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Universal Power Spectra for Wave Turbulence: Applications to Wind Waves, Flicker Noise, Solar Wind Spectrum, and Classical Second Sound

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To Aída Mostkoff,

to the memory of my father Ing. Quim. Pascual Larraza S.,

to my mother, brothers, and sisters

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ABSTRACT OF THE DISSERTATION

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Stationary solutions to the kinetic equation describing wave-wave interactions are obtained by means of dimensional estimates and in exact form from the collision operator. The solutions are interpreted as the turbulent spectrum in the inertial range and are shown to be local. At lowest nonlinear order one obtains the results of weak turbulence theory. As the low frequency power input is increased, the power spectrum of wave motion converges to a universal 1/f noise for non-dispersive waves in more than two dimensions, and to $1/\omega^5$ noise for deep gravity waves. It is also shown that a wave turbulent system is elastic and it can support a propagating energy mode with similarities to second sound. The conditions from parabolic to hyperbolic energy tranport are discussed. A parallel connection to He^4 for drift wave turbulence in plasmas is made and order of magnitude estimates for the plasma diffusivity are obtained. It is also suggested that a wave turbulence picture can be used to understand the magnetic field fluctuations in the solar wind.

INTRODUCTION

In contrast with the problem of turbulence of an incompressible flow, the theory of wave turbulence which describes systems of driven nonlinear interacting random waves, offers a tractable systematic statistical closure. The statistical problem is formulated in terms of a Boltzmann equation for waves which forms the basis for the description of wave turbulence.

The physical understanding of turbulence in an incompressible flow is still far from complete. Predictions on the onset of turbulence when a system is driven far from equilibrium or prescriptions for dealing with higher order correlations and the turbulent transport which dominates the molecular transport are not available.

By turbulence, one means those processes in open driven systems which are dominated by a random redistribution of energy among the effectively infinite degrees of freedom. The turbulent redistribution of energy (generally due to time reversible nonlinear processes) dominates those precesses (such as linear transport) which would bring about thermodynamic equilibrium. Turbulent steady states can arise when there exists a mechanism whereby energy can be transferred from the degrees of freedom coupled to excitation to the degrees of freedom coupled to decay.

Turbulence of incompressible flow is governed by the Navier- Stokes equation for the flow velocity $\vec{v}$

$$\frac{\partial \vec{v}}{\partial t} = - \vec{v} \cdot \nabla \vec{v} + \nu \nabla^2 \vec{v} - \frac{1}{\rho} \nabla p \qquad (0.1)$$

with the extra condition

$$\nabla \cdot \vec{v} = 0 \tag{0.2}$$

for incompressibility.

The first term in (0.1) is the nonlinear inertial force (convective term) as required by Galilean covariance, the second term is the viscosity force with kinematic viscosity ν, and the third term is the force arising from pressure gradients. The flow is characterized by the Reynolds number $R=u_0L/\nu$, where u_0 is a typical flow velocity (mean flow) with a typical length scale L of variations of the mean flow. The Reynolds number represents the ratio of the inertial force to the viscosity force and can also be viewed as the ratio of the excitation mechanism to the stabilization mechanism.

Taking the curl of (0.1) yields

$$\frac{\partial}{\partial t} \nabla \times \vec{v} = \nabla \times [\vec{v} \times \nabla \times \vec{v}] + \nu \nabla^2 \nabla \times \vec{v} \tag{0.3}$$

which with (0.2) yields two equations for the divergence and the curl of $\vec{v}$.

In principle (rotational) turbulence in a classical fluid is fully characterized by (0.2) and (0.3) but the general solution is far from clear. In order to elucidate some qualitative properties of the flow it is convenient to decompose the flow velocity into Fourier components with wave vector $\vec{k}$.

$$\vec{v}(\vec{x},t) = \sum_{k} \vec{v}(\vec{k},t) \exp(i\vec{k} \cdot \vec{x}) \tag{0.4}$$

The convective term excites flow modes with different length scales (sum and difference of wave vectors), and as R increases shorter length scales modes are excited. Viscosity tends to damp the excitations and provides a mechanism for stabilizing the flow. When R passes a critical value R_C the excited modes become unstable and the flow becomes turbulent.

With low Reynolds number $R \sim R_C$, only a few modes with long length scales are excited and a finite number of ODE's are said to provide a description of the flow. For large Reynolds number $R \gg R_C$, however infinitely many modes with short length scales are excited and closure schemes for the statistical description are required. In this case, turbulence is described by the energy cascade and the vortex stretching (Kraichnan, 1974).

Measurements of the energy spectrum of rotational turbulence over more than three decades done by Grant, Stewart, and Moilliet (1962) in a tidal channel flow, showed a power law dependence with an exponent very close to the value -5/3 predicted by Kolmogorov (1941) and Obukhov (1941). This was a high Reynolds number experiment ($\sim 10^8$). The Kolmogorov-Obukhov law is true in almost every high Reynolds number homogeneous isotropic turbulent flow.

By turbulence, we are refering to a variety of phenomena observed in high Reynolds number flow. The specific application that we have in mind, here, considers homogeneous isotropic vortex turbulence where one can take full advantage of the symmetries of the Navier-Stokes equations from which the qualitative picture for local turbulence provided by Kolmogorov can be obtained. In arriving at the Kolmogorov-Obukhov law, we will adopt the point of view described by Frisch (1980) which, as we will see later can be adapted to the problem of wave turbulence.

Frisch considers four main ideas. First, the redistribution of energy among length scales is dominated by inertial nonlinearities and not by the kinematic viscosity. This in turn implies that there are three regions in phase space. The energy containing region where energy is introduced at a rate ε at low wave numbers; the inertial range, where nonlinearities will carry the energy to higher and higer wave numbers until a point it meets the dissipation range here real viscous effects dominate and energy is eventually converted into heat. The inertial region is the range of scales over which direct energy injection and viscosity are negligible.

Stationarity, the second idea, is established through the cascade mechanism just described above. By stationarity we are not implying a stagnant inertial equilibrium. Dynamical quantities of a particular length scale fluctuate rapidly about a characteristic value. Average measurements will yield a typical value of a dynamical quantity in a given scale.

The third idea into the Kolmogorov picture is locality, that is, that the rate at which energy rolls over from one length scale to the next in the inertial region is a function of the energy contained in that length scale.

Finally, the most subtle of all the ideas is translational invariance. We will now use this framework to arrive at the -5/3 law and point out how translational invariance has been used.

Let E_n be the energy in length scale ℓ_n, where $\ell_n=2^{-n}\ell_o$, $k_n \sim \ell_n^{-1}$, and ℓ_o is a reference length. The characteristic velocity variation over the structure of size ℓ_n is, in the inertial range,

$$v_n \sim (E_n/\rho)^{1/2} \tag{0.5}$$

The characteristic time over which the energy in eddy n goes to eddy (n+1), the turnover time, is

$$t_n \sim \ell_n/v_n \tag{0.6}$$

According to the locality hypothesis, the energy transfer rate from ℓ_n to ℓ_{n+1} is

$$\varepsilon_n = E_n/t_n = \varepsilon \tag{0.7}$$

and is a constant for the entire inertial range. This follows from stationarity and because we are dealing with inertial forces, energy is conserved. Thus using (0.5), (0.6), and (0.7), one readily gets the discrete spectral intensity

$$E_n \sim (\rho\varepsilon^2)^{1/3} \ell_n^{2/3} \tag{0.8}$$

To get the spectral energy density we note that

$$E = \sum_n E_n = \int u(k)dk = \sum_n u(k_n)\Delta k_n \tag{0.9}$$

where $\Delta k_n = k_{n+1} - k_n = 1/\ell_n \sim k_n$. Thus

$$e(k) = E(k)/k \sim (\rho\varepsilon^2)^{1/3} k^{-5/3} \tag{0.10}$$

The translational invariance idea has been used in the derivation, for we have assumed that smaller and smaller structures are space filling. In principle it is possible to have a self similar response which is not space filling. This latter idea has been invoked to explain intermittency (Frisch, 1980). We will not discuss this here.

The above presentation is only phenomenological. One would need to solve the statistical problem from the Navier-Stokes equation where only the inertial range is concerned, so that the results would be independent from the excitation and dissipation. By doing so a hierarchy of equations for the cumulants is obtained, whereby the rate of change of the n-th order cumulant depends on cumulants of order (n+1). This leads to the fundamental difficulty of closing the system to describe the high wave number behaviour with low wave number forcing. Closure procedures compatible with Kolmogorov's law have been developed (Kraichnan, 1971). However, those closure schemes are not systematically derived from the Navier-Stokes equation but based on some ad hoc assumptions about the nature of the statistics. A consistent derivation of (0.10) from (0.2) and (0.3) is still lacking.

The bottom line of the problem is that the hypothesis of locality has not been strictly proved. Locality enters the problem as a convergence test of the integrals involved in the closure problem. Convergence can be shown for some closure procedures for low order cumulants but it is not readily guaranteed for higher order cumulants. Indeed, even if the distribution is initially Gaussian or quasi-Gaussian, the energy exchange between modes may set up a statistical correlation between them. By the time a significant change of energy has taken place, a significant non-Gaussian distribution may have been set up, making the locality hypothesis a questionable issue.

The origin of all these fundamental difficulties is that in the theory of turbulence of an incompressible fluid as described by Navier- Stokes, there is no small parameter available. Indeed, in the reversible high amplitude limit (0.2) and (0.3) have no parameters with dimensions. The only time scale available is the nonlinear turnover time (in the inertial range) or the diffusion time (in the dissipation range).

The turbulence problem for waves has properties analogous in many respects to those of ordinary hydrodynamic turbulence (Zakharov, 1984 and references therein), but without all its difficulties. Namely, regions of wave number can be separated in k-space such that the turbulence has a universal power law spectrum determined only by the magnitude of the energy flux in the region of large k. Formation of the turbulent spectrum takes place as a result of the nonlinear process of interaction of waves in the inertial region.

If $\mathcal{E}_k$ denotes the energy spectrum function for waves of wave vector $\vec{k}$, and $n_k = \mathcal{E}_k/\omega_k$ denotes the action density, where $\omega_k=\omega(\vec{k})$ is the frequency of linear waves with wave vector $\vec{k}$, Litvak (1960) and Hasselman (1963) obtained the equation for the slow rate of change of $\mathcal{E}_k$ or n_k due to nonlinear interactions

$$\dot{n}_k = I\{n_k\} \tag{0.11}$$

where $I\{n_k\}$ is an integral operator, the collision operator, and is quadratic in n_k for the case of magnetohydrodynamic waves considered by Litvak, and cubic in n_k for gravity waves on water derived in Hasselman's paper. Both Litvak and Hasselman used as a closure problem the ad hoc assumption that the random waves were statistically Gaussian. It was later shown by Benney and Saffman (1966) that this hypothesis was unnecessary when they considered a system of random, spatially homogeneous dispersive waves. Using a multiple time scale method, they established a closed integro-differential equation like (0.11) for the energy spectrum. Newell and Aucoin (1971) extended the ideas of Benney and Saffman for the case of sound waves in two or higher dimensions, where the group velocity depends on direction of the wave, and provides a semidispersive nature.

There are three time scales in a system of interacting waves, namely the period of the wave, T; the nonlinear interaction time, τ_c; and the lifetime of the wave determined by molecular transport, τ_v. In order to fix ideas let us consider $\tau_v \gg \tau_c$. This situation is realized in a turbulent system. Thus we can think of describing the problem of weakly interacting spatially homogeneous random waves for a conservative system, for which $\tau_c \gg T$. Closure of the hierarchy of equations for the cumulants for dispersive and semidispersive systems comes from assuming that there is an initial instant at which they are smooth and factorize into products of lower ones. Then on a time scale τ_c, factorization of higher cumulants by products of lower ones prevails. It is these terms that describe the energy transfer mechanism which occurs in a system of random waves.

. The formation of the turbulent spectrum characterized by nonlinear wave-wave interaction considers the inertial range. In this region the principal term of the kinetic equation (0.9) is the collision integral implying that the specific form of the source and the sink is quite unimportant. The stationary solutions of the Kolmogorov type are solutions of $I\{n_k\}=0$. The properties of locality can be verified directly. Translational invariance is a consequence of wave number conservation in the process of resonant interaction of waves.

Unlike the Kolmogorov spectrum that is determined from considerations of dimensionality and similarity, the spectral energy density $\omega_k n_k$ of local isotropic turbulence requires an additional relation between the flux and the spectrum. This is because of the appearance of an additional local characteristic which is the phase velocity of the wave. This additional relation is imposed by the kinetic equation. To determine this relation, consider $u(\vec{k})=\omega_k n_k k^{d-1}$. Energy conservation requires that in the inertial range

$$\frac{\partial u}{\partial t} + \frac{\partial Q}{\partial k} = 0$$

which we obtain by multiplying (0.11) by the frequency and the density of states with d the dimensionality of $\vec{k}$ space. The energy flux, Q, over the spectrum is determined by the equation

$$\frac{\partial Q}{\partial k} = -\,\omega_k k^{d-1} I\{n_k\}$$

Stationarity requires $I\{n_k\}=0$ and Q=constant non zero corresponds to a Kolmogorov type spectrum.

The organization of this work is as follows. Chapter I deals with the spectrum for wave turbulence of two particular systems: gravity waves on water and acoustic waves, with applications to wind waves and 1/f noise. The obtained spectra is based on dimensional estimates and on the Kolmogorov picture for turbulence adapted to waves. The kinetic description for waves, although not explicit is used by invoking the basic nonlinear resonant interactions among random waves. Following Larraza and Putterman (1987), we consider first the underlying equations that describe the system of waves and establish a set of basic realizability inequalities for an inertial range. The dimensional estimates give spectra proportional to a power of the frequency, that because of the resonant nature of the wave-wave interactions are particular to the system. For deep water waves the leading order nonlinearities yield a power spectrum proportional to ω^{-4} whereas for sound the off equilibrium spectrum is $\omega^{-3/2}$ (Zakharov, 1984).

As the strengh of the external drive is increased one expects that higher order nonlinearities become important in determining the steady state. These higher order terms modify the power spectrum and in the limit of infinite nonlinearity lead to a saturation phenomenon where the spectrum of deep water waves goes as ω^{-5}, the Phillips spectrum for wind driven ocean waves (Phillips, 1977), and the spectrum of sound waves goes as ω^{-1} (Larraza, Putterman, and Roberts, 1985). Based on estimates of the global Mach number for the Phillips spectrum we motivate these universal power spectra.

Chapter II follows the treatment of Larraza, Putterman and Roberts, (1985) and considers universal spectra for waves from the formal point of view of a kinetic equation. A set of similarity transformations is used to arrive at these spectra and the properties of locality are verified directly within the kinetic equation.

It is found that there are two solutions that give a stationary power spectrum. The first corresponds to equipartition of energy in a closed system. The other solution is a consequence of self-similarity and has a nonzero constant flux of energy over the spectrum, and a 1/f dependence on frequency. The two solutions are physically differentiated by a boundary condition in $\vec{k}$ space, the flux of energy at long wavelength.

Chapter III shows that a wave turbulent medium can support a propagating energy mode with similarities to second sound (Larraza and Putterman, 1986). Following the ideas of Putterman and Roberts (1982, 1983) consideration of a complete theory of wave interaction is made. This includes scattering of sound by sound to create sum and difference frecuencies represented by a collision operator; refraction effects due to a slowly varing sound intensity and background; and reaction of the background flow due to variations in the distribution of acoustic noise. With this picture adapted to wave turbulence we establish the conditions for the transition from parabolic to hyperbolic energy tranport. The consequences of this turbulent second sound for plasma confinement are discussed at the end of the chapter.

Finally Chapter IV is an overlook at future problems. In particular it elaborates about how wave turbulence might be a possible mechanism of observed phenomena, for example magnetic field fluctuations driven by the solar wind.

Speculations about 1/f noise as the ultimate low frecuency spectrum for any wave turbulent system are made based on the renormalization of the dispersion law.

We comment on the possibility of turbulent second sound in the ocean with applications to predictions on storm variations.

CHAPTER I

BASIC EQUATIONS AND SCALING ARGUMENTS

For a barotropic fluid, conservation of mass and Newton's law take the form:

$$\partial\rho/\partial t + \nabla\cdot\rho\vec{v} = 0 \tag{1.1}$$

$$\rho[\partial\vec{v}/\partial t + (\vec{v}\cdot\nabla)\vec{v}] = -\nabla p + \rho\vec{g} + \eta\nabla^2\vec{v} + (\tfrac{1}{3}\eta+\zeta)\nabla(\nabla\cdot\vec{v}) \tag{1.2}$$

where ρ, v, p are density, velocity and pressure and where the first and second viscosities η, ζ are assumed constant. The irreversible viscous forces are derivable from the divergence of a viscous stress tensor

$$-\partial\tau_{ij}/\partial r_j = \eta\nabla^2\vec{v}_i + (\tfrac{1}{3}\eta+\zeta)[\nabla(\nabla\cdot\vec{v})]_i \tag{1.3}$$

In describing surface water waves, we use the incompressibility condition ρ=constant and $\nabla\cdot\vec{v}=0$. However, the motion should be supplemented with boundary conditions at the free surface, $z = \xi(x,y,t)$, where (x,y) labels the coordinates of the surface. One boundary condition is the continuity of the stress across the free surface, that is

$$p n_i + \tau_{ij} n_j + \sigma \kappa(x,y,t) n_i = p_o n_i \qquad (1.4)$$

where $\hat{n}$ is the normal vector to the surface, $\kappa(x,y,t)$ is the curvature of the surface with surface tension σ, and p_o is the ambient pressure. The second boundary condition

$$\frac{\partial \xi}{\partial t} + \vec{v} \cdot \nabla \xi = v_z \qquad (1.5)$$

states that particles on the surface move with the local fluid velocity.

The small amplitude dispersion law implied by (1.2), (1.4), and (1.5), in the case of an inviscid fluid of depth h is

$$\omega^2 = (gk + \sigma k^3/\rho)\tanh(kh)$$

For deep water ($kh \gg 1$) and long wavelengths ($g\rho \gg \sigma k^2$) the dispersion law reduces to $\omega^2 = gk$ which corresponds to deep gravity waves. It turns out, as we will see later, that most of the energy for surface waves in the ocean is found in deep gravity waves. Corrections to the phase velocity due to surface tension results in 3 parts in 10^4 for $\omega =$ 10 rad/sec and 3 parts in 10^8 for $\omega = 1$ rad/sec.

For sound waves the motion away from a boundary is irrotational, yet compressible. Neglecting g and taking time derivative of (1.1) and subtracting the divergence of (1.2) yields

$$\partial^2 \rho/\partial t^2 - \nabla^2 p = \partial^2 (\rho v_i v_j + \tau_{ij})/\partial r_i \partial r_j \qquad (1.6)$$

Equation (1.6) should be supplemented with an equation of state relating the thermodynamic quantities involved. For a barotropic fluid, $p=p(\rho)$. Expanding p in a

Taylor series

$$p = p_o + c^2\delta\rho + \tfrac{1}{2}(\partial^2 p/\partial\rho^2)_o(\delta\rho)^2 + \ldots$$

where $\rho = \rho_o + \delta\rho$ enables (1.6) to be rewritten in the form

$$\frac{\partial^2\delta\rho}{\partial t^2} - c^2\nabla^2\delta\rho = \frac{\partial^2}{\partial r_i\partial r_j}(\rho v_i v_j + \tau_{ij}) + \nabla^2\sum_{m=2}\frac{1}{m!}\frac{\partial^m p}{\partial\rho^m}(\delta\rho)^m \qquad (1.7)$$

The small amplitude dispersion law for a lossless medium implied by (1.7) is $\omega=ck$.

For wave propagation, one requires that the system undergoes several oscillations before it decays to a value 1/e its original amplitude. This implies that the period of the wave must be much smaller that the wave lifetime due to molecular viscosity or

$$\omega \gg \alpha k^2 \qquad (1.8)$$

where $\alpha = \eta/\rho$ for gravity waves and $\alpha = (4\eta/3 + \zeta)/\rho$ for sound waves in a barotropic fluid.

When the medium receives a large energy imput at long wavelength, modes with higher wavenumber are going to be excited due to nonlinear interactions. As in the Kolmogorov picture, an inertial range occurs when the reversible nonlinear terms in the equations of motion dominate the damping due to the linear irreversible terms such as molecular viscosity. Turbulent transport dominates molecular transport and the latter can be neglected. The requirements for an inertial region is

$$\omega a \gg \alpha k \qquad (1.9)$$

where a is the amplitude of the wave. In order to keep a wave description for the system of interacting waves one has to require that the Mach number for waves, the ratio a of the amplitude of the wave to the wavelength, be small

$$ka < 1 \tag{1.10}$$

For deep water waves with $\omega^2 = gk$, (1.10) implies that the acceleration of a fluid particle on the surface should be less than gravity in order to preserve a wave description. Wave breaking ("white-caps" formation) occurs in high wave number regions and is responsible in the end for energy and momentum losses in a gravity wave system. For sound waves (1.10) implies that the ratio v/c of the particle velocity to the speed of sound be small.

Consistency of (1.8), (1.9), and (1.10) implies that

$$\omega > k\omega a \gg \alpha k^2 \tag{1.11}$$

For gravity waves (1.11) requires that the rotational region of the flow be small compared to 1/k. Then throughout the rest of the fluid we have potential flow as for an ideal fluid. Note that the flow obtained from solving the equations for an ideal fluid do not satisfy the boundary conditions (1.4) which require that certain combinations of the space derivatives of the velocity should vanish. Thus within a thin surface layer the corresponding velocity derivatives decrease rapidly and by (1.8) this does not imply large velocity gradients. For these approximations, the energy dissipation in a deep gravity wave is (Landau and Lifshitz, 1959)

$$\gamma = 2\eta k^2/\rho = 2\eta\omega^4/\rho g^2$$

For clean water and $\omega = 1$ rad/sec, $1/\gamma \sim 3.1$ years and the viscous decay length $\omega/\gamma k \sim 10^6$ Km, while for $\omega = 10$ rad/sec, $1/\gamma \sim 2.67$ hours and $\omega/\gamma k \sim 10$ Km. Condition (1.8) is very well satisfied in this frequency range. For $\omega = 1$rad/sec, $\omega a/k\alpha \sim 10^5$ a/cm, so for a~1m we get an amplitude 7 orders of magnitude higher than the critical amplitude of 10^{-5}cm for which $\omega a/k\alpha = 1$, and two orders of magnitude less than the corresponding wavelength; while for $\omega \sim 10$ rad/sec, $\omega a/k\alpha \sim 10^4$ a/cm and consistency with (1.10) demands amplitudes no larger than 60 cm which are 6 orders of manitude bigger than the critical amplitude.

Under the strong assumptions (1.11), the fluid equations describing the inertial range for gravity waves in deep water are

$$\nabla^2\phi = 0, \quad \nabla\phi \to 0 \text{ as } z \to -\infty \tag{1.12a}$$

$$\left.\begin{aligned} &\frac{\partial\xi}{\partial t} + \nabla\phi\cdot\nabla\xi = \partial\phi/\partial z \\ &\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + g\xi = 0 \end{aligned}\right\} \text{ at } z = \xi(x,y,t) \tag{1.12b}$$

where $\vec{v} = \nabla\phi$ and $\nabla\cdot\vec{v} = 0$ for incompressible potential flow.

For sound waves (1.11) results in the equation

$$\frac{\partial^2\delta\rho}{\partial t^2} - c^2\nabla^2\delta\rho = \frac{\partial^2}{\partial r_i\partial r_j}(\rho v_i v_j) + \nabla^2\sum_{m=2}\frac{1}{m!}\frac{\partial^m p}{\partial\rho^m}(\delta\rho)^m \tag{1.13}$$

Although kinetic equations for wave turbulence can be derived from (1.12a,b) for gravity waves and (1.13)for sound waves, we will follow here the simpler Kolmogorov dimensional approach and study the process whereby wave energy cascades from one length scale to the next. We label the properties of successive scales with subscript n so that we have the wavenumbers

$$k_n = 2^n/\ell_o \tag{1.14}$$

For sound waves the energy per unit volume on this length scale is:

$$E_n = (c^2/\rho_o)(\delta\rho_n)^2 \tag{1.15}$$

The key to the cascade argument is that the rate at which energy rolls over from one length scale to the next is a function of the energy contained in that length scale (locality). For sound waves the lack of dispersion implies that the basic nonlinear interaction is a three wave resonance so that waves with frequencies ω_1 and ω_2 scatter to produce waves with frequency $\omega_3 = \omega_1 \pm \omega_2$. To leading order this effect is produced by the term in (1.13) with m = 2 and therefore yields a rollover time t_n for the wave energy given by

$$1/t_n \cong \omega_n G^2 E_n/\rho c^2 \tag{1.16}$$

where $G = 1 + (\rho/c)dc/d\rho$ is the macroscopic Gruneisen coefficient (the symbol $\cong$ means equality except for a numerical factor). The stationary state then follows from setting the rollover rate equal to the input rate Q

$$E_n/t_n = Q \tag{1.17}$$

The discrete stationary spectrum then is

$$E_n \cong [Q\rho c^2/\omega_n G^2]^{1/2} \tag{1.18}$$

so that the continuous power spectral density is

$$e(\omega) = [Q\rho c^2/G^2]^{1/2}/\omega^{3/2} \tag{1.19}$$

The energy per unit area of water waves is

$$U_n = \rho g \xi_n^2 \tag{1.20}$$

These waves differ from sound wave in that the strong downward dispersion requires that the leading interaction effect be produced by a four wave process. Thus instead of (1.16) one has

$$\frac{1}{t_n} \cong \omega_n k_n^4 \xi_n^4 \tag{1.21}$$

Seeking a steady state with

$$U_n/t_n = q \tag{1.22}$$

yields the power spectral density

$$u(\omega) \cong g^2[q\rho^2]^{1/3}/\omega^4 \tag{1.23}$$

For weak wave turbulence one finds (1.19) for acoustics and (1.23) for surface waves on deep water. From (1.16) and (1.21) one can establish a weak condition for an inertial range which will require that the turnover time t_n be less than the lifetime of the wave. In about the time t_n, which is larger than the characteristic period of the wave, an effective energy transfer takes place; hence one has the time scales satisfying

$$T \ll t_n \ll \tau_\alpha$$

where T is the period of the wave and τ_α is the lifetime of the wave due to molecular viscosity.

For many years ocean waves have been described by the Phillips spectrum (Phillips, 1977) wherein

$$u(\omega) \cong \rho g^3/\omega^5 \tag{1.24}$$

which differs from (1.23) as regards to the power of ω and its independence of the value of the energy input q. For this reason (1.24) is thought of as corresponding to some kind of saturation regime. Numerous technical causes for saturation have been proposed especially white- capping. We would now like to argue that the passage from (1.23) to (1.24) can be understood in terms of higher order nonlinearities, i.e. in terms of higher order wave processes.

Figure 1.1 shows the stationary frecuency spectrum of wind generated waves. The logarithmic vertical scale covers six decades. Plotted is the frequency spectrum of the mean square surface displacement, which is proportional to the power spectral density and is fitted to

$$\Phi(\omega) = \beta g^2 \omega^{-5}$$

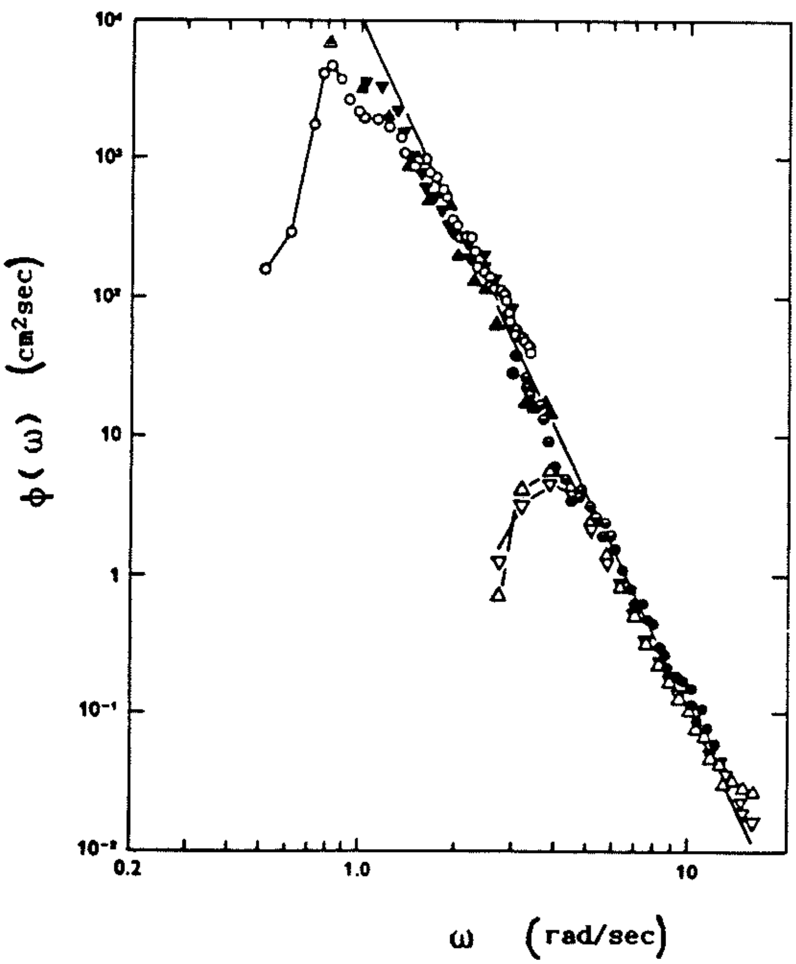

Figure 1.1 Frequecy spectrum of the mean square surface displacement for deep gravity waves.

where $\beta \simeq 10^{-2}$ is a numerical constant. One can see that over the frequency range considered in Figure 1.1 the requirements (1.11) are very well satisfied. An estimation of the square of the global Mach number over this frequency range gives

$$M^2 = \int \Phi(\omega) k^2 d\omega \simeq 2.3 \times 10^{-2}$$

which is a considerable number. This motivates one to consider higher order nonlinearities as the saturation mechanism.

We can arrive at an expression for the rollover time from energy transfer considerations and from the nature of the dispersion law. Since we are dealing with energy tranfer among waves, the contribution from each interacting wave varies as the product $(k_n \xi_n)^2$. Dispersion for gravity waves requires that the number of interacting waves be greater than three. For a four wave process the expression for the rollover time reduces to (1.21), which we can now generalize. The cascade time for an m wave process will be, therefore,

$$1/t_n \cong \omega_n k_n^{2(m-2)} \xi_n^{2(m-2)}$$

which with (1.20,1.22) yields (1.24) in the limit $m \to \infty$.

This suggests the following picture for the transition from (1.23) to (1.24). As the power input to wave motion is increased higher nonlinear effects come into play and shift the exponent of ω in the steady state power spectrum. In practice one should observe a response somewhere between (1.23) and (1.24) depending upon which range of m dominates.

In the acoustic case an m wave (i.e. m phonon) process leads to

$$1/t_n \cong \omega_n G_m^2 (E_n/\rho c^2)^{m-2}$$

where G_m is determined by the nonlinear equation of state in (1.13). Now one finds

$$e_m(\omega) \cong \frac{\rho c^2}{\omega} \left(Q/\rho c^2 \omega G_m^2 \right)^{1/(m-1)} \quad (1.25)$$

which in the limit of large m goes over to 1/f noise.

For semidispersive waves the redistribution of externally imposed energy by high order nonlinearities leads to 1/f noise. The type of nonlinear effect which saturates the nondispersive spectrum at 1/f noise also saturates the deep water spectrum at ω^{-5} noise (1.24).

Recently Phillips (1985) has called into question the use of the saturated spectrum (1.24). He implies that the fit of experimental data taken over many years to (1.24) rather than (1.23) may have been motivated by a lack of appreciation of the theory of energy balance of various processes in the weak turbulence limit. To this extent it should be emphasized that the generalization of Hasselmann's kinetic equation (Hasselmann, 1963) to higher order nonlinear effects will yield spectra with exponents between those of (1.23) and (1.24).

CHAPTER II

KINETIC THEORY OF WAVE TURBULENCE

The results of the previous Chapter were obtained based on dimensional estimates. They give the right power law dependence for deep water waves in the ocean and provide a framework to generalize to other systems like 1/f noise. However they do not account for the transition amplitude from weak turbulence (1.19) or (1.23) to the stronger wave turbulent regime as represented by (1.24) or (1.25), nor provide a proof for the locality hypothesis.

In regards to the first point we should say that this is still an open question. A similar problem exists in the theory of phase transitions. Namely, the determination of the critical exponents is directly drawn from a set of thermodynamic restrictions supplemented with a scaling hypothesis, but the actual value of the critical point cannot be computed for most systems.

Most real systems will have nonlinearities of all orders present so that the equation of motion will include a sum over m, the order of the nonlinearity. For water waves the nonlinearities come from the boundary conditions at the free surface (1.12b), while for sound waves, the convective term and the equation of state are the sources of nonlinearities as implied by (1.13). The connection which we propose between the mathematical results (1.24) or (1.25) and the real power spectra is that as the input power Q is increased, higher order (higher m) nonlinear terms will dominate the response because the kinetic equation for e_m is proportional to (m-1) powers of e_m. Although the actual response may be due to the effects of many different nonlinear coefficients, the frequency dependence of the spectrum will accumulate (at large drive) at a 1/f spectrum for sound waves or at a $1/f^5$ spectrum for deep water waves.

The second point, locality, will be discussed at the end of this Chapter.

For concreteness consider sound waves of amplitude $\delta\rho$ in a medium for which $\omega = ck$ so that the nonlinear wave equation takes the form

$$\partial^2\delta\rho/\partial t^2 - c^2\nabla^2\delta\rho = \nabla^2\{(G_{m+1}c^2/\rho^{m-1})\delta\rho^m\} \qquad (2.1)$$

for an "m+1" wave (or so-called "m+1" phonon process) characterized by the nonlinear coupling coefficient G_{m+1}. The equilibrium density is ρ and the ∇^2 on the right-hand side of (2.1) expresses conservation of momentum as would, for instance, be implied by the Euler fluid mechanics (Landau and Lifshitz, 1959). For the three phonon process $G_3 = 1 + (\rho/c)dc/d\rho$. The equation of state for pressure $p(\rho)$ has nonlinearities of all order so that for $m > 2$, $G_{m+1}c^2/\rho^{m-1} \sim (\partial^m p/\partial\rho^m)/m!$.

Our main result is that from Eq. (2.1) the stationary isotropic spectral intensity for m+1 wave processes is (within a numerical coefficient) given by (1.25):

$$e_{m+1}(\omega) \sim \frac{\rho c^2}{\omega}\left(Q/\rho c^2\omega G_{m+1}\right)^{1/m} \qquad (2.2)$$

where Q is the rate at which wave energy is supplied per unit volume at long wavelength. For the sound field the power spectrum is the Fourier transform of the square correlation of $\delta\rho$ so that $\int e(\omega)d\omega = (c^2/\rho)\langle(\delta\rho)^2\rangle$ is the total wave energy density.

As the order of the phonon process increases ($m \to \infty$), $e_m(\omega)$ smoothly approaches a 1/f power spectrum. The stationary wave spectra for m+1 = 3,4 have been known for some time (Zakharov, 1984, and references therein); what appears to have been overlooked is the fact that the self-similar spectrum can be calculated for all m and has 1/f noise as an accumulation point. This result is independent of dimension and applies for $d > 2$.

The kinetic equation for the wave action $n(\vec{k})$, defined by $(c^2/\rho) \langle \delta\rho^2 \rangle = \int \omega n(\vec{k}) d\vec{k}$ for general m will have the form (see Appendix):

$$\dot{n}(k) = I\,\{n(k)\} = \sum_{\{s\}} \int d\tau\; W(\vec{k})\; f(\vec{k}) \tag{2.3}$$

where

$$d\tau \equiv d\vec{k}_1 d\vec{k}_2 \ldots d\vec{k}_m; \quad s = (s_1, \ldots, s_m)\;; \quad s_i = \pm 1 \tag{2.4}$$

$$W(\vec{k}) = H(\vec{k}, s_i \vec{k}_i)\; \delta\left(\vec{k} + \sum_{i=1}^{m} s_i \vec{k}_i\right)\; \delta\left(\omega + \sum_{i=1}^{m} s_i \omega_i\right)\;; \quad \omega, \omega_i > 0 \tag{2.5}$$

$$f(\vec{k}) = n n_1 n_2 \ldots n_m \left\{ \frac{1}{n} + \sum_{i=1}^{m} \frac{s_i}{n_i} \right\}; \quad n_i = n(k_i)\;. \tag{2.6}$$

For a nonlinear wave equation of the form (2.1)

$$H \sim [c^2 G_{m+1}^2 / (\rho c)^{m-1}]\; k k_1 k_2 \ldots k_m\;. \tag{2.7}$$

Consider a typical term in the collision integral (2.3) which corresponds to two incoming waves and m-1 outgoing waves:

$$\tilde{I}(n) = \pm P_{(012\ldots m)} \int d\tau\; \tilde{W}(\vec{k}, \vec{k}_1 | \vec{k}_2 \ldots \vec{k}_m)\; \tilde{f}(\vec{k}, \vec{k}_1 | \vec{k}_2 \ldots \vec{k}_m) \tag{2.8}$$

Here $P_{(012\ldots m)}$ denotes summation over cycles of (012...m), the index "0" denotes "$\vec{k}$", and the minus sign occurs whenever $\vec{k}$ appears as an outgoing wave:

$$\tilde{W}(\vec{k},\vec{k}_1|\vec{k}_2...\vec{k}_m) = \tilde{H}(\vec{k},\vec{k}_1|\vec{k}_2...\vec{k}_m)\ \delta(\vec{k}+\vec{k}_1-\vec{k}_2-...-\vec{k}_m) \times$$

$$\times\ \delta(\omega+\omega_1-\omega_2-...-\omega_m) \tag{2.9}$$

$$\tilde{f}(\vec{k},\vec{k}_1|\vec{k}_2...\vec{k}_m) = nn_1n_2...n_m\left(\frac{1}{n}+\frac{1}{n_1}-\frac{1}{n_2}-\cdots-\frac{1}{n_m}\right) \tag{2.10}$$

The functions $\tilde{H}$ and $\tilde{f}$ (and $\tilde{W}$) are symmetric with respect to exchange of arguments, among the incoming waves or among the outgoing waves. For isotropic media, $\tilde{H}$, $\tilde{f}$, $\tilde{W}$ are invariant under simultaneous rotation of all the m+1 vector arguments through the same angle, and $\omega(\vec{k}) = \omega(k)$. Furthermore, we consider a system for which ω and $\tilde{H}$ are homogeneous functions of k so that

$$\omega(\lambda k) = \lambda^{\beta}\omega(k) \tag{2.11}$$

$$\tilde{H}(\lambda\vec{k},\lambda\vec{k}_1|\lambda\vec{k}_2...\lambda\vec{k}_m) = \lambda^{\ell}\ \tilde{H}(\vec{k},\vec{k}_1|\vec{k}_2...\vec{k}_m) \tag{2.12}$$

and therefore the degree of homogeneity of $\tilde{W}$ is $\ell-d-\beta$. For Eq. (2.1) [and therefore H given by (2.7)]

$$\ell = m+1\ , \quad \beta = 1 \tag{2.13}$$

All the processes in (2.8) can be put into the form of the first term by an appropriate variable transformation. To this end consider the process

$$(q_iq_{i+1}|q_{i+2}...q_mkq_1...q_{i-1})$$

where we have provisionally denoted the integration variables by "q" instead of "k". Generalizing Kats and Kontorovich (1973) we make the change of variable:

$$\vec{q}_i = (\lambda\bar{g})_{m-i+1}\,\vec{k}_{m-i+1}$$

$$\vec{q}_{i+j} = (\lambda\bar{g})_{m-i+1}\,\vec{k}_j \qquad 1 \le j \le m\text{-}i$$

$$\vec{q}_s = (\lambda\bar{g})_{m-i+1}\,\vec{k}_{m-i+1+s} \qquad 1 \le s \le i\text{-}1 \tag{2.14}$$

where $\bar{g}$ is a rotation and λ an extension given by:

$$\bar{g}_n\hat{k}_n = \hat{k}, \quad \lambda_n = k/k_n, \quad \hat{k}_n = \vec{k}_n/k_n \tag{2.15}$$

so that $\bar{g}\vec{k}$ is shorthand for the matrix multiplication $g_{i\alpha}k_\alpha$ where $g_{i\alpha}$ is a right orthogonal matrix; $\bar{g}^2$ stands for two rotations. This set of transformations is meaningful only in the inertial range where decay and direct energy injection can be ignored. To the extent that inertial effects dominate (generally referred to as locality), the domain of integration is left unchanged by the transformations (2.14) (Kats and Kontorovich, 1973).

From the symmetry and homogeneity properties of $\tilde{H}$ and the Jacobian implied by (2.14):

$$\tilde{H}(\vec{q}_i\vec{q}_{i+1}|...\vec{q}_{i-1})\;\delta(\vec{q}_i+\vec{q}_{i+1}-...-\vec{k}-...-\vec{q}_{i-1}) \times$$

$$\times\;\delta(\omega_i+\omega_{i+1}-...-\omega-...-\omega_{i-1})\;\tilde{f}(\vec{q}_i\vec{q}_{i+1}|...\vec{q}_{i-1})\;d\vec{q}_1...d\vec{q}_m \;=$$

$$= (\lambda_{m-i+1})^r \tilde{H}(\vec{k},\vec{k}_1|...\vec{k}_m)\ \delta(\vec{k}+\vec{k}_1-\vec{k}_2-...-\vec{k}_m) \times$$

$$\times\ \delta(\omega+\omega_1-\omega_2-...-\omega_m)\ \tilde{f}_{m-i+1}(\Lambda_{m-i+1},k)\ d\vec{k}_1...d\vec{k}_m \tag{2.16}$$

where $r = \ell+md-\beta$, $\Lambda_n = (\lambda\bar{g})_n$, and

$$\tilde{f}_n(\Lambda_n,k) = \tilde{f}(\Lambda_n^2\vec{k}_n,\Lambda_n\vec{k}_1|...\Lambda_n\vec{k}_{n-1},\vec{k},\Lambda_n\vec{k}_{n+2}...\Lambda_n\vec{k}_m) \tag{2.17}$$

Applying this transformation to all the terms in (2.8) yields:

$$\tilde{I}(n) = \int d\tau\ \tilde{W}(\vec{k},\vec{k}_1|...\vec{k}_m)\ \{\tilde{f}(\vec{k},\vec{k}_1|...\vec{k}_m) + \lambda_1^r\tilde{f}_1(\Lambda_1,k)$$

$$- \lambda_2^r\tilde{f}_2(\Lambda_2,k) - ... - \lambda_m^r\tilde{f}_m(\Lambda_m,k)\} \tag{2.18}$$

We seek a self-similar solution to $\tilde{I}(n) = 0$ of the form

$$n = A\omega^{\gamma} \tag{2.19}$$

for which

$$\tilde{f}(\vec{k},\vec{k}_1|...\vec{k}_m) = A^m(\omega\omega_1...\omega_m)^{\gamma}\ [\omega^{-\gamma} + \omega_1^{-\gamma} - \omega_2^{-\gamma} - ... - \omega_m^{-\gamma}]$$

$$\tilde{f}_n(\Lambda_n,k) = \lambda_n^{\ m\beta\gamma}\ \tilde{f}(\vec{k},\vec{k}_1|...\vec{k}_m) \tag{2.20}$$

so that (2.18) becomes

$$\tilde{I}(n) = \omega^{\nu} \int d\tau\ \tilde{W}(\vec{k},\vec{k}_1|...\vec{k}_m)\ \tilde{f}(\vec{k},\vec{k}_1|...\vec{k}_m) \times$$

$$\times\ \{\omega^{-\nu} + \omega_1^{-\nu} - \omega_2^{-\nu} - ... - \omega_m^{-\nu}\} \quad (2.21)$$

where

$$\nu = m\gamma + (\ell + md - \beta)/\beta' \quad (2.22)$$

There are now two solutions to $\tilde{I}(n) = 0$. The first corresponds to equipartition of energy in a closed system and is realized by $\gamma = -1$. It leads to the vanishing of f as is seen from (2.20) and the "δ" function restriction on the conservation of energy (or ω) as given by (2.5) or (2.9). The other solution is a consequence of self similarity as well as energy conservation and corresponds to

$$\nu = -1\ , \qquad \gamma = -(\ell+md)/m\beta \quad (2.23)$$

The spectral intensity is given by

$$e(\omega) = n\omega k^{d-1}/(d\omega/dk) \quad .$$

This approach can be used to scale "n" for very general interactions in isotropic homogeneous media. Application to Eq. (2.1), thus using (2.13) for ℓ and β, yields for (2.23)

$$\gamma+d = -(m+1)/m \quad (2.24)$$

so that e(ω) has the frequency dependence (2.2). The complete expression (2.2) follows from the method of Kats (1976) by setting the "k" space energy flux Q equal to a constant where

$$Q(k) \sim \int^{k} \omega(k')(k')^{d-1}\, I(k')\, dk' \tag{2.25}$$

and corresponds to flow from low k to high k. From (2.19,2.21) we find

$$I = D(\nu)k^{\beta\nu-d} \quad \text{where} \quad D(-1) = 0$$

Therefore (2.25) yields

$$Q(k) = \frac{D(\nu)}{\beta(1+\nu)} k^{\beta(1+\nu)} (\omega/k^{\beta})$$

so that as $\nu \to -1$ and the self-similar solution is approached, we find

$$Q(k) = (dD/d\nu)_{\nu=-1} (\omega/\beta k^{\beta}) \tag{2.26}$$

For Eq. (2.1) (thus taking $\beta = 1$)

$$D = A^{m} c^{\gamma m+1} \varepsilon(\nu)G^{2}_{m+1}/(\rho c)^{m-1} \tag{2.27}$$

where ε is a dimensionless numeric and $\varepsilon(-1) = 0$. Placing (2.27) into (2.26) yields Eq. (2.2). Equipartition, $\gamma = -1$, leads to $Q = 0$.

The change of variable (2.14) transforms the argument of the collision integral (2.8) as well as the domain of integration. The stationary self- similar solution (2.19) is obtained from (2.8) by neglecting contributions which arise from the transformations of the various endpoints of integration. Physically this means that for these turbulent states the nonlinear development of a sound wave at wave number k is dominated by its interaction with waves k_i near k ("locality"). Thus the validity of (2.19) is contingent upon the convergence of all contributions to I(k) that are due to intermediate arguments of integration k_i that are far from k. Denote the endpoints by k_o (the energy containing region) and k_d (the dissipative region), so that $k_o < k < k_d$. As we will see below a direct calculation of the contributions to I(k) from these two regions yields respectively $(k_o/k)^{1/m}\, k^{-4}$ and $(k/k_d)^{1/m}\, k^{-4}$. These endpoint contributions vanish as $k_o \to 0$ and $k_d \to \infty$. Thus for an arbitrary but finite number of waves the similarity range is effectively infinite.

To prove this, one has to realize that the conservation laws allow one to explore many possibilities. Let us consider the convergence of the integral (2.3) for the small wave number region. In particular consider

$$k_1, \ldots, k_{m-1} \ll k, k_m$$

Bearing in mind that the distribution $n(\vec{k}_i) \to \infty$ when $k_i \to 0$, we establish the fact that the divergent terms are $n_1 n_2 \cdots n_{m-1}(n_m - n)$ in the first component of (2.8) and $n_1 n_2 \cdots n_{m-1}(n - n_m)$ in the (m −1)th component. Calling $\vec{q} = \vec{k}_1 - \vec{k}_2 - \cdots - \vec{k}_{m-1}$ and $\vec{p} = \vec{k}_{m-1} - \vec{k}_1 - \vec{k}_2 - \cdots - \vec{k}_{m-2}$, and integrating with respect to $\vec{k}_m$ by using the momentum conservation law, gives for these terms

$$\tilde{I} \cong \int d\tau' \Big(\tilde{H}(\vec{k},\vec{k}_1 | \vec{k}_2 \cdots \vec{k}_{m-1}, \vec{k}+\vec{q})\, n_1 n_2 \cdots n_{m-1} (n(\vec{k}+\vec{q}) - n(\vec{k})) \times$$

$$\times\, \delta(\omega + \omega_1 - \omega_2 - \cdots - \omega_{m-1} - \omega(\vec{k}+\vec{q})) - \tilde{H}(\vec{k}_{m-1}, \vec{k}-\vec{p} | \vec{k}, \vec{k}_1 \cdots \vec{k}_{m-2}) \times$$

$$n_1 n_2 \cdots n_{m-1} (n(\vec{k}) - n(\vec{k}-\vec{p}))\, \delta(\omega_{m-1} + \omega(\vec{k}-\vec{p}) - \omega - \omega_1 - \cdots - \omega_{m-2}) \Big) \quad (2.28)$$

where

$$d\tau' = d\vec{k}_1 \cdots d\vec{k}_{m-1}$$

For small $k_1, \cdots, k_{m-1}$

$$\tilde{H}(\vec{k}, \vec{k}_1 | \vec{k}_2 \cdots \vec{k}_{m-1}, \vec{k}+\vec{q}) \cong \tilde{H}(\vec{k}_{m-1}, \vec{k}-\vec{p} | \vec{k}, \vec{k}_1 \cdots \vec{k}_{m-2}) \cong$$

$$\cong (k_1 k_2 \cdots k_{m-1}) k^2,$$

and

$$\delta(\omega + \omega_1 - \omega_2 - \cdots - \omega_{m-1} - \omega(\vec{k}+\vec{q})) \cong \delta(\omega_{m-1} + \omega(\vec{k}-\vec{p}) - \omega - \omega_1 - \cdots - \omega_{m-2}) \cong$$

$$\cong \frac{1}{c}\, \delta(k_1 - k_2 - \cdots k_{m-1} - \hat{k} \cdot \vec{q}) \equiv \tilde{\delta}$$

for $\vec{q} \cong \vec{p}$. With account of this, the integral (2.28) reduces to the form

$$\tilde{I} \cong k^2 \frac{\partial^2 n}{\partial k_i \partial k_j} \int d\tau' k_1 k_2 \cdots k_{m-1} q_i q_j n_1 n_2 \cdots n_{m-1}\, \tilde{\delta}$$

By evaluating the powers of wave numbers $k_1, \cdots, k_{m-1}$, we establish that the integral converges at zero if

$$d(m-1) + m + \gamma(m-1) > 0$$

The left hand side of the inequality is equal to 1/m and thus we have the convergence at low k guaranteed.

By considering the convergence of (2.3) at large wavenumbers, let us look at the range

$$k_1, k_2 \gg k, k_3, \cdots, k_m$$

The divergent terms as $k_1, k_2 \to \infty$ are those linear in n_1 and n_2 in the first and third components of (2.8). Calling $\vec{p} = \vec{k} - \vec{k}_3 - \cdots - \vec{k}_m$, and $\vec{q} = \vec{k}_3 - \vec{k} - \vec{k}_4 - \cdots - \vec{k}_m$, by the same token, integrating over $\vec{k}_2$ we obtain an integral of the form

$$\tilde{I} \cong \int d\tau' (k k_3 k_4 \cdots k_m) k_1^2 \; \delta(k - k_3 - k_4 - \cdots - \hat{k}_1 \cdot \vec{p}) \times$$

$$\times \, n n_3 n_4 \cdots n_m \left(\vec{p} \cdot \frac{\partial n_1}{\partial \vec{k}_1}\right)$$

where

$$d\tau' = d\vec{k}_1 d\vec{k}_3 d\vec{k}_4 \cdots d\vec{k}_m$$

The convergence condition as $k_1 \to \infty$ reads

$$1 + d + \gamma < 0$$

with the left hand side of the inequality being equal to $-1/m$ the inequality is satisfied.

The nonlinear processes considered here lead not only to the scattering of wave energy from one frequency to another but also to the renormalization of the dispersion law. In general these effects will maintain the linear relation between ω and k, but c will become a function of Q. Thus the frequency dependence of the spectrum (2.2) will be unaltered although c will depend upon total noise power.

Although the result (2.2) is independent of d, it should not be applied to $d < 2$ as Newell and Aucoin (1971) have argued that (2.3), (2.6) cannot be justified for $d < 2$.

Many systems receive a significant input of energy at low frequency. Consider for instance the oceans which are driven by large tidal forces on a daily, monthly, yearly period. The self-consistent limiting power spectrum for waves in these systems is 1/f noise.

Although wave turbulence is quite different from vortex turbulence, the corresponding controlled wave experiments have not yet been carried out. We think that it would yield valuable insight into the 1/f problem if, in a controlled apparatus, the transition into and out of 1/f noise could be measured as Q is varied. Depending upon the manner in which a given system is excited, the total power input may be different from Q, the net power input to the wave motion.

Equation (2.21) has two power law solutions: equipartition of energy in a closed system and nonlinear self-similarity in a driven open system. Each of these responses follow from conservation of energy. Following the discusion of the previous chapter, the strength of the input energy Q will determine the power law for the spectrum. For large energy input a 1/f noise spectrum at low frequency will prevail, and at higher frequencies the wave amplitude will be such that the weak wave turbulence conditions will determine the power law which in the acoustic turbulence case is $\omega^{-3/2}$. For even higher frequencies the thermal bath dictates the nature of the spectrum. The schematic representation of the most general power spectrum of a driven system is shown in Figure 2.1. The transition frequency for the $\omega^{-3/2}$ region to equipartition should occur when the basic dimensionless variable $Q\rho c^8/T^2\omega^7 G^2$ is of order unity.

Thus we see that 1/f noise is as basic to off equilibrium response as equipartition is to equilibrium. The extent to which many phenomena exhibit 1/f noise may be a measure of how far the enviroment is from equilibrium.

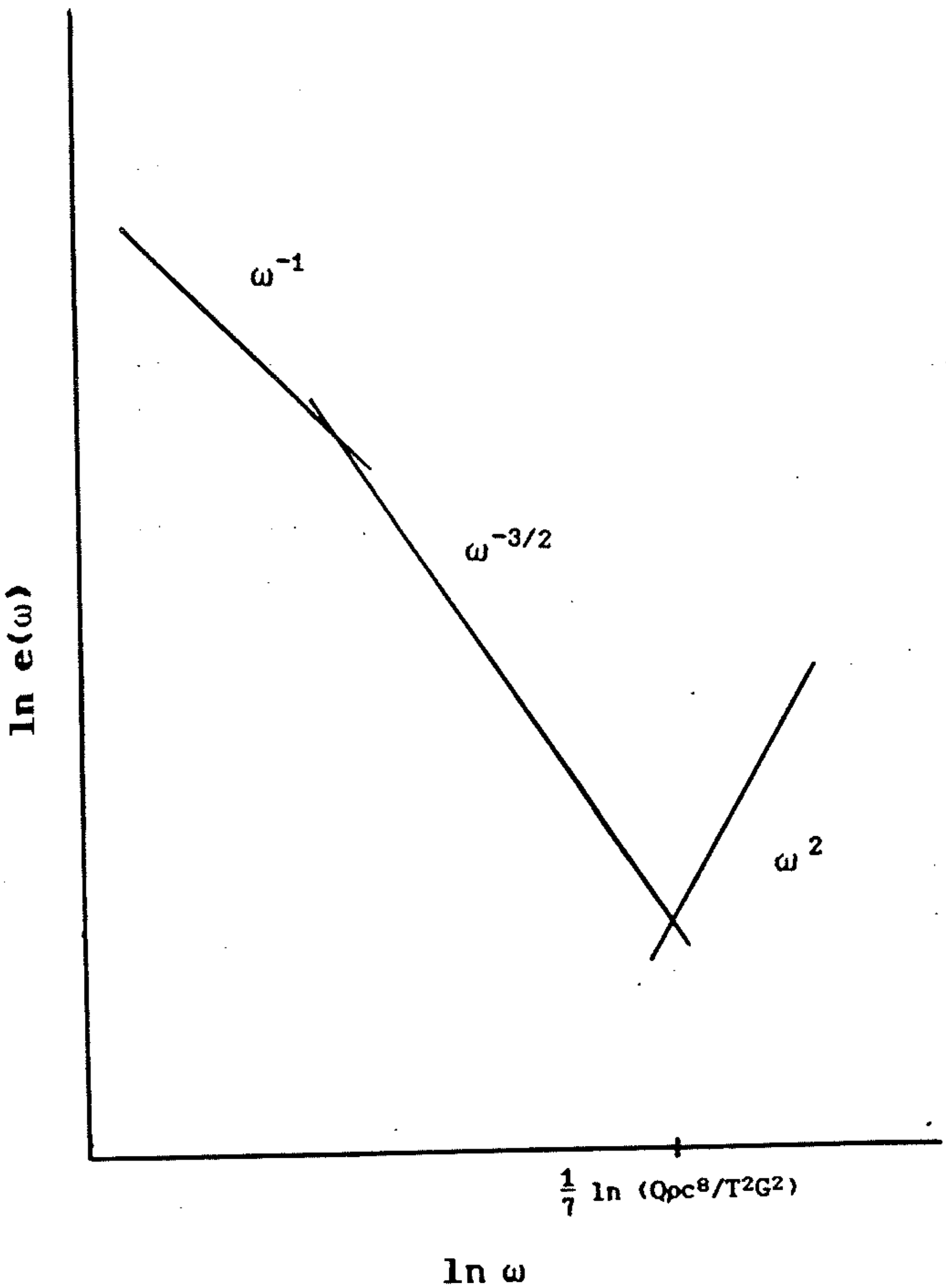

Figure 2.1 Power spectra for wave turbulence

CHAPTER III

ELASTICITY OF WAVE TURBULENCE

In this chapter we emphasize that the wave turbulence is elastic so that its energy fluctuations, instead of diffusing (parabolic motion), propagate as waves (hyperbolic motion). The transport of energy in a wave turbulent medium is therefore greatly enhanced so that it should be very difficult to confine energy to localized regions. As a particular application of these ideas we propose that the elasticity of wave turbulence (turbulent second sound) is the cause of the anomalous transport which significantly limits the effectiveness of devices designed to achieve controlled plasma fusion.

Liquid He^4 exhibits the most striking transition to a state with anomalously high transport. As it is cooled through $T_\lambda = 2.17^\circ K$ a 10^7 increase in the transport is made apparent through a sudden cessation of boiling (Atkins, 1959; London, 1954) It is misleading to refer to this loss of energy confinement, as He^4 transforms from the worst to the best known thermal conductor, as being due to an anomalous diffusivity since the equation of heat flow has become hyperbolic (Landau, 1941) rather than diffusive. The phenomenon whereby fluctuations in the power spectrum of the excitations (e.g. phonons) of He^4 propagate as waves is called second sound. In this state the effective transport is increased by a factor of a^2/ℓ^2 relative to the normal state (London, 1954; Putterman, 1974). Here a/ℓ is the ratio of a characteristic vessel size to the excitation mean free path.

We will show that the second sound is not limited to the near equilibrium He^4 superfluid and that it is also a property of classical wave turbulence. Although superfluidity is usually ascribed to quantum effects such as Bose condensation (London, 1954), off diagonal long range order (Penrose, 1951; Yang, 1962), and quantized excitations (Landau, 1954), a number of the macroscopic properties of superfluidity including second sound and the two fluid theory can be derived from the single component classical Euler hydrodynamics (Putterman and Roberts, 1982 and 1983). In this approach the normal fluid motion is characterized by the geometrical acoustics of the high frequency solution of a one component barotropic fluid and the superfluid motion is the long wavelength solution. This is the method to be applied below to the far-off equilibrium wave turbulent state of a classical fluid.

All fluids other than helium have a viscosity at long wavelength and should therefore be described by irreversible equations at the Navier-Stokes level. Our use of reversible equations to derive turbulent second sound is based upon the fact that in the turbulent state the linear irreversible transport processes are small compared to the reversible nonlinearities. Thus to the extent that the wave motion in the medium is driven far from equilibrium a real classical fluid will mimic superfluidity and yield second sound in the wave turbulence.

In a plasma wave turbulence and anomalous transport are probably due to the highly anisotropic drift wave motion (Horton, 1984; Liewer, 1985). Prior to commenting upon the plasmas, we first facilitate insight into the general properties of wave turbulence by studying the much simpler acoustic field described by the Euler hydrodynamics for a fluid of density ρ and velocity $\vec{v}$:

$$\partial\rho/\partial t + \nabla\cdot\rho\vec{v} = 0 \tag{3.1}$$

$$\partial\vec{v}/\partial t + (\vec{v}\cdot\nabla)\vec{v} = -(1/\rho)\nabla p(\rho) \tag{3.2}$$

where "p" is the pressure. Viscosity "η" has been dropped from (3.2) since in the turbulent state $\rho(\vec{v}\cdot\nabla)\vec{v} >> \eta\nabla^2\vec{v}$. We have taken the expansion coefficient to zero so that pressure is a function of density alone. Following Putterman and Roberts (1982, 1983) one seeks a solution to (3.1,3.2) of the form

$$\rho = \rho_o(\vec{r},t) + \int \rho_1(\vec{k},\vec{r},t)[\exp i\theta(\vec{k},\vec{r},t)]d\vec{k} + \text{c.c.}$$

$$\vec{v} = \vec{v}_o(\vec{r},t) + \int \vec{v}_1(\vec{k},\vec{r},t)[\exp i\theta(\vec{k},\vec{r},t)]d\vec{k} + \text{c.c.}$$

where the subscript zero denotes the slowly varying background and the integrals are over the shorter wavelength sound waves. We have explicitly allowed for the possibility that the intensity of sound at wave number k might be a slowly varying function of position. This refraction effect is one of the three basic nonlinear processes implied by (3.1,3.2). The other processes include the scattering of sound by sound to create sum and difference frequencies and the reaction of the background flow due to variations in the distribution of noise $\rho_1(\vec{k},\vec{r},t)$. The equations which describe these processes follow from (3.1,3.2) and are

$$\frac{\partial n}{\partial t} + \frac{\partial \omega}{\partial k_i}\frac{\partial n}{\partial r_i} - \frac{\partial \omega}{\partial r_i}\frac{\partial n}{\partial k_i} = I\{n\} \tag{3.3}$$

$$\frac{\partial \rho_0}{\partial t} = -\nabla\cdot\left[\rho_o\vec{v}_o + \int \rho_1\vec{v}_1{}^* d\vec{k} + \text{c.c.}\right] \tag{3.4}$$

$$\frac{\partial \vec{v}_o}{\partial t} + (\vec{v}_o\cdot\nabla)\vec{v}_o = -\nabla\left[\phi_o + \frac{U}{c}\frac{dc}{d\rho}\right] \tag{3.5}$$

where $n(\vec{k},\vec{r},t) = \varepsilon_k/ck$, $\omega = ck + \vec{v}_o\cdot\vec{k}$ is the dispersion law with sound velocity $c = (\partial p/\partial\rho)^{1/2}$; $\phi_o = \int dp/\rho$; $U = \int nck\, d\vec{k}$; $\varepsilon_k = (c^2/\rho_o)|\rho_1{}^2(k)| + \rho_o|v_1{}^2(k)|$. The collision integral I is some quadratic functional of n and describes the production of sum and difference frequencies.

Formation of the turbulent spectrum takes place as a result of the interaction of the waves in the inertial region of phase space where nonlinearities dominate viscosity. In this region the kinetic equation (3.3) yields a stationary isotropic solution n_o corresponding to a constant flux of energy $Q_o = \int I\{n_o\}\omega d\vec{k}$ from the input at low frequency to the heat reservoir at high frequency (Zakharov,1984) with

$$n_o = A_o/\omega_o^{\gamma}; \;\; A_o = b\rho c^5 \left[Q_o/\rho c^2 G^2\right]^{1/2} \qquad (3.6)$$

where $b = 1.37$, $\gamma = 9/2$, $\omega_o = ck$ and $G \equiv 1 + (\rho/c)dc/d\rho$ is the macroscopic Gruneisen coefficient. As we saw in the last chapter higher order nonlinearities can modify the exponent γ of the wave turbulence (3.6).

Due to the conservation of $\vec{k}$ in a collision of sound waves $I\{n\}=0$ has a slightly anisotropic generalized solution (Kats and Kontorovich, 1973)

$$n = n_o - (\vec{k}\cdot\vec{w} + \theta\omega_o)\partial n_o/\partial\omega_o \qquad (3.7)$$

where $\vec{w}$, θ are constant. For the turbulent state this solution is valid only to first order in $\vec{w}$. The near steady state motion is obtained by letting $\vec{w}$, θ vary slowly with $\vec{r}$, t. The equations for the relaxation of $\rho_o, \vec{v}_o, \vec{w}, \theta$ come from (3.4,3.5) and the moments of (3.3) with respect to $\omega, \vec{k}$. To calculate these moments one needs to note that to the right order

$$\frac{\partial n}{\partial t} = -\left(\vec{k}\cdot\frac{\partial \vec{w}}{\partial t} + \omega_o\frac{\partial\theta}{\partial t}\right)\frac{\partial n_o}{\partial\omega_o}$$

$$\frac{\partial n}{\partial r_i} = -\left(k_j\frac{\partial w_j}{\partial r_i} + \omega_o\frac{\partial\theta}{\partial r_i}\right)\frac{\partial n_o}{\partial\omega_o}$$

$$\frac{\partial n}{\partial k_i} = -\frac{\partial n}{\partial k}\hat{k}_i = -\frac{\partial n_o}{\partial\omega_o}c\hat{k}_i$$

The collision integral being a quadratic functional of n will, due to the anisotropic solution

(3.7) result in

$$I\{n\} = I\{n_o\}\left(1 - 2\gamma\theta - 2\gamma\frac{\vec{w}}{c}\cdot\hat{k} \right)$$

where we have in mind the evaluation of the moments $\int\left\{\begin{matrix}\omega\\ \vec{k}\end{matrix}\right\} I\{n\}d\vec{k}$, which by (2.25) are non-zero. The last term comes from considering the integral

$$\mathcal{J} = \int d\Omega_1 d\Omega_2 \hat{k}_1 \delta(\vec{k} - \vec{k}_1 - \vec{k}_2) = \frac{8\hat{k}}{kk_1k_2}\left(-\frac{\pi}{4} - \frac{\pi}{8kk_1}[k_2^2 - (k+k_1)^2] \right)$$

which because of the dispersion relation $\omega_o = ck$ and the frequency restriction in the three wave precesses reduces to

$$\mathcal{J} = \frac{2\pi\hat{k}}{kk_1k_2} = \hat{k}\int d\Omega_1 d\Omega_2 \delta(\vec{k} - \vec{k}_1 - \vec{k}_2)$$

Here $d\Omega_i$ is the element of solid angle for $\vec{k}_i$ and $\hat{k}_1$ is a unit vector in the direction of $\vec{k}_1$. With these considerations the moments of the kinetic equation with respect to ω and $\vec{k}$ and to the right order in the perturbed quantities yield

$$B\frac{\partial\theta}{\partial t} + \frac{1}{3}B\nabla\cdot(\vec{w} + \vec{v}_o) = -2\gamma\theta Q_o \tag{3.8}$$

$$\frac{B}{3c^2}\frac{\partial\vec{w}}{\partial t} + \frac{B}{3}\nabla\theta + \frac{B}{3c}\nabla c = -\frac{2\gamma\vec{w}Q_o}{3c^2} \tag{3.9}$$

where $B = -\int\omega_o^2(\partial n_o/\partial\omega_o)d\vec{k}$ and we have retained only those terms which are at most linear in $\vec{w}$, $\vec{v}_o$ and spatial gradients. In this manner (3.4,3.5) become

$$\partial\rho_o/\partial t + \nabla\cdot(\rho_o\vec{v}_o + \rho_t\vec{w}) = 0 \tag{3.10}$$

$$\frac{\partial \vec{v}_o}{\partial t} + \frac{c^2}{\rho_o} \nabla \rho_o + B \frac{\beta}{\rho} \nabla \theta = U \left(\frac{\beta}{\rho} \right)^2 \nabla \rho_o \qquad (3.11)$$

where we have set $\beta = (\rho/c)dc/d\rho$ and used $|d^2c/d\rho^2| << (dc/d\rho)^2/c$ and

$$\rho_t \vec{w} = \int n\vec{K} d\vec{K}$$

so that $\rho_t = B/3c^2$ is the turbulent density to be thought of in parallel with the normal fluid density of He^4. Substitution of the solutions

$$\theta = \theta' \exp i(\vec{K}\cdot\vec{r} - \Omega t)\,; \quad \vec{w} = \vec{w}' \exp i(\vec{K}\cdot\vec{r} - \Omega t)$$

$$\rho_0 = \rho_{00} + \rho' \exp i(\vec{K}\cdot\vec{r} - \Omega t)\ ; \quad \vec{v}_0 = \vec{v}' \exp i(\vec{K}\cdot\vec{r} - \Omega t)$$

into equations (3.8,3.9,3.10,3.11) leads to a system of four homogeneous equations with four unknowns. The secular solution gives a fourth order dispersion law in the frequency Ω and wave number K:

$$u^4[1+6iQ_o/\rho_t c^2\ \Omega] - u^2[2\rho_t\beta/\rho + 4/3 - 3\rho_t\beta^2/\gamma\rho + (6iQ_o/\rho_t c^2\ \Omega) \times$$
$$\times (1+\beta\rho_t/\rho - 2\beta^2\rho_t/3\rho)] + (1-\rho_t/\rho)[1/3 - \beta^2(1+1/\gamma)\rho_t/\rho] = 0$$

where $u \equiv \Omega/cK$ and squares of the imaginary terms have been neglected. The dispersion law describes two different propagating modes u_1, u_2. One is the mechanical first sound mode and the other is the propagating energy mode. The real and imaginary parts of the dispersion law determine the speed and damping of these waves. The key parameters are β, ρ_t/ρ and $Q_o/\rho_t c^2$. If one introduces the frequency ω_m at which energy is being injected into the wave turbulent system then (3.6) yields

$$\frac{Q_o}{\rho_t c^2 \ \Omega} = \frac{\rho_t}{\rho} G^2 \left(\frac{3}{8\pi\gamma b}\right)^2 \frac{\omega_m}{\Omega} \tag{3.12}$$

which is approximately the inverse quality factor for the second sound. We observe that turbulent second sound cannot propagate at low Ω. Above some minimum Ω the damping will, however, be sufficiently small so as to allow propagation as shown in Figure 3.1 where $\mathrm{Im}\, u_2/\mathrm{Re} u_2$ is plotted as a function of K/k_m ($k_m c = \omega_m$) for $\beta = 1/2$ and $\rho_t/\rho = .5$. In line with the inertial range approximation,

$$\left(Q_o/\rho c^2 \omega_m G^2 \right)^{1/4} >> \eta\omega_m/\rho c^2 \tag{3.13}$$

which trivially follows from (1.9), (1.15), and (1.18);we have neglected damping due to the background viscosity which contributes to $\mathrm{Im} u_2/\mathrm{Re} u_2$ as $\eta\ \Omega/\rho c^2$. Figure (3.2) shows $\mathrm{Re} u_{1,2}$ versus ρ_t/ρ_o. Figure (3.1) shows $\mathrm{Im} u_{1,2}/\mathrm{Re} u_{1,2}$ versus ρ_t/ρ_o for $k_m/K = 1$.

Second sound in equilibrium in liqiud helium corresponds to the case $Q_0 = 0$ and n given by a Planck spectrum. Thus the r.h.s. of (3.8) and (3.9) would vanish in He^4 and second sound would propagate without damping in the limit of low frequency "Ω". The off equilibrium cascade sets up its own intrinsic damping (in addition to η) which leads to a low frequency cutoff below which turbulent second sound is over damped.

As in the near equilibrium theory of superfluidity we expect that the equations for the reaction of the background only make sense when the background is not totally depleted or $\rho_t < \rho_o$. Combined with the requirement that the dimensionless damping (3.12) be small we obtain the basic inequality which governs the appearance of second sound in acoustic wave turbulence:

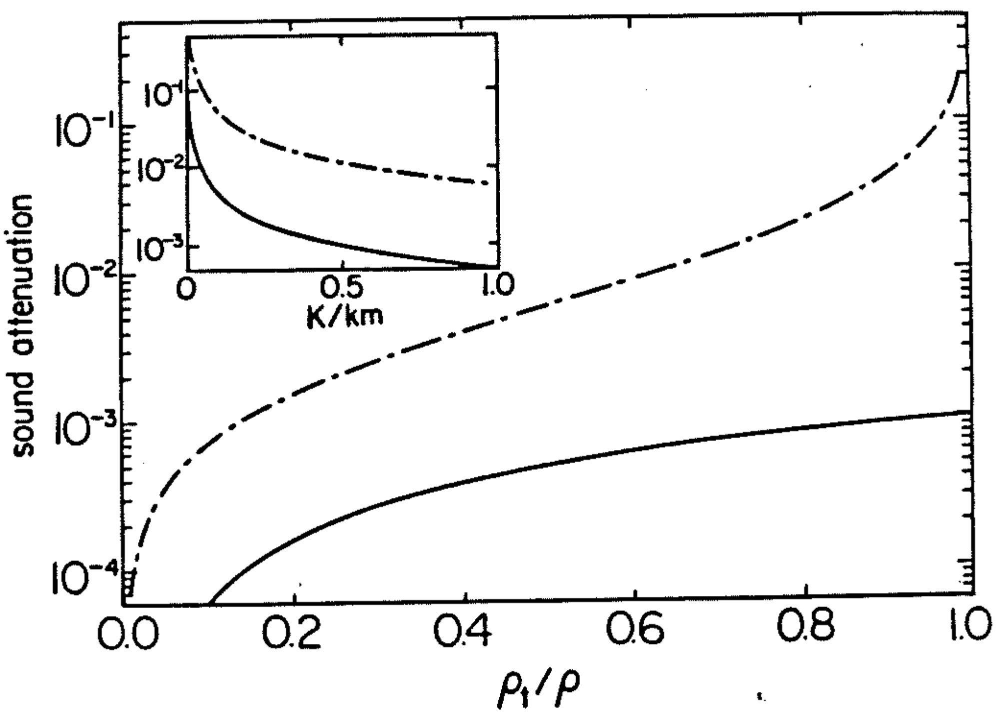

Figure 3.1 Attenuation per cycle of sound and second sound (dashed line) in wave turbulence.

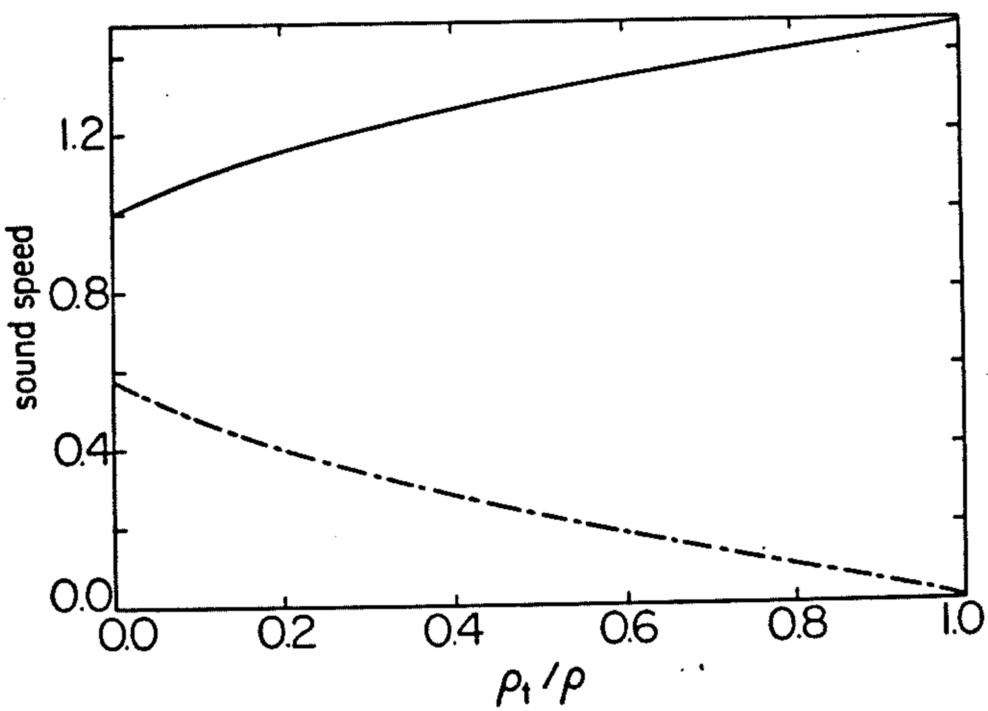

Figure 3.2 Speed of sound and turbulent second sound (dashed line), normalized to the speed of sound in the equilibrium fluid.

$$\frac{Q_o}{\rho c^2\,\Omega} < \frac{\rho_t}{\rho} = \frac{8\pi}{3}\gamma b\left(\frac{Q_o}{\rho c^2\,\omega_m G^2}\right)^{1/2} < 1 \qquad (3.14)$$

The entropy density of this wave is $k_B\int\gamma\theta d\vec{k}$. This integral must be cut off at the high end of the inertial region which is determined by the value of ω which yields equality in (3.13).

The general stationary off equilibrium solution to $I\{n\} = 0$ is $n_O(Q_o,T,\omega)$. For low ω this solution becomes (3.6) and for high ω it asymptotes to the equilibrium spectrum $n_{eq}(T,\omega) = n_o(O,T,\omega)$. If a local equilibrium ansatz like (3.7) is now applied to n_{eq} by assuming that T is slowly varying one is led to equations that are similar to (3.8,3.9) but have $Q_o = 0$. In this case the inertia of the noise is labelled ρ_n(instead of ρ_t) and is given by $B(n_{eq})/3c^2$. Second sound in the thermal distribution can occur only when $\rho_n<\rho$. In almost all systems n_{eq} is significantly large out to very high ω and unless some natural cutoff exists ρ_n is much grater than ρ (for equipartition $n_{eq} = k_BT/\omega$). For dielectrics (e.g. He^4) at sufficiently low T, Wien's law provides the cutoff. At ordinary T it therefore appears highly improbable that systems could have thermal second sound. On the other hand it is remarkable that the inertia ρ_t of the turbulent noise is convergent ($\gamma> 4$) and that a fully classical system at finite T should exhibit second sound if it is sufficiently far off equilibrium. In this case it is the variations in the phase space energy flux $Q = Q_o + \delta Q$ which lead to variations in $n_o(Q,T,\omega)$; Eq. (3.7) corresponds to $\delta Q/Q = 2\gamma(\theta+\hat{k}\cdot\vec{w}/c)$, and to the turbulent inertia

$$\rho_t\vec{w} = \int [\partial n_o(Q,T,\omega)/\partial Q]\delta Q\vec{k}d\vec{k}$$

The differentiation of n_o with respect to Q provides for convergence at high ω. Thermal noise is still present but its role is only to dress the background properties of the fluid (Eq. 3.1,3.2) relative to which the kinetic theory of the turbulent waves is developed.

The concept of second sound in plasmas has been addressed in the literature (Tsytovich, 1977 and references therein). These works fail to include the reaction of the background

(3.10,3.11) and the intrinsic damping due to the cascade. Thus their results are invalid at long wavelength. In the absence of background reaction there is no counterflow and therefore no second sound. The fact that the key inequality (3.14) plays no role in these references casts doubt on the relevance of those calculations to second sound. Furthermore none of these papers connect the second sound with the long-standing problem of anomalous transport. This situation is partly reminiscent of Peshkov's initially unsuccessful attempt to excite He^4 second sound with a pressure transducer (Peshkov, 1944).

While the above calculation predicts second sound in classical acoustic turbulence this phenomenon is general and thus it should also appear in other situations such as turbulent wind driven waves on the ocean and drift waves in a plasma. Regarding inhomogeneous plasmas we expect that via various instabilities a discharge injects energy into the drift wave spectrum at some low frequency ω_m. This energy then cascades, due to nonlinear drift wave interactions, to higher and higher frequencies until at some ω_∞ it is dissipated by transport processes. The physical means whereby energy is injected or dissipated are completely different and in order to have an extended inertial range they must be well separated in ω. The steady state drift wave spectrum corresponding to (3.6) would follow either from setting the drift wave energy flux Q = constant, or by balancing the sources (injection), sinks (dissipation) and drift wave collisions "I". It is not yet clear whether the observed spectrum (Brower et al., 1985) can be explained by leading order nonlinearity or whether higher order effects are essential; but the observation of a power law dependence in Brower et. al. gives some reason to believe that there may be an inertial range in the drift wave response. Fluctuations in this spectrum could then propagate as second sound and be an efficient vehicle for the escape of energy, thus working against confinement. Furthermore the second sound is itself nonlinear and can display a spectrum of harmonics. This is a possible explanation for the temperature-like fluctuations observed in an already wave turbulent plasma (Arunasalam et al., 1977).

The parallel between wave turbulence and superfluidity enables one to motivate an approximate expression for an effective diffusivity. We consider the response of the medium to a small variation in energy density described by a $\nabla\theta$. The system responds with an internal convection wherein the center of mass stays at rest (cf. 3.10)

$$\rho_0 \vec{v}_0 + \rho_t \vec{w} = 0 \tag{3.15}$$

and the spatial energy flow is (cf. 3.8)

$$\frac{1}{3} B \, (\vec{w} + \vec{v}_0) \tag{3.16}$$

Now in the steady state the dynamical equations (3.9) and (3.11) yield

$$\frac{B}{3} \nabla\theta + \frac{B}{3\rho} \beta \nabla\rho_0 + \frac{2\gamma \vec{w} Q_0}{3c^2} = \eta_t \nabla^2 (\vec{v}_0 + \vec{w}) \tag{3.17}$$

$$\frac{c^2}{\rho_0} \nabla\rho_0 + B \frac{\beta}{\rho} \nabla\theta = U \left(\frac{\beta}{\rho} \right)^2 \nabla\rho_0 \tag{3.18}$$

where, in parallel with the superfluid case we have included the effects of viscosity η_t in the equation of motion of the turbulent waves. This term would follow from our basic kinetic equation in a Chapman-Enskog type expansion. From (3.17) and (3.18) we find

$$B \, \nabla\theta + \frac{2\gamma}{c^2} Q_0 \vec{w} = 3\eta_t \nabla^2 (\vec{v}_0 + \vec{w}) \tag{3.19}$$

where terms of order ρ_t/ρ have been assumed small. In the case in which the irreversible term in Q_0 dominates; the energy flow is

$$\frac{1}{3} B(\vec{w} + \vec{v}_0) = -(1 - \frac{\rho_t}{\rho})\frac{c^2 B^2}{6\gamma Q_0} \nabla\theta \tag{3.20}$$

which in view of (3.8) implies a wave turbulent diffusivity

$$D_t = (1 - \frac{\rho_t}{\rho})\frac{c^2 B^2}{6\gamma Q_0} \cong (1 - \frac{\rho_t}{\rho})\frac{c^2}{\omega_m} (\frac{\rho_t}{\rho})^{-1} \tag{3.21}$$

where we have used (3.12).

In the case where irreversibility is dominated by η_t the solution to (3.19) follows the He^4 case directly and yields

$$D_t = \frac{a^2 c^2 \rho_t}{12\eta_t} = \frac{a^2 c^2 \rho_t}{12 D_C \rho_0} \tag{3.22}$$

where "a" is determined by some geometrical confining length. By D_c we mean a classical diffusivity i.e. η_t/ρ_0.

Background reaction is a key to the realization of this transport due to "internal convection" (London, 1954; Putterman, 1974) where the center of mass stays at rest. The appearance of a characteristic geometric size "a" in the diffusivity (3.22) indicates that a hyperbolic response has been forced into a parabolic framework (Putterman, 1974). For the near equilibrium superfluid $D_C = \eta_n/\rho_o$ (η_n is the normal fluid viscosity) and $\int e(\omega)d\omega = \rho_n c^2$ is roughly the thermal energy density. For the wave turbulent plasma $\int e(\omega)d\omega \cong \rho_t c^2$ is determined by the drift wave fluctuations. The energy content of drift waves appears to be an involved issue (Horton, 1984). Rather than attempt a detailed picture we will scale the parameters by setting $U \cong \langle(\delta\rho)^2\rangle c_D^2/\rho$ which is surely an underestimate. Here c_D is the drift wave velocity and $\langle\,\rangle$ indicates an average. The classical diffusivity of the plasma is determined by the magnetic field limited random walk so $D_c = 6\omega_p^4/\omega_c^2 N V_{th}$ where ω_p is the plasma frequency, ω_c the electron cyclotron frequency, N the plasma number density and V_{th}

the thermal velocity. Taking B=20kG $N=2\times10^{13}\,cm^{-3}$, $k_B T = 400ev$ yields $D_c \sim 8cm^2/sec$. For these parameters the drift wave fluctuations were measured (Mazzucato, 1976, 1982) as $<(\delta\rho/\rho_o)^2> = 2.5\times10^{-5}$ with "a" = 16 cm. and $c_D \sim 10^5$ cm/sec. This yields for (3.22) $D_t \sim 6.\times10^5\ cm^2/sec$ which is about a factor of 10 higher than the experimentally determined anomalous diffusivity corresponding to a confinement time of 5 ms. Unlike parabolic estimates of transport which yield a lower bound, hyperbolic estimates normally yield an upper bound. Since the injection frequency ω_m will scale to c_D/a the diffusivity determined by the cascade (3.21) appears for these numbers to be about the same as (3.22).

In some experiments (Grieger et al., 1985) it is found that there is a drop in confinement time as density is increased. We think it worthwhile to interpret these effects in terms of the inequalities (3.13, 3.14) required for turbulent second sound.

Finally we must emphasize that as the energy content of He^4 is increased by increasing the temperature, the diffusivity increases at first and then precipitously falls to zero at T_λ. In parallel with this fact we propose that as the wave turbulence in a plasma is increased a point will be reached where $\rho_t \to \rho_0$ and second sound will be localized to diffusion and the confinement time will rapidly rise. In order to predict the parameters at which controlled fusion will suddenly be made possible requires detailed experiments on the power spectrum of drift wave turbulence for many regimes as well as an elucidation of the energy content of drift waves of given amplitude.

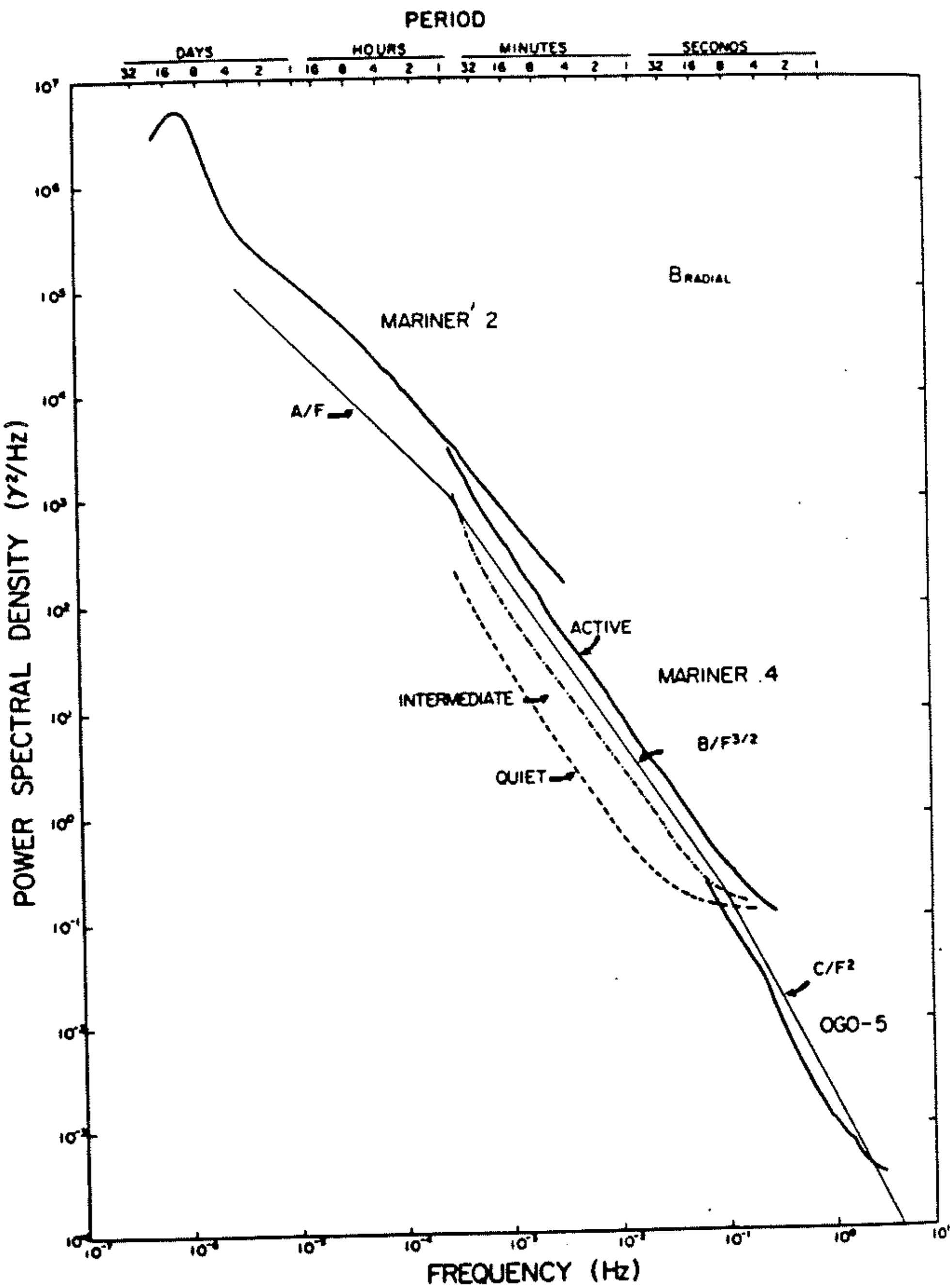

Figure 4.1 Composite spectrum of radial component of the interplanetary magnetic field.

Sun. Superimposed on the large scale field structure are numerous small scale variations which result from waves and discontinuities in the solar wind (Burlaga, 1972).

No clear picture nor a unique interpretation of the variable slopes of the spectrum for solar wind has yet emerged. The spectra vary with solar wind conditions and apparently represents fluctuations in Alfvén waves. These are incompressible excitations of the fluid with a small amplitude dispersion law $\omega = c_A k$, where $c_A = B/(4\pi m_p n)$ is the Alfvén phase velocity with B the magnetic field, m_p the protons mass, and n the density of protons. Typical values of the parameters at 1AU are n=6 protons/cm^3, and B=6γ, so that $c_A \sim 50$ km/sec. For weakly nonlinear Alfvén waves, equipartition of potential energy of the field fluctuations and kinetic energy of the flow occurs. So a measurement of the magnetic field fluctuations could correspond to an Alfvén waves spectrum.

A possible interpretation of the spectra in terms of waves is to consider the solar wind as a turbulent medium. One could argue that the power spectra of variations in the magnetic field might be due to MHD turbulence where the two characteristic lengths of the turbulent spectrum are the size of the turbulent region itself and, at the other extreme, the proton gyroradius (which might be the characteristic length for dissipative processes). The structural similarities of the MHD equations and the hydrodynamic equations might lead one to consider adapt the wave turbulence picture just developed in the previous chapters. In fact, Iroshnikov (1963) and Kraichnan (1965) incompressible MHD turbulence theory consists of a superposition of weakly nonlinear Alfvén waves and the spectrum they obtained is proportional to $f^{-3/2}$ in the inertial range of wave numbers. This could explain the central piece of the spectrum taken by Mariner 4 (Siscoe et al., 1968). As for the low frequency part of the spectrum, it is assumed that the Mariner 2 data were obtained during an unusually disturbed period of time (Russell, 1972). This might indicate that the energy input into the waves was so large that the spectral energy density accumulated at 1/f noise.

A wave turbulence description for a system of interacting waves has to satisfy the basic inequalities (1.11). For an ambient magnetic field at 1AU of approximately 6γ, the corresponding wave Mach numbers

$$M = \frac{1}{6\gamma}\left(\int B^2(\omega)d\omega\right)^{1/2}$$

range from 10^{-1}, in the 1/f region, to 10^{-2} in the $f^{-3/2}$ region. As for the dissipation mechanism for Alfvén waves it could very well due to the statistical acceleration of particles at the cyclotron frequency which is of the order of 10^{-1} Hz for protons at 1AU. For this time scale, a nonlinear kinetic description of the waves can be written as

$$\frac{\partial e(\omega)}{\partial t} + \frac{\partial Q}{\partial \omega} = -\gamma(\omega)e(\omega) \qquad (4.1)$$

Here $\partial Q/\partial\omega = -\omega k^2 (\partial k/\partial\omega) I\{n\}$, and for the damping process in question $\gamma(\omega)$ = constant. Stationarity requires that

$$\frac{\partial Q}{\partial \omega} = -\gamma(\omega)e(\omega) \qquad (4.2)$$

A solution $e(\omega) \sim \omega^{\alpha}$ and the three wave process implied by the nature of the dispersion law yield for α the relation

$$2\alpha + 2 = \alpha \qquad (4.3)$$

and thus $\alpha = -2$ in agreement with the observed power law spectrum for the frequency range 10^{-1} Hz to 1Hz of Fig. 4.1

The explanation of the spectra for solar wind in terms of wave turbulence although plausible, will require further theoretical and experimental study. A complete theoretical where B_0 is the ambient magnetic field and $<B^2>$ is the integrated power spectral density. In parallel with the acoustic turbulence we may write the spectral density as

$$B^2(\omega) \cong B_0^2 \left(\frac{Q_0}{B_o^2}\right)^{1/2} f^{-3/2}$$

An inequality like (3.14) would read

$$\frac{Q_0}{B_o^2\,\Omega} < \frac{<B^2>}{B_o^2} \cong \left(\frac{Q_0}{B_o^2}\right)^{1/2} < 1$$

Using the parameters in figure 4.1 for this frequency range we find a value $(Q_0/B_0^2) \cong 10^{-8}\ \mathrm{sec}^{-1}$, for which $<B^2>/B_0^2 \cong 10^{-2} < 1$. It is in principle possible to have turbulent second sound in weak wave turbulence magnetic field fluctuations in the solar wind.

In conclusion, wave turbulence appears to be relevant to a broad spectrum of natural phenomena. Furthermore, wave turbulence is a more tractable theoretical subject at this time because of the existence of a close kinetic theory which presumably describes the approach to, and fluctuations about, this state. Future theoretical work should focus on higher order correlation functions e.g. $<n_1 n_2> - <n_1><n_2>$. Especially we also look forward to the first controlled laboratory experiments on wave turbulence.

APPENDIX

In this appendix we give the general guide for the formulation of Eq. (2.3). Starting with equation (2.1)

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2 \nabla^2 \delta\rho = \nabla^2 \frac{G_{m+1} c^2}{\rho_o^{m-1}} (\delta\rho)^m$$

we fourier transform with respect to the coordinates

$$\frac{\partial^2 \rho_k}{\partial t^2} + \omega_k^2 \rho_k = \frac{G_{m+1} c^2}{\rho_o^{m-1}} k^2 \int d\tau_k \, \delta(\vec{k}_1 + \vec{k}_2 + \cdots + \vec{k}_m - \vec{k}) \rho_{k_1} \cdots \rho_{k_m}$$

where

$$\delta\rho = \int \rho_k e^{i\vec{k}\cdot\vec{x}} d\vec{k} \quad ; \quad \rho_{-k} = \rho_k^*$$

$$\omega_k = ck$$

We now introduce canonical variables a_k and a_k^* through

$$\rho_k = \frac{k\rho_o^{1/2}}{(2\omega_k)^{1/2}} \left(a_k + a_{-k}^* \right)$$

from which we obtain the equation

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i\int d\tau_k \; V_{kk_1k_2\cdots k_m} \; \delta(\vec{k}_1+\vec{k}_2+\cdots+\vec{k}_m - \vec{k}) \times$$

$$\times \, (a_{k_1} + a^*_{-k_1})\cdots(a_{k_m} + a^*_{-k_m}) \qquad \text{(A.1)}$$

where

$$V_{kk_1k_2\cdots k_m} = \left(\frac{G^2_{m+1}\, c^2}{2^{m+1}\,(\rho c)^{m-1}} \right)^{1/2} (kk_1\cdots k_m)^{1/2}$$

We now express (A.1) in the compact form

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i\sum_{\{s\}} \int d\tau_k \; V_{\vec{k}\,|s_i\vec{k}_i} \; \delta\Big(\vec{k} - \sum_{i=1}^{m} s_i k_i \Big) \; a^{s_1}_{k_1}\cdots a^{s_m}_{k_m} \qquad \text{(A.2)}$$

where $a^{+1}_k = a_k$, $a^{-1}_k = a^*_k$. We now consider the evolution for the square correlation function

$$\langle a^s_k a^{s'}_{k'} \rangle = n_k \delta(\vec{k} - \vec{k}')\delta_{s,-s'}$$

for which we multiply (A.2) by $a^*_{k'}$ and add it to the complex conjugate equation

$$\frac{\partial n_k}{\partial t}\,\delta(\vec{k} - \vec{k}') = -i\sum_{\{s\}} \int d\tau_k \Big\{ V_{\vec{k}\,|s_i\vec{k}_i} \; \delta\Big(\vec{k} - \sum_{i=1}^{m} s_i\vec{k}_i \Big) \; \langle a^{s_1}_{k_1}\cdots a^{s_m}_{k_m} \, a^*_{k'} \rangle$$

$$- V^*_{\vec{k}'\,|s_i\vec{k}_i} \; \delta\Big(\vec{k}' - \sum_{i=1}^{m} s_i\vec{k}_i \Big) \; \langle a^{-s_1}_{k_1}\cdots a^{-s_m}_{k_m} a_k \rangle \qquad \text{(A.3)}$$

The mean value $\langle a^{s_1}_{k_1}\cdots a^{s_p}_{k_p} \rangle$ decomposes into the sum of products of spectral cumulants

$$\delta\Big(\sum_{i=1}^{p} s_i\vec{k}_i\Big)\, I^{s_1 s_2 \cdots s_p}_{k_1 k_2 \cdots k_p} + \sum_{r_1 r_2 \cdots r_p} \delta\Big(\sum_{i=1}^{r_1} s_i\vec{k}_i\Big)\, \delta\Big(\sum_{r_1}^{r_2} s_i\vec{k}_i\Big) \cdots I^{s_1 \cdots s_{r_1} \cdots}_{k_1 \cdots k_{r_1}}$$

The sum term reduces the available phase space giving zero contribution to the evolution of the energy spectrum. Rather, that term introduces small nonlinear modifications to the phase velocity of the waves. Equation (A.3) becomes

$$\frac{\partial n_k}{\partial t}\,\delta(\vec{k}-\vec{k}') = -2\delta(\vec{k}-\vec{k}')\,\mathrm{Im}\sum_{\{s\}} \int d\tau_k\, \delta\Big(\vec{k}-\sum_{i=1}^{m} s_i\vec{k}_i\Big) \times$$

$$\times\, V_{\vec{k}\,|\,s_i\vec{k}_i}\, I^{-1\, s_1 s_2 \cdots s_p}_{k k_1 k_2 \cdots k_p} \qquad \text{(A.4)}$$

At this point, motivated by the asymptotic analysis of Benney and Saffman (1966) we use the cumulant discard approach, which states that correlations higher than the second develop at a very low rate. Odd correlation functions are assumed to decrease rapidly with an increase in order, while the even ones are factorized with improved accuracy via square correlations at t=0. The time variations of $\langle a^{s_1}_{k_1} \cdots a^{s_p}_{k_p} \rangle$ are slow and can be neglected. From (A.2) we find that

$$-i\Big(\omega_k - \sum_{i'=1}^{m} s_{i'}\omega_{k_{i'}}\Big)\, \langle a^{-1}_{k}\, a^{s_{1'}}_{k_{1'}} \cdots a^{s_{m'}}_{k_{m'}} \rangle =$$

$$= i\sum_{\{s\}} \int d\tau_k \Big\{ V_{\vec{k}\,|\,s_i\vec{k}_i}\, \delta\Big(\vec{k}-\sum_{i=1}^{m} s_i\vec{k}_i\Big) \langle a^{-s_1}_{k_1} \cdots a^{-s_m}_{k_m}\, a^{s_{1'}}_{k_{1'}} \cdots a^{s_{m'}}_{k_{m'}} \rangle$$

$$-\sum_{i'=1}^{m} s_{i'}\, V^{s_{i'}}_{\vec{k}_{i'}\,|\,s_i\vec{k}_i}\, \delta\Big(\vec{k}_i-\sum_{i=1}^{m} s_i\vec{k}_i\Big) \langle a^{-1}_{k}\, a^{s_{i'}s_1}_{k_1} \cdots a^{s_{i'}s_m}_{k_m}\, a^{s_{1'}}_{k_{1'}} \cdots \not{a}^{s_{i'}}_{k_{i'}} \cdots a^{s_{m'}}_{k_{m'}} \rangle$$

where the slashed quantity is to be deleted from the correlator. The RHS of the above expresion gives, by use of the symetries of $V_{\vec{k}\,|\,s_i\vec{k}_i}$

$$\text{RHS} = 2i\Big\{ V^{-1}_{\vec{k}\,|\,s_i'\,\vec{k}_i'}\, \delta\Big(\vec{k} - \sum_{i=1}^{m} s_i'\vec{k}_i' \Big)\, n_{k_1'}\, n_{k_2'} \cdots n_{k_m'}$$

$$- \sum_{i'=1}^{m} s_{i'}\, V^{s_{i'}}_{\vec{k}\,|\,s_i'\,\vec{k}_i'}\, \delta\Big(\vec{k} - \sum_{i=1}^{m} s_i'\vec{k}_i' \Big)\, n_k n_{k_1'}\, n_{k_2'} \cdots \not{n}_{k_i'} \cdots n_{k_m'}$$

+ other terms that reduce the available phase space.

Finally, using the identity $\mathrm{Im}(x+i\varepsilon) = \pi\delta(x)$ we arrive at (2.3) where $H(\vec{k}, s_i\vec{k}_i) = |V_{\vec{k}\,|\,s_i\vec{k}_i}|^2$.

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