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Abstract. The vortex method for two dimensional inviscid, incompressible flow with highly oscillatory vorticity fields is studied. A homogenization result for Euler equation in velocity-vorticity formulation is obtained and weak continuity of the equation is proved. Convergence of vortex methods is analyzed in the case when continuous vorticity is not well resolved by the computational particles. Numerical results are given. Comparisons are made with the corresponding finite difference approximation.

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1. Introduction

Vortex methods have received a great deal of attention in the past decade and have found numerous applications in turbulent combustion([11],[22]), boundary layer theory [6], aerodynamics calculation([5],[15]), and flow through heart valve[16] among others. In the vortex methods, the interaction of the computational vortices mimic the physical mechanisms in the actual flow so that the effort is focused on the variables of the most interest and the region of the most interest.

As we know, vortex methods are primarily used to simulate the incompressible flow at high Reynolds numbers. Usually such flow will develop certain fine scale structure after some time. Due to the limited capacity of the available computing facilities, it is practically impossible to resolve all the small scale features in the flow. Naturally, one would like to know whether or not it is possible to obtain long time convergence in flow variables such as averaged velocity profile and mean stream-line distribution if the small scale structure is not well resolved by the computational particles. Accuracy of vortex methods over a long time interval has been studied by Perlman [20] and Beale [3] for 2-D Euler equations with piecewise smooth vorticity. Recently, Sethian and Ghoniem [23] have studied experimentally the convergence of vortex methods for two dimensional incompressible flow at high Reynolds numbers. They observe qualitative numerical convergence in such flow variables as eddy sizes and average velocity profiles even with a relatively few number of particles.

In this paper, we study 2-D Euler equations with highly oscillatory vorticity. We make a simplified assumption that at certain time t_0 the vorticity distribution is of two scales, i.e. $\omega(x,t_0) = \omega_0(x; \frac{x}{\varepsilon})$. We then analyze the convergence of the vortex methods for the Euler equation in the case when the oscillatory wave is not well resolved by the computational particles. The classical convergence analysis for smooth solutions (see e.g. [1], [4], [7], [12], [21]) can not be applied directly here. The error constants will grow inversely proportional to the oscillatory wave length ε. In order to show convergence, one needs to take into account the cancellation among high frequency vorticity components at different space locations. Ideally it would be desirable to require that the error is small if the grid size is below a limit which is independent of the small scales. This is not possible in general. However, we can show that the convergence is essentially independent of the oscillatory wave length ε . The concept of convergence essentially independent of ε was first introduced by Engquist in [9]. It means that the numerical solutions converge as long as the grid size does not belong to a set which has an arbitrarily small measure independent of ε (see Definition 1 in section 3). Thus for almost all samplings of the grid, we can obtain convergence independent of the oscillatory wave length ε .

We would like to point out that for the oscillatory problems, convergence essentially independent of the wave length ε is the best we can hope for. The fact that we need to avoid a set in the ratio between the wave length ε and the grid size reflects the nature of the oscillatory problems. This is not limited by the techniques we use. In the vortex method, since velocity depends on the moving space average of initial vorticity (see Theorem 2.2), it is necessary that the moving space average of the initial numerical values approximate that of the exact initial values. It is obvious that this can not be achieved for all samplings of grid independent of ε .

The main results in this paper are stated and proved in section 2, 3 and 5. In section 2, we prove that at least up to a finite time the velocity field and flow map have uniformly bounded first order derivatives independent of the small scale ε. Moreover we show that the homogenized velocity and flow map are governed by the same Euler equation but with homogenized initial vorticity. More precisely, the velocity field corresponding to the oscillatory vorticity converges strongly to the velocity corresponding to the averaged initial vorticity. The rate of convergence is also given. These results are used in the section 3 and 4 where the vortex method is shown to be convergent essentially independent of ε . This is done by using the techniques in the convergence analysis for smooth flows and taking into account cancellation in the oscillatory component. Following the same line but using only the Hölder continuity of the velocity field, we also prove the convergence of the vortex method for any finite time. In section 5, the weak continuity of the Euler equation is proved using the classical compactness argument. As a consequence, the homogenization result mentioned before is established for any finite time.

Our numerical experiments in section 6 give qualitative convergence results for the vortex method. We observe that the size of the error for oscillatory vorticity in vortex methods is of the same order of magnitude as that for the smooth vorticity. This indicates that the error constants in the error estimates for vortex methods are essentially independent of the high frequency vorticity components. On the other hand, the size of the error in finite difference approximation for oscillatory vorticity is $O(1/\varepsilon)$ larger that that for the smooth vorticity. Also vorticity at later time is dissipated considerably in the finite difference methods, in which case the numerical solution could potentially converge to wrong solution in problems when the small scale vorticity plays an important role.

We mention that similar convergence analysis of particle methods for the Carleman model with highly oscillatory data has been given by Engquist[9], and homogenization theory for semilinear hyperbolic systems with two scales initial data has been studied by McLaughlin, Papanicolaou and Tartar ([19], [24]). See also [10] and [13] for more discussions along these lines.

2. Homogenization results

The two dimensional inviscid, incompressible flow of constant density is governed by the following Euler equation:

$$(2.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla p\,, \\ \nabla\cdot u = 0\,, \quad x, u \in R^2, \end{cases}$$

where u is velocity, p is pressure.

For smooth initial velocity, smooth solution is known to exist for all time and is unique ([14], [17]).

The particle trajectory, denoted as $X(t, \alpha)$, satisfies

$$(2.2)\qquad \begin{cases} \dfrac{dX(t, \alpha)}{dt} = u(X(t, \alpha),t), \\ X(0, \alpha) = \alpha. \end{cases}$$

We are interested in the case when the initial vorticity is of the form

$$\omega(\alpha,0) = \omega_0(\alpha;\frac{\alpha}{\varepsilon}).$$

We assume that $\omega_0(\alpha,\beta)$ is a smooth function in α and β, $\omega_0(\alpha;\beta)$ is 1-periodic in each component of β. Moreover we assume that ω_0 has finite support contained in $\{ \alpha ; |\alpha| \le R \}$.

The vorticity transport equation is given by

$$(2.3)\qquad \begin{cases} \dfrac{\partial\omega}{\partial t} + (u\cdot\nabla)\omega = 0, \\ \omega(\alpha,0) = \omega_0(\alpha;\dfrac{\alpha}{\varepsilon}). \end{cases}$$

Thus we have

$$(2.4)\qquad \omega(X(t, \alpha),t) = \omega(\alpha,0) = \omega_0(\alpha;\frac{\alpha}{\varepsilon}).$$

On the other hand, the velocity can be expressed in terms of the vorticity by the Biot-Savart law

$$(2.6)\qquad u(x,t) = \int K(x-z)\omega(z,t)dz = \int K(x-X(t, \alpha))\omega_0(\alpha;\frac{\alpha}{\varepsilon})d\alpha$$

with K given by

$$(2.7)\qquad K(x) = \frac{1}{2\pi |x|^2}(-x_2, x_1).$$

Throughout this paper we will use C and c to denote generic constants which are independent of parameters ε, h and δ. We will also use the following notations.

Denote $X^{-1}(t,x)$ as the inverse mapping of $X(t,\alpha)$, that is

$$X(t, X^{-1}(t,x)) \equiv x , \quad X^{-1}(t, X(t,\alpha)) \equiv \alpha.$$

Define

$$\overline{\omega}_0(x) = \int_{[0,1]^2} \omega_0(x; y)dy , \quad \tilde{\omega}_0(x; \frac{x}{\varepsilon}) = \omega_0(x; \frac{x}{\varepsilon}) - \overline{\omega}_0(x),$$

and

$$\overline{\omega}(x,t) = \overline{\omega}_0(X^{-1}(t,x)) , \quad \tilde{\omega}(x,t) = \tilde{\omega}_0(X^{-1}(t,x); \frac{X^{-1}(t,x)}{\varepsilon}).$$

An important observation is that the velocity field has bounded first order partial derivatives independent of ε , although the vorticity is highly oscillatory. This is described by the following proposition:

Proposition 2.1 *$\max_{1\le i\le 2}\{\|\partial_i u(t,\cdot)\|_\infty\}$, $\max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\}$ and $\|\frac{d^2}{dt^2}X(t,\cdot)\|_\infty$ are bounded independent of ε up to a finite time T, where T depends on initial vorticity but is independent of ε . In particular, if the solution of the Euler equation is steady , then T can be arbitrarily large.*

The proof will be given at the end of this section.

Theorem 2.1 *For $0 \le t \le T$, we have*

$$| u(z,t) - \int K(z-X(t,\alpha)\overline{\omega}_0(\alpha)\, d\alpha \,| \le C\varepsilon \,|\log(\varepsilon)\,|$$

where T is given in Proposition 2.1 .

Proof. This follows immediately from Lemma 2.1 below and Proposition 2.1 .

Theorem 2.2 *Let $\overline{u}(x,t)$ be the velocity field corresponding to the mean initial vorticity $\overline{\omega}_0(x)$. Then we have for any $1>s>0$*

$$|\, u(x,t) - \overline{u}(x,t)\,| \le C\ (\varepsilon\,|\log(\varepsilon)\,|)^{1-s}\ , \quad \text{for } 0 \le t \le T.$$

The proof will be given at the end of section 3.

Let $R_* = R\,(1+1/(2\|\omega_0\|_\infty))\exp(2\|\omega_0\|_\infty T)$. It can be shown easily that for $0 \le t \le T$, $\omega(x,t)$ has support contained in the region $\{\ x;\ |x| \le R_*\ \}$ and

$$\max_{|\alpha|\le R} |X(t,\alpha)| \le R_* .$$

Lemma 2.1 *For $z= X(t,\alpha)$ with $|\alpha| \le R$, we have*

$$(2.8)\qquad |\int \partial_i K(z-y)(\tilde{\omega}(y,t) - \tilde{\omega}(z,t))dy\ | \le C\ (1 + \max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\}),$$

$$(2.9)\qquad |\int K(z-y)\tilde{\omega}(y,t)\ dy\ | \le C\varepsilon\, |\log(\varepsilon)|\ \max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\},$$

where ∂_i is the partial derivative with respect to the ith space variable, and the integration is carried out on a circular disk $\{x;\ |x| \le 2R_*\ \}$.

Proof. We give the proof for (2.8) only. (2.9) can be proved similarly.

First we decompose the right hand side of (2.8) as

$$(2.10)\qquad \int \partial_i K(z-y)(\tilde{\omega}(y,t) - \tilde{\omega}(z,t))dy = \int_{|z-y|\le\varepsilon} \partial_i K(z-y)(\tilde{\omega}(z,t) - \tilde{\omega}(y,t))dy$$

$$- \tilde{\omega}(z,t)\int_{|z-y|>\varepsilon} \partial_i K(z-y)\ dy + \int_{|z-y|>\varepsilon} \partial_i K(z-y)\tilde{\omega}(y,t)dy.$$

Since the flow is incompressible, the determinant of the Jacobian matrix of $X(t,\alpha)$ is identically equal to one. We have

$$[\ \partial_i X_j^{-1}(t,z)\ ] = [\ \partial_i X_j(t,\alpha)\ ]^{-1} = \begin{bmatrix} \partial_2 X_2 & -\partial_2 X_1 \\ -\partial_1 X_2 & \partial_1 X_1 \end{bmatrix} \tag{2.11}$$

where α is evaluated at $X^{-1}(t,z)$.

Thus we have

$$\max_{1\le i\le 2}\{\|\partial_i X^{-1}(t,\cdot)\|_\infty\} \le \max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\}, \tag{2.12}$$

which implies

$$\Big|\int_{|z-y|\le\varepsilon} \partial_i K(z-y)(\tilde\omega(y,t) - \tilde\omega(z,t))\ dy\ \Big|$$

$$\le \frac{c_1}{\varepsilon} \max_{1\le i\le 2}\{\|\partial_i X^{-1}(t,\cdot)\|_\infty\} \int_{|z-y|\le\varepsilon} |\ \partial_i K(z-y)\ |\ |\ z-y\ |\ dy \tag{2.13}$$

$$\le \frac{c_1}{\varepsilon} \max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\} \int_{|\eta|\le\varepsilon} |\partial_i K(\eta)|\ |\eta| d\eta = c_1 \max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\},$$

where the constant c_1 is bounded by the first order partial derivatives of $\tilde\omega_0(x;y)$.

Now let's study the second term on the right hand side of (2.10)

$$I_2 \equiv \tilde\omega(z,t) \int_{|y-z|>\varepsilon} \partial_i K(z-y)\ dy. \tag{2.14}$$

Note that $|\ z\ | \le R_*$ for $|\ \alpha\ | \le$ R . Recall that the integration is over $\{y;\ |\ y\ | \le 2\ R_*\ \}$. Thus we can split the integration into two parts

$$I_2 = \tilde\omega(z,t) \int_{R_*\ge|y-z|>\varepsilon} \partial_i K(z-y) dy + \tilde\omega(z,t) \int_{\substack{R_*<|z-y| \\ |y|\le 2R_*}} \partial_i K(z-y) dy. \tag{2.15}$$

Clearly, the first term on the right hand side of (2.15) is equal to zero. The second term is bounded by

$$\|\tilde\omega\|_\infty \int_{R_*<|z-y|\le 3R_*} |\ \partial_i K(z-y)\ | dy\ = 2\|\omega_0\|_\infty \log(3).$$

Therefore we obtain that

$$|\ I_2\ | \le 2\|\omega_0\|_\infty log\,(3). \tag{2.16}$$

By change of variable, the last term on the right hand side of (2.10) becomes

$$I_3 \equiv \int_{|z-y|>\varepsilon} \partial_i K(z-y)\tilde\omega(y,t) dy\ = \int_{|X(t,\alpha)-X(t,\beta)|>\varepsilon} \partial_i K(X(t,\alpha)-X(t,\beta))\tilde\omega_0(\beta;\frac{\beta}{\varepsilon}) d\beta.$$

Rewrite

$$\tilde{\omega}_0(q;\frac{q}{\varepsilon}) \equiv w_1(q;\frac{q_1}{\varepsilon},\frac{q_2}{\varepsilon}) + w_2(q;\frac{q_1}{\varepsilon}),$$

where w_1, w_2 are defined by

$$w_1(q;p_1,p_2) \equiv \omega_0(q;p_1,p_2) - \int_0^1 \omega_0(q;p_1,p_2)dp_2,$$

(2.17)

$$w_2(q;p_1) \equiv \int_0^1 \omega_0(q;p_1,p_2)dp_2 - \int_0^1\int_0^1 \omega_0(q;p_1,p_2)dp_1dp_2.$$

Note that

$$\int_0^1 w_1(q;p_1,p_2)dp_2 = 0 \ , \quad \int_0^1 w_2(q;p_1)dp_1 = 0 \ . \tag{2.18}$$

Furthermore we express w_1 as

$$w_1(q;\frac{q_1}{\varepsilon},\frac{q_2}{\varepsilon}) = \frac{d}{dq_2}\int_{a(q_1)}^{q_2} w_1(q;\frac{q_1}{\varepsilon},\frac{s}{\varepsilon})ds - \int_{a(q_1)}^{q_2} \partial_2 w_1(q;\frac{q_1}{\varepsilon},\frac{s}{\varepsilon})ds, \tag{2.19}$$

where $\partial_2 w_1(q;p)$ is the partial derivative with respect to the second variable in q, $a(q_1)$ is any function of q_1.

Obviously we have from (2.18)

$$\left| \int_{a(q_1)}^{q_2} w_1(q;\frac{q_1}{\varepsilon},\frac{s}{\varepsilon})ds \right| \leq c_2\varepsilon, \tag{2.20}$$

$$\left| \int_{a(q_1)}^{q_2} \partial_2 w_1(q;\frac{q_1}{\varepsilon},\frac{s}{\varepsilon})ds \right| \leq c_3\varepsilon. \tag{2.21}$$

By substituting w_1 in (2.19) into the last term in (2.10), and performing integration by parts for the first term on the right hand side of (2.19) in q_2 in each appropriate subdomain, we can show using (2.20-21)

$$\left| \int_{|X(t,x)-X(t,q)|>\varepsilon} \partial_i K(X(t,x)-X(t,q))w_1(q;\frac{q}{\varepsilon})dq \right|$$

$$\leq c_4\varepsilon \max_{1\leq i\leq 2}\{\|\partial_i X(t,\cdot)\|_\infty\} \int_{\varepsilon<|\eta|\leq 3R.} |\partial_2\partial_i K(\eta)|d\eta + c_5 \leq c_4 \max_{1\leq i\leq 2}\{\|\partial_i X(t,\cdot)\|_\infty\} + c_5,$$

where we have made use of the fact that ω_0 has bounded support. Consequently there is no contribution from the boundary terms when performing integration by parts.

Similar argument applies to w_2 term. Hence we have proven that the last term on the right hand side of (2.10) is bounded independent of ε . Combining this with the results in (2.13) and (2.16), we complete the proof of the lemma.

Now we can give the proof of Proposition 2.1.

Proof of Proposition 2.1:

Differentiating both sides of (2.2) with respect to α_i, we have

$$\frac{d}{dt}(\partial_i X(t,\alpha)) = \partial_i X_1(t,\alpha)\partial_1 u(z,t) + \partial_i X_2(t,\alpha)\partial_2 u(z,t) \tag{2.22}$$

where z is evaluated at X(t, α).

We need to obtain a bound on $\partial_i u(z,t)$.

$$\partial_i u(z,t) = \frac{1}{2}\begin{bmatrix}-\delta_{2,i}\\ \delta_{1,i}\end{bmatrix}\omega(z,t) + \int \partial_i K(z-y)(\omega(y,t) - \omega(z,t))dy \tag{2.23}$$

where the integration is on the circular disk { y; | y | ≤ $2R_*$ }, $\{\delta_{i,j}\}$ is the Kronecker delta.

To bound the last term in (2.23), we first study the case when ω is replaced by $\overline{\omega}$. We get

$$\begin{aligned}
(2.24)\quad & \left| \int \partial_i K(z-y)(\overline{\omega}(y,t) - \overline{\omega}(z,t))dy \right| \\
&\le \left| \int_{|y-z|\le 1} \partial_i K(z-y)(\overline{\omega}(y,t) - \overline{\omega}(z,t))dy \right| + \left| \int_{1<|y-z|\le 3R_*} \partial_i K(z-y)(\overline{\omega}(y,t) - \overline{\omega}(z,t))dy \right| \\
&\le c_7 \|\omega_0\|_{1,\infty} \max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\} \int_0^1 \frac{1}{r^2}\cdot r\cdot r dr + c_8\|\omega_0\|_\infty \int_1^{3R_*} \frac{1}{r^2}\cdot r dr \\
&\le c_7 \|\omega_0\|_{1,\infty} \max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\} + c_8\|\omega_0\|_\infty \log(3R_*),
\end{aligned}$$

where $\|\omega_0\|_{1,\infty} = \max_{1\le i\le 2}\|\partial_i \omega_0(\alpha;\ \beta)\|_\infty$.

For the term with ω replaced by $\tilde{\omega}$ in (2.23), Lemma 2.1 applies. Thus we have proven

$$|\partial_i u(z,t)| \le c\|\omega_0\|_{1,\infty} \max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\} + d\ log(3R_*)\ \|\omega_0\|_\infty, \tag{2.25}$$

where c and d are universal constants.

Let $M(t) = \max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\}$, then M(t) satisfies

$$\frac{d}{dt}M(t) \le M(t)(C\cdot M(t) + D) \tag{2.26}$$

with $C = c\|\omega_0\|_{1,\infty}$, $D = d\ |\ log(3R_*)\ |\ \|\omega_0\|_\infty$.

As a result of (2.26), we obtain

$$M(t)(1 - \frac{C}{C+D}exp(Dt)) \le \frac{D}{C+D}exp(Dt). \tag{2.27}$$

Hence we conclude that for $T < \frac{1}{D}\cdot log((C+D)/C)$,

(2.28) $\max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\}$ is bounded independent of ε for $0 \le t \le T$.

Differentiating (2.2) with respect to t, we get

$$\frac{d}{dt}\left(\frac{d}{dt}X(t,\alpha)\right) = \frac{d}{dt}X_1(t,\alpha)\partial_1 u(z,t) + \frac{d}{dt}X_2(t,\alpha)\partial_2 u(z,t). \tag{2.29}$$

Since $\max_{1\le i\le 2}\{\|\partial_i u(t,\cdot)\|_\infty\}$ is bounded, application of Gronwall inequality to (2.29) implies that $\|\frac{d}{dt}X(t,\cdot)\|_\infty$ is bounded independent of ε up to time T. It follows then from (2.29) that $\|\frac{d^2}{dt^2}X(t,\cdot)\|_\infty$ is also bounded independent of ε .

In the case when the solution of the Euler equation is steady, we can easily modify the proof above to show that $\max_{1\le i\le 2}\{\|\partial_i u(t,\cdot)\|_\infty\}$ is bounded independent of ε and $\max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\}$ up to arbitrarily large time T. Thus it follows immediately that $\max_{1\le i\le 2}\{\|\partial_i X(t,\cdot)\|_\infty\}$ and $\|\frac{d^2}{dt^2}X(t,\cdot)\|_\infty$ are bounded, because the right hand side of (2.26) is now a linear function in M(t). This completes the proof of Proposition 2.1 .

Remark 2.1 The difficulty in obtaining a long time estimate in the general case is due to the nonlinear nature in the characteristic equation (2.2). We end up with a quadratic form in the right hand side of (2.26).

3. Convergence of the semi-discrete vortex methods

Solving Euler equation (2.1) is equivalent to solving the following system of equations by coupling the velocity equation (2.6) into the characteristic equation (2.2),

$$\begin{cases} \dfrac{dX(t,\alpha)}{dt} = \int K(X(t,\alpha)-X(t,\beta))\omega_0(\beta;\frac{\beta}{\varepsilon})d\beta \\ X(0,\alpha) = \alpha. \end{cases} \tag{3.1}$$

Let $\Omega_h = \{\alpha_j;\ \alpha_j = (j_1h, j_2h),\ \alpha_j \in \textit{support of } \omega_0\}$, $X_j(t) \equiv X(t,\alpha_j)$. We denote by $X_i^h(t)$, $u_i^h(t)$ the approximations of $X(t,\alpha_i)$, $u(X(t,\alpha_i),t)$ respectively.

In the vortex (blob) method, we approximate the integral in (3.1) by finite summation and solve (3.1) for a finite collection of vortex blobs ([6], [4] and [1]). In the two dimensional case, the algorithm is as follows

$$\begin{cases} \dfrac{dX_i^h(t)}{dt} = \sum_{jh\in\Omega_h} K_\delta(X_i^h(t)-X_j^h(t))\omega_j h^2 \\ X_i^h(0) = \alpha_i, \end{cases} \tag{3.2}$$

$$u_i^h(t) = \sum_{jh\in\Omega_h} K_\delta(X_i^h(t)-X_j^h(t))\omega_j h^2, \tag{3.3}$$

where $\omega_j = \omega_0(x_j)$, $K_\delta = K * \zeta_\delta$, ζ_δ is a cut-off function satisfying

$$\zeta_\delta(x) = \frac{1}{\delta^2}\zeta\left(\frac{x}{\delta}\right) \quad , \quad \delta = h^\sigma \ \textit{with} \ \sigma<1. \tag{3.4}$$

Assumption 1 We assume that $\omega_0(x; \cdot) \in C_0^r$ with $r \geq 3$. $\zeta(x)$ belongs to C^2 and is radial symmetric, satisfying $\int_{R^2} \zeta(x)dx = 1$ and $|\zeta(x)| \leq L_0|x|^{-7/2}$ for $|x| > 1$.

For smooth flow, convergence properties of vortex method have been established in [1], [4], [7], [12] and [21]. In our context, these analyses implies convergence of the vortex method if the small scale features are well resolved by the computational particles. However, this is often times unrealistic. It is generally expected that vortex method would still perform reasonably well even when the small scale features are not well resolved. Theoretically it would be desirable to prove convergence even in this case.

The classical concept of convergence only takes into account the size of the grid. In our problem, errors in the sampling of the grid is also significant. Therefore we need to modify the classical concept of convergence.

Definition 1: (Engquist [9]) The approximation u^n converges to u as $h \to 0$ essentially independent of ε if for any $\tau > 0$, $T > 0$ there exists a set $s(\varepsilon,h_0) \subset (0,h_0)$ with measure $(s(\varepsilon,h_0)) \geq (1-\tau)h_0$ such that

$$\|u(\cdot,t_n) - u^n\| \leq \tau , \quad 0 \leq t_n \leq T$$

is valid for all $h \in s(\varepsilon,h_0)$ and where h_0 is independent of ε.

Define the set $s(\varepsilon,h_0)$ as follows:

$$s(\varepsilon,h_0) = \{0<h\leq h_0;\ (kh/\varepsilon) \notin (2i-\frac{\tau}{|k|^{3/2}},\ 2i+\frac{\tau}{|k|^{3/2}}) \text{ for } i=0,1,\ldots,[h_0/(2\varepsilon)]+1,\ 0\neq k \in Z\}.$$

For $0 < h_1 < h_2$, we define the set $s_r(\varepsilon,h_1,h_2)$ by

$$s_r(\varepsilon,h_1,h_2) = \{h_1\leq h\leq h_2;\ (kh/\varepsilon) \notin (2i-\frac{\tau}{|k|^{3/2}},\ 2i+\frac{\tau}{|k|^{3/2}}) \text{ for } 0<|k|\leq(1/h_1)^{1/(r-1)},$$
$$\text{for } i=[h_1/(2\varepsilon)],\ldots,[h_2/(2\varepsilon)]+1,\ k\in Z\}$$

where r is related to the degree of regularity of ω_0 as in the Assumption 1.

Then the measure of $s(\varepsilon,h_0)$ is bounded from below by

$$\textit{measure of } s(\varepsilon,h_0) \geq h_0(1-\tau\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}) \geq h_0(1-3\tau).$$

Similarly, the measure of $s_r(\varepsilon,h_1,h_2)$ is bounded from below by $(h_2-h_1)(1-3\tau)$.

Theorem 3.1 *The numerical solutions of vortex method (3.2-3) converge to the exact solutions of the Euler equation essentially independent of* ε . *More precisely, there exists a positive constant* h_0 *such that for* $0 \le t \le T$, *all* $h \in s(\varepsilon, h_0)$ *and any positive s, we have*

$$\| X_i^h(t) - X(t,\alpha_i) \|_{l_\infty} \le C\delta^{1+s},$$

$$\| u_i^h(t) - u(X(t,\alpha_i),t) \|_{l_\infty} \le C\delta^{1+s},$$

provided that δ *and h satisfy the relation*

(3.5) $$h\,|\,log(h)\,| = c\,\delta^{1+2s} \quad \text{and} \quad \varepsilon \le h.$$

Corollary 3.1 *All the estimates in Theorem 3.1 are still valid if we replace the set* $s(\varepsilon,h_0)$ *by* $s_r(\varepsilon,h_1,h_2)$ *in Theorem 3.1.*

Remark 3.1 Theorem 3.1 is very encouraging. It tells us that we do not need to resolve all the small scale features in the flow and we are still able to obtain useful approximation to certain flow variables for almost all samplings of particles. One disadvantage of Theorem 3.1 is that the set $s(\varepsilon,h_0)$ excludes almost all the rational numbers for ε small. Since computers can only generate rational numbers, Theorem 3.1 can not be used directly to practical computation. On the other hand, the number of particles we use in practical computation is always finite. And the grid size usually has a lower bound ($h \ge h_1$) which is limited by the computer capacity. Thus we can use Corollary 3.1 in which the set $s_r(\varepsilon,h_1,h_2)$ excludes only those numbers near $\{i/k\}$ with $|k| \le (1/h_1)^{1/(r-1)}$. The value of $(1/h_1)^{1/(r-1)}$ is small in general. For instance, $(1/h_1)^{1/(r-1)} = 10$ for r=3 and h_1=0.001. Therefore there are more than enough numbers available in a computer to make Corollary 3.1 meaningful.

In order to prove theorem 3.1, we need to use a number of technical lemmas.

Lemma 3.1 (Beale and Majda) *With time t fixed, we have*

$$\sum_{|jh|<R} \max_{|y_j| \le C_0\delta} |\,D^\beta K_\delta(z - X_j(t) + y_j)\,|\,h^2 \le \begin{cases} C, & \beta = 0 \\ C|\log\delta|, & |\beta| = 1 \\ C\delta^{-1}, & |\beta| = 2 \end{cases}$$

for all z with $|\,z\,| < R_*$, *provided that h is small enough. Here C depends only on* R_*, C_0 .

See [4] for a proof.

Lemma 3.2 *Assume that* $|\,\zeta(r)\,| \le L_0 r^{-7/2}$, *we have*

$$|\,K_\delta(x)\,| \le L_1\delta^{-1}, \quad \text{for } 0 < r < \infty$$

$$|\,K_\delta(x) - K(x)\,| \le L_1\delta^{3/2} r^{-5/2}, \quad \text{for } \delta < r < \infty$$

This was proved by Hald [12].

Lemma 3.3 *Suppose that $f(y) \in C^r(R)$ $(r \geq 3)$ is 1-periodic in y and $\int_{[0,1]} f(y)dy = 0$. Then for all $h \in s(\varepsilon, h_0)$, we have*

$$| \sum_{a \leq kh \leq b} f(\frac{x_k}{\varepsilon}) | \leq C/\tau$$

where $x_k = kh$, and C is independent of h and ε.

Proof. We follow closely the argument given by Engquist in [9]. Since f(y) is periodic, it can be expanded in a Fourier series

$$f(y) = \sum_m a_m \exp(2\pi i m y).$$

The Fourier coefficient a_0 is equal to zero since $\int_0^1 f(y)dy = 0$. Moreover since $f(y) \in C^r$, we have $|a_m| \leq C|m|^{-r}$. Thus we obtain

$$|D| = | \sum_{a \leq kh \leq b} f(\frac{x_i}{\varepsilon}) | = | \sum_{a \leq kh \leq b} \sum_m a_m \exp(2\pi i m x_k \varepsilon^{-1}) |$$

$$= | \sum_{m \neq 0} a_m \sum_{a \leq kh \leq b} \exp(2\pi i m x_k \varepsilon^{-1}) | \leq \sum_{m \neq 0} \frac{C}{|m|^r |1 - \exp(2\pi i m h/\varepsilon)|}.$$

For $h \in s(\varepsilon, h_0)$, we have

$$|1 - \exp(2\pi i m h/\varepsilon)| = |2\sin(\pi m h/\varepsilon)| \geq 4\tau/(m^{3/2}), \qquad m \neq 0.$$

Hence we obtain for $h \in s(\varepsilon, h_0)$ the following estimate

$$|D| \leq \sum_{m \neq 0} \frac{C|m|^{3/2}}{|m|^r \ (4\tau)} \leq C_*/\tau \qquad \text{, for } r \geq 3.$$

This completes the proof of lemma 3.3 .

Corollary 3.2 *Lemma 3.3 is still true if we replace the set $s(\varepsilon, h_0)$ by $s_r(\varepsilon, h_1, h_2)$.*

Proof. Let M = $(1/h_1)^{1/(r-1)}$. Arguing as in the proof of Lemma 3.3, we have for $h \in s_r(\varepsilon, h_1, h_2)$

$$|D| = | \sum_{m \neq 0} a_m \sum_{a \leq kh \leq b} \exp(2\pi i m x_k \varepsilon^{-1}) |$$

$$\leq \sum_{0<|m|<M} \frac{C}{|m|^r |1 - \exp(2\pi i m h/\varepsilon)|} + \sum_{|m|>M} \frac{C}{|m|^r \cdot h_1} \leq C_1/\tau + \frac{C_2}{M^{r-1} \cdot (h_1)} \leq C_1/\tau + C_2.$$

Thus the proof of Corollary 3.2 is complete.

Lemma 3.4 *For all $h \in s(\varepsilon, h_0)$, we have*

$$\left| \sum_{jh \in \Omega_h} \partial^\beta K_\delta(x - X_j(t)) \tilde{\omega}_0(\alpha_j; \frac{\alpha_j}{\varepsilon}) h^2 \right| \le \begin{cases} Ch|\log(\delta)|, & |\beta|=0, \\ Ch/\delta, & |\beta|=1. \end{cases}$$

Proof: Rewrite

$$\tilde{\omega}_0(q; \frac{q}{\varepsilon}) \equiv w_1(q; \frac{q_1}{\varepsilon}, \frac{q_2}{\varepsilon}) + w_2(q; \frac{q_1}{\varepsilon})$$

where w_1, w_2 are defined in (2.17).

Denote the two dimensional index j = (i,k), $\alpha_j = (ih, kh)$. Then

$$\sum_{jh \in \Omega_h} \partial^\beta K_\delta(x - X_j(t)) w_1(\alpha_j; \frac{\alpha_j}{\varepsilon}) h^2 = \sum_{i=a}^{b} \sum_{k=c(i)}^{d(i)} \partial^\beta K_\delta(x - X_{i,k}(t)) w_1(\alpha_{i,k}; \frac{\alpha_{i,k}}{\varepsilon}) h^2.$$

It can be verified directly that

$$\sum_{k=c(i)}^{d(i)} \partial^\beta K_\delta(x - X_{i,k}(t)) w_1(\alpha_{i,k}; \frac{\alpha_{i,k}}{\varepsilon}) = \sum_{k=c(i)}^{d(i)} \partial^\beta K_\delta(x - X_{i,d(i)}(t)) \sum_{k=c(i)}^{d(i)-1} w_1(\alpha_{i,d(i)}; \frac{\alpha_{i,k}}{\varepsilon})$$

$$(3.6) \qquad - \sum_{k=c(i)}^{d(i)-1} (\partial^\beta K_\delta(x - X_{i,k+1}(t)) - \partial^\beta K_\delta(x - X_{i,k}(t))) \sum_{l=c(i)}^{k} w_1(\alpha_{i,k+1}; \frac{\alpha_{i,l}}{\varepsilon})$$

$$- \sum_{k=c(i)}^{d(i)-1} \partial^\beta K_\delta(x - X_{i,k}(t)) \sum_{l=c(i)}^{k} (w_1(\alpha_{i,k+1}; \frac{\alpha_{i,l}}{\varepsilon}) - w_1(\alpha_{i,k}; \frac{\alpha_{i,l}}{\varepsilon})).$$

The first term on the right hand side of (3.6) vanishes since ω_0 has bounded support. Observe that

$$w_1(\alpha_{i,k+1}; \frac{\alpha_{i,l}}{\varepsilon}) - w_1(\alpha_{i,k}; \frac{\alpha_{i,l}}{\varepsilon}) = h \partial_2 w_1(\alpha_{i,k}; \frac{\alpha_{i,l}}{\varepsilon}) + O(h^2),$$

$$\text{and} \int_{[0,1]} \partial_2 w_1(\alpha_{i,k}; \beta_1, \beta_2) d\beta_2 = 0 = \int_{[0,1]} w_1(\alpha_{i,k}; \beta_1, \beta_2) d\beta_2,$$

where $\partial_j w_1(\alpha;\beta)$ represents the partial derivative in the j-th component of α variable.

Furthermore, since $X(t,\alpha)$ has bounded first order partial derivatives in α, we find.

$$|\partial^\beta K_\delta(x - X(t, \alpha_{i,k+1})) - \partial^\beta K_\delta(x - X(t, \alpha_{i,k}))| \le ch \max_{|\alpha|=1; |y_{i,k}| < c_0 h} |\partial^\alpha \partial^\beta K_\delta(x - X(t, \alpha_{i,k}) + y_{i,k})|.$$

Therefore, by using Lemma 3.3 we obtain

$$| \sum_{jh \in \Omega_h} \partial^\beta K_\delta(x - X_j(t)) w_1(\alpha_j; \frac{\alpha_j}{\varepsilon}) h^2 | \le$$

$$\le ch \sum_{|jh| < R} \max_{\substack{|\alpha|=1 \\ |y_j| < c_0 h}} | \partial^\alpha \partial^\beta K_\delta(x - X(t, \alpha_j) + y_j)| + ch \sum_{|jh| < R} | \partial^\beta K_\delta(x - X(t, \alpha_j))|$$

In light of Lemma 3.1, we have proven Lemma 3.5 when $\bar{\omega}_0$ is replaced by w_1. Similarly we can show that Lemma 3.5 is true when $\bar{\omega}_0$ is replaced by w_2 .

Consistency Lemma *Suppose that* $\varepsilon \le h$. *For all* $h \in s(\varepsilon, h_0)$, *there exists a constant* C_c *such that*

(3.7) $$\| u(X_i(t), t) - \sum_{jh \in \Omega_h} K_\delta(X_i(t) - X_j(t)) \omega_j h^2 \|_{l_\infty} \le C_c (\delta^{1+1/3} + h\, |log(h)|)$$

provided that $h < \delta$, *where* C_c *is independent of* h *and* ε . *The estimate (3.7) is still valid if we replace the set* $s(\varepsilon, h_0)$ *by* $s_r(\varepsilon, h_1, h_2)$.

Proof. We first study the case with ω_0 in (3.7) replaced by $\bar{\omega}_0$.

Define $\overline{em}$ as

$$\overline{em} \equiv \int K(X_i(t) - X(t,\alpha)) \bar{\omega}_0(\alpha) d\alpha - \int K_\delta(X_i(t) - X(t,\alpha)) \bar{\omega}_0(\alpha) d\alpha,$$

where the integration is over a circular disk containing the support of ω_0.

Since $\int_{|y| \le R.} K(y) dy = \int_{|y| \le R.} K_\delta(y) dy = 0$, we get

$$\overline{em} = \int (K(z-y) - K_\delta(z-y))(\bar{\omega}_0(X^{-1}(t,y)) - \bar{\omega}_0(X^{-1}(t,z))) dy \quad , \quad with \quad z = X_i(t).$$

From this, we find

(3.8) $$|\overline{em}| \le 2\pi \int_{|z-y| \le \delta} \frac{2L_1}{r} \max_i \|\partial_i X^{-1}(t, \cdot)\|_\infty \max_i \|\partial_i \bar{\omega}_0(\cdot)\|_\infty\, r\, dr$$
$$+ 2\pi \int_{|z-y| > \delta} \frac{L_1 \delta^{3/2}}{r^{5/2}} H_{1/3}(\bar{\omega}_0) (\max_i \|\partial_i X^{-1}(t, \cdot)\|_\infty\, r)^{1/3}\, r\, dr,$$

where we have used Lemma 3.2 , and the Hölder norm $H_\lambda(\bar{\omega}_0)$ is defined by

(3.9) $$H_\lambda(\bar{\omega}_0) = \sup_{x \ne y} \frac{|\bar{\omega}_0(x) - \bar{\omega}_0(y)|}{|x-y|^\lambda}.$$

Since $X^{-1}(t,y)$ has bounded space derivatives, inequality (3.8) gives

(3.10) $$| \overline{em} | \le c_1 \delta^2 + c_2 \delta^{1+1/3}$$

Define $\overline{ed}$ as

$$\overline{ed} \equiv \int K_\delta(X_i(t)-X(t,\alpha))\overline{\omega}_0(\alpha)d\alpha - \sum_{jh\in\Omega_h} K_\delta(X_i(t)-X(t,\alpha_j))\overline{\omega}_0(\alpha_j)h^2$$

$$= \sum_{jh\in\Omega_h} \left(\int_{B_j} K_\delta(X_i(t)-X(t,\alpha))\overline{\omega}_0(\alpha)d\alpha_1 d\alpha_2 - K_\delta(X_i(t)-X(t,\alpha_j))\overline{\omega}_0(\alpha_j)h^2 \right)$$

where $B_j = \{ (\alpha_1, \alpha_2) \,;\, |\alpha_1 - j_1 h| \le h/2 \,,\, |\alpha_2 - j_2 h| \le h/2 \}$. Then we obtain

$$|\,\overline{ed}\,| \le Ch \sum_{jh\in\Omega_h} \max_{|y_j| \le C_0 h} |\, D^\beta K_\delta(X_i(t)-X_j(t)+y_j)\,|$$

where we have used Proposition 2.1 and $|\,\beta\,| = 1$.

Lemma 3.1 then gives

(3.11) $$|\,\overline{ed}\,| \le Ch \,|\, log(\delta)\,|.$$

It follows from (3.10) and (3.11) that

(3.12) $$|\, \int K(X_i(t)-X(t,\alpha))\overline{\omega}_0(\alpha)d\alpha - \sum_{jh\in\Omega_h} K_\delta(X_i(t)-X_j(t))\overline{\omega}_0(\alpha_j)h^2 \,|$$

$$= |\,\overline{em} + \overline{ed}\,| \le C_1(\delta^{1+1/3} + h|\, log(\delta)\,|).$$

Now let's consider the case when ω_0 in (3.7) is replaced by $\tilde{\omega}_0$. Theorem 2.1 implies

(3.13) $$|\, \int K(X_i(t)-X(t,\alpha))\tilde{\omega}_0(\alpha;\frac{\alpha}{\varepsilon})d\alpha \,| \le C\varepsilon \,|\, log(\varepsilon)\,|.$$

On the other hand, Lemma 3.4 implies that for all $h \in s(\varepsilon, h_0)$,

(3.14) $$|\, \sum_{jh\in\Omega_h} K_\delta(X_i(t)-X_j(t))\tilde{\omega}_0(\alpha_j;\frac{\alpha_j}{\varepsilon})h^2 \,| \le Ch|\, log(\delta)\,|$$

Thus we have proven that for all $h \in s(\varepsilon, h_0)$

(3.15) $$\|\, \int K(X_i(t)-X(t,\alpha))\tilde{\omega}_0(\alpha;\frac{\alpha}{\varepsilon})d\alpha - \sum_{jh\in\Omega_h} K_\delta(X_i(t)-X_j(t))\tilde{\omega}_0(\alpha_j;\frac{\alpha_j}{\varepsilon})h^2 \,\|_{l_\infty}$$

$$\le C_2(\varepsilon|\, log(\varepsilon)\,| + h \,|\, log(\delta)\,|)$$

Hence the Lemma follows from (3.12) and (3.15).

Stability Lemma *Assume for some* $T_* \ge 0$

(3.16) $$\max_{\substack{jh\in\Omega_h \\ 0\le t\le T_*}} |\, X_i^h(t) - X_i(t)\,| \le \delta.$$

Then there exists a constant C_s such that for $0 \le t \le T_*$, the estimate

$$\| u_i^h(t) - \sum_{jh\in\Omega_h} K_\delta(X_i(t)-X_j(t))\omega_j h^2 \|_{L_h^\mu} \le C_s \| X_i^h(t) - X_i(t) \|_{L_h^\mu} \tag{3.17}$$

holds for all $h \in s(\varepsilon, h_0)$ *provided that* $h \le \delta$ *, where*

$$\| a_j \|_{L_h^\mu} \equiv (\sum_{jh\in\Omega_h} |a_j|^\mu h^2)^{1/\mu}. \tag{3.18}$$

Remark 3.2: The estimate (3.17) is still valid if we replace the set $s(\varepsilon, h_0)$ by $s_r(\varepsilon, h_1, h_2)$.

Proof. We first consider the case when ω_0 in (3.17) is replaced by $\overline{\omega}_0$. Since $\overline{\omega}_0(\alpha)$ and $X(t,\alpha)$ have bounded first order partial derivatives in α, the argument in the proof of the 2-D stability lemma by Beale and Majda [4] applies. Therefore (3.17) is valid in the case when ω_0 is replaced by $\overline{\omega}_0$.

Now let's study the case when ω_0 in (3.17) is replaced by $\tilde{\omega}_0$.

Define $es\,1(i)$ by

$$es\,1(i) \equiv \sum_{jh\in\Omega_h} K_\delta(X_i(t)-X_j(t))\tilde{\omega}_0(\alpha_j;\frac{\alpha_j}{\varepsilon})h^2 - \sum_{jh\in\Omega_h} K_\delta(X_i(t)-X_j^h(t))\tilde{\omega}_0(\alpha_j;\frac{\alpha_j}{\varepsilon})h^2.$$

Since $\tilde{\omega}_0$ is uniformly bounded independent of ε , Beale and Majda's argument in the 2–D stability lemma [4] can be applied to the term es1(i). We then obtain

$$\| es\,1(i) \|_{L_h^\mu} \le C_3 \| X_i^h(t) - X_i(t) \|_{L_h^\mu} \tag{3.19}$$

under the assumption (3.16).

It remains to estimate the term $es\,2(i)$ defined below

$$es\,2(i) \equiv \sum_{jh\in\Omega_h} K_\delta(X_i(t)-X_j^h(t))\tilde{\omega}_0(\alpha_j;\frac{\alpha_j}{\varepsilon})h^2 - \sum_{jh\in\Omega_h} K_\delta(X_i^h(t)-X_j^h(t))\tilde{\omega}_0(\alpha_j;\frac{\alpha_j}{\varepsilon})h^2.$$

Rewrite $es\,2(i)$ as

$$es\,2(i) = (X_i(t)-X_i^h(t)) \sum_{jh\in\Omega_h} (\int_0^1 DK_\delta(X_i^h(t)-X_j^h(t)+\theta(X_i(t)-X_i^h(t)))d\theta)\ \tilde{\omega}_0(\alpha_j;\frac{\alpha_j}{\varepsilon})h^2.$$

We are left to estimate

$$| \sum_{jh\in\Omega_h} \frac{\partial K_\delta}{\partial x_i}(z-X_j^h(t))\tilde{\omega}_0(\alpha_j;\frac{\alpha_j}{\varepsilon})h^2 | \le | \sum_{jh\in\Omega_h} \frac{\partial K_\delta}{\partial x_i}(z-X_j(t))\tilde{\omega}_0(\alpha_j;\frac{\alpha_j}{\varepsilon})h^2 | +$$

$$+ | \sum_{jh\in\Omega_h} (\frac{\partial K_\delta}{\partial x_i}(z-X_j^h(t)) - \frac{\partial K_\delta}{\partial x_i}(z-X_j(t)))\tilde{\omega}_0(\alpha_j;\frac{\alpha_j}{\varepsilon})h^2 |. \tag{3.20}$$

Lemma 3.4 implies that the first term on the right hand side of (3.20) is bounded by a constant for h $\in s(\varepsilon, h_0)$ if h $< \delta$.

On the other hand, the last term in (3.20) is bounded by

$$(3.21)\qquad \sum_{jh\in\Omega_h} \max_{\substack{|y_j|\le C_0\delta \\ |\beta|=2}} |D^\beta K_\delta(z-X_j(t)+y_j)|\ |X_j^h(t)-X_j(t)|\ \|\tilde{\omega}_0\|_\infty h^2.$$

By Lemma 3.1 and assumption (3.16), (3.21) is bounded by $C\delta/\delta = C$. Thus we have proven

$$\| es\,2(i)\|_{L_h^\mu} \le C_4 \| X_i^h(t) - X_i(t)\|_{L_h^\mu}.$$

Therefore we have

$$\| u_i^h(t) - \sum_{jh\in\Omega_h} K_\delta(X_i(t)-X_j(t))\omega_j h^2 \|_{L_h^\mu}$$

$$= \| es\,1(i) + es\,2(i)\|_{L_h^\mu} \le C_s \ \| X_i^h(t) - X_i(t)\|_{L_h^\mu}.$$

And this completes the proof of the Stability Lemma.

Now we can present the proof of Theorem 3.1 .

Proof of Theorem 3.1 . Let $e_i(t) = X_i^h(t) - X_i(t)$. From equations (2.2) and (3.2), we obtain

$$\dot{e}_i(t) = u_i^h(t) - u(X_i(t),t)$$

$$(3.22)\qquad =[\, u_i^h(t) - \sum_{jh\in\Omega_h} K_\delta(X_i(t)-X_j(t))\omega_j h^2 \,]$$

$$+ [\, \sum_{jh\in\Omega_h} K_\delta(X_i(t)-X_j(t))\omega_j h^2 - u(X_i(t),t)\,].$$

Assuming (3.16) and (3.5) hold, we can apply the Stability and Consistency Lemma to the first and second terms in the right hand side of (3.22) respectively. We then obtain for all h $\in s(\varepsilon,h_0)$

$$(3.23)\qquad \| \dot{e}_i(t)\|_{L_h^\mu} \le C_s \ \| e_i(t)\|_{L_h^\mu} + C_c\ \delta^{1+2s} \quad , \quad \text{for } 0\le t\le T_*,$$

where T_* is defined in the Stability Lemma. The existence of such a positive T_* was shown in [3].

It follows from (3.23) that

$$\| \dot{e}_i(t)\|_{L_h^\mu} \le w(t)\ , \quad 0\le t\le T_*,$$

where w is the solution of $w' = C_s w + C_c\delta^{1+2s}$, w(0)=0.

Thus, for all h $\in s(\varepsilon,h_0)$

$$(3.24)\qquad \| e_i(t)\|_{L_h^\mu} \le C_0\delta^{1+2s} \ , \ \text{for } 0\le t\le T_*$$

provided that (3.16) and (3.5) hold. Here C_0 is bounded by exp((C_s+C_c))T). Definition (3.18) then gives

$$\max_i | e_i | \le h^{-2/\mu} \| e_i(t) \|_{L_h^\mu} \le C_0 \delta^{1+2s-2(1+3s)/\mu)}. \tag{3.25}$$

Choose μ large enough so that $\mu > 2(1+3s)/s$. Then for h small, we have

$$\max_i | e_i | \le C_0 \delta^{1+s} \le \delta/2.$$

Hence $T_* = T$ and (3.25) is valid for $0 \le t \le T$. This completes the proof of Theorem 3.1.

proof of Corollary 3.1 : The proof is exactly the same as that of Theorem 3.1.

Now we can present the proof of Theorem 2.2 .

Proof of Theorem 2.2 . Denote $\overline{X}(t,\alpha)$ as the flow map corresponding to the mean initial vorticity $\overline{\omega}_0$.

$$\frac{d\overline{X}(t,\alpha)}{dt} = \int K(\overline{X}(t,\alpha) - \overline{X}(t,\beta))\overline{\omega}_0(\beta)d\beta \ ; \quad \overline{X}(0,\alpha) = \alpha.$$

Let $\alpha_i = (i_1 h, i_2 h)$. Then we have

$$\frac{d}{dt}(X(t,\alpha_i) - \overline{X}(t,\alpha_i)) = I_c + I_s$$

where I_c and I_s are defined as follows

$$I_c = \int K(X(t,\alpha_i) - X(t,\beta))\omega_0(\beta; \frac{\beta}{\varepsilon})d\beta - \int K(X(t,\alpha_i) - X(t,\beta))\overline{\omega}_0(\beta)d\beta,$$

$$I_s = \int K(X(t,\alpha_i) - X(t,\beta))\overline{\omega}_0(\beta)d\beta - \int K(\overline{X}(t,\alpha_i) - \overline{X}(t,\beta))\overline{\omega}_0(\beta)d\beta.$$

In light of Theorem 2.1, we have

$$| I_c | \le C\varepsilon \,| log(\varepsilon) |. \tag{3.26}$$

By Proposition 2.1 , $X(t,\alpha)$ has bounded first order partial derivatives. Moreover $\overline{X}(t,\alpha)$ and $\overline{\omega}_0$ are smooth. Following the proof in the Consistency Lemma, we can show

$$\left| \int K(X(t,\alpha_i) - X(t,\beta))\overline{\omega}_0 \, d\beta - \sum_{jh \in \Omega_h} K_\delta(X(t,\alpha_i) - X(t,\beta_j))(\overline{\omega}_0)_j h^2 \right| \le C \; (\delta^{1+1/3} + h \,| \log(h) |),$$

$$\left| \int K(\overline{X}(t,\alpha_i) - \overline{X}(t,\beta))\overline{\omega}_0 d\beta - \sum_{jh \in \Omega_h} K_\delta(\overline{X}(t,\alpha_i) - \overline{X}(t,\beta_j))(\overline{\omega}_0)_j h^2 \right| \le C \; (\delta^{1+1/3} + h \,| \log(h) |).$$

Arguing as Beale and Majda did in their proof of Stability lemma [4], we can show

$$\left\| \sum_{jh \in \Omega_h} K_\delta(X(t,\alpha_i) - X(t,\beta_j))(\overline{\omega}_0)_j h^2 - \sum_{jh \in \Omega_h} K_\delta(\overline{X}(t,\alpha_i) - \overline{X}(t,\beta_j))(\overline{\omega}_0)_j h^2 \right\|_{L_h^\mu} \tag{3.27}$$

$$\le C \; \| X(t,\alpha_i) - \overline{X}(t,\alpha_i) \|_{L_h^\mu},$$

provided that $\| X(t,\alpha_i) - \overline{X}(t,\alpha_i) \|_\infty \le \delta$ for $0 \le t \le T_*$.

Choose $h = \varepsilon$ and $\delta^{1+2s} = h \mid log(h) \mid$ (s>0). Combining above results, we arrive at

$$\| \frac{d}{dt}(X(t,\alpha_i)-\overline{X}(t,\alpha_i)) \|_{L_h^\mu} \le C_s \, \| X(t,\alpha_i)-\overline{X}(t,\alpha_i) \|_{L_h^\mu} + C_c \, \delta^{1+2s} \quad , \quad \text{for } 0 \le t \le T_* .$$

We end up with exactly the same differential inequality as in the proof of Theorem 3.1. We obtain by the same argument

$$\| X(t,\alpha_i)-\overline{X}(t,\alpha_i) \|_\infty \le C \, \delta^{1+s} \quad , \text{ for } 0 \le t \le T.$$

Since both $X(t,\alpha)$ and $\overline{X}(t,\alpha)$ have bounded first order partial derivatives, we conclude that for $0 \le t \le T$

$$\| X(t,\alpha)-\overline{X}(t,\alpha) \|_\infty \le C \, \delta^{1+s} = (\varepsilon \mid C \, \log(\varepsilon) \mid)^{1-s}$$

and

$$\| u(t,z)-\overline{u}(t,z) \|_\infty \le C \, \delta^{1+s} = (\varepsilon \mid C \, \log(\varepsilon) \mid)^{1-s},$$

This completes the proof of Theorem 2.2 .

In the proof of Theorem 3.1 , we made use of the fact that the flow map and the velocity field have uniformly bounded first order partial derivatives. This was established only for a short time. Regularities of the solution to the Euler equation have been studied in the early work of Kato [14] and McGrath [17]. Among other things, they proved that the velocity field is almost Lipschitz continuous and the flow map is Hölder continuous under the assumption that the initial vorticity is bounded. More precisely, they showed that given any $T>0$, there exists a constant C such that for $0 \le t \le T$,

(3.28) $$| u(x,t) - u(y,t) | \le C \, \|\omega_0\|_\infty \, | x - y | \, (1 + \chi(|x-y|)),$$

(3.29) $$| X(t,\alpha) - X(s,\beta) | \le C \, (|\alpha-\beta| + |t-s|)^\lambda ,$$

where $0<\lambda<1$, C and λ depend only on T and $\|\omega_0\|_\infty$, $\chi(s)$ is defined by

$$\chi(s) = \begin{cases} \log(1/s) , & 0 \le s \le 1, \\ 0 , & s \ge 1. \end{cases}$$

Using these results, we can extend the convergence results in Theorem 3.1 to any finite time. This is the content of the following theorem:

Theorem 3.2 *The numerical solutions of the vortex method converge to the exact solutions of the Euler equation essentially independent of* ε . *More precisely, given any* $T > 0$, *there exists a positive number* h_0 *such that for* $\delta > h^\beta$, $h \in s(\varepsilon, h_0)$, *we have*

$$\|X_j(t) - X_j^h(t)\|_{l_\infty} \le C \, \delta^{1+\lambda},$$

where $\beta = \dfrac{4+3\lambda+\lambda^2}{2\lambda}$, *$C$ depends only on T.*

The proof of Theorem 3.2 goes along the same line as our proof of Theorem 3.1 . Instead of using the results in section 2, we use only the Hölder continuity of the velocity field and the flow map. Details of the proof are omitted here.

4. Convergence of the Time-Discrete Vortex Methods

In practical computation, we need to discretize the characteristic equation

$$(4.1)\qquad \begin{cases} \dfrac{dX_i^h(t)}{dt} = \sum_{jh\in\Omega_h} K_\delta(X_i^h(t)-X_j^h(t))\omega_0(\alpha_j;\dfrac{\alpha_j}{\varepsilon})h^2 \\ X_i^h(0) = \alpha_i, \end{cases}$$

by some finite difference methods.

Here we are only interested in proving convergence for the simplest time differencing scheme, since the semi-discrete vortex methods do not converge faster than O(h) if $\varepsilon \le h$.

Denote by X_i^n the approximation of $X(t_n, \alpha_i)$. We use the simple forward Euler method to approximate (4.1). We obtain an equation for X_i^n

$$(4.2)\qquad \begin{cases} X_i^{n+1} = X_i^n + \Delta t \sum_{jh\in\Omega_h} K_\delta(X_i^n-X_j^n)\omega_j h^2 \\ X_i^0 = \alpha_i, \end{cases}$$

where $\omega_j \equiv \omega_0(\alpha_j;\dfrac{\alpha_j}{\varepsilon})$.

For simplicity, we will present only a short time convergence proof for the fully discretized vortex method. A long time result can be proved similarly.

Let $C_1 = \max_{0\le t\le T} \| \dfrac{d^2X}{dt^2}(t,\cdot) \|_\infty$. Denote by $C_4 = \max\{C_1, C_c, C_s\}$, and $C_5 = exp(C_4 T)$, where T is defined in Proposition 2.1, C_c and C_s are constants in the Consistency and Stability Lemmas respectively.

Theorem 4.1 *Suppose that $\mu > 3$ is a large constant. Then for $n\Delta t \le T$ and $h \in s(\varepsilon, h_0)$, we have*

$$(4.3)\qquad \begin{aligned} \| X_i^n - X_i(t_n) \|_{L_h^\mu} &\le C_5(\Delta t + \delta^{1+s}), \\ \| u_i^n - u_i(t_n) \|_{L_h^\mu} &\le C_5(\Delta t + \delta^{1+s}), \end{aligned}$$

for all Δt , h and δ satisfying the relation

$$(4.4)\qquad C_5(\Delta t + \delta^{1+s})/h^{2/\mu} < \delta.$$

Proof: We follow closely the argument by Anderson and Greengard in [1] and prove the theorem by induction.

The case n=0 is trivially true.

Suppose that for n ≥ 1, we have

$$\| X_i^n - X_i(t_n) \|_{L_h^\mu} \le C_4 \Delta t \Big(\sum_{j=0}^{n-1} (1+C_4\Delta t)^j \Big)(\Delta t + \delta^{1+s}). \tag{4.5}$$

Note that

$$C_4 \Delta t \sum_{j=0}^{n-1} (1+C_4\Delta t)^j = (1+C_4\Delta t)^n - 1 \le exp\,(C_4 T) = C_5.$$

Then (4.4) and (4.5) imply

$$\| X_i^n - X_i(t_n) \|_{L_\infty} < \delta. \tag{4.6}$$

Therefore the Stability Lemma applies at the (n+1)-st step. At the (n+1)-st step, we find

$$\| X_i^{n+1} - X_i(t_{n+1}) \|_{L_h^\mu} = \| X_i^n + \Delta t \sum_{jh \in \Omega_h} K_\delta(X_i^n - X_j^n)\omega_j h^2 - X_i(t_n + \Delta t) \|_{L_h^\mu}$$

$$\le \| X_i^n - X_i(t_n) \|_{L_h^\mu} + \| X_i(t_n) + \Delta t \frac{dX_i}{dt}(t_n) - X_i(t_n + \Delta t) \|_{L_h^\mu}$$

$$+ \| \Delta t \sum_{jh \in \Omega_h} K_\delta(X_i^n - X_j^n)\omega_j h^2 - \Delta t \sum_{jh \in \Omega_h} K_\delta(X_i(t_n) - X_j(t_n))\omega_j h^2 \|_{L_h^\mu}$$

$$+ \| \Delta t \sum_{jh \in \Omega_h} K_\delta(X_i(t_n) - X_j(t_n))\omega_j h^2 - \Delta t \int K(X_i(t_n) - X(t_n,\alpha))\omega_0(\alpha; \frac{\alpha}{\varepsilon}) d\alpha \|_{L_h^\mu} .$$

By Proposition 2.1, $\frac{d^2}{dt^2} X(t,\alpha)$ is bounded independent of ε . Applying the Stability and Consistency Lemma to the last two terms in above inequality, we obtain for $h \in s(\varepsilon, h_0)$

$$\| X_i^{n+1} - X_i(t_{n+1}) \|_{L_h^\mu} \le (1+\Delta t C_s) \| X_i^n - X_i(t_n) \|_{L_h^\mu} + C_1 \Delta t^2 + C_c \Delta t\, \delta^{1+s}$$

$$\le (1+\Delta t C_4) C_4 \Delta t \Big(\sum_{j=0}^{n-1} (1+C_4\Delta t)^j \Big)(\Delta t + \delta^{1+s}) + C_4 \Delta t (\Delta t + \delta^{1+s})$$

$$= C_4 \Delta t \Big(\sum_{j=0}^{n} (1+C_4\Delta t)^j \Big)(\Delta t + \delta^{1+s}),$$

where we have used the induction assumption.

By induction principle, (4.5) is true for all $n \le T/\Delta t$. This proves Theorem 4.1 .

5. Weak continuity of the Euler equation in velocity-vorticity formulation

Our main result in this section is the following theorem:

Theorem 5.1 *Consider system (2.1) with initial data $\omega_0(x,\varepsilon)$, where ε is a small parameter. Assume $\|\omega_0\|_{L^\infty} \le C$,*

(5.1) $$\omega_0(x,\varepsilon) \to \overline{\omega}_0(x) \in L^\infty(R^2) \quad \text{weak } *,$$

and the support of $\omega_0(x,\varepsilon)$ is contained in a ball $B(0,R)$. Let $\{u(x,t,\varepsilon), \omega(x,t,\varepsilon)\}$ and $\{\overline{u}(x,t), \overline{\omega}(x,t)\}$ denote the solutions of (2.1) with $\omega(x,0,\varepsilon) = \omega_0(x,\varepsilon)$ and $\overline{\omega}(x,0) = \overline{\omega}_0(x)$ respectively. Then

(5.2) $$u(x,t,\varepsilon) \to \overline{u}(x,t) \quad \text{pointwise in } (x,t)$$

and for any $t \in [0, +\infty)$

(5.3) $$\omega(x,t,\varepsilon) \to \overline{\omega}(x,t) \quad \text{weak } * \text{ in } L^\infty(R^2).$$

Remark 5.1 As an easy consequence, we see that part of Theorem 2.2 is valid up to any finite time, i.e. the homogenized equation is still the original Euler equation. But we don't know whether the estimate still holds or not.

Remark 5.2 Examples given by Diperna and Majda [8] show that for the 3-D Euler equation, the homogenized equation will not be the original Euler equation if the initial velocity is oscillatory. In this direction, some homogenization results have been obtained by McLaughlin et al [18] using formal asymptotic analysis.

Proof of Theorem 5.1 . The theorem will be proved by a classical compactness argument on $u(x,t,\varepsilon)$. Let T be any fixed number in $(0, +\infty)$. Denote by $\|\cdot\|$ the norm in $C_b([0,T]\times R^2)$, the space of continuous bounded functions. First from

$$u(x,t,\varepsilon) = \int_{B(0,R)} K(x-y)\omega(y,t,\varepsilon)dy,$$

we have

$$|u(x,t,\varepsilon)| \le C \|\omega(\cdot,t,\varepsilon)\|_\infty \int_{B(0,R)} |K(x-y)|\, dy \le C \,\|\omega(\cdot,t,\varepsilon)\|_\infty \le C.$$

Next we prove equi-continuity. From (3.28), we have

$$|u(x,t,\varepsilon) - u(y,t,\varepsilon)| \le C\,|x-y|\,(\chi(|x-y|) + 1).$$

Therefore for any $0<\lambda<1$, there exists a constant C independent of ε such that

$$|u(x,t,\varepsilon) - u(y,t,\varepsilon)| \le C|x-y|^\lambda.$$

On the other hand, by a change of variable, we obtain

$$| u(x,t,\varepsilon) - u(x,s,\varepsilon) | = |\int K(x-y)(\omega(y,t,\varepsilon) - \omega(y,s,\varepsilon))\, dy \,|$$

$$= | \int \{K(x-X_\varepsilon(t,\alpha)) - K(x-X_\varepsilon(s,\alpha))\} \ \omega_0(\alpha,\varepsilon) d\alpha \,|,$$

where $X_\varepsilon(t,\alpha)$ denotes the flow map associated with $\{u(x,t,\varepsilon), \omega(x,t,\varepsilon)\}$.

Let $d = | X_\varepsilon(t,\alpha) - X_\varepsilon(s,\alpha) |$, $B(x,2d)$ be the disk with center x and radius $2d$, then

$$| \int_{B(x,2d)} \{K(x-X_\varepsilon(t,\alpha)) - K(x-X_\varepsilon(s,\alpha))\} \ \omega_0(\alpha,\varepsilon) d\alpha \,|$$

$$\le C \int_{B(x,2d)} \left(\frac{d\alpha}{|x-X_\varepsilon(t,\alpha)|} + \frac{d\alpha}{|x-X_\varepsilon(s,\alpha)|} \right) \le C\ d.$$

On $R^2 \setminus B(x,2d)$, we have

$$| K(x-X_\varepsilon(t,\alpha)) - K(x-X_\varepsilon(s,\alpha)) | \le | \nabla K(x-\tilde{x}) | \ | X_\varepsilon(t,\alpha) - X_\varepsilon(s,\alpha) |$$

$$\le C\ d \,| x - X_\varepsilon(t,\alpha) |^2,$$

where $\tilde{x}$ is on the segment connecting $X_\varepsilon(t,\alpha)$ and $X_\varepsilon(s,\alpha)$. Thus

$$| \int_{R^2 \setminus B(x,2d)} \{K(x-X_\varepsilon(t,\alpha)) - K(x-X_\varepsilon(s,\alpha))\} \ \omega_0(\alpha,\varepsilon) d\alpha \,|$$

$$\le cd \int_{2d \le |x-X_\varepsilon(t,\alpha)| \le R_0} \frac{d\alpha}{| x - X_\varepsilon(t,\alpha) |} \le Cd \ \log(R_* /2d).$$

where we have used the fact that $\omega_0(\alpha,\varepsilon)$ has compact support, and R_* is defined in section 2.

Using (3.29) we get

$$| u(x,t,\varepsilon) - u(x,s,\varepsilon) | \le cd \ \log(1/d) \le cd^\lambda \le c|t-s|^{\lambda^2}.$$

Notice that $u(x,t,\varepsilon)$ decays as $|x| \to +\infty$ uniformly with respect to ε ,

$$| u(x,t) | \le C \int_{B(0,R)} \frac{dy}{| x-y |}.$$

By Ascoli-Arzela Theorem, we can extract a subsequence $\{u(x,t,\varepsilon_k)\}$ such that

(5.5) $$u(x,t,\varepsilon_k) \to \bar{u}(x,t) \ \ in \ \ C([0,T] \times R^2) \ as \ k \to +\infty.$$

Let $\bar{\omega}(x,t) = \partial_1 \bar{u}_2 - \partial_2 \bar{u}_1$ where $(\bar{u}_1, \bar{u}_2) = \bar{u}(x,t)$. As a consequence of (5.5), we have

(5.6) $$\int \omega(x,t,\varepsilon_k)\phi(x,t) dxdt \to \int \bar{\omega}(x,t)\phi(x,t) dxdt \ , \ \ as \ k \to +\infty,$$

where $\phi(x,t) \in C_0(R^2 \times [0,T))$, the space of continuous functions in $R^2 \times [0,T)$ with compact support. For $\phi \in C^1(R^2 \times [0,T))$, this can be proved directly using integration by parts. A standard density argument proves (5.6) for any $\phi \in C_0(R^2 \times [0,T))$.

The next step is to prove that $\{\overline{u}(x,t), \overline{\omega}(x,t)\}$ is a weak solution of (2.1) with initial condition $\overline{\omega}(x,0) = \overline{\omega}_0(x)$. More precisely, we will show that for any $\phi \in C^1([0,T]\times R^2)$ with compact support in $[0,T)\times R^2$,

$$(5.7) \qquad \int_{R^2\times[0,T]} \overline{\omega}\Big(\frac{\partial\phi}{\partial t} + (\overline{u}(x,t)\cdot\nabla\phi) \Big)dxdt - \int_{R^2} \phi(x,0)\overline{\omega}_0(x)dx = 0.$$

As $\{u(x,t,\varepsilon), \omega(x,t,\varepsilon)\}$ are classical solutions to (2.1), (5.7) holds with $\overline{u}(x,t)$, $\overline{\omega}(x,t)$ replaced by $u(x,t,\varepsilon)$ and $\omega(x,t,\varepsilon)$ respectively.

Then from (5.6), we have

$$(5.8) \qquad \int_{R^2\times[0,T]} \omega(x,t,\varepsilon_k)\frac{\partial\phi}{\partial t}dxdt \to \int_{R^2\times[0,T]} \overline{\omega}(x,t)\frac{\partial\phi}{\partial t}dxdt,$$

and

$$\int_{R^2\times[0,T]} \omega(x,t,\varepsilon_k)(\overline{u}\cdot\nabla\phi)dxdt \to \int_{R^2\times[0,T]} \overline{\omega}(x,t)(\overline{u}\cdot\nabla\phi)dxdt.$$

From (5.5) we also have,

$$\Big| \int_{R^2\times[0,T]} \omega(x,t,\varepsilon_k)(u(x,t,\varepsilon_k) - \overline{u}(x,t))\cdot\nabla\phi dxdt \Big|$$

$$\leq C \int_{R^2\times[0,T]} | u(x,t,\varepsilon_k) - \overline{u}(x,t) | \, |\nabla\phi| \, dxdt \to 0,$$

Therefore, we obtain

$$(5.9) \qquad \int_{R^2\times[0,T]} \omega(x,t,\varepsilon_k)(u(x,t,\varepsilon_k)\cdot\nabla\phi)dxdt \to \int_{R^2\times[0,T]} \overline{\omega}(\overline{u}\cdot\nabla\phi)dxdt.$$

(5.1), (5.8) and (5.9) together imply (5.7). It is easy to see $\nabla\cdot\overline{u} = 0$ in weak sense. Hence we have shown that $\{\overline{u}(x,t), \overline{\omega}(x,t)\}$ is a weak solution of (2.1).

From classical results on elliptic equations, we know

$$\| \nabla u(\cdot,t,\varepsilon_k) \|_{L^2} \leq C \, \| \omega(\cdot,t,\varepsilon_k) \|_{L^2} \leq C.$$

On the other hand, by Young's inequality we have

$$\|u(\cdot,t,\varepsilon_k)\|_{L^2(R^2)} = \|(K*\omega)(\cdot,t,\varepsilon_k)\|_{L^2(R^2)} \leq \|K\|_{L^1(B(0,R))} \|\omega(\cdot,t,\varepsilon_k)\|_{L^2} \leq C.$$

Therefore we obtain

$$\| \overline{u}(\cdot,t) \|_{H^1(R^2)} \leq liminf \, \| u(\cdot,t,\varepsilon_k) \|_{H^1(R^2)} \leq C.$$

Hence $\overline{u}(x,t) \in L^\infty([0,T],H^1(R^2))$.

Similarly we can show $\overline{u}(x,t) \in C([0,T], L^2(R^2))$.

It is proved in [2] that the weak solution of (2.1) with $\omega_0(x) \in L^\infty$ is unique in the class

$u(x,t) \in L^\infty([0,T], H^1(R^2)) \cap C([0,T], L^2(R^2))$. We have shown that $\bar{u}(x,t)$ is indeed in this class. Consequently the limiting function $\bar{u}(x,t)$ is unique. From this we conclude that the whole sequence $\{u(x,t,\varepsilon)\}$ converges in $C([0,T]\times R^2)$ to $\bar{u}(x,t)$, and $\{\omega(x,t,\varepsilon)\}$ converges weakly to $\bar{\omega}(x,t)$, where $\{\bar{u}(x,t), \bar{\omega}(x,t)\}$ is the unique weak solution of (2.1) with $\omega(x,0) = \bar{\omega}_0(x)$.

It remains to show (5.3). Let $X(t,\alpha)$ denote the flow map of the limiting flow. Then from (5.5) we have

$$X_\varepsilon(t,\alpha) \to X(t,\alpha) \quad as \ \varepsilon \to 0 \tag{5.10}$$

uniformly in $B(0,R)$ for any fixed $t \in [0,T]$. Now for any function $\phi(x) \in C_0(R^2)$ and $t \geq 0$, by a change of variable, we get

$$\int_{B(0,R_*)} \omega(x,t,\varepsilon)\phi(x) = \int_{B(0,R)} \omega_0(\alpha,\varepsilon)\phi(X_\varepsilon(t,\alpha))d\alpha$$

$$= \int_{B(0,R)} \omega_0(\alpha,\varepsilon)\{\phi(X_\varepsilon(t,\alpha)) - \phi(X(t,\alpha))\}d\alpha + \int_{B(0,R)} \omega_0(\alpha,\varepsilon)\phi(X(t,\alpha))d\alpha.$$

The first term goes to zero for $\phi(x) \in C_0(R^2)$ because of (5.10). The second term goes to $\int \bar{\omega}_0(\alpha)\phi(X(t,\alpha))d\alpha = \int \bar{\omega}(x,t)\phi(x)dx$. This proves (5.3) and the proof of the theorem is complete.

6. Numerical results

The numerical experiments are carried out for the 2-D Euler equation with radial symmetric vorticity distribution. We test the convergence of the vortex method when the highly oscillatory vorticity is not well resolved by the computational particles. To compare with the finite difference method, we also compute the solution using a Lax-Wendroff type of scheme for the Euler equation in streamline vorticity formulation. An advantage of using a radial symmetric vorticity is that we can obtain the exact solution explicitly.

We choose the initial vorticity distribution to be

$$\omega(x,0) = \begin{cases} 5.0(1-|x|)^3(0.5+sin(2\pi\frac{|x|}{\varepsilon})), & |x| < 1, \\ 0, & |x| \geq 1. \end{cases} \tag{6.1}$$

In the vortex method calculation, we distribute the particles at time $t = 0$ by polar coordinates, and approximate the ordinary differential equation (3.2) by the forward Euler method.

Let $j=(j_1,j_2)$, $r_j = j_2 h$ and $\Delta\theta(j) = 2\pi h/(6r_j)$. Define

$$\theta_j = j_1\Delta\theta(j_2), \ \omega_j = \omega(r_j,0).$$

Denote by X_i^n the approximation of $X(t_n,\alpha_i)$, and u_i^n the approximation of $u(X(t_n,\alpha_i),t_n)$, where $\alpha_i=(\theta_i,r_i)$, $t_n = n\Delta t$. The fully discretized vortex method is given by

$$(6.2)\qquad \begin{cases} X_i^{n+1} = X_i^n + \Delta t \sum\limits_{jh\in\Omega_h} K_\delta(X_i^n-X_j^n)\omega_j r_j h\Delta\theta(j) \\ X_i^0 = \alpha_i \ , \end{cases}$$

$$(6.3)\qquad u_i^n = \Delta t \sum_{jh\in\Omega_h} K_\delta(X_i^n-X_j^n)\omega_j r_j h\Delta\theta(j).$$

In our experiment, we choose $\delta=h^{0.8}$, $\varepsilon = \pi/213$. In table 1 and 2, we give the numerical errors in velocity at t=0 and t=4 respectively. We observe the numerical stability and convergence essentially independent of ε .

For the finite difference approximation, we use the Lax-Wendroff type of scheme to approximate the Euler equation in streamline vorticity formulation

$$(6.4)\qquad \begin{cases} \dfrac{\partial\omega}{\partial t} + u\dfrac{\partial\omega}{\partial x} + v\dfrac{\partial\omega}{\partial y} = 0 \\ \omega((x,y),0) = \omega_0((x,y)), \end{cases}$$

where the velocity is given by

$$(6.5)\qquad u = \frac{\partial\phi}{\partial y}\ ,\quad v = -\frac{\partial\phi}{\partial x}.$$

The stream function ϕ is coupled to the vorticity by

$$(6.6)\qquad \Delta\phi = -\omega.$$

Denote by $\omega_{i,j}^n$, $u_{i,j}^n$ and $\phi_{i,j}^n$ the approximations of $\omega(x,t)$, $u(x,t)$ and $\phi(x,t)$ at x =(ih,jh), t= $n\Delta t$ respectively.

The finite difference scheme is given by

$$(6.7)\qquad \begin{cases} \omega_{i,j}^{n+1} = \omega_{i,j}^n + \Delta t(\omega_t)_{i,j}^n + \Delta t(\omega_{tt})_{i,j}^n \\ \omega_{i,j}^0 = \omega_0(ih,jh) \end{cases}$$

where

$$(6.8)\qquad (\omega_t)_{i,j}^n = -u_{i,j}^n D_0^x\phi_{i,j}^n - v_{i,j}^n D_0^y\phi_{i,j}^n,$$

$$(6.9)\qquad (\omega_{tt})_{i,j}^n = (u^2\omega_{xx}+uu_x\omega_x+2uv\,\omega_{xy}+uv_x\omega_y+vu_y\omega_x+v^2\omega_{yy}+vv_y\omega_y)_{i,j}^n.$$

We approximate each term in the right hand side of (6.9) by central difference scheme, e.g.

$$(\omega_x)_{i,j}^n = D_0^x\omega_{i,j}^n\ ,\quad (\omega_{xx})_{i,j}^n = D_+^x D_-^x\omega_{i,j}^n\ ,\quad (\omega_{xy})_{i,j}^n = D_0^x D_0^y\omega_{i,j}^n.$$

The streamline function ϕ is updated by

(6.10) $$D_+^x D_-^x \phi_{i,j}^{n+1} + D_+^y D_-^y \phi_{i,j}^{n+1} = -\omega_{i,j}^{n+1},$$

and the velocity is computed by

(6.11) $$u_{i,j}^{n+1} = D_0^y \phi_{i,j}^{n+1}$$
$$v_{i,j}^{n+1} = -D_0^x \phi_{i,j}^{n+1},$$

where the difference operators D_0^x etc. are defined by

$$D_0^x f_{i,j}^n = (f_{i+1,j}^n - f_{i-1,j}^n)/2h,$$

$$D_+^x f_{i,j}^n = (f_{i+1,j}^n - f_{i,j}^n)/h \quad , \quad D_-^x f_{i,j}^n = (f_{i,j}^n - f_{i-1,j}^n)/h.$$

In the above scheme, we have made use of the fact that the velocity is time independent. To solve the Poisson equation, we use the known exact stream function to give the boundary condition for (6.10).

Table 3 displays the errors at t=4 in velocity using the algorithm (6.7-11) for smooth vorticity (ε =1.4). We observe a second order convergence in space and time. In table 4, we compute the errors at t= 4. in velocity with ε =0.0148. As we see, the sizes of the errors grow, indicating that the error constants become larger if $\varepsilon < h$. Moreover the rate of convergence reduces to first order.

In figure 1, we plot the vorticity obtained by the finite difference algorithm (6.7-11). In comparison with the exact vorticity distribution in figure 3, we see that the vorticity is considerably dissipated at t=4 away from the origin. In figure 2, we display the vorticity distribution for the vortex method calculation for t=4 at the approximate particle locations. There is no numerical dissipation in vorticity for vortex methods. The exact initial vorticity has been built into the algorithm since the vorticity is conserved along streamlines.

We can see that in the model considered in this paper, velocity behaves quite well although the vorticity field is highly oscillatory. Thus it is not surprising that the finite difference method could still provide good approximation for the average vorticity. This can also be seen from the analysis of linear problems by Engquist [9]. Our homogenization result indicates that the velocity only depends on the average vorticity. This may explain why the results in table 4 are still reasonable. However, the dissipation error in the vorticity could potentially damage the solutions in problems when the higher order powers of vorticity play an important role. In this case we could expect that the vortex method will outperform the traditional finite difference methods if the small scale features are not well resolved on the computational grid.

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Table 1 : Vortex method calculation: $\varepsilon \cong 0.0148$, at time t=0

Errors in velocity field in [0,1] on x-axis			
Gridsize	L1-norm	L2-norm	Max-norm
0.200	0.034255	0.046638	0.085578
0.125	0.014554	0.015486	0.021995
0.050	0.003391	0.005391	0.014075

Table 2 : Vortex method calculation: $\varepsilon \cong 0.0148$, at time t=4

Errors in velocity field in [0,1] on x-axis			
Gridsize	L1-norm	L2-norm	Max-norm
0.200	0.039724	0.052350	0.093256
0.125	0.017793	0.023809	0.042253
0.050	0.007579	0.010647	0.024781

* In the calculation above, we have chosen timestep size =space grid size

Table 3 : Finite difference calculation: $\varepsilon \cong 1.4$, at time t=4

Errors in velocity field in [0,1] on x-axis			
Gridsize	L1-norm	L2-norm	Max-norm
0.200	0.005148	0.005991	0.009136
0.125	0.001267	0.001528	0.002852
0.050	0.000197	0.000241	0.000436

Table 4 : Finite difference calculation: $\varepsilon \cong 0.0148$, at time t=4

Errors in velocity field in [0,1] on x-axis			
Gridsize	L1-norm	L2-norm	Max-norm
0.200	0.063400	0.081830	0.123438
0.125	0.044858	0.055485	0.082049
0.050	0.016700	0.026616	0.081283

* In the calculation above, we have chosen timestep size = space grid size

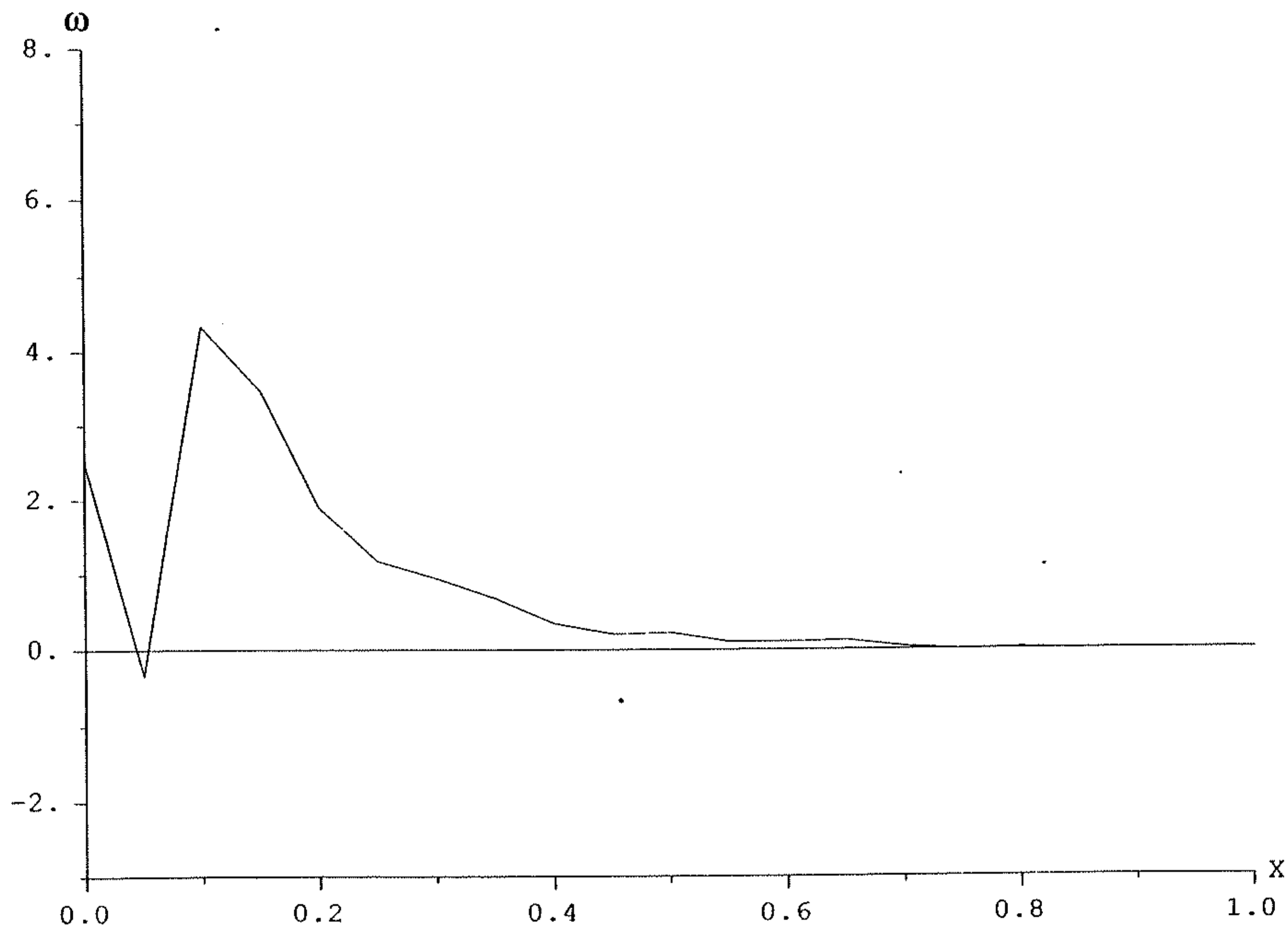

Figure 1: Finite difference solution ω on $[0,1]$, $h=0.05$, $\varepsilon=0.013$, $t=4$.

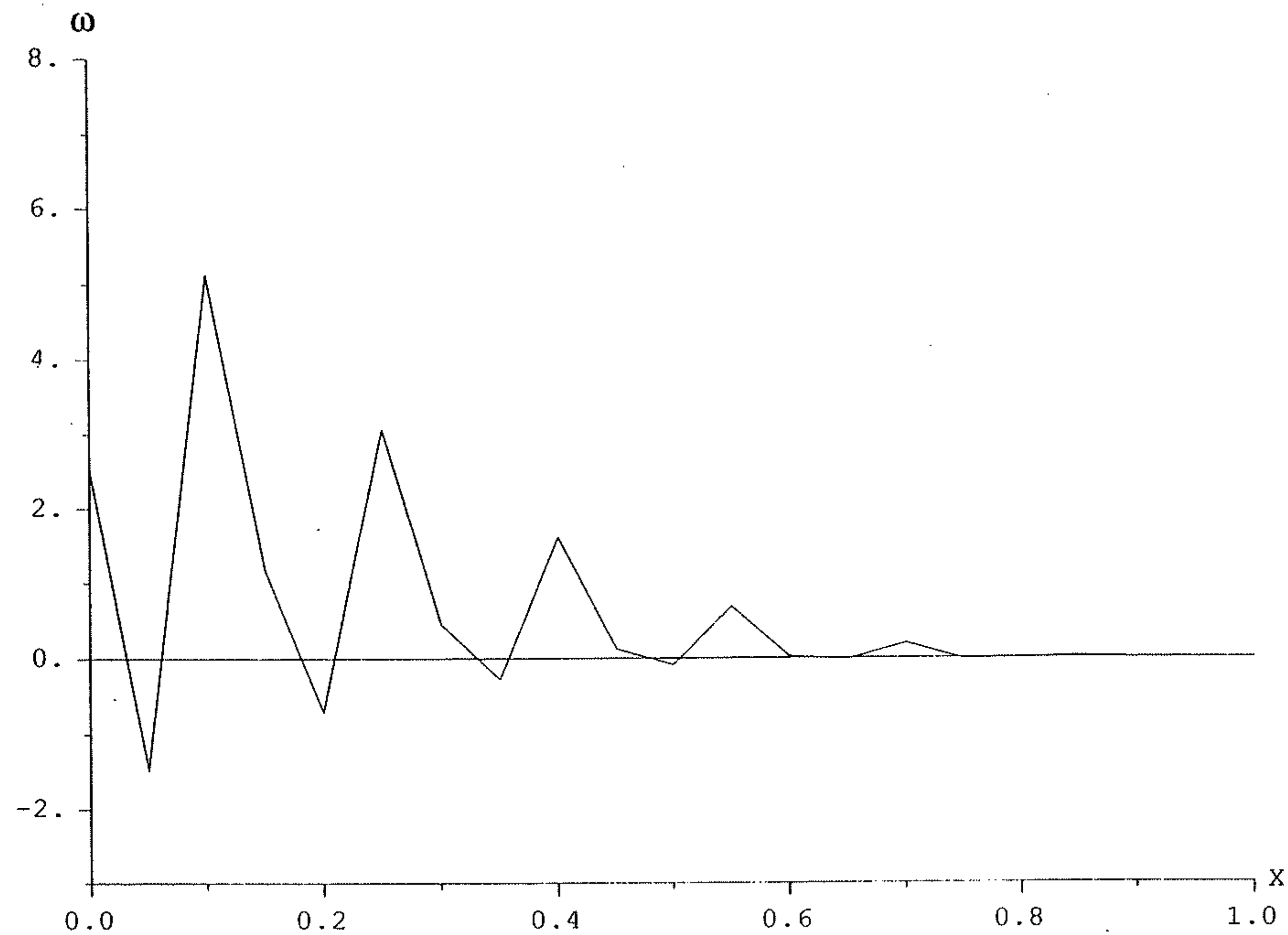

Figure 2: Vortex method solution ω on [0,1], h=0.05, $\delta = h^{0.8}$, ε=0.013, t=2.

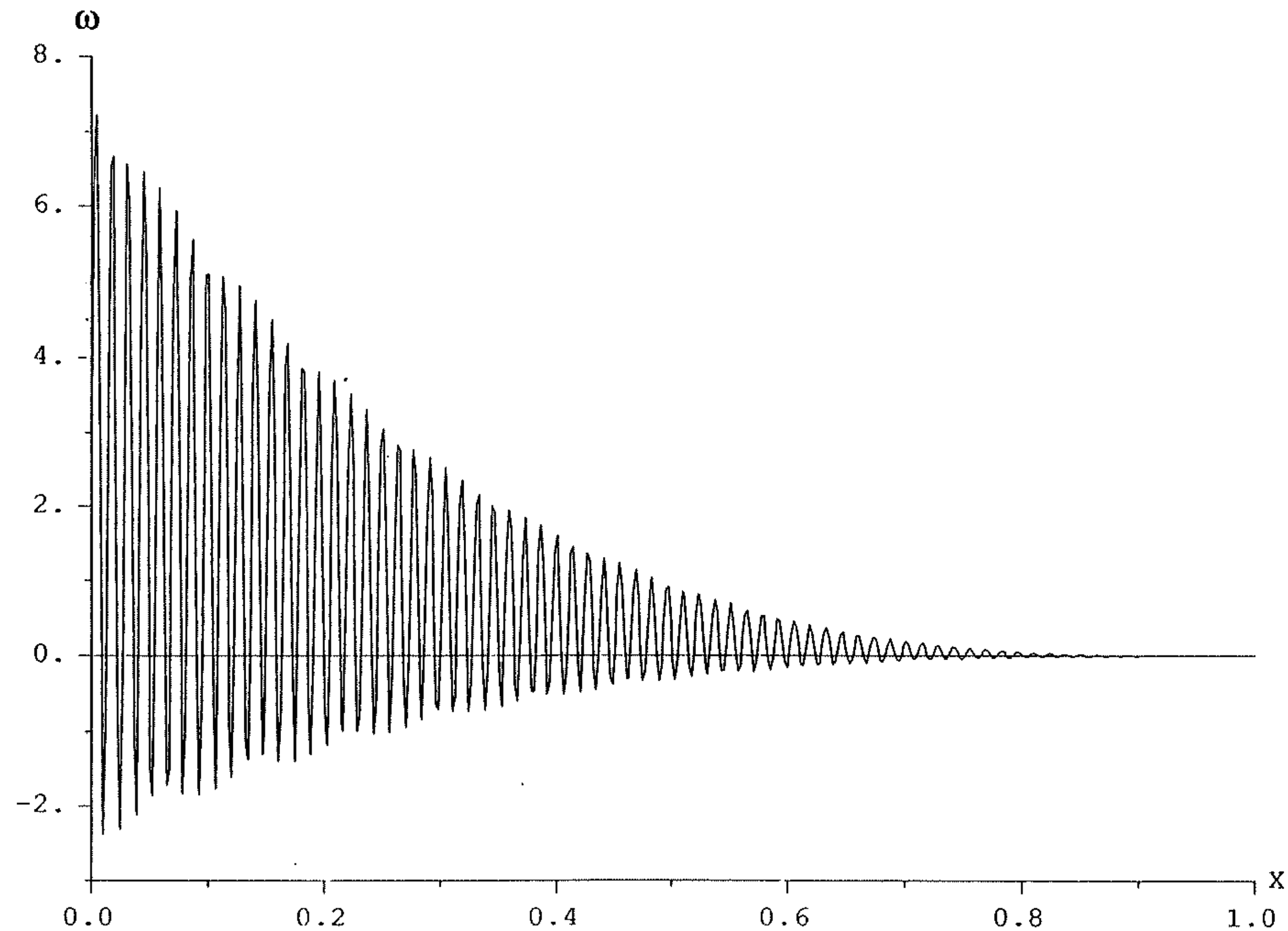

Figure 3: Vorticity field ω on $[0,1]$, $\varepsilon = 0.013$.