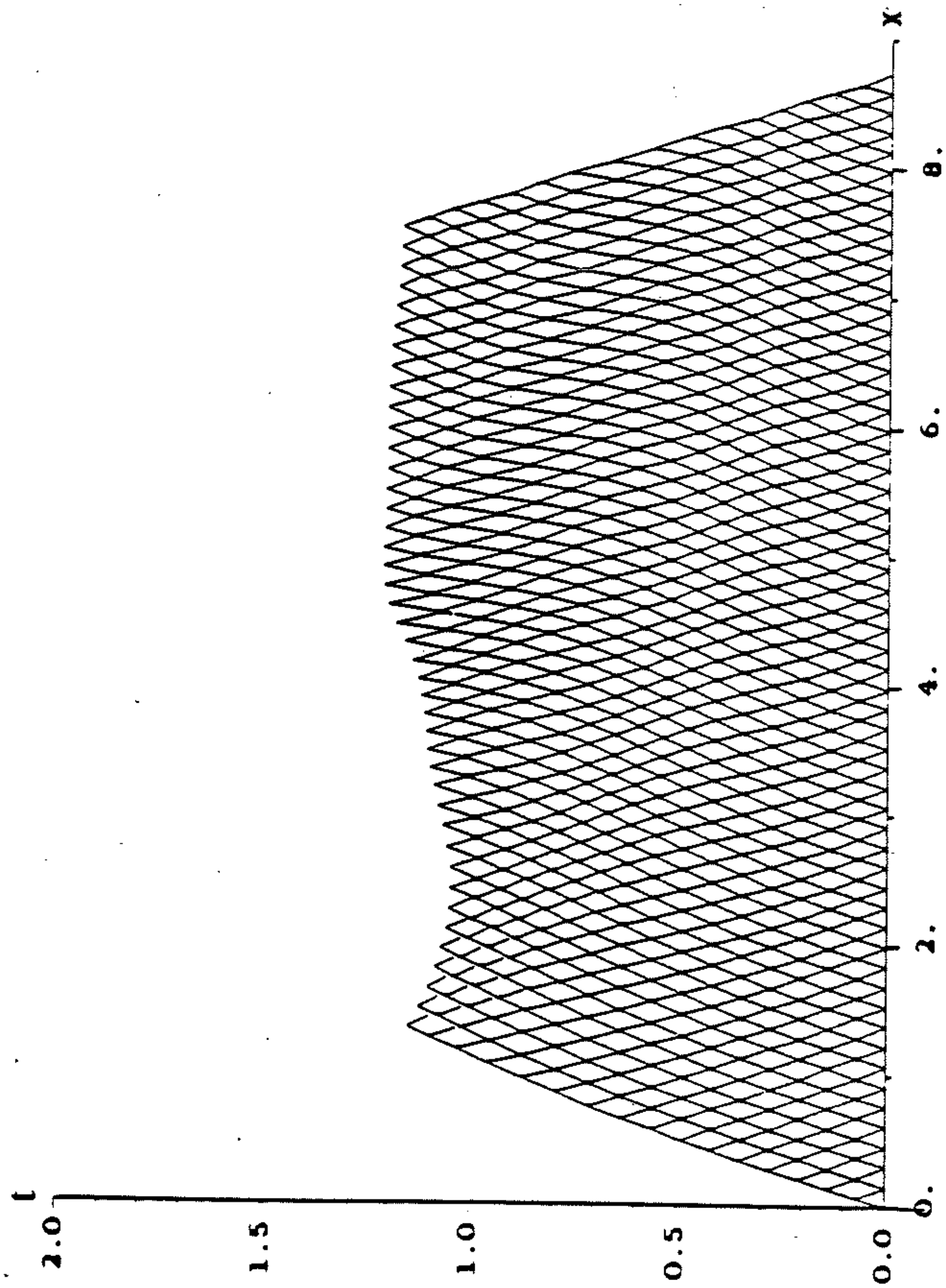


Figure 4

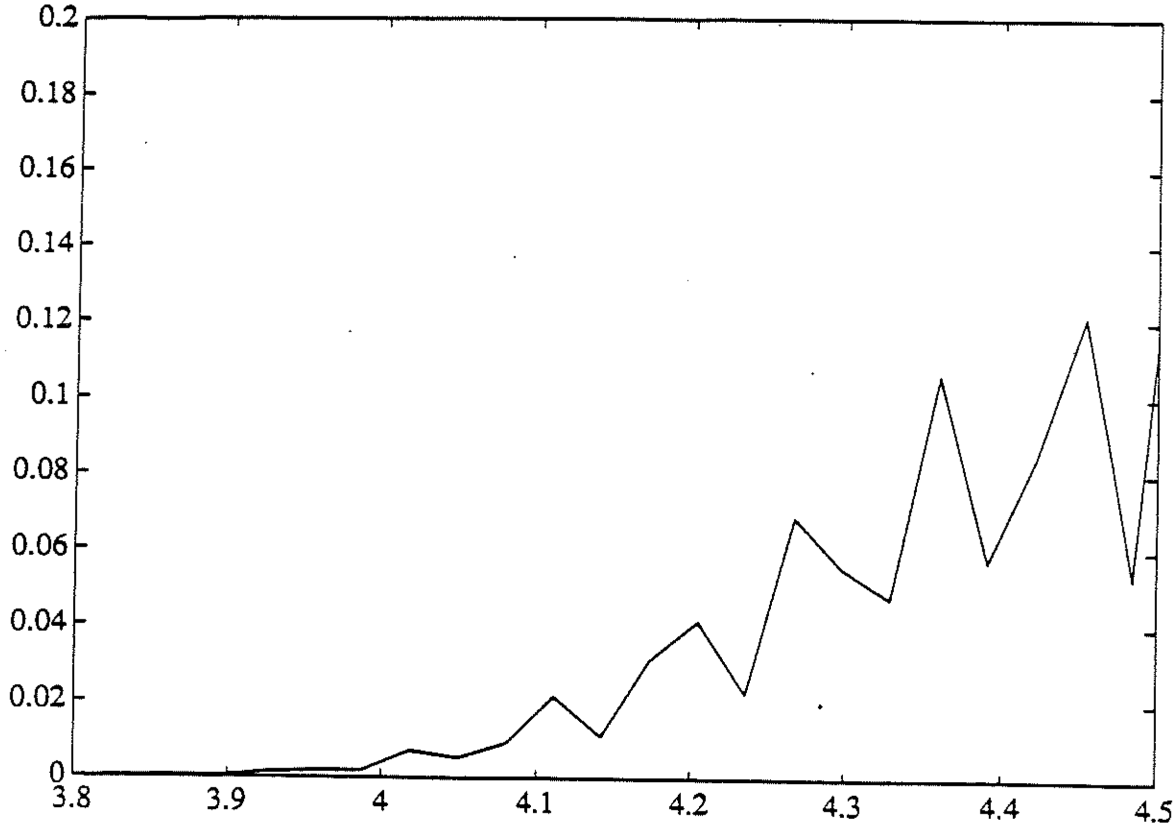
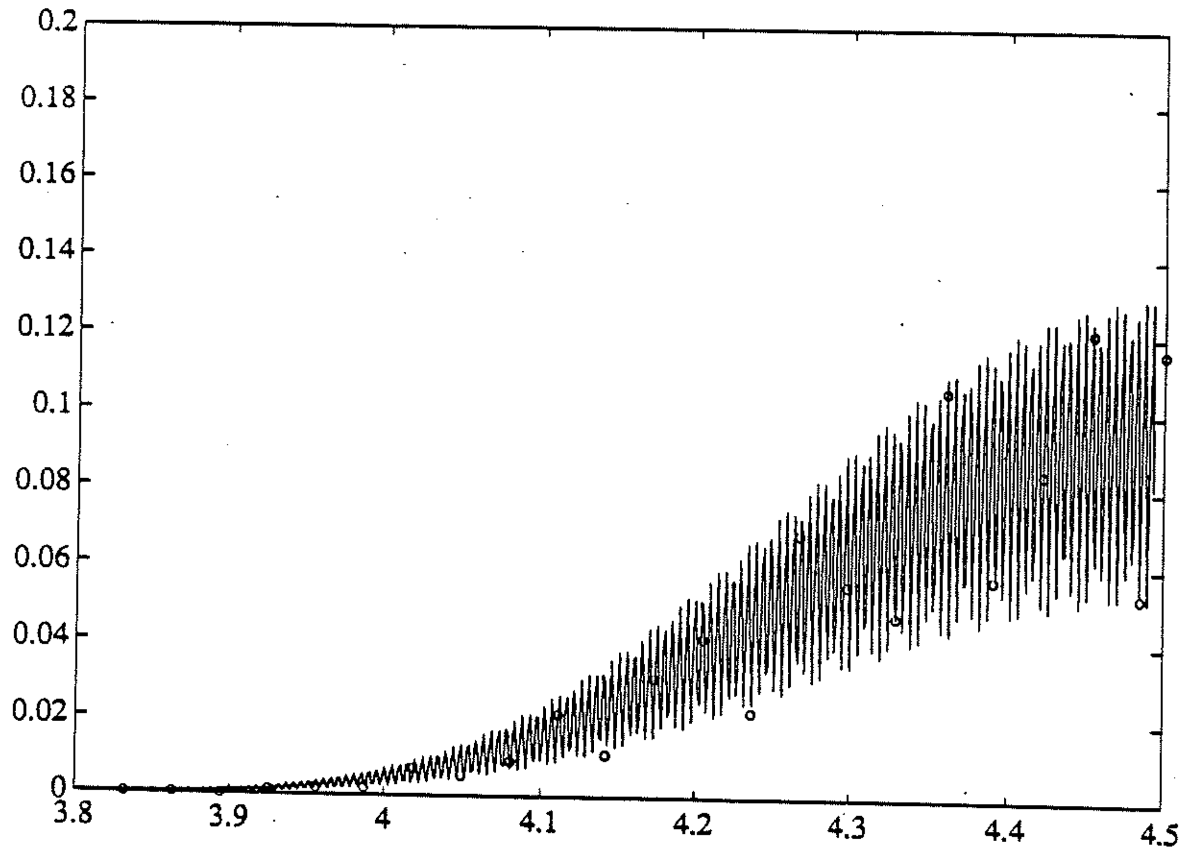


Figure 5a



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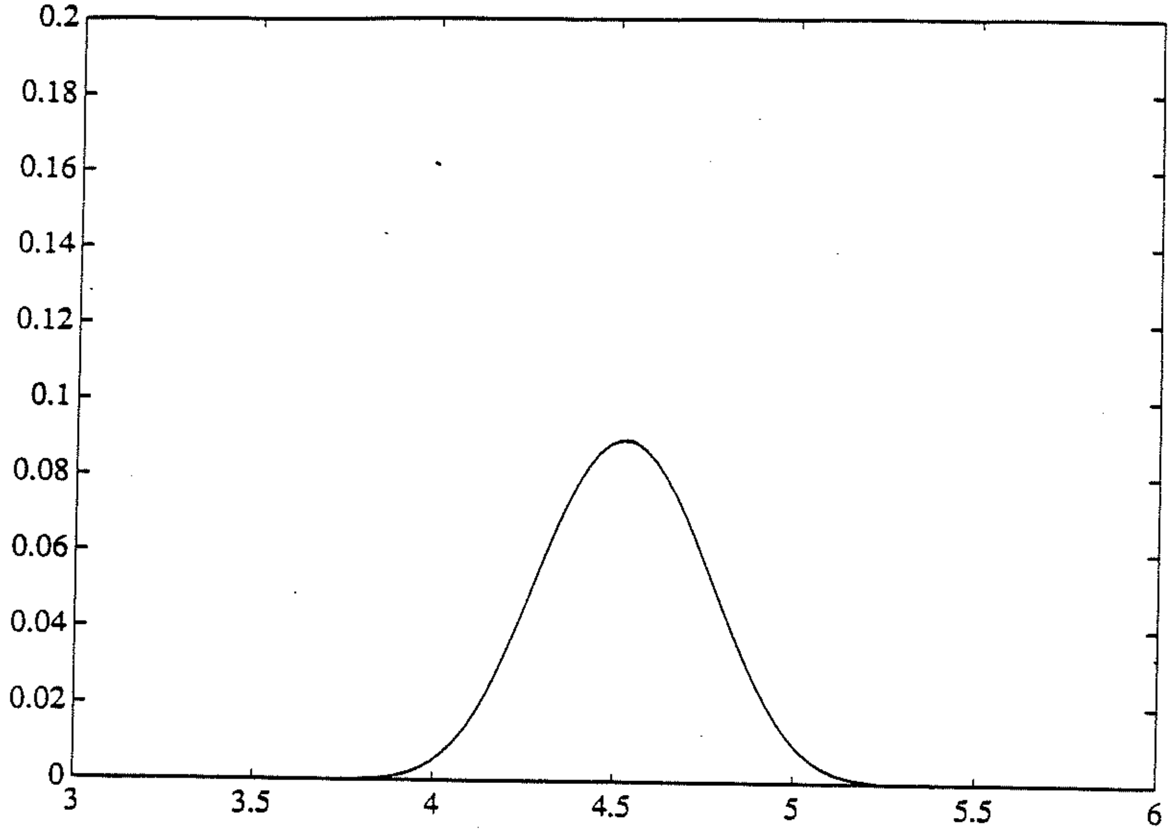

Figure 6a

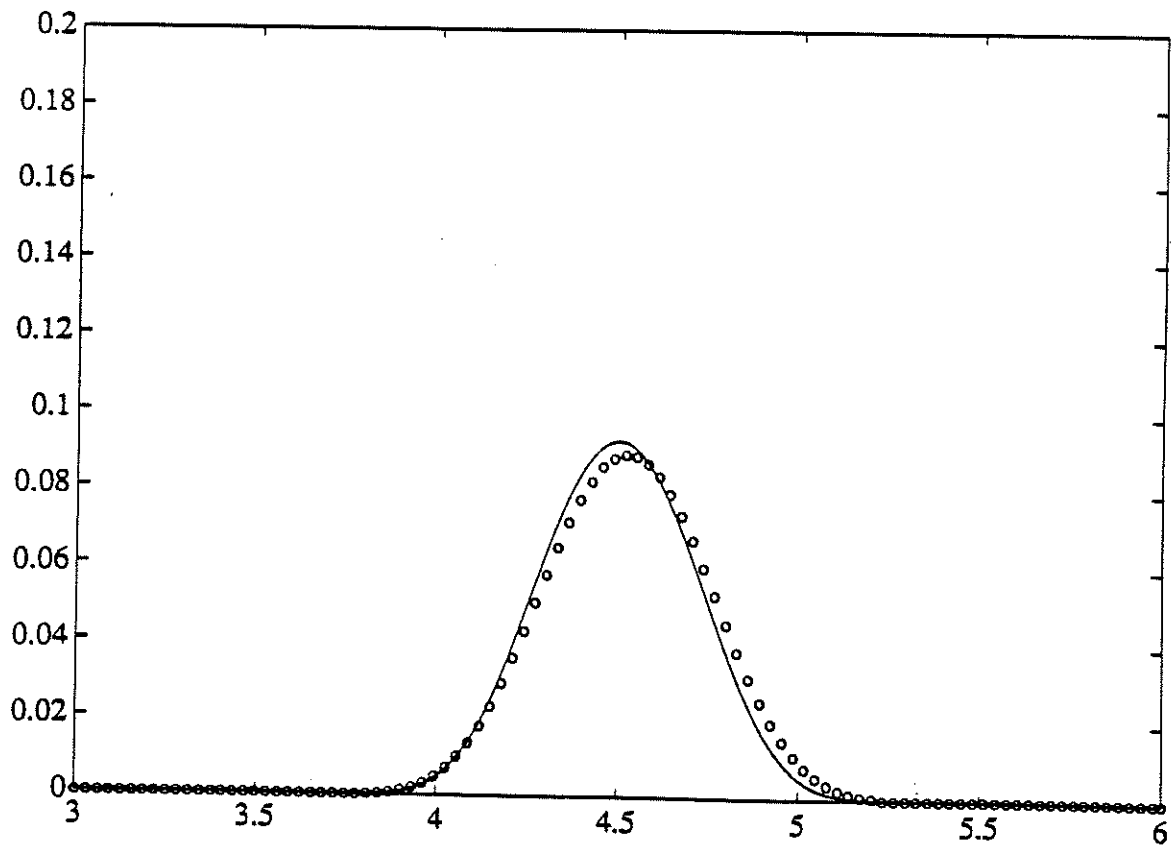

Figure 6b